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By CLAUDIA KLÜPPELBERG, GABRIEL KUHN

Center for Mathematical Sciences, Munich University of Technology, D-85747 Garching, Germany.

cklu@ma.tum.de gabriel@ma.tum.de

and LIANG PENG

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, USA.

peng@math.gatech.edu

Summary

Recently there has been an increasing interest in applying elliptical distributions to risk management. Under weak conditions, Hult and Lindskog (2002) showed that a random vector with an elliptical distribution is in the domain of attraction of a multivariate extreme value distribution. In this chapter we study two estimators for the tail dependence function, which are based on extreme value theory and the structure of an elliptical distribution, respectively. After deriving second order regular variation estimates and proving asymptotic normality for both estimators, we show that the estimator based on the structure of an elliptical distribution is better than that based on extreme value theory in terms of both asymptotic variance and optimal asymptotic mean squared error. Our theoretical results are confirmed by a simulation study.

Keywords: asymptotic normality, elliptical distribution, regular variation, tail copula, tail dependence function.

1 Introduction

Let $(X, Y), (X_1, Y_1), (X_2, Y_2), \cdots$ be independent random vectors with common distribution function F and continuous marginals F_X and F_Y. Define the *tail copula*

$$\lambda(x, y) \quad := \quad \lim_{t \to 0} \frac{1}{t} P\left(1 - F_X(X) \le tx,\, 1 - F_Y(Y) \le ty\right)$$

for $x, y \geq 0$. Then $\lambda(1,1)$ is called the *upper tail dependence coefficient*, see e.g. Joe (1997) and, by Huang (1992), $l(x, y) := x + y - \lambda(x, y)$ is called the *tail dependence function*. Assuming that (X, Y) is in the domain of attraction of a bivariate extreme value distribution, there exist several estimators for estimating the tail dependence function $l(x, y)$, see Huang (1992), Einmahl, de Haan and Huang (1993) and de Haan and Resnick (1993). The optimal rate of convergence for estimating $l(x, y)$ is given by Drees and Huang (1998). An alternative method for estimating $l(x, y)$ is via estimating the spectral measure, see Einmahl, de Haan and Sinha (1997) and Einmahl, de Haan and Piterbarg (2001). For modeling dependence of extremes parametrically, we refer to Tawn (1988) and Ledford and Tawn (1997).

Triggered by financial risk management problems we observe an increasing interest in elliptical distributions as natural extensions of the normal family allowing for the modeling of heavy tails and extreme dependence. The vector (X, Y) is *elliptically distributed*, if

$$(X, Y)^T = \boldsymbol{\mu} + G\mathbf{A}\boldsymbol{U}^{(2)}, \tag{1.1}$$

where $\boldsymbol{\mu} = (\mu_X, \mu_Y)^{\mathrm{T}}$, $G > 0$ is a random variable, called *generating variable*, $\mathbf{A} \in \mathbb{R}^{2\times 2}$ is a deterministic matrix with

$$\mathbf{A}\mathbf{A}^T =: \boldsymbol{\Sigma} := \begin{pmatrix} \sigma^2 & \rho\sigma\upsilon \\ \rho\sigma\upsilon & \upsilon^2 \end{pmatrix}$$

and $\mathrm{rank}(\boldsymbol{\Sigma}) = 2$, $\boldsymbol{U}^{(2)}$ is a 2-dimensional random vector uniformly distributed on the unit hyper-sphere $\mathcal{S}_2 := \{\boldsymbol{z} \in \mathbb{R}^2 : \|\boldsymbol{z}\| = 1\}$, and $\boldsymbol{U}^{(2)}$ is independent of G.

Note that ρ is termed as the linear correlation coefficient of $\boldsymbol{\Sigma}$. Under certain conditions, Hult and Lindskog (2002) showed that regular variation of $1-G$ with index $\alpha > 0$, i.e., $\lim_{t\to\infty}(1 - G(tx))/(1 - G(t)) = x^{-\alpha}$ for all $x > 0$, (notation: $1 - G \in \mathcal{R}_{-\alpha}$) is equivalent to regular variation of (X, Y) with the same index $\alpha > 0$ (see Resnick (1987) for the definition of multivariate regular variation). Moreover, if $1 - G \in \mathcal{R}_{-\alpha}$, then

$$\lambda(1,1) = \left(\int_{(\pi/2-\arcsin\rho)/2}^{\pi/2} (\cos\phi)^\alpha \,\mathrm{d}\phi\right) \Big/ \left(\int_0^{\pi/2} (\cos\phi)^\alpha \,\mathrm{d}\phi\right). \tag{1.2}$$

Here we are interested in estimating the dependence function $\lambda(x, y)$ by assuming that $1 - G \in RV_{-\alpha}$ for some $\alpha > 0$. Since $1 - G \in RV_{-\alpha}$ implies that

(X, Y) is in the domain of attraction of an extreme value distribution, a naive estimator is to apply Huang's estimator by ignoring the structure of the elliptical distribution, i.e.,

$$\widehat{\lambda}^{\text{Hu}}_{k_{\text{Hu}},n}(x,y) \quad := \quad \frac{1}{k_{\text{Hu}}} \sum_{i=1}^{n} \boldsymbol{I}\left(X_i \geq X_{(n-\lfloor xk_{\text{Hu}}\rfloor,n)},\ Y_i \geq Y_{(n-\lfloor yk_{\text{Hu}}\rfloor,n)}\right), \tag{1.3}$$

where $X_{(1,n)} \leq \cdots \leq X_{(n,n)}$ and $Y_{(1,n)} \leq \cdots \leq Y_{(n,n)}$ denote the order statistics of $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$, respectively, $k_{\text{Hu}} = k_{\text{Hu}}(n) \stackrel{n\to\infty}{\longrightarrow} \infty$ and $k_{\text{Hu}}/n \stackrel{n\to\infty}{\longrightarrow} 0$. The same estimator has been analysed by Schmidt and Stadtmüller (2005); see their equation (4.14). The aim of this chapter is two-fold. Firstly, we suggest a new estimator, which exploits the structure of an elliptical distribution similar to (1.2). Secondly, we aim at determining the optimal number of order statistics to be used in both estimators. The choice will be based on the asymptotic mean squared error of the estimators.

Our chapter is organized as follows. We first derive an expression for $\lambda(x, y)$, which generalizes equation (1.2), and then construct a new estimator for $\lambda(x, y)$ via this expression; see section 2 for details. After deriving the second order behavior for elliptical distributions and the limiting distributions of both estimators in section 2, we show that the new estimator is better than the naive empirical estimator from Huang in terms of both asymptotic variance and optimal asymptotic mean squared error in section 3. More importantly, the optimal choice of the sample fraction for the new estimator is the same as that for Hill's estimator (Hill (1975)). That is, all data-driven methods for choosing the optimal sample fraction for Hill's estimator can be applied to our new estimator directly. A simulation study is provided in section 3 as well. All proofs are summarized in section 4.

2 Methodology and Main Results

The following theorem gives an expression for $\lambda(x, y)$, which will be employed to construct an estimator.

Theorem 2.1. *Suppose (X, Y) defined in (1.1) holds with $\sigma > 0$, $\upsilon > 0$, $|\rho| < 1$ and $1 - G \in \mathcal{R}_{-\alpha}$ for some $\alpha > 0$. Further, define*

$$g(t) \quad := \quad \arctan\left((t-\rho)/\sqrt{1-\rho^2}\right) \in [-\arcsin\rho, \pi/2], \quad t \in \mathbb{R}.$$

Then

$$\lambda(x,y) = \left(\int_{-\pi/2}^{\pi/2} (\cos\phi)^\alpha \, \mathrm{d}\phi\right)^{-1} \left(\int_{g\left((x/y)^{1/\alpha}\right)}^{\pi/2} x(\cos\phi)^\alpha \, \mathrm{d}\phi \right.$$
$$\left. + \int_{-\arcsin\rho}^{g\left((x/y)^{1/\alpha}\right)} y\left(\sin(\phi+\arcsin\rho)\right)^\alpha \, \mathrm{d}\phi\right). \qquad \square$$

In order to derive the asymptotic normality of $\widehat{\lambda}^{\mathrm{Hu}}_{k_{\mathrm{Hu}},n}(x,y)$, it is known that a second order condition is needed. Here we seek the relation of the second order behavior among the tail copula $\lambda(x,y)$, $\sqrt{X^2+Y^2}$ and G; see the next two theorems for details.

In the setting of (1.1) assume that there exists $A(t) \to 0$ such that for all $x > 0$ and some $\beta \le 0$

$$\lim_{t\to\infty} \frac{P(G \ge tx)/P(G \ge t) - x^{-\alpha}}{A(t)} = x^{-\alpha}\frac{x^\beta - 1}{\beta}, \tag{2.1}$$

where $\beta \le 0$ is called a *second order regular variation parameter*, see de Haan and Stadtmüller (1996). Additionally, we assume

$$\lim_{t\to\infty} t^2 A(t) =: a \in [-\infty, \infty]. \tag{2.2}$$

Since $A \in \mathcal{R}_\beta$, $a = 0$ for $\beta < -2$ and $|a| = \infty$ for $\beta \in (-2, 0]$.

The following two theorems derive the corresponding second order condition for $\sqrt{X^2+Y^2}$ and the tail copula $\lambda(x,y)$.

Theorem 2.2. *Assume that the conditions of Theorem 2.1, (2.1) and (2.2) hold. Further, define*

$$d_1(\phi) = \sigma^2\cos^2\phi + \upsilon^2\sin^2(\phi+\arcsin\rho),$$
$$d_2(\phi) = \mu_X\sigma\cos\phi + \mu_Y\upsilon\sin(\phi+\arcsin\rho).$$

Then, for all $x > 0$,

$$\lim_{t\to\infty} \frac{P\{\sqrt{X^2+Y^2} \ge tx\}/P\{\sqrt{X^2+Y^2} \ge t\} - x^{-\alpha}}{t^{-2} + |A(t)|}$$
$$= x^{-\alpha}\left(\int_{-\pi}^{\pi} (d_1(\phi))^{\alpha/2}\,\mathrm{d}\phi\right)^{-1}\left\{\frac{a}{1+|a|}\frac{x^\beta - 1}{\beta}\left(\int_{-\pi}^{\pi} (d_1(\phi))^{(\alpha-\beta)/2}\,\mathrm{d}\phi\right)\right.$$
$$+ \frac{1}{1+|a|}\frac{\alpha}{2}\left(x^{-2}-1\right)\int_{-\pi}^{\pi} (d_1(\phi))^{\alpha/2-1} \times$$
$$\left. \times \left[\alpha(d_2(\phi))^2 + d_1(\phi)(\mu_X^2+\mu_Y^2)\right]\mathrm{d}\phi\right\}. \tag{2.3}$$

