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Sergey Kuniavsky:

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Department of Economics
University of Munich

Volkswirtschaftliche Fakultät
Ludwig-Maximilians-Universität München

Reserve Price in Search Models*

Sergey Kuniavsky[†]

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Abstract

Search models are used in a variety of fields. One of those is consumer search and in it the Stahl model is one of the most popular search models. The literature in search models concentrates mainly on equilibria with a consumer reserve price. This is a simplifying condition, which narrows down severely the freedom of the consumer. For example, once the model has a finite number of sellers who select different strategies a situation may arise where reserve price may not exist. This paper addresses the possibility of equilibria without reserve price, when sellers can use asymmetric strategies.

Here a condition is given which ensures existence of reserve price in all equilibria. The condition involves prices where sellers set mass points, and undercutting those prices. The condition states that if a searcher is satisfied with such price, she should be satisfied also when a small discount is offered to that price. Assuming this, it is possible to concentrate only on equilibria with reserve price, and investigate also situations where the sellers are heterogeneous, or equilibria are not symmetric.

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1 Introduction

Search models are an extensive and important field, with a large literature. The field, originally developed due to diamond paradox, Diamond [4], had since grown and covers

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[†]Munich Graduate School of Economics, Kaulbachstr. 45, Munich. Email: Sergey.Kuniavsky@lrz.uni-muenchen.de. Financial support from the Deutsche Forschungsgemeinschaft through GRK 801 is gratefully acknowledged.

many topics, such as consumer search, for example Stahl [8], Varian [10], and many other areas. One of the most popular models applied in the search literature is a model introduced by Stahl [8]. It describes a simple consumer search with two types of consumers, one with an endogenous reserve price. Additional research, for example Janssen et al. [5], shows that Stahl model performs very well in reality and predicts pricing of goods very well. Therefore, this paper concentrates on the Stahl model. Additional empiric paper by Baye et al. [2] empirically shows existence of the two types of consumers from the Stahl model.

Recently there develops literature on an extension of the Stahl model, where the sellers are heterogeneous, for example Astorne-Figari and Yankelevitsch [1], or Burdett and Smith [3]. Note that in the original paper by Stahl [8], it is explicitly noted that the paper concentrates on symmetric strategy choices of sellers. One of the results of such models is that a symmetric NE is no longer possible when the sellers are heterogeneous. This is a very important extension, as in reality sellers vary in size and popularity. Therefore, it is important to note that asymmetric NE plays an important role in this extension.

However, once asymmetric NE arises, it creates an additional complexity. Throughout the search literature there is a crucial assumption of a searcher reserve price, or of seller symmetry, where sellers use same pricing strategy. If sellers use the same strategy then clearly reserve price exists (see Rothschild [7]). Reserve price is a setting where consumers set a threshold price, which determines whether the good is purchased. The condition is simple - a transaction takes place if and only if the price is weakly (or strictly) below the threshold price. This allows easier analysis of the game and is used in proving many of the results. However, as noted in Rothshild [7], once an asymmetric strategies are in place, additional NE, without a consumer reserve price, may exist. The reason behind it is simple - once a price was revealed to the searcher, she can interpret this price and gain information regarding prices in other stores. The price observed combined with the initial Bayesian beliefs of the searcher, lead to beliefs update regarding strategies used by other sellers. This can lead to situations where no reserve price exists, which are very complex to analyze.

One can wonder whether in search models equilibria without reserve price prevail. A working paper by Olszewski and Wolinsky [6], provide a search model with equilibria that have no reserve price. Clearly, their model differs significantly from the Stahl search model. Yet if certain search models have no reserve price equilibria, it may be the case that also Stahl search model has such equilibria.

This paper provides a condition on the game which ensures existence of a reserve price. The result has a very important application allowing the search literature to remain in the reserve price world, once this condition is met. Here a wider set of possible strategies is discussed, satisfying a reasonable condition. The main property of the condition is allowing free undercut for sellers, namely, at certain prices it is possible to offer arbitrarily small discounts without losing consumers, and therefore can be seen as a trembling hand perfection condition. The paper shows that in this space of consumers search strategies, all NE have a reserve price.

Similar result can hold for many other search models, which have a similar problem, where asymmetries can lead to asymmetric choice of strategies and to additional equilibria, without reserve price. Then an observed price can imply additional insights on the market. However, similar results can be made for additional search models, and can reduce the possibility for a non reserve price NE.

The structure of the paper is as follows: first the Stahl model is introduced. Then the problem of no reserve price is illustrated in a couple of examples. Afterwards, 'undercut proof' condition is introduced and lastly, it is shown that such a condition is a sufficient one for reserve price to exist in NE.

2 Model

The model is similar to the Stahl model, introduced by Stahl in Stahl [8].

In the model there are N sellers, selling an identical good. Seller i owns n_i stores, where n_i can vary among sellers. The production cost is normalized to 0, and seller can meet demand. Additionally, there are consumers, each of whom wishes to buy a unit of the good. The mass of consumers is normalized to 1. This implies that there are many small, strategically insignificant consumers.

