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Sonderforschungsbereich 386, Paper 480 (2006)

Online unter: http://epub.ub.uni-muenchen.de/

First published in *J. Appl. Probab.* **44** (4) ©2007 by The Applied Probability Trust

AN EXPONENTIAL CONTINUOUS TIME GARCH PROCESS

STEPHAN HAUG* **   CLAUDIA CZADO* ***

Abstract

In this paper we introduce an exponential continuous time GARCH(p, q) process. It is defined in such a way that it is a continuous time extension of the discrete time EGARCH(p, q) process. We investigate stationarity, mixing and moment properties of the new model. An instantaneous leverage effect can be shown for the exponential continuous time GARCH(p, p) model.

Keywords: exponential continuous time GARCH process; EGARCH, Lévy process; stationarity; stochastic volatility

2000 Mathematics Subject Classification: Primary 60G10, 60G12, 91B70
Secondary 91B84

1. Introduction

GARCH type processes have become very popular in financial econometrics to model returns of stocks, exchange rates and other series observed at equidistant time points. They have been designed (see Engle [9] and Bollerslev [3]) to capture so-called *stylised facts* of such data, which are e.g. volatility clustering, dependence without correlation and tail heaviness. Another characteristic is that stock returns seem to be negatively correlated with changes in the volatility, i.e. that volatility tends to increase after negative shocks and to fall after positive ones. This effect is called *leverage effect* and can not be modeled by a GARCH type process without further extensions. This finding led Nelson [19] to introduce the exponential GARCH process, which is able to model this asymmetry in stock returns. The log-volatility of the EGARCH(p, q) process was modeled as an ARMA$(q, p-1)$ process. We also like to mention another popular model the LARCH process, which explains besides a long memory property also the leverage effect as shown in Giraitis et al. [10].
The availability of high frequency data, which increased enormously in the last years, is one reason to consider continuous time models with similar behaviour as discrete time GARCH models. The reason for this is of course that at the highest available frequency the observations of the price process occur at irregularly spaced time points and therefore it is kind of natural to assume an underlying continuous time model. Different approaches have been taken to set up a continuous time model, which has the same features as discrete time GARCH processes. Recently Klüppelberg et al.[13] developed a continuous time GARCH$(1, 1)$ model, shortly called COGARCH$(1, 1)$. Their approach differs fundamentally from previous attempts, which could be summarized as

* Postal address: Chair for Mathematical Statistics, Centre for Mathematical Sciences, Munich University of Technology, Boltzmannstrae 3, D-85747 Garching, Germany.
** Email: haug@ma.tum.de
*** Email: cczado@ma.tum.de

diffusion approximations (see e.g. Nelson [18]), by the fact that their model is driven by only one source of randomness (like discrete time GARCH) instead of two (like in the diffusion approximations). They replaced the noise process of discrete time GARCH by the jumps of a Lévy process. The COGARCH$(1,1)$ was then extended by Brockwell et al.[5] to a continuous time GARCH(p,q) process for general orders $p,q \in \mathbb{N}$, $q \geq p$, henceforth called COGARCH(p,q).
In this paper a continuous time analogue of the EGARCH(p,q) model is introduced. The noise processes will also be modeled by the increments of a Lévy process. As in the discrete time case we describe the log-volatility process as a linear process, more preciseley a continuous time ARMA$(q,p-1)$ process.
The paper is now organized as follows. In Section 2 we review the definition of the discrete time EGARCH process. After a short review of elementary properties of Lévy processes we define the *exponential continuous time* GARCH(p,q) process at the beginning of Section 3. In addition we state stationarity conditions for the log-volatility and volatility process of our model. Afterwards the leverage effect in our model is considered. We close the section with an investigation of the mixing properties of the (log)volatility and return process. In Section 4 we derive second order properties of the volatility process. Section 5 is devoted to the analysis of the second order behaviour of the return process. We derive expressions for the first and second moment of the return process. The stylised fact of zero correlation in the return process but correlation of the squared returns is also shown.

2. The discrete time EGARCH process

Motivated by empirical evidence that stock returns are negatively correlated with changes in returns volatility Nelson [19] defined the exponential GARCH process (EGARCH) to model this effect, which is called *leverage effect* (see also Section 3.1).

The process $(X_n)_{n\in\mathbb{Z}}$ of the form $X_n = \sigma_n \epsilon_n \,, n \in \mathbb{Z}$, where $(\epsilon_n)_{n\in\mathbb{Z}}$ is an i.i.d. sequence with $\mathbb{E}(\epsilon_1) = 0$ and $\mathbb{V}\mathrm{ar}(\epsilon_1) = 1$, is called an EGARCH process*, if the volatility process $(\sigma_n^2)_{n\in\mathbb{Z}}$ satisfies*

$$\log(\sigma_n^2) = \mu + \sum_{k=1}^{\infty} \beta_k f(\epsilon_{n-k}) \,,$$

where $f : \mathbb{R} \to \mathbb{R}$ is some measurable real valued deterministic function, $\mu \in \mathbb{R}$ and $(\beta_k)_{k\in\mathbb{N}}$ are real coefficients such that $\mathbb{E}(|f(\epsilon_n)|) < \infty \,, \mathbb{V}\mathrm{ar}(f(\epsilon_n)) < \infty$ and $\sum_{k=1}^{\infty} |\beta_k| < \infty$.

Nelson [19] also suggested a finite parameter model by modeling the log-volatility as an ARMA$(q,p-1)$ process instead of an infinite moving average process. This leads to the EGARCH(p,q) model, which is defined in the following way.

Let $p,q \in \mathbb{N}, \mu, \alpha_1, \ldots, \alpha_q, \beta_1, \ldots, \beta_p \in \mathbb{R}$, suppose $\alpha_q \neq 0 \,, \beta_p \neq 0$ and that the autoregressive polynomial $\phi(z) := 1 - \alpha_1 z - \cdots - \alpha_q z^q$ *and the* moving average polynomial $\psi(z) := \beta_1 + \beta_2 z + \cdots + \beta_p z^{p-1}$ *have no common zeros and that $\phi(z) \neq 0$ on $\{z \in \mathbb{C} \mid |z| \leq 1\}$. Let $(\epsilon_n)_{n\in\mathbb{Z}}$ be an i.i.d. sequence with $\mathbb{E}(\epsilon_1) = 0$ and $\mathbb{V}\mathrm{ar}(\epsilon_1) = 1$,* *and let* $f(\cdot)$ *be such that* $\mathbb{E}(|f(\epsilon_n)|) < \infty$ *and* $\mathbb{V}\mathrm{ar}(f(\epsilon_n)) < \infty$. *Then* $(X_n)_{n\in\mathbb{Z}}$, *where* $X_n = \sigma_n \epsilon_n$ *and*

$$\log(\sigma_n^2) = \mu + \sum_{k=1}^{p} \beta_k f(\epsilon_{n-k}) + \sum_{k=1}^{q} \alpha_k \log(\sigma_{n-k}^2)$$

is called an EGARCH(p,q) process.