Also, for all $x > 0$ *and* $V(x) := \inf\{y : P(\sqrt{X^2+Y^2} > y) \le x^{-1}\}$,

$$\begin{aligned}
&\lim_{t\to\infty} \frac{V(tx)/V(t) - x^{1/\alpha}}{(F_Y^{\leftarrow}(1-t^{-1}))^{-2} + |A(F_Y^{\leftarrow}(1-t^{-1}))|} \\
&= x^{1/\alpha}\left(\int_{-\pi}^{\pi}(d_1(\phi))^{\alpha/2}\,\mathrm{d}\phi\right)^{-1}\left(\frac{\int_{-\pi}^{\pi}(d_1(\phi))^{\alpha/2}\,d\phi}{\upsilon^{\alpha}\int_0^{\pi}(\sin\phi)^{\alpha}\,d\phi}\right)^{-(2\wedge|\beta|)/\alpha} \times \\
&\quad \times \left\{\frac{a}{1+|a|}\,\frac{x^{\beta/\alpha}-1}{\alpha\beta}\left(\int_{-\pi}^{\pi}(d_1(\phi))^{(\alpha-\beta)/2}\,\mathrm{d}\phi\right)\right. \\
&\quad + \frac{1}{1+|a|}\,\frac{1}{2}\left(x^{-2/\alpha}-1\right)\int_{-\pi}^{\pi}(d_1(\phi))^{\alpha/2-1}\times \\
&\qquad \left.\times\left[\alpha(d_2(\phi))^2 + d_1(\phi)(\mu_X^2+\mu_Y^2)\right]\mathrm{d}\phi\right\} =: x^{1/\alpha}\mathcal{B}_{(2.4)}(x). \qquad (2.4)
\end{aligned}$$

Especially, when $\mu_X = \mu_Y = 0$, *we have for all* $x > 0$

$$\begin{aligned}
\lim_{t\to\infty}\frac{V(tx)/V(t) - x^{1/\alpha}}{A(F_Y^{\leftarrow}(1-t^{-1}))} &= x^{1/\alpha}\frac{x^{\beta/\alpha}-1}{\upsilon^{\beta}\alpha\beta}\left(\int_{-\pi}^{\pi}(d_1(\phi))^{\alpha/2}\,\mathrm{d}\phi\right)^{-1}\times \\
&\quad \times\left(\int_{-\pi}^{\pi}(d_1(\phi))^{(\alpha-\beta)/2}\,\mathrm{d}\phi\right)\left(\frac{\int_{-\pi}^{\pi}(d_1(\phi))^{\alpha/2}\,d\phi}{\int_0^{\pi}(\sin\phi)^{\alpha}\,d\phi}\right)^{\beta/\alpha} \\
&=: x^{1/\alpha}\mathcal{B}_{(2.5)}(x). \qquad (2.5)
\end{aligned}$$

□

Theorem 2.3. *Assume that the conditions of Theorem 2.1 and (2.1) hold. Further, define*

$$\begin{aligned}
\mathcal{S}_2^+ &:= \left\{\boldsymbol{z}\in\mathbb{R}^2 : \boldsymbol{z} \ge \mathbf{0} \text{ and } \|\boldsymbol{z}\| = 1\right\} \quad \text{and} \\
\mathcal{B}_{(2.6)}(x) &:= -x\frac{x^{-\beta/\alpha}-1}{\beta}\left(\int_0^{\pi}(\sin\phi)^{\alpha}\mathrm{d}\,\phi\right)^{-1}\left(\int_0^{\pi}(\sin\phi)^{\alpha-\beta}\,\mathrm{d}\phi\right) \qquad (2.6)
\end{aligned}$$

Then,

$$
\begin{aligned}
&\lim_{t\to 0}\frac{t^{-1}P\left(F_X(X)\geq 1-tx,\ F_Y(Y)\geq 1-ty\right)-\lambda(x,y)}{A\left(F_Y^{\leftarrow}(1-t)\right)}\\
&= \quad \upsilon^{-\beta}\left\{\frac{x}{\beta}\int_{g((x/y)^{1/\alpha})}^{\pi/2}\left[x^{-\beta/\alpha}(\cos\phi)^{\alpha-\beta}-(\cos\phi)^{\alpha}\right]\mathrm{d}\phi\right.\\
&\quad +\frac{y}{\beta}\int_{-\arcsin\rho}^{g((x/y)^{1/\alpha})}\left[y^{-\beta/\alpha}(\sin(\phi+\arcsin\rho))^{\alpha-\beta}-(\sin(\phi+\arcsin\rho))^{\alpha}\right]\mathrm{d}\phi\\
&\quad +\mathcal{B}_{(2.6)}(x)\int_{g((x/y)^{1/\alpha})}^{\pi/2}(\cos\phi)^{\alpha}\,d\phi\\
&\quad +\mathcal{B}_{(2.6)}(y)\int_{-\arcsin\rho}^{g((x/y)^{1/\alpha})}(\sin(\phi+\arcsin\rho))^{\alpha}\,d\phi\\
&\quad \left.-\lambda(x,y)\frac{1}{\beta}\int_{-\pi/2}^{\pi/2}(\cos\phi)^{\alpha}\left((\cos\phi)^{-\beta}-1\right)\mathrm{d}\phi\right\}\left(\int_{-\pi/2}^{\pi/2}(\cos\phi)^{\alpha}\,\mathrm{d}\phi\right)^{-1}\\
&=: \quad \mathcal{B}_{(2.7)}(x,y)
\end{aligned}
\tag{2.7}
$$

holds for all $x, y \geq 0$ and uniformly on $\mathcal{S}_2^+$. □

Now we are ready to define our new estimator. Put $Z_i = \sqrt{X_i^2+Y_i^2}$ for $i = 1, \dots, n$ and let $Z_{(1,n)} \leq \cdots \leq Z_{(n,n)}$ denote their order statistics. First we estimate the index α by Hill's estimator, which is defined as

$$
\widehat{\alpha}^{\mathrm{H}}_{k_{\mathrm{El}},n} \quad := \quad \left(\frac{1}{k_{\mathrm{El}}}\sum_{i=1}^{k_{\mathrm{El}}}\log Z_{(n-i+1,n)}-\log Z_{(n-k_{\mathrm{El}},n)}\right)^{-1},
$$

where $k_{\mathrm{El}} = k_{\mathrm{El}}(n) \to \infty$ and $k_{\mathrm{El}}/n \to 0$ as $n \to \infty$. Now let (X, Y) and $(\widetilde{X}, \widetilde{Y})$ be iid with elliptical distribution. Then, it follows from Hult and Lindskog (2002) that $\tau = (2/\pi)\arcsin\rho$, where τ is called *Kendall's tau* and defined by

$$
\tau \quad := \quad P\left(\left(X-\widetilde{X}\right)\left(Y-\widetilde{Y}\right)>0\right)-P\left(\left(X-\widetilde{X}\right)\left(Y-\widetilde{Y}\right)<0\right).
$$

As usual, we estimate Kendall's tau by

$$
\widehat{\tau}_n \quad := \quad \frac{2}{n(n-1)}\sum_{1\leq i,j\leq n}\operatorname{sign}\left((X_i-X_j)(Y_i-Y_j)\right),
$$

which results in estimating ρ by

$$\widehat{\rho}_n = \sin\left(\frac{\pi}{2}\widehat{\tau}_n\right).$$

Hence, we can estimate $\lambda(x,y)$ by replacing ρ and α in Theorem 2.1 by $\widehat{\rho}_n$ and $\widehat{\alpha}^{\mathrm{H}}_{k_{\mathrm{El}},n}$, respectively. Let us denote this estimator by

$$\widehat{\lambda}^{\mathrm{El}}_{k_{\mathrm{El}},n}(x,y). \tag{2.8}$$

We remark that $\widehat{\lambda}^{\mathrm{El}}_{k_{\mathrm{El}},n}(1,1)$ was mentioned by Schmidt (2003), but without further study. The following theorem shows the asymptotic normalities of $\widehat{\lambda}^{\mathrm{Hu}}_{k_{\mathrm{Hu}},n}(x,y)$ and $\widehat{\lambda}^{\mathrm{El}}_{k_{\mathrm{El}},n}(x,y)$, which allows us to compare these two estimators theoretically.

Theorem 2.4. *Assume that the conditions of Theorem 2.1 and (2.1) hold. Suppose* $k_{\mathrm{Hu}} = k_{\mathrm{Hu}}(n) \overset{n\to\infty}{\longrightarrow} \infty$, $k_{\mathrm{Hu}}/n \overset{n\to\infty}{\longrightarrow} 0$ *and*

$$\sqrt{k_{\mathrm{Hu}}}A(F_Y^{\leftarrow}(1-k_{\mathrm{Hu}}/n)) \overset{n\to\infty}{\longrightarrow}: \mathcal{K}_{\mathrm{Hu}},$$

for $|\mathcal{K}_{\mathrm{Hu}}| < \infty$. *Then, as* $n \to \infty$,

$$\begin{aligned}&\sup_{0\le x,y\le T}\left|\sqrt{k_{\mathrm{Hu}}}\left(\widehat{\lambda}^{\mathrm{Hu}}_{k_{\mathrm{Hu}},n}(x,y)-\lambda(x,y)\right)-\mathcal{K}_{\mathrm{Hu}}\mathcal{B}_{(2.7)}(x,y)-B(x,y)\right| \\ &= o_p(1),\end{aligned} \tag{2.9}$$

for any $T>0$, *where* $\mathcal{B}_{(2.7)}(x,y)$ *is defined in Theorem 2.3,*

$$B(x,y) = W(x,y) - \left(1-\frac{\partial\lambda(x,y)}{\partial x}\right)W(x,0) - \left(1-\frac{\partial\lambda(x,y)}{\partial y}\right)W(0,y),$$

and $W(x,y)$ *is a Wiener process with zero mean and covariance structure*

$$\begin{aligned}&E\left(W(x_1,y_1)W(x_2,y_2)\right) \\ &= x_1\wedge x_2 + y_1\wedge y_2 - \lambda(x_1\wedge x_2, y_1) - \lambda(x_1\wedge x_2, y_2) - \lambda(x_1, y_1\wedge y_2) \\ &\quad -\lambda(x_2, y_1\wedge y_2) + \lambda(x_1,y_2) + \lambda(x_2,y_1) + \lambda(x_1\wedge x_2, y_1\wedge y_2).\end{aligned}$$

Therefore, for any fixed $x,y>0$,

$$\sqrt{k_{\mathrm{Hu}}}\left(\widehat{\lambda}^{\mathrm{Hu}}_{k_{\mathrm{Hu}},n}(x,y)-\lambda(x,y)\right) \overset{d}{\longrightarrow} \mathcal{N}\left(\mathcal{K}_{\mathrm{Hu}}\mathcal{B}_{(2.7)}(x,y), \sigma^2_{\mathrm{Hu}}\right)$$

as $n\to\infty$, *where*

$$\begin{aligned}\sigma^2_{\mathrm{Hu}} &= x\left(\frac{\partial}{\partial x}\lambda(x,y)\right)^2 + y\left(\frac{\partial}{\partial y}\lambda(x,y)\right)^2 + 2\lambda(x,y)\times \\ &\times\left(\frac{1}{2} - \frac{\partial}{\partial x}\lambda(x,y) - \frac{\partial}{\partial y}\lambda(x,y) + \left(\frac{\partial\lambda(x,y)}{\partial x}\right)\left(\frac{\partial\lambda(x,y)}{\partial y}\right)\right),\end{aligned} \tag{2.10}$$

$$\frac{\partial}{\partial x}\lambda(x,y) = \left(\int_{-\pi/2}^{\pi/2}(\cos\phi)^\alpha\,\mathrm{d}\phi\right)^{-1}\int_{g((x/y)^{1/\alpha})}^{\pi/2}(\cos\phi)^\alpha\,\mathrm{d}\phi \quad \textit{and} \tag{2.11}$$

$$\frac{\partial}{\partial y}\lambda(x,y) = \left(\int_{-\pi/2}^{\pi/2}(\cos\phi)^\alpha\,\mathrm{d}\phi\right)^{-1}\times \tag{2.12}$$

$$\times\int_{-\arcsin\rho}^{g((x/y)^{1/\alpha})}(\sin(\phi+\arcsin\rho))^\alpha\,\mathrm{d}\phi.$$

□

Theorem 2.5. *Assume that the conditions of Theorem 2.1 and (2.1) hold. Further assume (2.2) holds when* $\boldsymbol{\mu}\neq 0$. *Suppose* $k_{\mathrm{El}} = k_{\mathrm{El}}(n,\boldsymbol{\mu}) \stackrel{n\to\infty}{\longrightarrow} \infty$, $k_{\mathrm{El}}/n \stackrel{n\to\infty}{\longrightarrow} 0$ *and*

$$\sqrt{k_{\mathrm{El}}}\Big(\,(F_Y^{\leftarrow}(1-k_{\mathrm{El}}/n))^{-2} + |A(F_Y^{\leftarrow}(1-k_{\mathrm{El}}/n))|\,\Big) \stackrel{n\to\infty}{\longrightarrow}:\ \mathcal{K}_{\mathrm{El}},\quad \boldsymbol{\mu}\neq 0,$$

$$\sqrt{k_{\mathrm{El}}}A(F_Y^{\leftarrow}(1-k_{\mathrm{El}}/n)) \stackrel{n\to\infty}{\longrightarrow}:\ \mathcal{K}_{\mathrm{El}},\quad \boldsymbol{\mu}=\mathbf{0},$$

for $|\mathcal{K}_{\mathrm{El}}|<\infty$ *Then, as* $n\to\infty$,

$$\sup_{0\le x,y\le T}\left|\sqrt{k_{\mathrm{El}}}\left(\widehat{\lambda}^{\mathrm{El}}_{k_{\mathrm{El}},n}(x,y)-\lambda(x,y)\right)-\mathcal{B}_{(2.15)}(x,y)Z_0\right| = o_p(1), \tag{2.13}$$

where $Z_0\sim\mathcal{N}\left(-\alpha^2\mathcal{K}_{\mathrm{El}}\mathcal{B}_{(2.14)},\,\alpha^2\right)$ *with*

$$\mathcal{B}_{(2.14)} := \begin{cases}\int_0^1\mathcal{B}_{(2.4)}(1/s)\,\mathrm{d}s, & \boldsymbol{\mu}\neq\mathbf{0},\\ \int_0^1\mathcal{B}_{(2.5)}(1/s)\,\mathrm{d}s, & \boldsymbol{\mu}=\mathbf{0},\end{cases} \tag{2.14}$$

$\mathcal{B}_{(2.4)}(s)$ *and* $\mathcal{B}_{(2.5)}(s)$ *are defined in Theorem 2.2 and*

$$\mathcal{B}_{(2.15)}(x,y) := \left\{\int_{g\left((x/y)^{1/\alpha}\right)}^{\pi/2}x(\cos\phi)^\alpha\ln(\cos\phi)\,\mathrm{d}\phi\right.$$

$$+\int_{-\arcsin\rho}^{g\left((x/y)^{1/\alpha}\right)}y(\sin(\phi+\arcsin\rho))^\alpha\ln(\sin(\phi+\arcsin\rho))\,\mathrm{d}\phi$$

$$\left.-\lambda(x,y)\left(\int_{-\pi/2}^{\pi/2}(\cos\phi)^\alpha\ln(\cos\phi)\,\mathrm{d}\phi\right)\right\}\left(\int_{-\pi/2}^{\pi/2}(\cos\phi)^\alpha\,\mathrm{d}\phi\right)^{-1} \tag{2.15}$$

Therefore, for any fixed $x,y>0$,

$$\sqrt{k_{\mathrm{El}}}\left(\widehat{\lambda}^{\mathrm{El}}_{k_{\mathrm{El}},n}(x,y)-\lambda(x,y)\right)$$

$$\stackrel{d}{\longrightarrow}\ \mathcal{N}\left(-\alpha^2\mathcal{K}_{\mathrm{El}}\mathcal{B}_{(2.14)}\mathcal{B}_{(2.15)}(x,y),\,\alpha^2\left(\mathcal{B}_{(2.15)}(x,y)\right)^2\right).$$

□

The next corollary gives the optimal choice of sample fraction for both estimators. As criterion we use the *asymptotic mean squared error* of $\widehat{\lambda}^{\mathrm{Hu}}_{k_{\mathrm{Hu}},n}$ and $\widehat{\lambda}^{\mathrm{El}}_{k_{\mathrm{El}},n}$, denoted by $\mathrm{amse}_{\mathrm{Hu}}(k_{\mathrm{Hu}})$ and $\mathrm{amse}_{\mathrm{El}}(k_{\mathrm{El}})$, respectively.

Corollary 2.6. *Assume that the conditions of Theorems 2.4 and 2.5 hold. Further, suppose that*

$$\begin{aligned} A(F_Y^{\leftarrow}(1-t)) &\sim b_0 t^{-\beta/\alpha}, \\ (F_Y^{\leftarrow}(1-t))^{-2} + |A(F_Y^{\leftarrow}(1-t))| &\sim b_1 t^{(2\wedge(-\beta))/\alpha} \end{aligned}$$

for some $b_0, b_1 > 0$ as $t \to 0$ and define

$$b_2 t^{-\beta_2/\alpha} := \begin{cases} b_1 t^{(2\wedge(-\beta))/\alpha} & \boldsymbol{\mu} \neq \mathbf{0}, \\ b_0 t^{-\beta/\alpha}, & \boldsymbol{\mu} = 0. \end{cases}$$

Then

$$\mathrm{amse}_{\mathrm{Hu}}(k_{\mathrm{Hu}}) = \sigma^2_{\mathrm{Hu}} k^{-1}_{\mathrm{Hu}} + \left(b(k_{\mathrm{Hu}}/n)^{-\beta/\alpha}\mathcal{B}_{(2.7)}(x,y)\right)^2$$

and

$$\mathrm{amse}_{\mathrm{El}}(k_{\mathrm{El}}) = (\mathcal{B}_{(2.15)}(x,y))^2 \left(\alpha^2 k^{-1}_{\mathrm{El}} + \left(\alpha^2 b_2 (k_{\mathrm{El}}/n)^{-\beta_2/\alpha}\mathcal{B}_{(2.14)}\right)^2\right).$$

Let $k^{\mathrm{opt}}_{\mathrm{Hu}}$ and $k^{\mathrm{opt}}_{\mathrm{El}}$ denote the optimal sample fraction in the sense of minimizing $\mathrm{amse}_{\mathrm{Hu}}$ *and* $\mathrm{amse}_{\mathrm{El}}$*, respectively. Then*

$$\begin{aligned} k^{\mathrm{opt}}_{\mathrm{Hu}} &= \left(\frac{-\alpha\sigma^2_{\mathrm{Hu}}}{2\beta b_0^2\left(\mathcal{B}_{(2.7)}(x,y)\right)^2}\right)^{\alpha/(\alpha-2\beta)} n^{-2\beta/(\alpha-2\beta)}, \\ k^{\mathrm{opt}}_{\mathrm{El}} &= \left(-2\beta_2\alpha b_2^2\left(\mathcal{B}_{(2.14)}\right)^2\right)^{-\alpha/(\alpha-2\beta_2)} n^{-2\beta_2/(\alpha-2\beta_2)}, \\ \mathrm{amse}^{\mathrm{opt}}_{\mathrm{Hu}} &:= \mathrm{amse}_{\mathrm{Hu}}\left(k^{\mathrm{opt}}_{\mathrm{Hu}}\right) = n^{2\beta/(\alpha-2\beta)} b_0^{2\alpha/(\alpha-2\beta)}\left(1-\frac{\alpha}{2\beta}\right)\times \\ &\quad \times \left(\left(\sigma^2_{\mathrm{Hu}}\right)^{-\beta/\alpha}\mathcal{B}_{(2.7)}(x,y)\sqrt{-2\beta/\alpha}\right)^{2\alpha/(\alpha-2\beta)} \quad \text{and} \\ \mathrm{amse}^{\mathrm{opt}}_{\mathrm{El}} &:= \mathrm{amse}_{\mathrm{El}}\left(k^{\mathrm{opt}}_{\mathrm{El}}\right) = n^{2\beta_2/(\alpha-2\beta_2)} b_2^{2\alpha/(\alpha-2\beta_2)}\left(1-\frac{\alpha}{2\beta_2}\right)\times \\ &\quad \times \alpha^2(\mathcal{B}_{(2.15)}(x,y))^2\left(\sqrt{-2\alpha\beta_2}\mathcal{B}_{(2.14)}\right)^{2\alpha/(\alpha-2\beta_2)}. \end{aligned}$$

□

Remark 2.7. Note that $k_{\rm El}^{\rm opt}$ is independent of x and y, but $k_{\rm Hu}^{\rm opt}$ depends on x and y. In case of $\boldsymbol{\mu} = \mathbf{0}$, both $\text{amse}_{\rm Hu}^{\rm opt}$ and $\text{amse}_{\rm El}^{\rm opt}$ depend on n, α, β, ρ, υ, x, y and b_0, $\text{amse}_{\rm El}^{\rm opt}$ additionally depends on σ, but the ratio $\text{amse}_{\rm Hu}^{\rm opt}/\text{amse}_{\rm El}^{\rm opt}$ is independent of n and b_0. Since the optimal $k_{\rm El}^{\rm opt}$ is the same as that for Hill's estimator, when $\boldsymbol{\mu} = 0$, all data-driven methods for choosing the optimal sample fraction for Hill's estimator can be applied to $\widehat{\lambda}_{k_{\rm El},n}^{\rm El}(x, y)$ directly. Note that $\boldsymbol{\mu}$ is the median of (X, Y) and the mean of (X, Y) when $\alpha > 1$. Hence, we could estimate $\boldsymbol{\mu}$ by the sample median, say $\widehat{\boldsymbol{\mu}} = (\widehat{\mu}_X, \widehat{\mu}_Y)$. Therefore, consider the new estimator $\widehat{\lambda}_{k_{\rm El},n}^{\rm El}(x, y)$ with $Z_i = \sqrt{X_i^2 + Y_i^2}$ replaced by $\sqrt{(X_i - \widehat{\mu}_X)^2 + (Y_i - \widehat{\mu}_Y)^2}$. Similar to the proofs in Ling and Peng (2004), we can show that Theorem 2.5 and Corollary 2.6 hold with $\boldsymbol{\mu} = 0$ for this new estimator. □

3 Comparisons and Simulation Study

First we compare $\sigma_{\rm Hu}^2, \sigma_{\rm El}^2$ given in Theorem 2.4 and 2.5. Note that both only depend on α, ρ, x and y. In Figure 1, we plot the ratio $\sigma_{\rm El}^2(\alpha)/\sigma_{\rm Hu}^2(\alpha)$ for $x = y = 1$ as a function of α, and each curve therein corresponds to a different correlation $\rho \in \{0.1, \ldots, 0.9\}$. From Figure 1, we conclude that $\widehat{\lambda}_{k,n}^{\rm El}$ is always better in terms of asymptotic variance.