To achieve the asymmetric relation between the stock returns and the volatility, $f(\epsilon_n)$ must be a function of the magnitude and the sign of ϵ_n as noted by Nelson [19]. Therefore he proposed the following function:

$$f(\epsilon_n) := \theta\epsilon_n + \gamma[|\epsilon_n| - \mathbb{E}(|\epsilon_n|)]\,, \tag{2.1}$$

with real coefficients θ and γ. We see that $f(\epsilon_n)$ is piecewise linear in ϵ_n and has slope $\theta+\gamma$ for positive shocks ϵ_n and slope $\theta-\gamma$ for negative ones. Therefore $f(\epsilon_n)$ allows the volatility process $(\sigma_n^2)_{n\in\mathbb{Z}}$ to respond asymmetrically to positive and negative jumps in the stock price.

3. Exponential COGARCH

The goal of this section is to construct a continuous time analogue of the discrete time EGARCH(p,q) process. Therefore we will use the idea of Klüppelberg et al. [13] to replace the noise variables ϵ_n by the increments of a Lévy process $L = (L_t)_{t\geq 0}$. Any Lévy process L on $\mathbb{R}$ has a characteristic function of the form $\mathbb{E}(e^{iuL_t}) = \exp\{t\psi_L(u)\}\,,\quad t \geq 0$, with

$$\psi_L(u) \quad := \quad i\gamma_L u - \frac{\tau_L^2}{2}u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iux\chi_{(-1,1)}(x))\nu_L(dx)\,, \quad u \in \mathbb{R},$$

where $\tau_L^2 \geq 0,\ \gamma_L \in \mathbb{R}$, the measure ν_L satisfies

$$\nu_L(\{0\}) = 0 \qquad \text{and} \qquad \int_{\mathbb{R}} \min(x^2, 1)\nu_L(dx) < \infty$$

and $\chi_A(\cdot)$ denotes the indicator function of the set $A \subset \mathbb{R}$. The measure ν_L is called the *Lévy measure* of L and the triplet $(\gamma_L, \tau_L^2, \nu_L)$ is called the *characteristic triplet* of L. The map ψ_L is called the *Lévy symbol* . For more details on Lévy processes we refer to Sato [21] or Applebaum [1].

We consider Lévy processes L defined on a probability space $(\Omega, \mathcal{F}, P)$ with jumps $\Delta L_t := L_t - L_{t-}$, zero mean and finite variance. In that case the Lévy-Itô decomposition (see e.g. Theorem 2.4.16 of Applebaum [1]) of L is

$$L_t = B_t + \int_0^t \int_{\mathbb{R}\setminus\{0\}} x\tilde{N}_L(dt, dx)\,, \qquad t \geq 0,$$

where B is a Brownian motion with variance τ_L^2 and $\tilde{N}_L(t, dx) = N_L(t, dx) - t\nu_L(dx)$, $t \geq 0$, is the compensated random measure associated to the Poisson random measure

$$N_L(t, A) = \#\{0 \leq s < t; \Delta L_s \in A\} = \sum_{0<s\leq t} \chi_A(\Delta L_s), \qquad A \in \mathcal{B}(\mathbb{R} \setminus \{0\}),$$

on $\mathbb{R}_+ \times \mathbb{R} \setminus \{0\}$, which is independent of B.

The driving noise process in this continuous time model will be constructed similar as in the discrete time case. In particular for a zero mean Lévy process L, with $\mathbb{E}(L_1^2) < \infty$, and parameters $(\theta, \gamma)^T \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ we define the driving process M of the log-volatility process by

$$M_t := \int_{\mathbb{R}\setminus\{0\}} h(x)\tilde{N}_L(t, dx)\,, \qquad t \geq 0, \tag{3.1}$$

with $h(x) := \theta x + \gamma |x|$.

Remark 3.1 *(i) The process M defined by (3.1) is by construction a process with independent and stationary increments and by Theorem 4.3.4 in Applebaum [1] well defined if*

$$\int_{\mathbb{R}} |h(x)|^2 \nu_L(dx) < \infty\,. \tag{3.2}$$

Condition (3.2) is satisfied since ν_L is a Lévy measure and L has finite variance. By equation (2.9) of Applebaum [1] the characteristic triplet of M is $(\gamma_M, 0, \nu_M)$, where $\nu_M := \nu_L \circ h^{-1}$ is the Lévy measure of M and $\gamma_M := -\int_{|x|>1} x\nu_M(dx)$. The precise form of ν_M depends on the sign and size of θ and γ and is given in the following formulas:

$$\nu_M((-\infty, -x]) = \begin{cases} \nu_L([-\frac{x}{\theta+\gamma}, \infty)) + \nu_L((-\infty, -\frac{x}{\theta-\gamma}])\,, & -\gamma > \theta > \gamma \\ \nu_L((-\infty, -\frac{x}{\theta-\gamma}])\,, & -\theta < \gamma < \theta \\ \nu_L([-\frac{x}{\theta+\gamma}, \infty))\,, & -\theta > \gamma > \theta \\ 0 & -\gamma < \theta < \gamma \end{cases}$$

and

$$\nu_M([x, \infty)) = \begin{cases} \nu_L([\frac{x}{\theta+\gamma}, \infty)) + \nu_L((-\infty, \frac{x}{\theta-\gamma}])\,, & -\gamma < \theta < \gamma \\ \nu_L((-\infty, \frac{x}{\theta-\gamma}])\,, & -\theta > \gamma > \theta \\ \nu_L([\frac{x}{\theta+\gamma}, \infty))\,, & -\theta < \gamma < \theta \\ 0 & -\gamma > \theta > \gamma \end{cases}$$

for $x > 0$. One recognises that M is a spectrally negative Lévy process for $\gamma < \theta < -\gamma$, i.e. M has only negative jumps, and a spectrally positive Lévy process for $-\gamma < \theta < \gamma$.
(ii) In case the jump part of L is of finite variation, M is a Lévy process of finite variation with Lévy-Itô decomposition

$$M_t := \sum_{0<s\leq t} [\theta \Delta L_s + \gamma |\Delta L_s|] - Ct\,, \qquad t > 0,$$

where $C := \gamma \int_{\mathbb{R}} |x| \nu_L(dx)$.

Now we define the *exponential continuous time* GARCH(p, q) *process* by specifying the log-volatility process as a continuous time ARMA$(q, p-1)$ process, henceforth called CARMA$(q, p-1)$ process (see e.g. Brockwell and Marquardt [6] for details on CARMA processes), which is the continuous time analogue of an ARMA$(q, p-1)$ process. The driving noise process of the CARMA$(q, p-1)$ process will be defined similarly to (2.1).

Definition 3.2 *Let $L = (L_t)_{t\geq 0}$ be a zero mean Lévy process with Lévy measure ν_L such that $\int_{|x|\geq 1} x^2 \nu_L(dx) < \infty$. Then we define the* exponential COGARCH(p, q) *process G, shortly* ECOGARCH(p, q)*, as the stochastic process satisfying,*

$$dG_t \quad := \quad \sigma_{t-} dL_t, \quad t > 0, \quad G_0 = 0,$$

where the log-volatility process $\log(\sigma^2) = (\log(\sigma_t^2))_{t\geq 0}$ *is a* CARMA$(q, p-1)$ *process, $1 \leq p \leq q$, with mean $\mu \in \mathbb{R}$ and state space representation*

$$\log(\sigma_t^2) \quad := \quad \mu + \mathbf{b}^T \mathbf{X}_t, \quad t > 0\,, \log(\sigma_0^2) = \mu + \mathbf{b}^T \mathbf{X}_0 \tag{3.3}$$

$$d\mathbf{X}_t \quad = \quad A\mathbf{X}_t + \mathbf{1}_q dM_t\,, \quad t > 0 \tag{3.4}$$

where $\mathbf{X}_0 \in \mathbb{R}^q$ is independent of the driving Lévy process M. The $q \times q$ matrix A and the vectors $\mathbf{b} \in \mathbb{R}^q$ and $\mathbf{1}_q \in \mathbb{R}^q$ are defined by

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_q & -a_{q-1} & -a_{q-2} & \cdots & -a_1 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{q-1} \\ b_q \end{bmatrix}, \ \mathbf{1}_q = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

with coefficients $a_1, \ldots, a_q, b_1, \ldots, b_p \in \mathbb{R}$, where $a_q \neq 0, b_p \neq 0$, and $b_{p+1} = \cdots = b_q = 0$.