Second, we compare the two estimators in terms of optimal asymptotic mean squared errors. Since the ratio of the optimal asymptotic mean squared error depends on α, β, $\boldsymbol{\Sigma}$, $\boldsymbol{\mu}$, x, y, we consider elliptical distributions with $\sigma = \upsilon = 1$, $\mu_X = \mu_Y = 0$. In Figure 2, we consider $G \sim$ Fréchet(α), i.e. $P(G > x) = \exp(-x^{-\alpha})$, $x > 0$, hence (2.1) is satisfied with $\beta = -\alpha$. In Figure 3, we consider $G \sim$ Pareto(α), i.e. $P(G > x) = (1 + x)^{-\alpha}$ for $x > 0$, therefore, (2.1) is satisfied with $\beta = -1$. Under the above setup, the ratio of optimal asymptotic mean squared errors only depends on α, ρ, x, y. Similar to Figure 1, we plot the ratio $\text{amse}_{\rm El}^{\rm opt}(\alpha)/\text{amse}_{\rm Hu}^{\rm opt}(\alpha)$ for $x = y = 1$ as a function of α for different ρ's in Figures 2 and 3. We conclude from both Figures that $\widehat{\lambda}_{k,n}^{\rm El}$ always performs better than $\widehat{\lambda}_{k,n}^{\rm Hu}$ in terms of optimal asymptotic mean squared errors as well.

Third, we examine the influence of x and y to the ratio of asymptotic mean squared error. We plot the ratio $\text{amse}_{\rm El}^{\rm opt}(\alpha)/\text{amse}_{\rm Hu}^{\rm opt}(\alpha)$ for $\|(x, y)\| = \sqrt{2}$ and $G \sim$ Pareto(α) in Figure 4, where each curve corresponds to a different pair of $(\alpha,\, \rho) \in \{(20, 0.9), (10, 0.6),\, (5, 0.3), (1, 0.1)\}$. This figure further confirms that $\widehat{\lambda}_{k,n}^{\rm El}$ always has a smaller optimal asymptotic mean squared error than $\widehat{\lambda}_{k,n}^{\rm Hu}$.

Finally, we study the finite sample behavior of the two estimators $\widehat{\lambda}^{\mathrm{El}}_{k,n}(x,y)$ and $\widehat{\lambda}^{\mathrm{Hu}}_{k,n}(x,y)$. As above, we consider two elliptical distributions with $\sigma = \upsilon = 1$, $\mu_X = \mu_Y = 0$, $G \sim$ Fréchet(α) in Figure 5 and $G \sim$ Pareto(α) in Figure 6. We generate 1000 random samples of size $n = 1000$ from these elliptical distributions with $(\alpha, \rho) \in \{(20, 0.9), (10, 0.6), (5, 0.3), (1, 0.1)\}$, and plot $\widehat{\lambda}^{\mathrm{El}}_{k,n}(1,1)$ and $\widehat{\lambda}^{\mathrm{Hu}}_{k,n}(1,1)$ against $k = 1, \ldots, 300$ for different pairs (α, ρ) in Figures 5 and 6, where the solid line corresponds to $\widehat{\lambda}^{\mathrm{El}}_{k,n}(1,1)$ and the dashed line to $\widehat{\lambda}^{\mathrm{Hu}}_{k,n}(1,1)$. This simulation study also confirms the better performance of $\widehat{\lambda}^{\mathrm{El}}_{k,n}$.

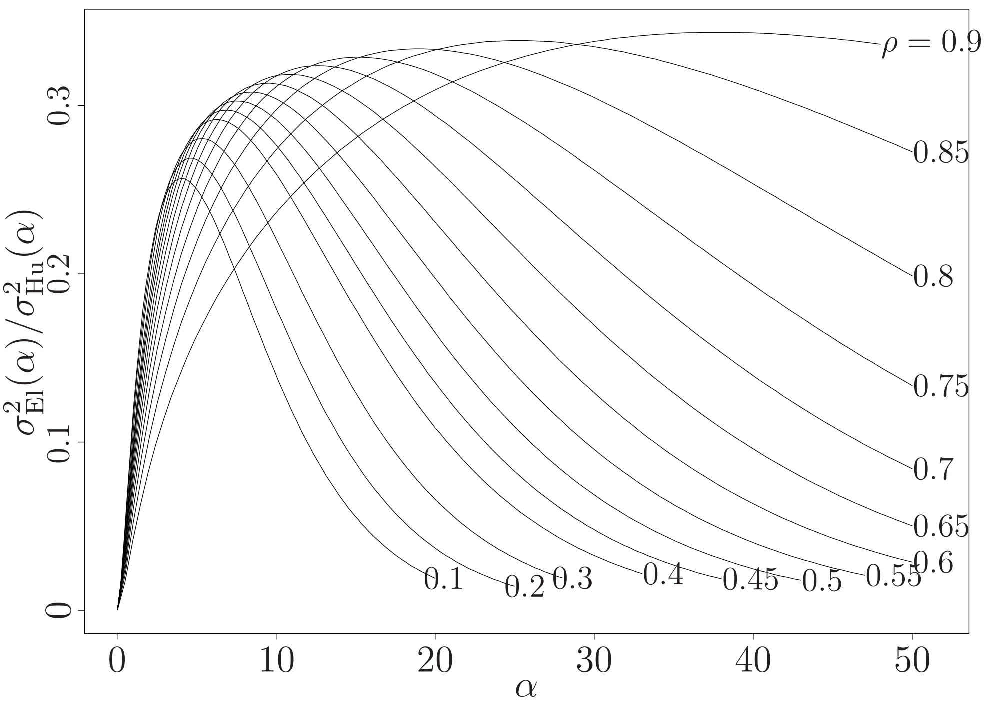

Figure 1: Ratio $\sigma^2_{\mathrm{El}}(\alpha)/\sigma^2_{\mathrm{Hu}}(\alpha)$ for different correlations ρ.

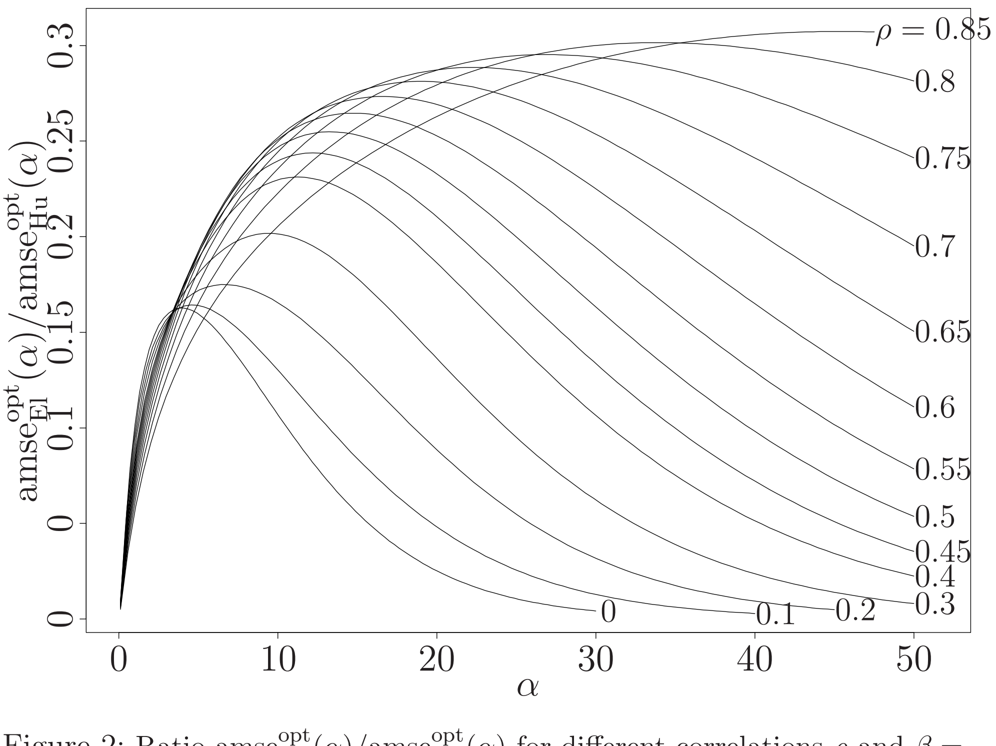

Figure 2: Ratio $\mathrm{amse}^{\mathrm{opt}}_{\mathrm{Hu}}(\alpha)/\mathrm{amse}^{\mathrm{opt}}_{\mathrm{El}}(\alpha)$ for different correlations ρ and $\beta = -\alpha$.

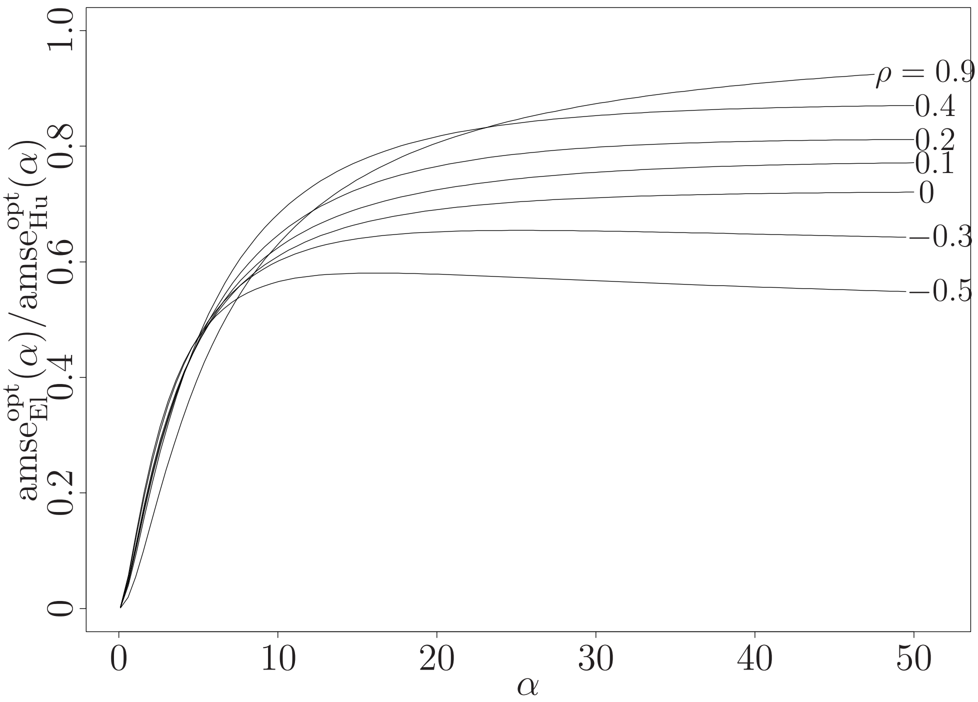

Figure 3: Ratio $\mathrm{amse}_{\mathrm{Hu}}^{\mathrm{opt}}(\alpha)/\mathrm{amse}_{\mathrm{El}}^{\mathrm{opt}}(\alpha)$ for different correlations ρ and $\beta = -1$.