Returns over a time interval of length $r > 0$ are described by the increments of G

$$G_t^{(r)} := G_t - G_{t-r} = \int_{(t-r,t]} \sigma_{s-}\, dL_s\,, \qquad t \geq r > 0\,. \tag{3.5}$$

Thus this gives us the possibility to model ultra high frequency data, which consists of returns over varying time intervals. On the other hand an equidistant sequence of such non-overlapping returns of length r is given by $(G_{nr}^{(r)})_{n\in\mathbb{N}}$.

In the sequel we refer to G and $G^{(r)}$ as the *(log-)price process* and *(log-)return process*, respectively. Also σ^2 and $\log(\sigma^2)$ will be called the *volatility process* and *log-volatility process*, respectively.

Proposition 3.3 *Let σ^2 and G be as in Definition 3.2, with θ and γ not both equal to zero. If the eigenvalues of A all have negative real parts and $\mathbf{X}_0$ has the same distribution as $\int_0^\infty e^{Au}\mathbf{1}_q dM_u$, then $\log(\sigma^2)$ and σ^2 are strictly stationary.*

Proof: The strict stationarity of $\log(\sigma^2)$ follows from Proposition 2 in Brockwell and Marquardt [6], since it is a CARMA$(q, p-1)$ process. Since strict stationarity is invariant under continuous transformations, σ^2 also has this property. □

Remark 3.4 *The solution of the continuous time state space model (3.3) and (3.4) has the representation*

$$\log(\sigma_t^2) = \mu + \mathbf{b}^T e^{At}\mathbf{X}_0 + \int_0^t \mathbf{b}^T e^{A(t-u)}\mathbf{1}_q dM_u, \qquad t > 0.$$

If we choose a second Lévy process $(\tilde{L}_t)_{t\geq 0}$ independent of L and with the same distribution as L, then we can define an extension $(L_t^*)_{t\in\mathbb{R}}$ of L to the real line by:

$$L_t^* := L_t\chi_{[0,\infty)}(t) - \tilde{L}_{-t-}\chi_{(-\infty,0)}(t), \qquad t \in \mathbb{R},$$

where $\chi_A(\cdot)$ denotes the indicator function of the set A. Using L^* instead of L in (3.1) we get an extension M^* of M. In the following we will write for simplicity L and M instead of L^* and M^*. In the strictly stationary case the log-volatility process can be defined on the whole real line

$$\log(\sigma_t^2) = \mu + \int_{-\infty}^t g(t-u)dM_u, \qquad t \in \mathbb{R}, \tag{3.6}$$

with kernel function

$$g(t) = \mathbf{b}^T e^{At}\mathbf{1}_q\chi_{(0,\infty)}(t) \tag{3.7}$$

(see section 2 of Brockwell and Marquardt [6] for more details).

From (3.5) it follows directly that the increments $G_\cdot^{(r)} = \int_{(\cdot-r,\cdot]}\sigma_{s-}dL_s$ of G are stationary if the volatility σ^2 is stationary, since the increments of L are stationary and independent by definition.

Corollary 3.5 *If σ^2 is strictly stationary, then G has strictly stationary increments.*

Remark 3.6 *(i) If $q \geq p+1$ the log-volatility process is $(q-p-1)$ times differentiable, which follows from the state space representation of $\log(\sigma^2)$, and hence the volatility process has continuous sample path. In particular the volatility will only contain jumps for $p = q$.*
(ii) The volatility of the ECOGARCH(p,q) process is positive by definition. Therefore the parameters do not need to satisfy any constraints to assure positivity of the volatility. This is not the case for the COGARCH(p,q) model. For higher order COGARCH(p,q) processes these condition become quite difficult to check (see Theorem 5.1 in Brockwell et al. [5]).

3.1. Leverage effect

In empirical return data researchers have found evidence (see e.g. Section 1 in Nelson [19]) that current returns are negatively correlated with future volatility. This means that a negative shock increases the future volatility more than a positive one of the same size or increases it while a positive one even decreases the volatility. This phenomenon is called *leverage effect* in the literature.
If we take a look at the shocks of the state process $\mathbf{X}$ in the ECOGARCH(p,q) model

$$\Delta M_t = \begin{cases} (\theta+\gamma)\Delta L_t, & \Delta L_t \geq 0 \\ (\theta-\gamma)\Delta L_t, & \Delta L_t < 0 \end{cases},$$

we see that:

(i) for $-\gamma < \theta < 0$ $(0 < \theta < \gamma)$ a positive jump ΔL_t leads to a smaller (greater) positive jump ΔM_t than a negative jump of the same size,

(ii) for $\theta > |\gamma|$ a positive jump ΔL_t leads to a positive jump ΔM_t, while a negative jump of the same size results in a negative jump ΔM_t,

(iii) for $0 < \theta < -\gamma$ $(\gamma < \theta < 0)$ a positive jump ΔL_t leads to a smaller (greater) negative jump ΔM_t than a negative jump of the same size,

(iv) for $\theta < -|\gamma|$ a positive jump ΔL_t leads to a negative jump ΔM_t, while a negative jump of the same size results in a positive jump ΔM_t.

If we compare this to the COGARCH(p,q) process, we see that in the COGARCH model the innovations of the volatility process at time t are given by the squared innovations of the log-price process (see Section 2 of Brockwell et al. [5]). Hence the volatility process of the COGARCH model reacts in the same way to positive and negative shocks. Now we will consider the *instantaneous leverage effect*, which is defined as

$$\mathbb{C}\mathrm{ov}(\Delta G_t, \sigma_t^2 \mid \ |\Delta L_t| > \epsilon)$$

being negative. Intuitively it is clear that this correlation can only be different from zero, if the sample paths of σ^2 exhibit jumps. But from Remark 3.6 (iv) we know that this is just the case for $p = q$. The reason is that for $p < q$ the parameter b_q will be zero and therefore the jump ΔL_t at time t just contributes to the $(q-1)$th derivative of the state process $\mathbf{X}$, but is not taken into account for the log-volatility at that time point. Thus we will expect an instantaneous leverage effect only for the ECOGARCH(p,p) models. This will be shown in the next proposition, in particular we will show that the sign of the correlation is equal to the sign of θb_q. This result is similar to the discrete time case (see Proposition 2.9 in Surgailis and Viano [22]).

Proposition 3.7 *Assume that the distribution of the jumps of L is symmetric, i.e. for all $\epsilon > 0$,*

$$P(\Delta L_t \in dx| \ |\Delta L_t| > \epsilon) = P(\Delta L_t \in -dx| \ |\Delta L_t| > \epsilon), \qquad t \geq 0.$$

Conditionally on the event that $|\Delta L_t| > \epsilon$, the sign of $\mathbb{C}\mathrm{ov}(\Delta G_t, \sigma_t^2)$ is equal to the sign of θb_q.

Proof: Since the distribution of the jumps of L is symmetric we get

$$\mathbb{E}(\Delta G_t \mid \; |\Delta L_t| > \epsilon) = \mathbb{E}(\sigma_{t-})\mathbb{E}(\Delta L_t \mid \; |\Delta L_t| > \epsilon) = 0\,.$$

This then implies

$$\begin{aligned}\mathbb{C}\mathrm{ov}(\Delta G_t, \sigma_t^2 \mid \; |\Delta L_t| > \epsilon) &= \mathbb{E}(\Delta G_t \sigma_t^2 \mid \; |\Delta L_t| > \epsilon) \\ &= \mathbb{E}\left(\Delta G_t \exp\left\{\log(\sigma_{t-}^2) + b_q \Delta M_t\right\} \mid \; |\Delta L_t| > \epsilon\right) \\ &= \mathbb{E}\left(\sigma_{t-}^3 \Delta L_t \exp\{b_q(\theta \Delta L_t + \gamma |\Delta L_t|)\} \mid \; |\Delta L_t| > \epsilon\right)\end{aligned}$$

Since ΔL_t is independent of σ_{t-}^3 we get

$$\begin{aligned}&\mathbb{C}\mathrm{ov}(\Delta G_t, \sigma_t^2 |\; |\Delta L_t| > \epsilon) \\ &= \mathbb{E}(\sigma_{t-}^3)\mathbb{E}(\Delta L_t \exp\{b_q(\theta \Delta Lt + \gamma |\Delta L_t|)\} \mid |\Delta L_t| > \epsilon) \\ &= \mathbb{E}(\sigma_{t-}^3) \int_{x>\epsilon} x \exp(b_q \gamma x)(\exp(\theta b_q x) - \exp(-\theta b_q x)) P(\Delta L_t \in dx |\; |\Delta L_t| > \epsilon)\,.\end{aligned}$$

From $\mathrm{sgn}(\exp(\theta b_q x) - \exp(-\theta b_q x)) = \mathrm{sgn}(\theta b_q)$ for all $x > \epsilon$ the desired result follows. $\square$

Example 3.8 *As a first illustrative example we consider an ECOGARCH$(1,1)$ process driven by a Lévy process L with Lévy symbol*

$$\psi_L(u) = -\frac{u^2}{2} + \int_{\mathbb{R}} (e^{iux} - 1)\lambda \Phi_{0,1/\lambda}(dx)\,,$$

where $\Phi_{0,1/\lambda}(\cdot)$ is the distribution function of a normal distribution with mean 0 and variance $1/\lambda$. This means that L is the sum of a standard Brownian motion W and the compound Poisson process $J_t = \sum_{k=1}^{N_t} Z_k$, $t \geq 0$, where $(N_t)_{t\in\mathbb{R}}$ is an independent Poisson process with intensity $\lambda > 0$ and jump times $(T_k)_{k\in\mathbb{Z}}$. The Poisson process N is also independent from the i.i.d. sequence of jump sizes $(Z_k)_{k\in\mathbb{Z}}$, with $Z_1 \sim N(0, 1/\lambda)$. The Lévy process M is in this case given by the following expression

$$M_t = \sum_{k=1}^{N_t} [\theta Z_k + \gamma |Z_k|] - Ct\,, \quad t > 0,$$

with $C = \gamma \int_{\mathbb{R}} |x| \lambda \Phi_{0,1/\lambda}(dx) = \sqrt{\frac{2\lambda}{\pi}}\gamma$. If we just consider the case that $\theta < -\gamma < 0$ then the Lévy measure ν_M of M is defined by

$$\nu_M((-\infty, -x]) = \lambda \Phi_{0,1/\lambda}\left(\left[-\frac{x}{\theta+\gamma}, \infty\right)\right)\,, \qquad x > 0,$$

on the negative half real line and by

$$\nu_M([x, \infty)) = \lambda \Phi_{0,1/\lambda}\left(\left(-\infty, \frac{x}{\theta-\gamma}\right]\right)\,, \qquad x > 0,$$

on the positive half real line. In the top row of Figure 1 a simulated sample path of the compound Poisson process J, with $N(0, 1/2)$ distributed jumps, can be seen over *three time scales. The corresponding Lévy process M, with parameters $\theta = -0.2$ and $\gamma = 0.1$, can be seen in the bottom row. Over all three time intervals one can recognise the desired asymmetry for this set of parameters. If J jumps up, then M jumps down and vice versa. If J does not move, then one observes the downwards drift of M, which can bee seen on the right hand side of Figure 1.*

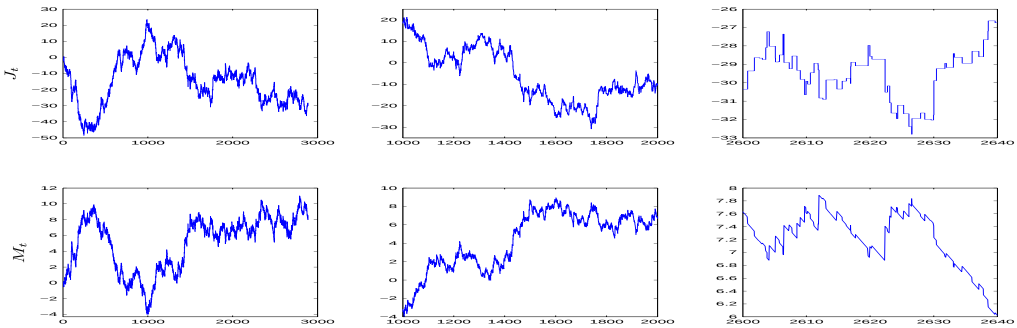

Figure 1: Simulated sample pathes of J (*top row*) and M (*bottom row*), with parameters $\theta = -0.2$ and $\gamma = 0.1$, over three different time scales.

The log-volatility process is then of the form

$$\begin{aligned}
\log(\sigma_t^2) &= \mu + b_1 e^{-a_1 t} X_0 + \int_0^t b_1 e^{-a_1(t-s)} dM_s \\
&= \mu + b_1 e^{-a_1 t} X_0 + \sum_{k=1}^{N_t} b_1 e^{-a_1(t-T_k)} [\theta Z_k + \gamma |Z_k|] - C \frac{b_1}{a_1} \left(1 - e^{-a_1 t}\right) ,
\end{aligned}$$

for $t > 0$, and the log-price process is given by

$$G_t = \int_0^t \sigma_{s-} dW_s + \sum_{k=1}^{N_t} \sigma_{T_k-} Z_k \,, \quad t > 0, \quad G_0 = 0 \,.$$

with jump times T_k, $k \in \mathbb{N}$.

Generally the simulation of a sample path of the log-price process G and the log-volatility process $\log(\sigma^2)$ over a time interval $[0, T]$ is done in the following steps.