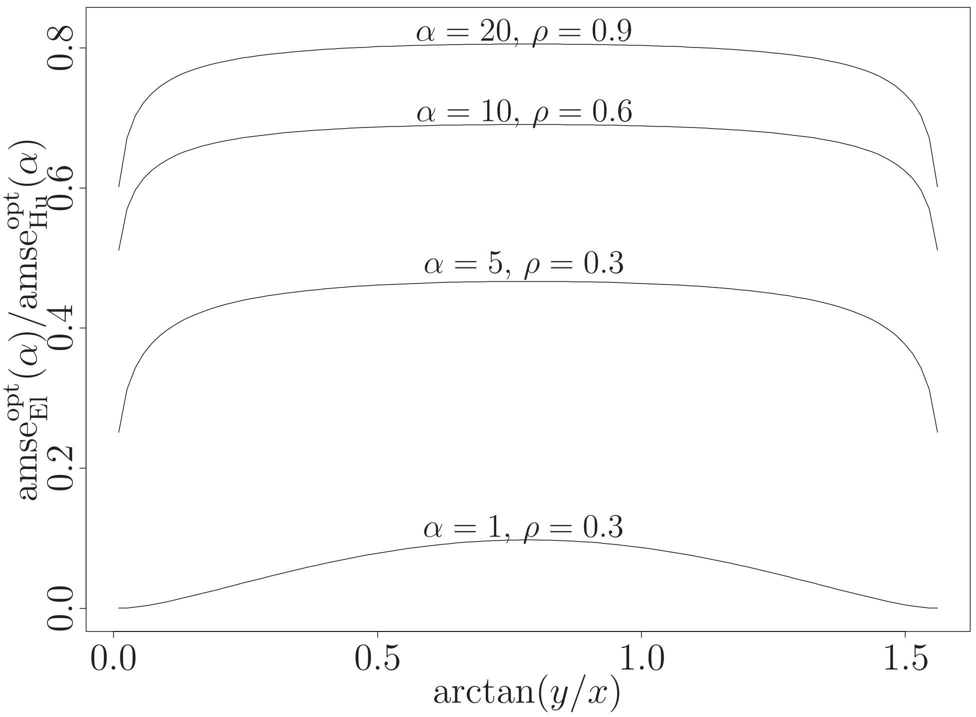

Figure 4: Ratio $\mathrm{amse}_{\mathrm{El}}^{\mathrm{opt}}(\alpha)/\mathrm{amse}_{\mathrm{Hu}}^{\mathrm{opt}}(\alpha)$, $x^2 + y^2 = 2$, for different $(\alpha,\ \rho)$ and $\beta = -1$.

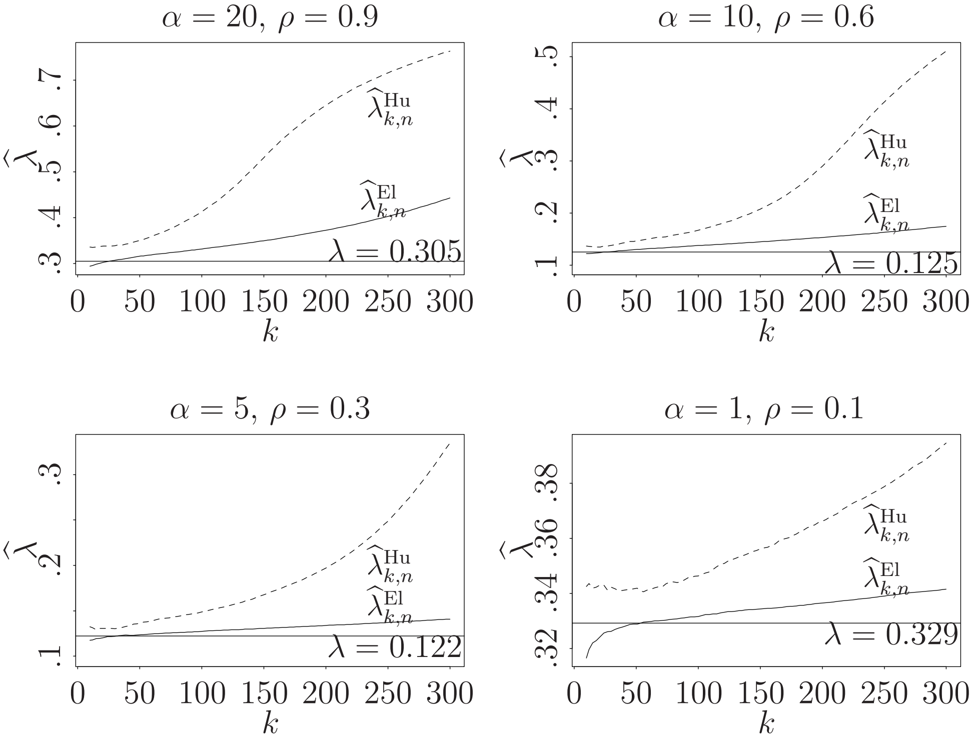

Figure 5: Mean of estimators $\widehat{\lambda}^{\mathrm{Hu}}_{k,n}(1,1)$ and $\widehat{\lambda}^{\mathrm{El}}_{k,n}(1,1)$ for 1 000 samples of length $n = 1\,000$ and different k with $\sigma = \upsilon = 1$, $\boldsymbol{\mu} = \mathbf{0}$, $G \sim$ Fréchet(α), and different (α, ρ).

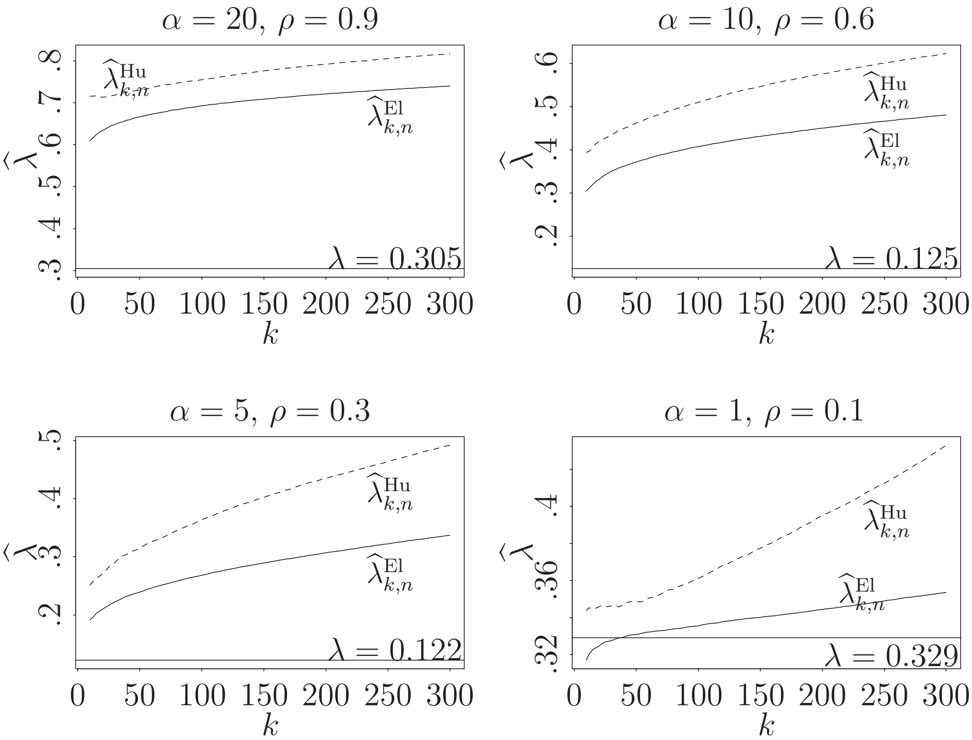

Figure 6: Mean of estimators $\widehat{\lambda}^{\mathrm{Hu}}_{k,n}(1,1)$ and $\widehat{\lambda}^{\mathrm{El}}_{k,n}(1,1)$ for 1 000 samples of length $n = 1\,000$ and different k with $\sigma = \upsilon = 1$, $\boldsymbol{\mu} = \mathbf{0}$, $G \sim$ Pareto(α) and different (α, ρ).

4 Proofs

Proof of Theorem 2.1: Without loss of generality, we assume $\boldsymbol{\mu} = \mathbf{0}$. Let $\Phi \sim \text{unif}(-\pi,\pi)$ be independent of G and $F_i^{\leftarrow}(x)$ denote the inverse of $F_i(x)$, $i = 1, 2$. Then, by Hult and Lindskog (2002),

$$\begin{aligned}
&F_X^{\leftarrow}(u) = \tfrac{\sigma}{\upsilon} F_Y^{\leftarrow}(u), && \text{for } 0 < u < 1,\\
&\lim_{t\to\infty}(1 - F_i(tx))/(1 - F_i(t)) = x^{-\alpha}, && \text{for } x > 0 \text{ and } i = 1, 2,\\
&(X, Y) \overset{\mathrm{d}}{=} (\sigma G \cos\Phi, \upsilon G \sin(\arcsin\rho + \Phi)).
\end{aligned} \tag{4.16}$$

Therefore,

$$\begin{aligned}
&t^{-1} P(F_X(X) \geq 1 - tx,\ F_Y(Y) \geq 1 - ty)\\
&= t^{-1} P(G\cos\Phi \geq F_Y^{\leftarrow}(1 - tx)/\upsilon, G\sin(\arcsin\rho + \Phi) \geq F_Y^{\leftarrow}(1 - ty)/\upsilon)\\
&= \frac{1}{2\pi t}\int_{-\arcsin\rho}^{\pi/2} P\left(G \geq \frac{F_Y^{\leftarrow}(1 - tx)}{\upsilon\cos\phi} \vee \frac{F_Y^{\leftarrow}(1 - ty)}{\upsilon\sin(\arcsin\rho + \phi)}\right) \mathrm{d}\phi.
\end{aligned} \tag{4.17}$$