1. *Choose observation times $0 = t_0 < t_1 < \cdots < t_n \leq T$, possibly random.*
2. *Simulate the jump times (T_k), $k = 1, \ldots, n_T$, with $n_T := \max\{k \in \mathbb{N} : T_k \leq T\}$, of the compound Poisson process J.*
3. *Approximate the state process (3.4) of the log-volatility by a stochastic Euler scheme.*

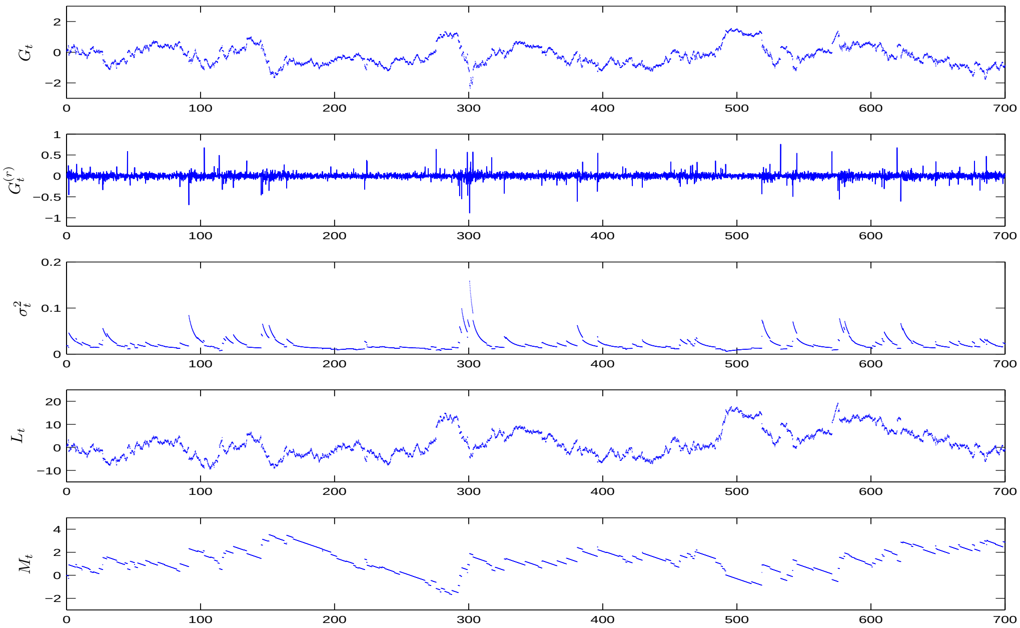

Figure 2: Observations of the log-price process G_t (*top row*), the return process $G_t^{(r)}$ (*second row*), the volatility process σ_t^2 (*third row*), with parameters $b_1 = 1, a_1 = 0.1, \mu = -4, \theta = -0.2$ and $\gamma = 0.1$ and the driving Lévy process L_t (*last row*) in the time interval $(0, 700]$.

4. Compute an approximation $\widehat{G}$ via the recursion

$$\widehat{G}_{t_i} = \widehat{G}_{t_{i-1}} + \sigma_{t_{i-1}} \tilde{W}_i + \sum_{k=N_{t_{i-1}}+1}^{N_{t_i}} \sqrt{\exp\{\mu + \mathbf{b}^T \widehat{\mathbf{X}}_{T_k-}\}} Z_k,$$

where $\tilde{W}_i \sim N(0, t_i - t_{i-1})$ and $\widehat{\mathbf{X}}_{T_k-}$ is the Euler approximation without the jump ΔM_{T_k}.

In Figure 2 the results of the above simulation procedure are shown. The jump rate λ is now chosen to be $1/4$, which implies a variance of the jump sizes Z_i of 4. For exponentially distributed interarrival times $\Delta t_i := t_i - t_{i-1} \sim expo(1)$ the sample path of the log-price G, the return process $G^{(\Delta t.)}$ and the volatility process σ^2 are displayed in the first three rows of Figure 2. The sample path of the driving Lévy process L is shown in the last row. From the plots of the return and volatility process we see the negative correlation between the two processes. We recognise on the one hand increases in the volatility after large negative returns and on the other a decrease in the volatility after a larger positive return. This displays the leverage effect explained in Section 3.1.

3.2. Mixing

Mixing properties are useful for a number of applications including asymptotic statistics as the central limit theorem is in place for mixing processes (cf. Doukhan [8] for a comprehensive treatment of mixing properties) . For an example in this continuous time GARCH setting compare Theorem 3 in Haug et al. [12]. Thus we will derive mixing properties of the strictly stationary volatility process and the return process in the ECOGARCH(p, q) model.

First we recall the definition of strong mixing, which is also called α-mixing for a process with continuous time parameter.

Definition 3.9 (Davydov [7]) *For a process $Y = (Y_s)_{s\geq 0}$ define the σ-algebras $\mathcal{F}^Y_{[0,u]} := \sigma((Y_s)_{s\in[0,u]})$ and $\mathcal{F}^Y_{[u+t,\infty)} := \sigma((Y_s)_{s\geq u+t})$ for all $u \geq 0$. Then Y is called strongly or α-mixing, if*

$$\begin{aligned} \alpha_Y(t) &= \sup_{u\geq 0} \alpha(\mathcal{F}^Y_{[0,u]}, \mathcal{F}^Y_{[u+t,\infty)}) \\ &:= \sup_{u\geq 0} \sup\{|P(A\cap B) - P(A)P(B)| : A \in \mathcal{F}^Y_{[0,u]}, B \in \mathcal{F}^Y_{[u+t,\infty)}\} \to 0, \end{aligned}$$

as $t \to \infty$.

Above we denote by $\sigma(\cdot)$ the generated completed σ-algebra. The strong mixing property with exponential rate of the log-volatility, volatility and return process is the subject of the next theorem. Here strong mixing with exponential rate (exponentially α-mixing) means that $\alpha(t)$ decays to zero exponentially fast for $t \to \infty$.

Theorem 3.10 *Let $\log(\sigma^2)$ be defined by (3.3) and (3.4) with θ and γ not both equal to zero. Assume that $\mathbb{E}(L_1^2) < \infty$, the eigenvalues of A all have negative real parts and $\mathbf{X}_0$ has the same distribution as $\int_0^\infty e^{Au}\mathbf{1}_q dM_u$, hence $\log(\sigma^2)$ and σ^2 are strictly stationary.*
(i) Then there exist constants $K > 0$ and $a > 0$ such that

$$\alpha_{\log(\sigma^2)}(t) \leq K \cdot e^{-at} \quad \text{and} \quad \alpha_{\sigma^2}(t) \leq K \cdot e^{-at}, \qquad \text{as} \quad t \to \infty, \tag{3.8}$$

where $\alpha_{\log(\sigma^2)}(t)$ and $\alpha_{\sigma^2}(t)$ are the α-mixing coefficients of the log-volatility and volatility process, respectively.
(ii) Then the discrete time process $(G^{(r)}_{nr})_{n\in\mathbb{N}}$, where $G^{(r)}_{nr}$ is defined in (3.5), is strongly mixing with exponential rate and ergodic.