Note that

$$\begin{aligned}
t = P(X > F_X^{\leftarrow}(1 - t)) &= P(G\cos\Phi > F_Y^{\leftarrow}(1 - t)/\upsilon)\\
&= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} P\left(G > \frac{F_Y^{\leftarrow}(1 - t)}{\upsilon\cos\phi}\right) \mathrm{d}\phi.
\end{aligned}$$

Further, $1 \geq P(G > x/\cos\phi)/P(G \geq x) \overset{x\to\infty}{\longrightarrow} (\cos\phi)^\alpha$. Hence, in the following formula we can apply the dominated convergence theorem and obtain

$$\begin{aligned}
\frac{1}{\mathcal{B}_{(4.18)}(t)} &:= \frac{1}{2\pi t} P(G \geq F_Y^{\leftarrow}(1 - t)/\upsilon)\\
&\overset{t\to 0}{\longrightarrow} \left(\int_{-\pi/2}^{\pi/2} (\cos\phi)^\alpha \,\mathrm{d}\phi\right)^{-1} =: \frac{1}{\mathcal{B}_{(4.18)}}.
\end{aligned} \tag{4.18}$$

Next, we obtain for $\phi \in (-\arcsin\rho, \pi/2)$

$$\frac{F_Y^{\leftarrow}(1 - tx)}{\upsilon\cos\phi} \geq \frac{F_Y^{\leftarrow}(1 - ty)}{\upsilon\sin(\arcsin\rho + \phi)} \quad\Leftrightarrow\quad \frac{F_Y^{\leftarrow}(1 - ty)}{F_Y^{\leftarrow}(1 - tx)} \leq \frac{\sin(\arcsin\rho + \phi)}{\cos\phi}.$$

Note that $\sin(\arcsin\rho+\phi)/\cos\phi$ is strictly increasing, hence its inverse exists and equals $\arctan\left((\cdot-\rho)/\sqrt{1-\rho^2}\right)$. Therefore,

$$\begin{aligned}
&\frac{F_Y^{\leftarrow}(1-tx)}{\upsilon\cos\phi} \geq \frac{F_Y^{\leftarrow}(1-ty)}{\upsilon\sin(\arcsin\rho+\phi)} \\
&\Leftrightarrow \quad \phi \geq \arctan\left(\frac{F_Y^{\leftarrow}(1-ty)/F_Y^{\leftarrow}(1-tx)-\rho}{\sqrt{1-\rho^2}}\right) \\
&\quad =: \ g\left(\frac{F_Y^{\leftarrow}(1-ty)}{F_Y^{\leftarrow}(1-tx)}\right) =: h(x,y,t). \qquad (4.19)
\end{aligned}$$

Since $1-F_Y\in\mathcal{R}_{-\alpha}$, by Proposition 1.7(9) of Geluk and de Haan (1987) $F_Y^{\leftarrow}(1-tx)/F_Y^{\leftarrow}(1-t)\stackrel{t\to 0}{\longrightarrow} x^{-1/\alpha}$, i.e.,

$$h(x,y,t)\stackrel{t\to 0}{\longrightarrow} g\left((x/y)^{1/\alpha}\right). \qquad (4.20)$$

It follows from (4.17) and (4.19) that

$$\begin{aligned}
&t^{-1}P\left(F_X(X)\geq 1-tx,\ F_Y(Y)\geq 1-ty\right) \\
&= \frac{1}{\mathcal{B}_{(4.18)}(t)}\int_{h(x,y,t)}^{\pi/2}\frac{P\left(G\geq \frac{F_Y^{\leftarrow}(1-t)}{\upsilon\cos\phi}\frac{F_Y^{\leftarrow}(1-tx)}{F_Y^{\leftarrow}(1-t)}\right)}{P\left(G\geq F_Y^{\leftarrow}(1-t)/\upsilon\right)}\,\mathrm{d}\phi \\
&\quad + \frac{1}{\mathcal{B}_{(4.18)}(t)}\int_{-\arcsin\rho}^{h(x,y,t)}\frac{P\left(G\geq \frac{F_Y^{\leftarrow}(1-t)}{\upsilon\sin(\arcsin\rho+\phi)}\frac{F_Y^{\leftarrow}(1-ty)}{F_Y^{\leftarrow}(1-t)}\right)}{P\left(G\geq F_Y^{\leftarrow}(1-t)/\upsilon\right)}\,\mathrm{d}\phi. \qquad (4.21)
\end{aligned}$$

Hence, the theorem follows from (4.18), (4.20) and Potter's inequality, e.g. see (1.20) in Geluk and de Haan (1987). □

Proof of Theorem 2.2: Since

$$(X,Y) \quad \stackrel{\mathrm{d}}{=} \quad \left(\mu_X+\sigma G\cos\Phi,\ \mu_Y+\upsilon G\sin(\Phi+\arcsin\rho)\right),$$

we have $X^2+Y^2\stackrel{\mathrm{d}}{=}G^2d_1(\Phi)+2Gd_2(\Phi)+\mu_X^2+\mu_Y^2$. Define

$$d_3(x,\phi) \quad := \quad \frac{1}{d_1(\phi)}\left(-d_2(\phi)+\sqrt{d_2^2(\phi)-d_1(\phi)\left(\mu_X^2+\mu_Y^2-x^2\right)}\right).$$

Since $P(X^2+Y^2\ge t)=P\left(G\ge d_3(t,\Phi)\right)$ holds for large t, we obtain

$$
\begin{aligned}
&\frac{P\left(X^2+Y^2\ge t^2x^2\right)}{P\left(X^2+Y^2\ge t^2\right)}-x^{-\alpha}\\
=\;& \left(\int_{-\pi}^{\pi}\frac{P\left(G\ge d_3(tx,\phi)\right)}{P(G\ge t)}\mathrm{d}\phi\right)\left(\int_{-\pi}^{\pi}\frac{P\left(G\ge d_3(t,\phi)\right)}{P(G\ge t)}\mathrm{d}\phi\right)^{-1}-x^{-\alpha}\\
=\;& \left\{\int_{-\pi}^{\pi}\Big[\frac{P\left(G\ge d_3(tx,\phi)\right)}{P(G\ge t)}-\left(\frac{1}{t}d_3(tx,\phi)\right)^{-\alpha}\Big]\mathrm{d}\phi\right.\\
&+\int_{-\pi}^{\pi}\Big[-x^{-\alpha}\frac{P\left(G\ge d_3(t,\phi)\right)}{P(G\ge t)}+x^{-\alpha}\left(\frac{1}{t}d_3(t,\phi)\right)^{-\alpha}\Big]\mathrm{d}\phi\\
&\left.+\int_{-\pi}^{\pi}\Big[\left(\frac{1}{t}d_3(tx,\phi)\right)^{-\alpha}-x^{-\alpha}\left(\frac{1}{t}d_3(t,\phi)\right)^{-\alpha}\Big]\mathrm{d}\phi\right\}\times\\
&\times\left(\int_{-\pi}^{\pi}\frac{P\left(G\ge d_3(t,\phi)\right)}{P(G\ge t)}\mathrm{d}\phi\right)^{-1}\\
=:\;& \left(\mathcal{J}_1(t)+\mathcal{J}_2(t)+\mathcal{J}_3(t)\right)\left(\int_{-\pi}^{\pi}\frac{P\left(G\ge d_3(t,\phi)\right)}{P(G\ge t)}\,\mathrm{d}\phi\right)^{-1}.
\end{aligned}
$$

Since $|\rho|<1$, it is straightforward to check that

$$
\begin{aligned}
&\lim_{t\to\infty}t^{-1}d_3(t,\phi)=(d_1(\phi))^{-1/2},\\
&0<\sup_{-\pi\le\phi\le\pi}d_1(\phi)<\infty,\quad\text{and}\\
&\sup_{-\pi\le\phi\le\pi}|d_2(\phi)|<\infty.
\end{aligned}
\tag{4.22}
$$

Hence, similarly to the proof of Theorem 2.1,

$$
\lim_{t\to\infty}\int_{-\pi}^{\pi}\frac{P\left(G\ge d_3(t,\phi)\right)}{P(G\ge t)}\,\mathrm{d}\phi=\int_{-\pi}^{\pi}(d_1(\phi))^{\alpha/2}\,\mathrm{d}\phi.
\tag{4.23}
$$

Similar to the proof of Lemma 5.2 of Draisma et al. (1999), for any $\varepsilon>0$, there exists $t_0>0$ such that for all $t\ge t_0, d_3(tx,\phi)\ge t_0$

$$
\begin{aligned}
&\left|\frac{\frac{P\left(G\ge d_3(tx,\phi)\right)}{P(G\ge t)}-\left(\frac{1}{t}d_3(tx,\phi)\right)^{-\alpha}}{A(t)}-\left(\frac{1}{t}d_3(tx,\phi)\right)^{-\alpha}\frac{\left(\frac{1}{t}d_3(tx,\phi)\right)^{\beta}-1}{\beta}\right|\\
\le\;& \varepsilon\left(1+\left(\frac{1}{t}d_3(tx,\phi)\right)^{-\alpha}+\left(\frac{1}{t}d_3(tx,\phi)\right)^{-\alpha+\beta}\exp\left\{\varepsilon\left|\ln\left(\frac{1}{t}d_3(tx,\phi)\right)\right|\right\}\right).
\end{aligned}
\tag{4.24}
$$

Using (4.22), for any fixed $x > 0$, we can choose t_0 large enough such that $d_3(tx, \phi) \geq t_0$ uniformly for $\phi \in [-\pi, \pi]$. That is, for any fixed $x > 0$, (4.24) holds uniformly for $\phi \in [-\pi, \pi]$. Therefore, by dominated convergence theorem and (4.23), for $x > 0$,

$$\lim_{t\to\infty} \frac{\mathcal{J}_1(t)}{A(t)} = \frac{1}{x^\alpha \beta} \int_{-\pi}^{\pi} \left(x^\beta \left(d_1(\phi)\right)^{(\alpha-\beta)/2} - \left(d_1(\phi)\right)^{\alpha/2} \right) \mathrm{d}\phi \quad \text{and} \tag{4.25}$$

$$\lim_{t\to\infty} \frac{\mathcal{J}_2(t)}{A(t)} = -\frac{1}{x^\alpha \beta} \int_{-\pi}^{\pi} \left(\left(d_1(\phi)\right)^{(\alpha-\beta)/2} - \left(d_1(\phi)\right)^{\alpha/2} \right) \mathrm{d}\phi. \tag{4.26}$$

It follows from (4.22) and a Taylor expansion, for $x > 0$, that

$$\begin{aligned} \mathcal{J}_3(t) &= \frac{\alpha}{tx^\alpha} \left(x^{-1} - 1\right) \int_{-\pi}^{\pi} \left(d_1(\phi)\right)^{\alpha/2-1/2} d_2(\phi)\, \mathrm{d}\phi \; + \; o\left(t^{-2}\right) \\ &+ \frac{\alpha}{2t^2 x^\alpha} \left(x^{-2} - 1\right) \int_{-\pi}^{\pi} \left(d_1(\phi)\right)^{\alpha/2-1} \left[\alpha (d_2(\phi))^2 + d_1(\phi)(\mu_X^2 + \mu_Y^2)\right] \mathrm{d}\phi. \end{aligned} \tag{4.27}$$

Note that $\sin(\phi + \arcsin \rho) = \sqrt{1-\rho^2} \sin \phi + \rho \cos \phi$. Then, splitting the integral into $[-\pi, -\pi/2)$, $[-\pi/2, 0)$, $[0, \pi/2)$, $[\pi/2, \pi]$ and using the symmetry of sin and cos, we obtain

$$\int_{-\pi}^{\pi} \left(d_1(\phi)\right)^{(\alpha-1)/2} d_2(\phi)\, \mathrm{d}\phi = 0. \tag{4.28}$$

Hence (2.3) follows from (4.25), (4.26), (4.27) and (4.28). Note that

$$\lim_{t\to\infty} P\left(\sqrt{X^2+Y^2} > t\right) / P(G > t) = \int_{-\pi}^{\pi} \left(d_1(\phi)\right)^{\alpha/2} d\phi$$

and, since $Y \overset{\mathrm{d}}{=} \mu_Y + \upsilon G \sin \Phi$ with $\Phi \sim \mathrm{unif}(-\pi, \pi)$ holds,

$$\lim_{t\to\infty} P(Y > t)/P(G > t) = \upsilon^\alpha \int_0^\pi (\sin \phi)^\alpha\, d\phi.$$

Therefore, we have

$$\begin{aligned} V(t) &\sim \inf \left\{ y : P(G > y) \leq t^{-1} \Big/ \int_{-\pi}^{\pi} \left(d_1(\phi)\right)^{\alpha/2} d\phi \right\} \quad \text{and} \\ F_Y^{\leftarrow}(1 - t^{-1}) &\sim \inf \left\{ y : P(G > y) \leq t^{-1} \Big/ \left(\upsilon^\alpha \int_0^\pi (\sin \phi)^\alpha\, d\phi \right) \right\}. \end{aligned}$$

Hence,

$$\lim_{t\to\infty}\frac{V(t)}{F_Y^{\leftarrow}(1-t^{-1})} = \left(\frac{\int_{-\pi}^{\pi}(d_1(\phi))^{\alpha/2}\,d\phi}{\upsilon^\alpha\int_0^\pi(\sin\phi)^\alpha\,d\phi}\right)^{1/\alpha},$$

i.e., since $t^2|A(t)|\overset{t\to\infty}{\longrightarrow}\infty$ for $-2<\beta\le 0$,

$$\lim_{t\to\infty}\frac{(V(t))^{-2}+|A(V(t))|}{(F_Y^{\leftarrow}(1-t^{-1}))^{-2}+|A(F_Y^{\leftarrow}(1-t^{-1}))|} = \left(\frac{\int_{-\pi}^{\pi}(d_1(\phi))^{\alpha/2}\,d\phi}{\upsilon^\alpha\int_0^\pi(\sin\phi)^\alpha\,d\phi}\right)^{-(2\wedge|\beta|)/\alpha}. \tag{4.29}$$

Note that, by Taylor expansion,

$$\left(\frac{V(tx)}{V(t)}\right)^{-\alpha} = \frac{1}{x}-\frac{\alpha}{x^{1+1/\alpha}}\left(\frac{V(tx)}{V(t)}-x^{1/\alpha}\right)+o\left(1/V(t)+|A(V(t))|\right). \tag{4.30}$$

Therefore, replacing t and x in (2.3) by $V(t)$ and $V(tx)/V(t)$, respectively, and using (4.29) and (4.30), we obtain (2.4). Let $\mu_X=\mu_Y=0$, then $\mathcal{J}_3(t)=0$ and we obtain (2.5). □

Proof of Theorem 2.3: In order to prove Theorem 2.3, we can assume $\mu_X=\mu_Y=0$ since $\lambda(x,y)$ is independent of margins. We also set $\upsilon=1$ and give the correction at the end of the proof. Using an upper-triangle decomposition of $\mathbf{\Sigma}$ yields $Y\overset{\mathrm{d}}{=}G\sin\Phi$, where $\Phi\sim\mathrm{unif}(-\pi,\pi)$ and is independent of G. Then, write

$$\begin{aligned}\frac{P(Y\ge tx)}{P(Y\ge t)}-x^{-\alpha} &= \frac{\int_0^\pi P\left(G\ge tx/\sin\phi\right)\mathrm{d}\phi}{\int_0^\pi P\left(G\ge t/\sin\phi\right)\mathrm{d}\phi}-x^{-\alpha}\\ &= \left(\int_0^\pi\frac{P\left(G\ge t/\sin\phi\right)}{P(G\ge t)}\mathrm{d}\phi\right)^{-1}\left\{\int_0^\pi\left[\frac{P\left(G\ge tx/\sin\phi\right)}{P(G\ge t)}-\left(\frac{x}{\sin\phi}\right)^{-\alpha}\right]\right.\\ &\quad\left.-x^{-\alpha}\left[\frac{P\left(G\ge t/\sin\phi\right)}{P(G\ge t)}-\left(\frac{1}{\sin\phi}\right)^{-\alpha}\right]\mathrm{d}\phi\right\}.\end{aligned}$$

Then, by (2.1), we have for $x>0$

$$\begin{aligned}&\lim_{t\to\infty}\left(\frac{P(Y\ge tx)}{P(Y\ge t)}-x^{-\alpha}\right)/A(t)\\ &= x^{-\alpha}\frac{x^\beta-1}{\beta}\left(\int_0^\pi(\sin\phi)^\alpha\,\mathrm{d}\phi\right)^{-1}\left(\int_0^\pi(\sin\phi)^{\alpha-\beta}\,\mathrm{d}\phi\right).\end{aligned}$$

Replacing t and x in the latter equation by $F_Y^{\leftarrow}(1-s)$ and $F_Y^{\leftarrow}(1-sy)/F_Y^{\leftarrow}(1-s)$, respectively, we obtain, for $y > 0$,

$$\lim_{s\to 0}\left(\left(\frac{F_Y^{\leftarrow}(1-sy)}{F_Y^{\leftarrow}(1-s)}\right)^{-\alpha} - y\right)/A\left(F_Y^{\leftarrow}(1-s)\right) \;=\; \mathcal{B}_{(2.6)}(y). \tag{4.31}$$

Denote $f(t) := F_Y^{\leftarrow}(1-t)$. Then, by (4.21), we can write

$$\begin{aligned}
& t^{-1}P\left(F_X(X) \geq 1-tx,\ F_Y(Y) \geq 1-ty\right) \\
= \;& \frac{1}{\mathcal{B}_{(4.18)}(t)}\left\{\int_{h(x,y,t)}^{\pi/2}\Big[\frac{P\left(G \geq \frac{f(t)}{\cos\phi}\frac{f(tx)}{f(t)}\right)}{P\left(G \geq f(t)\right)} - \left(\frac{f(tx)}{f(t)\cos\phi}\right)^{-\alpha}\Big]\,\mathrm{d}\phi\right. \\
& + \int_{h(x,y,t)}^{\pi/2}\Big[\left(\frac{f(tx)}{f(t)\cos\phi}\right)^{-\alpha} - x(\cos\phi)^{\alpha}\Big]\,\mathrm{d}\phi \;+\; \int_{h(x,y,t)}^{h(x,y,0)} x(\cos\phi)^{\alpha}\,\mathrm{d}\phi \\
& + \int_{-\arcsin\rho}^{h(x,y,t)}\Big[\frac{P\left(G \geq \frac{f(t)}{\sin(\arcsin\rho+\phi)}\frac{f(ty)}{f(t)}\right)}{P\left(G \geq f(t)\right)} \\
& \qquad\qquad - \left(\frac{f(ty)}{f(t)\sin\left(\arcsin\rho+\phi\right)}\right)^{-\alpha}\Big]\,\mathrm{d}\phi \\
& + \int_{-\arcsin\rho}^{h(x,y,t)}\Big[\left(\frac{f(ty)}{f(t)\sin\left(\arcsin\rho+\phi\right)}\right)^{-\alpha} - y\left(\sin\left(\arcsin\rho+\phi\right)\right)^{\alpha}\Big]\,\mathrm{d}\phi \\
& + \int_{h(x,y,0)}^{h(x,y,t)} y\left(\sin\left(\arcsin\rho+\phi\right)\right)^{\alpha}\,\mathrm{d}\phi \;+\; \int_{h(x,y,0)}^{\pi/2} x(\cos\phi)^{\alpha}\,\mathrm{d}\phi \\
& \left. + \int_{-\arcsin\rho}^{h(x,y,0)} y\left(\sin\left(\arcsin\rho+\phi\right)\right)^{\alpha}\,\mathrm{d}\phi\right\} \\
=: \;& \frac{1}{\mathcal{B}_{(4.18)}(t)}\left(\sum_{i=1}^{6}\mathcal{J}_i(t) + \mathcal{J}_7 + \mathcal{J}_8\right).
\end{aligned} \tag{4.32}$$

Note that $1/|\cos\phi| \geq 1$ and υ is given, using Potter's bound and similar arguments as in the proof of Lemma 5.2 of Draisma et al. (1999), for any $\varepsilon > 0$, there exists some small $t_0 > 0$ such that for all $f(t) \geq f(t_0)$, $f(tx) \geq f(t_0)$ and

$\phi \in [-\pi/2, \pi/2]$

$$\left| \frac{P\left(G \geq \frac{f(tx)}{\cos\phi}\right) / P\left(G \geq f(t)\right) - \left(\frac{f(tx)}{f(t)\cos\phi}\right)^{-\alpha}}{A(f(t))} \right.$$

$$\left. - \left(\frac{f(tx)}{f(t)\cos\phi}\right)^{-\alpha} \frac{\left(\frac{f(tx)}{f(t)\cos\phi}\right)^{\beta} - 1}{\beta} \right|$$

$$\leq \varepsilon\left(1+\left(\frac{f(tx)}{f(t)\cos\phi}\right)^{-\alpha}+\left(\frac{f(tx)}{f(t)\cos\phi}\right)^{-\alpha+\beta}\exp\left\{\varepsilon\left|\ln\frac{f(tx)}{f(t)\cos\phi}\right|\right\}\right), \tag{4.33}$$

and for all $t \leq t_0$ and $tx \leq t_0$,

$$(1-\varepsilon)x^{-1/\alpha}\exp(-\varepsilon|\log x|) \leq \frac{f(tx)}{f(t)} \leq (1+\epsilon)x^{-1/\alpha}\exp(\varepsilon|\log x|). \tag{4.34}$$

Since $f(t) \geq t_0$ and $t \leq t_0$ imply that $f(tx) \geq t_0$ and $tx \leq t_0$ for all $0 \leq x \leq 1$, respectively, by (4.33), (4.34), (4.20) and dominated convergence, we have

$$\lim_{t\to 0}\frac{\mathcal{J}_1(t)}{A(f(t))} = \frac{x}{\beta}\int_{h(x,y,0)}^{\pi/2}\left[x^{-\beta/\alpha}(\cos\phi)^{\alpha-\beta} - (\cos\phi)^{\alpha}\right]\mathrm{d}\phi \tag{4.35}$$

holds for all $x, y \geq 0$ and uniformly on $\mathcal{S}_2^+$. Similarly,

$$\lim_{t\to 0}\frac{\mathcal{J}_4(t)}{A(f(t))} = \frac{y}{\beta}\int_{-\arcsin\rho}^{h(x,y,0)}\left[y^{-\beta/\alpha}(\sin(\phi+\arcsin\rho))^{\alpha-\beta} - (\sin(\phi+\arcsin\rho))^{\alpha}\right]\mathrm{d}\phi \tag{4.36}$$

holds for all $x, y \geq 0$ and uniformly on $\mathcal{S}_2^+$.