Proof: (i) The log-volatility process is a CARMA$(q, p-1)$ process, which is equal to the first component of the q-dimensional OU process $\mathbf{V} := (V^1, \ldots, V^q)^T \in \mathbb{R}^q$ (see e.g. Section 4 of Brockwell [4]) where for fixed t

$$\mathbf{V}_t = e^{BAB^{-1}(t-s)}\mathbf{V}_s + \int_s^t e^{A(t-u)} B\mathbf{1}_q dM_u \qquad a.s., \tag{3.9}$$

with

$$B = \begin{bmatrix} b_1 & b_2 & b_3 & \cdots & b_q \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Since L, hence M, has finite second moment $\mathbf{V}$ also has finite second moment. Therefore the condition (4.5) in Masuda [15] is satisfied. By Theorem 4.3 in Masuda [15] $\mathbf{V}$ is then exponentially α-mixing. Since every component of a multidimensional exponentially strong mixing process is exponentially strong mixing, the log-volatility process is also exponentially α-mixing. The property of α-mixing is invariant under continuous transformations, which implies that σ^2 also has this property.
(ii) Define the σ-algebras $\mathcal{F}_I^{\sigma^2,dL} := \sigma(\sigma_t^2, L_t - L_s \, : \, s,t \in I)$ for $I \subset \mathbb{R}$ and $\mathcal{F}_J^{G^{(r)}} := \sigma(G_{kr}^{(r)} : k \in J)$ for $J \subset \mathbb{N}$. From (3.5) it follows that

$$\mathcal{F}_{\{1,2,\dots,l\}}^{G^{(r)}} \subset \mathcal{F}_{[0,lr]}^{\sigma^2,dL} \quad \text{and} \quad \mathcal{F}_{\{k+l,k+l+1,\dots\}}^{G^{(r)}} \subset \mathcal{F}_{[(k+l-1)r,\infty)}^{\sigma^2,dL} . \tag{3.10}$$

To show the strong mixing property of the return process we will use the following relation

$$\alpha(\mathcal{F}_1, \mathcal{F}_2) \le \tilde{\alpha}(\mathcal{F}_1, \mathcal{F}_2) \le 6\alpha(\mathcal{F}_1, \mathcal{F}_2) , \tag{3.11}$$

where $\mathcal{F}_1$ and $\mathcal{F}_2$ are σ-algebras,

$$\tilde{\alpha}(\mathcal{F}_1, \mathcal{F}_2) := \sup\left\{ \|\mathbb{E}(f|\mathcal{F}_1) - \mathbb{E}(f)\|_{L^1(P)} : \; f \in b\mathcal{F}_2, \, \|f\|_\infty \le 1 \right\}$$

and $b\mathcal{F}$ denotes the set of all bounded $\mathcal{F}$-measurable random variables. The left-hand inequality is easy to see (cf. Lemma B.2 in Haug et al. [12]) and the right-hand inequality follows from Lemma 3.5 in McLeish [17]. For a stochastic process Y the corresponding $\tilde{\alpha}$-mixing coefficient is defined as $\tilde{\alpha}_Y(t) := \sup_{s\in\mathbb{R}_+} \tilde{\alpha}(\mathcal{F}_{[0,s]}^T, \mathcal{F}_{[s+t,\infty)}^Y)$, $t \in \mathbb{R}_+$ (see e.g. Section 2.1 in Masuda [16]). Now since $(G_{nr}^{(r)})_{n\in\mathbb{N}}$ is strictly stationary we have the following

$$\begin{aligned} \tilde{\alpha}_{G^{(r)}}(k-l) &= \sup\left\{ \|\mathbb{E}(f|\mathcal{F}_{\{1,2,\dots,l\}}^{G^{(r)}}) - \mathbb{E}(f)\|_{L^1(P)} : \; f \in b\mathcal{F}_{\{k,k+1,\dots\}}^{G^{(r)}}, \, \|f\|_\infty \le 1 \right\} \\ &\le \sup\left\{ \|\mathbb{E}(f|\mathcal{F}_{[0,lr]}^{\sigma^2,dL}) - \mathbb{E}(f)\|_{L^1(P)} : \; f \in b\mathcal{F}_{[(k-1)r,\infty)}^{\sigma^2,dL}, \, \|f\|_\infty \le 1 \right\} , \end{aligned}$$

where the inequality follows from (3.10) and an application of Jensen's inequality (see also Remark 1 in Masuda [16]). From the exponentially α-mixing property of σ^2 and relation (3.11) we get that there exists a constant $K_\sigma > 0$ such that

$$\begin{aligned} \tilde{\alpha}_{\sigma^2}(t-s) &= \sup\left\{ \|\mathbb{E}(f|\mathcal{F}_{[0,s]}^{\sigma^2}) - \mathbb{E}(f)\|_{L^1(P)} : f \in b\mathcal{F}_{[t,\infty)}^{\sigma^2,dL}, \, \|f\|_\infty \le 1 \right\} \\ &\le K_{\sigma^2} e^{-a(t-s)} , \end{aligned}$$

for all $0 \le s \le t < \infty$ and $\|f\|_\infty \le 1$. Now it follows analogously to the proof of Lemma 1 in Kusuoka and Yoshida [14] that

$$\|\mathbb{E}(f|\mathcal{F}_{[0,lr]}^{\sigma^2,dL}) - \mathbb{E}(f)\|_{L^1(P)} \le K_{\sigma^2} e^{-a((k-1-l)r)} \|f\|_\infty$$

for all $f \in b\mathcal{F}_{[(k-1)r,\infty)}^{\sigma^2,dL}$. The only difference is that we do not have a Markov process, hence we have to condition on the information over the whole time interval $[0, lr]$ and not just on the information at the time point lr. This implies that we have

$$\tilde{\alpha}_{G^{(r)}}(k-l) \le K_{\sigma^2} e^{-a((k-1-l)r)} ,$$

which means that $(G_{nr}^{(r)})_{n\in\mathbb{N}}$ is exponentially α-mixing by (3.11). Since strict stationarity and strong mixing imply ergodicity the result follows. □

4. Second order properties of the volatility process

In this section we derive moments and the autocovariance function of the volatility process σ^2. Since it is a non-linear transformation of a CARMA$(q, p-1)$ process, we will first recall the moment structure and conditions for weak stationarity of a CARMA$(q, p-1)$ process.

Proposition 4.1 *If* $\mathbf{X}_0$ *has the same mean vector and covariance matrix as* $\int_0^\infty e^{Au}\mathbf{1}_q dM_u$, *then* $\log(\sigma^2)$ *is weakly stationary. In the weakly stationary case the mean and autocovariance function of* $\log(\sigma^2)$ *are given by*

$$\mathbb{E}(\log(\sigma_t^2)) = \mu \quad \textit{and} \quad \mathrm{Cov}(\log(\sigma_{t+h}^2), \log(\sigma_t^2)) = \mathbb{E}(M_1^2)\mathbf{b}^T e^{Ah}\Sigma\mathbf{b}\,, \qquad t, h \geq 0,$$

where $\Sigma := \int_0^\infty e^{As}\mathbf{1}_q\mathbf{1}_q{}^T e^{A^T s} ds$.

Proof: The condition for weak stationarity of $\log(\sigma^2)$ is given in Proposition 1 in Brockwell and Marquardt [6]. The moment expressions follow from Remark 4 in Brockwell and Marquardt [6] and the fact that $\int_{\mathbb{R}} g(u-h)g(u)du = \mathbf{b}^T e^{Ah}\Sigma\mathbf{b}$, with g defined in (3.7). □

The moments of the strictly stationary volatility process are exponential moments of the stationary distribution of the log-volatility process. In Proposition 3.3 we gave conditions for the existence of a stationary distribution F of the log-volatility process. In the following proposition we want to further characterise this distribution.