Using (4.31) and a way similar to the proof of Lemma 5.2 of Draisma et al. (1999), for any $\varepsilon > 0$, there exists $t_0 > 0$ such that for all $t \leq t_0$ and $tx \leq t_0$

$$\left|\frac{(F_Y^{\leftarrow}(1-tx)/F_Y^{\leftarrow}(1-t))^{-\alpha} - x}{A(F_Y^{\leftarrow}(1-s))} - \mathcal{B}_{(2.6)}(x)\right| \leq \varepsilon\left(C_1 + C_2 x + C_3 x^{1-\beta/\alpha}\exp(\varepsilon|\ln x|)\right), \tag{4.37}$$

where the constants $C_1 > 0, C_2 > 0, C_3 > 0$ are independent of x and t. Hence, it follows from (4.20) and (4.37) that

$$\lim_{t\to 0}\frac{\mathcal{J}_2(t)}{A\left(F_Y^{\leftarrow}(1-t)\right)} = \mathcal{B}_{(2.6)}(x)\int_{h(x,y,0)}^{\pi/2}(\cos\phi)^\alpha\, d\phi \quad \text{and} \tag{4.38}$$

$$\lim_{t\to 0}\frac{\mathcal{J}_5(t)}{A\left(F_Y^{\leftarrow}(1-t)\right)} = \mathcal{B}_{(2.6)}(y)\int_{-\arcsin\rho}^{h(x,y,0)}(\sin(\phi+\arcsin\rho))^\alpha\, d\phi \tag{4.39}$$

holds for all $x, y \geq 0$ and uniformly on $\mathcal{S}_2^+$.

Note that

$$\begin{aligned}
&\frac{1}{A(f(t))}\left(\left(\frac{f(ty)}{f(tx)}\right)^{-\alpha}-\frac{y}{x}\right)\\
&= \frac{1}{A(f(t))}\left[\frac{1}{x}\left(\left(\frac{f(ty)}{f(t)}\right)^{-\alpha}-y\right)-\frac{1}{x}\left(\frac{f(ty)}{f(tx)}\right)^{-\alpha}\left(\left(\frac{f(tx)}{f(t)}\right)^{-\alpha}-x\right)\right]\\
&\overset{t\to 0}{\longrightarrow} \frac{1}{x}\mathcal{B}_{(2.6)}(y)-\frac{y}{x^2}\mathcal{B}_{(2.6)}(x).
\end{aligned}$$

Similar to the proof of Lemma 5.2 of Draisma et al. (1999), for any $\varepsilon > 0$, there exists $t_0 > 0$ such that for all $t \leq t_0, tx \leq t_0, ty \leq t_0$

$$\begin{aligned}
&\left|\frac{1}{A(f(t))}\left(\left(\frac{f(ty)}{f(tx)}\right)^{-\alpha}-\frac{y}{x}\right)-\frac{1}{x}\mathcal{B}_{(2.6)}(y)+\frac{y}{x^2}\mathcal{B}_{(2.6)}(x)\right|\\
&\leq \frac{1}{x}\varepsilon\left(C_1+C_2y+C_3y^{1-\beta/\alpha}e^{\varepsilon|\log y|}\right)\\
&\quad +\frac{1}{x}\left(\frac{y}{x}\right)\left(C_1+C_2x+C_3x^{1-\beta/\alpha}\exp(\varepsilon|\log x|)\right)\\
&\quad +\frac{1}{x}\left(\frac{y}{x}\right)\exp(\varepsilon|\log(y/x)|)\left(C_1+C_2x+C_3x^{1-\beta/\alpha}\exp(\varepsilon|\log x|)\right),
\end{aligned}\tag{4.40}$$

where constants $C_1 > 0, C_2 > 0, C_3 > 0$ are independent of t, x, y. Using (4.40),

$$\begin{cases}
\limsup_{z\to 0}|g'(z^{-1/\alpha})z^{2/\alpha}| < \infty\\
\limsup_{z\to\infty}|g'(z^{-1/\alpha})| < \infty\\
\limsup_{z\to\infty}[\sin(g(z^{-1/\alpha})+\arcsin\rho)]^\alpha z < \infty
\end{cases}$$

and applying a Taylor expansion to $g(z^{-1/\alpha})$, we can show that

$$\begin{aligned}
\lim_{t\to 0}\frac{\mathcal{J}_3(t)}{A(f(t))} &= \lim_{t\to 0}\frac{1}{A(f(t))}\int_{g(f(ty)/f(tx))}^{g\left((x/y)^{1/\alpha}\right)} x(\cos\phi)^\alpha\,\mathrm{d}\phi \\
&= \frac{x}{\alpha}\left[\cos\left(g\left((x/y)^{1/\alpha}\right)\right)\right]^\alpha g'\left((x/y)^{1/\alpha}\right)\left(\frac{\mathcal{B}_{(2.6)}(y)}{y}-\frac{\mathcal{B}_{(2.6)}(x)}{x}\right)\left(\frac{x}{y}\right)^{1/\alpha}
\end{aligned} \tag{4.41}$$

and

$$\begin{aligned}
\lim_{t\to 0}\frac{\mathcal{J}_6(t)}{A(f(t))} &= \lim_{t\to 0}\frac{1}{A(f(t))}\int_{g\left((x/y)^{1/\alpha}\right)}^{g(f(ty)/f(tx))} y(\sin(\phi+\arcsin\rho))^\alpha\,\mathrm{d}\phi \\
&= -\frac{y}{\alpha}\left[\sin\left(g\left((x/y)^{1/\alpha}\right)+\arcsin\rho\right)\right]^\alpha g'\left((x/y)^{1/\alpha}\right)\times \\
&\qquad \times\left(\frac{\mathcal{B}_{(2.6)}(y)}{y}-\frac{\mathcal{B}_{(2.6)}(x)}{x}\right)\left(\frac{x}{y}\right)^{1/\alpha}
\end{aligned} \tag{4.42}$$

holds for all $x, y \geq 0$ and uniformly on $\mathcal{S}_2^+$. Since

$$x\left[\cos\left(g\left((x/y)^{1/\alpha}\right)\right)\right]^\alpha = y\left[\sin\left(g\left((x/y)^{1/\alpha}\right)+\arcsin\rho\right)\right]^\alpha, \tag{4.43}$$

we obtain $\lim_{t\to 0}(\mathcal{J}_3(t)+\mathcal{J}_6(t))/A(f(t)) = 0$.

By Theorem 2.1, $\lambda(x,y) = (\mathcal{J}_7+\mathcal{J}_8)/\mathcal{B}_{(4.18)}$, hence

$$\begin{aligned}
&\lim_{t\to 0}\frac{1}{A(f(t))}\left(\frac{1}{\mathcal{B}_{(4.18)}(t)}(\mathcal{J}_7+\mathcal{J}_8)-\lambda(x,y)\right) \\
&= \lim_{t\to 0}\frac{1}{A(f(t))}\left(-\frac{\lambda(x,y)}{\mathcal{B}_{(4.18)}(t)}\left(\mathcal{B}_{(4.18)}(t)-\mathcal{B}_{(4.18)}\right)\right) \\
&= -\lambda(x,y)\frac{1}{\beta}\left(\int_{-\pi/2}^{\pi/2}(\cos\phi)^\alpha\left((\cos\phi)^{-\beta}-1\right)\,\mathrm{d}\phi\right)\left(\int_{-\pi/2}^{\pi/2}(\cos\phi)^\alpha\,\mathrm{d}\phi\right)^{-1},
\end{aligned} \tag{4.44}$$

which obviously holds uniformly on $\mathcal{S}_2^+$ since $\sup_{\mathcal{S}_2^+}\lambda(x,y) < \infty$. Note that

$$A\left(F_Y^{\leftarrow}(1-t)\right)/A\left(F_{\upsilon Y}^{\leftarrow}(1-t)\right) \overset{t\to 0}{\longrightarrow} \upsilon^{-\beta}. \tag{4.45}$$

Hence the theorem follows from (4.35), (4.36), (4.38), (4.39), (4.41), (4.42), (4.44) and (4.45). □

Proof of Theorem 2.4: Similar to Huang (1992) or Einmahl, de Haan and Li (2004), we have, as $n \to \infty$,

$$\sup_{0\le x,y\le T}\left|\sqrt{k_{\mathrm{Hu}}}\left\{x+y-\widehat{\lambda}^{Hu}_{k_{\mathrm{Hu}},n}(x,y)-l(x,y)\right\}-\mathcal{K}_{\mathrm{Hu}}\mathcal{B}_{(2.7)}(x,y)-B(x,y)\right| = o_p(1),$$

where

$$B(x,y) = W(x,y)-\left(1-\frac{\partial\lambda(x,y)}{\partial x}\right)W(x,0)-\left(1-\frac{\partial\lambda(x,y)}{\partial y}\right)W(0,y),$$

and $W(x,y)$ is a Wiener process with zero mean and covariance structure

$$\begin{aligned} E\left(W(x_1,y_1)W(x_2,y_2)\right) &= l(x_1\wedge x_2,y_1)l(x_1\wedge x_2,y_2)-l(x_1,y_1\wedge y_2) \\ &+ l(x_2,y_1\wedge y_2)-l(x_1,y_2)-l(x_2,y_1)-l(x_1\wedge x_2,y_1\wedge y_2). \end{aligned}$$

Hence (2.9) follows from $\lambda(x,y)=x+y-l(x,y)$. It is straightforward to check that (2.10), (2.11) and (2.12) hold. Note that the result can also be obtained from Schmidt and Stadtmüller (2005) by taking the bias into account. □

Proof of Theorem 2.5: The result follows directly from

$$\sqrt{k_{\mathrm{El}}}\left(\widehat{\alpha}^{\mathrm{H}}_{k_{\mathrm{El}},n}-\alpha\right) \xrightarrow{d} \mathcal{N}\left(-\alpha^2\mathcal{K}_{\mathrm{El}}\mathcal{B}_{(2.14)},\alpha^2\right)$$

(see de Haan and Peng (1998)), $\widehat{\tau}_n-\tau=o_p\left(k_{\mathrm{El}}^{-1/2}\right)$ and the delta method for the expression of $\lambda(x,y)$ given in Theorem 2.1. □

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