Proposition 4.2 *Let* $(\gamma_M, 0, \nu_M)$ *be the characteristic triplet of the Lévy process* M, *where* M *is defined in (3.1), and* F *is the stationary distribution of the log-volatility process. Then* F *is infinitely divisible with characteristic triplet* $(\gamma_\infty, 0, \nu_\infty)$, *where*

$$\begin{aligned}\gamma_\infty &= \mu + \int_0^\infty g(s)\gamma_M ds + \int_0^\infty \int_{\mathbb{R}} g(s)x[\chi_{(-1,1)}(g(s)x) - \chi_{(-1,1)}(x)]\nu_M(dx)ds \\ \nu_\infty(B) &= \int_0^\infty \int_{\mathbb{R}} \chi_B(g(s)x)\nu_M(dx)ds, \qquad B \in \mathcal{B}(\mathbb{R})\,,\end{aligned}$$

with $g(s) = \mathbf{b}^T e^{As}\mathbf{1}_q\chi_{(0,\infty)}(s)$.

Proof: In the strictly stationary case the log-volatility process is the continuous time moving average process (3.6). Since M has finite variance, the kernel g and the driving Lévy process M satisfy the conditions in Theorem 2.7 in Rajput and Rosiński [20] which are:

- $\int_{\mathbb{R}} \left|\gamma_M g(s) + \int_{\mathbb{R}} xg(s)\left[\chi_{(-1,1)}(xg(s)) - \chi_{(-1,1)}(x)\right]\nu_M(dx)\right| ds < \infty$
- $\int_{\mathbb{R}}\int_{\mathbb{R}} \min(|g(s)x|^2, 1)\nu_M(dx)ds < \infty$.

Therefore the stationary distribution F of the log-volatility process is infinitely divisible with characteristic triplet $(\gamma_\infty, 0, \nu_\infty)$. □

Let $\log(\sigma_\infty^2)$ be a random variable with distribution F. Since F is infinitely divisible, we can now apply Theorem 25.17 of Sato [21] to calculate the exponential moments of $\log(\sigma_\infty^2)$, i.e. the moments of σ_∞^2, in the next Proposition.

Proposition 4.3 *Let F be the stationary distribution of $\log(\sigma^2)$ with characteristic triplet $(\gamma_\infty, 0, \nu_\infty)$. Then the k-th moment of σ_t^2 is finite, if*

$$\begin{aligned} k \in K_\infty &= \{s \in \mathbb{R} : \int_{|x|>1} e^{sx} \nu_\infty(dx) < \infty\} \\ &= \{s \in \mathbb{R} : \int_0^\infty \int_{x\in\mathbb{R}, |h(x)|>1} e^{sg(u)x} \nu_L(dx)du < \infty\} . \end{aligned}$$

In this case

$$\Psi_\infty(k) := \gamma_\infty k + \int_{\mathbb{R}} \left(e^{kx} - 1 - kx\chi_{(-1,1)}(x)\right) \nu_\infty(dx) , \tag{4.1}$$

is well defined and

$$\mathbb{E}(\sigma_t^{2k}) = e^{\Psi_\infty(k)} , \qquad \forall\ t \geq 0 . \tag{4.2}$$

Proof: The k-th exponential moment of a Lévy process $(X_t)_{t\geq 0}$ is computed in Theorem 25.17 of Sato [21]. Hence we can apply the Theorem for a Lévy process X with infinitely divisible distribution F at time one to get the k-th exponential moment of $\log(\sigma_t^2)$. It is then given by $\mathbb{E}(\exp(\log(\sigma_t^2))^k) = e^{\Psi_\infty(k)}, \quad \forall\ t \geq 0$, with $\Psi_\infty(k) = \gamma_\infty k + \int_{\mathbb{R}}(e^{kx} - 1 - kx\chi_{(-1,1)}(x))\nu_\infty(dx)$ (see equation (25.11) in Sato [21]). $\square$

Proposition 4.4 *Let $\log(\sigma_t^2)$ be the strictly stationary solution of (3.3) and (3.4). Assume that $\mathbb{E}(\sigma_t^4) < \infty$ for all $t \geq 0$. Let $\Psi_\infty^h(1)$ and $\Psi^h(1)$ be defined by (4.1) with kernel function g replaced by $g_\infty^h(s) = \mathbf{b}^T(I_q + e^{Ah})e^{As}\mathbf{1}_q$ and $g^h(s) = \mathbf{b}^T e^{As}\mathbf{1}_q\chi_{(0,h)}(s)$, respectively. Then the autocovariance function of σ^2 is given by the following expression*

$$\mathbb{C}\mathrm{ov}(\sigma_{t+h}^2, \sigma_t^2) = e^{\Psi_\infty^h(1)} e^{\Psi^h(1)} - e^{2\Psi_\infty(1)} , \qquad h > 0,\, t \geq 0 . \tag{4.3}$$

Proof: Let $\mathcal{F}_t^M = \sigma(M_s\,, -\infty < s \leq t)$ be the σ-algebra generated by the Lévy process M up to time t, then

$$\begin{aligned} \mathbb{E}(\sigma_{t+h}^2|\mathcal{F}_t^M) &= \mathbb{E}\left(\exp\left\{\mu + \int_{-\infty}^{t+h} g(t+h-s)dM_s\right\}\Big|\mathcal{F}_t^M\right) \\ &= \exp\left\{\mu + \int_{-\infty}^{t} \mathbf{b}^T e^{Ah} e^{A(t-s)}\mathbf{1}_q dM_s\right\} \mathbb{E}\left(\exp\left\{\int_t^{t+h} g(t+h-s)dM_s\right\}\right) . \end{aligned}$$

Therefore we get

$$\begin{aligned} &\mathbb{E}(\sigma_{t+h}^2\sigma_t^2) = \mathbb{E}(\mathbb{E}(\sigma_{t+h}^2\sigma_t^2|\mathcal{F}_t^M)) = \mathbb{E}(\sigma_t^2\mathbb{E}(\sigma_{t+h}^2|\mathcal{F}_t^M)) \\ &= \mathbb{E}\left(\sigma_t^2 \exp\left\{\mu + \int_{-\infty}^{t} \mathbf{b}^T e^{Ah} e^{A(t-s)}\mathbf{1}_q dM_s\right\} \mathbb{E}\left(\exp\left\{\int_t^{t+h} g(t+h-s)dM_s\right\}\right)\right) \end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\left(\exp\left\{2\mu + \int_{-\infty}^{t} \mathbf{b}^T (I_q + e^{Ah}) e^{A(t-s)} \mathbf{1}_q dM_s\right\}\right) \mathbb{E}\left(\exp\left\{\int_0^h g(s) dM_s\right\}\right) \\
&= \mathbb{E}\left(\exp\left\{\mu + \int_0^{\infty} \mathbf{b}^T (I_q + e^{Ah}) e^{As} \mathbf{1}_q dM_s\right\}\right) \\
&\qquad\qquad \times \mathbb{E}\left(\exp\left\{\mu + \int_0^{\infty} \mathbf{b}^T e^{As} \mathbf{1}_q \chi_{(0,h)}(s) dM_s\right\}\right) \\
&= e^{\Psi_\infty^h(1)} e^{\Psi^h(1)},
\end{aligned}
$$

where the last equality follows from (4.2) when we substitute the kernel g in (3.6) by $g_\infty^h(s)$ and g^h, respectively. This together with (4.2) yields (4.3). □

5. Second order properties of the return process

In this section we derive the moment structure of the *return process*

$$
G_t^{(r)} = G_t - G_{t-r} = \int_{(t-r,t]} \sigma_{s-}\, dL_s\,, \qquad t \geq r > 0\,.
$$

We will only consider the case of a strictly stationary volatility process.

5.1. Moments and autocovariance function of the return process

Proposition 5.1 *Let L be a Lévy process with $\mathbb{E}(L_1) = 0$ and $\mathbb{E}(L_1^2) < \infty$. Assume that the volatility process σ^2 is strictly stationary with finite mean. Then $\mathbb{E}(G_t^2) < \infty$ for all $t \geq 0$, and for every $t, h \geq r > 0$ it holds*

$$
\begin{aligned}
\mathbb{E}G_t^{(r)} &= 0 && (5.1)\\
\mathbb{E}(G_t^{(r)})^2 &= e^{\Psi_\infty(1)} r \mathbb{E}(L_1^2) && (5.2)\\
\mathbb{C}\mathrm{ov}(G_t^{(r)}, G_{t+h}^{(r)}) &= 0. && (5.3)
\end{aligned}
$$

If further $\mathbb{E}(L_1^4) < \infty$ and the volatility process has finite second moment, then $\mathbb{E}(G_t^4) < \infty$ for all $t \geq 0$ and for every $t, h \geq r > 0$ we have

$$
\mathbb{C}\mathrm{ov}((G_t^{(r)})^2, (G_{t+h}^{(r)})^2) = \mathbb{E}(L_1^2) \int_h^{h+r} \mathbb{C}\mathrm{ov}(G_r^2, \sigma_s^2) ds\,. \tag{5.4}
$$

Proof: If L has no Brownian component the proof of (5.1) - (5.3) is analogously to the proof of Proposition 5.1 in Klüppelberg et al. [13] and can be extended in the same way as in the proof of Proposition 2.1 in Haug et al. [12] in case L has a Brownian component. Since G is a square integrable martingale we get

$$
\mathbb{E}((G_r^{(r)})^2 (G_{h+r}^{(r)})^2) = \mathbb{E}(G_r^2 (G_{h+r} - G_h)^2) = \mathbb{E}(G_r^2 (G_{h+r}^2 - G_h^2))\,.
$$

Using this result, $G_t^2 = 2\int_0^t G_{s-}\sigma_{s-} dL_s + \int_0^t \sigma_{s-}^2 d[L,L]_s$, $t \geq 0$, and the compensation formula (see e.g. Section 0.5 in Bertoin [2]) we get

$$
\begin{aligned}
\mathbb{E}((G_r^{(r)})^2(G_{h+r}^{(r)})^2) &= \mathbb{E}\left(2\int_h^{h+r} G_r^2 G_{s-}\sigma_{s-}dL_s + \int_h^{h+r} G_r^2\sigma_{s-}^2 d[L,L]_s\right) \\
&= \mathbb{E}\left(\int_h^{h+r} G_r^2\sigma_{s-}^2 d[L,L]_s\right) = \int_h^{h+r}\mathbb{E}(G_r^2\sigma_s^2)\tau_L^2 ds + \int_h^{h+r}\mathbb{E}(G_r^2\sigma_s^2)ds\int_{\mathbb{R}} x^2\nu_L(dx) \\
&= \mathbb{E}(L_1^2)\int_h^{h+r}\mathbb{E}(G_r^2\sigma_s^2)ds
\end{aligned}
$$

Hence the covariance is equal to

$$
\begin{aligned}
\mathbb{C}\mathrm{ov}((G_r^{(r)})^2,(G_{h+r}^{(r)})^2) &= \mathbb{E}((G_r^{(r)})^2(G_{h+r}^{(r)})^2) - (\mathbb{E}(G_t^{(r)})^2)^2 \\
&= \mathbb{E}(L_1^2)\int_h^{h+r}\left(\mathbb{C}\mathrm{ov}(G_r^2,\sigma_s^2) + \mathbb{E}(G_r^2)\mathbb{E}(\sigma_s^2)\right)ds - (\mathbb{E}(G_t^{(r)})^2)^2 \\
&= \mathbb{E}(L_1^2)\int_h^{h+r}\mathbb{C}\mathrm{ov}(G_r^2,\sigma_s^2)ds\,.
\end{aligned}
$$

The covariance is finite if $\mathbb{E}(G_t^4) < \infty, \quad \forall\, t \geq 0$, and this follows with $\mathbb{E}(L_1^4) < \infty$ and $2 \in K_\infty$ analogously as in Proposition 1.1 in Haug et al. [12]. □

Example 5.2 *Let us consider again Example 3.8. From 50 000 equidistant observations of the simulated log-price we computed the empirical autocorrelation function of the returns and squared returns. In Figure 3 the first* 40 *lags of both empirical autocorrelation functions are shown. One recognises the GARCH like behaviour of zero correlation of the returns and significant correlation of the squared returns.*

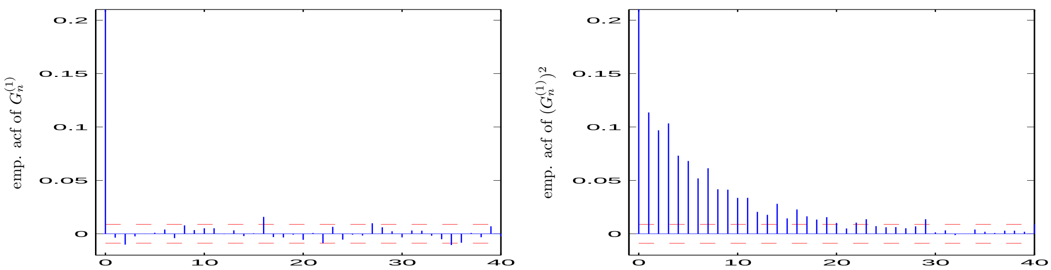

Figure 3: The first 40 lags of the empirical autocorrelation function of the return (*left*) and squared return (*right*) process.

Remark 5.3 *In Theorem 3.10 we have seen that volatility and return process are strongly mixing with exponential rate. A consequence of this property (see e.g. Section 1.2.2 in Doukhan [8]) is that there exist constants $K_1, K_2 > 0$ such that*

$$
|\mathbb{C}\mathrm{ov}(\sigma_{t+h}^2,\sigma_t^2)| \leq K_1\cdot e^{-ah} \quad \textit{and} \quad |\mathbb{C}\mathrm{ov}((G_{(n+h)r}^{(r)})^2,(G_{nr}^{(r)})^2)| \leq K_2\cdot e^{-ah}\,,
$$

for all $h > 0$, with $a > 0$ as in Theorem 3.10. In particular this means that the autocovariance function of the volatility and squared returns will decay to zero at an exponential rate*. Therefore we will speak of short memory process in both cases. The model can be extended to incorporate long memory effects, by specifing the log-volatility process by a fractionally integrated CARMA$(q, p-1)$ process. For more details we refer to Haug and Czado [11].*

Acknowledgements

We are indebted to two anonymous referees for a number of helpful suggestions and corrections. Further we like to thank Claudia Klüppelberg, Alexander Lindner and Robert Stelzer for fruitful and inspiring discussions. This work was supported by the Deutsche Forschungsgemeinschaft, Sonderforschungsbereich 386, Statistical Analysis of Discrete Structures.

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