The sellers set their price at the first stage of the game. Sellers cannot price discriminate. If the seller mixes then the distribution is selected simultaneously, and only at a later stage the realizations take place.

The consumers are of two types, both evaluate the good at some large finite value M, known to sellers. A fraction μ of consumers are shoppers, who know where the cheapest price is, and buy there. In case of a draw they randomize over all cheapest stores, spreading equally among them. The rest are searchers, who sample prices sequentially. Sampling price in the first, randomly and uniformly selected, **store** [1] is free. The second (or any later) store is randomly and uniformly selected from stores of previously unvisited sellers, and bears a cost c. The searcher may be satisfied, or search further on. When a searcher is satisfied, she has a perfect and free recall. This implies she will buy the item at the cheapest store she had encountered, randomizing in case of a draw.

[1] note that it is not the same as 'uniformly selected seller'

The reason for the two types of consumers is given already in the original paper introducing the model, Stahl [8]. If only shoppers exist ($\mu = 0$) then the setting is of Bertrand competition, and if only shoppers exist ($\mu = 1$) then the Diamond paradox, Diamond [4] is encountered. In the former setting the sellers will drive the price down to zero and in the latter up to infinity.

Before going on, a technical assumption is required:

Remark 2.1 *To avoid measure theory problems it is assumed that mixing is possible by setting mass points or by selecting distribution over full measure dense subsets of intervals. This includes all commonly used distributions and finite combinations thereof*

Additionally, it is easy to see the following two basic results:

- Sellers do not offer a price above some finite bound M in any NE. This is due to the maximal valuation of a consumer for the good.
- Searchers accept any price below c. The logic behind it is any price below my further search cost will be accepted, as it is not possible to reduce the cost by searching further.

2.1 Knowledge and Beliefs

Searchers have beliefs regarding the prices sellers set. For each possible (pure or mixed) strategy s of the model a belief is attached, stating how many stores are priced according to this strategy, denoted $n(s)$ (clearly the sum of $n(s)$ over all possible strategies s is N, the number of stores). Searchers do not know beforehand which realizations took place, nor which seller chose which strategy. Each strategy has an expected price, denoted $e(s)$. This structure is similar to the one applied in Stahl [8, 9].

Suppose the searcher observes the price p. Let the probability that this price p came from strategy s be denoted as $prob(p, s)$. To calculate $prob(p, s)$ first calculate chance that s is selected by some seller, according to searchers beliefs. Additionally the probability that p is the realization of strategy s (relevant for mixed strategies) is weighted in. One needs to note that if some strategies (with positive $n(s)$) have a mass point on p only those will be considered, otherwise the densities will play a role. Formally:

$$prob(p, s) = \frac{n(s)f(p)}{\sum_{p \in s'} n(s')f(p)} \tag{1}$$

Now, if the searcher thinks that strategy s was selected, searching further will yield (in expected terms) the expected price in stores belonging to other sellers. Namely, it is the expected price, only that $n(s)$ is now lower by $EN(s)$, where the expected size of seller using strategy s is denoted $EN(s)$, (as s was observed in one of the stores). Then the calculation of the expected price searching is as follows:

$$EPS = \frac{\sum_{s':n(s')>0,s'\neq s} n(s')e(s') + (n(s) - EN(s))^{+})e(s)}{\sum_{s'} n(s')} \tag{2}$$

Clearly if $n(s) < EN(s)$ is zero or below the corresponding element would be zero, or this situation would not be considered at all. For example, if only a single seller uses a price already observed elsewhere, then this seller would not be calculated for future visits.

In the process of searching the structure of beliefs can be probabilistic and more complicated, based on beliefs from which strategies previously observed prices came. Obviously, searchers search further only when the expected price in a search is at least c lower than the lowest price observed so far. Below is an example of how to calculate an expected search price, and additionally illustrates that sometimes no reserve price exists.

2.2 Satisfaction Sets

When one goes beyond reserve price, one needs to describe searchers behavior. A natural way is obtained by introducing satisfaction sets. Those describe stopping condition, and are a natural extension to reserve price. This is the most general setting for a stopping condition, allowing pinning down exactly all condition which make a searcher satisfied.

Searchers in the course of a search observe prices, and after some price vectors observed the search ends. Let the search ending vector sets be defined as buying satisfaction sets. Formally, a satisfaction set BS_k consists of all vectors of length k, such that after observing the k prices denoted in the vector, in the corresponding order, the searcher is satisfied, and does not search further. Note that since any price below c is satisfactory BS_k is non empty for all k.

Let v be a vector in BS_k for some $k \geq 1$. The lowest coordinate value in v is denoted as v_m. The set of coordinates in v with v_m will be denoted $Min(v)$. From the definitions v_m is the price paid by the searcher after visiting k stores, observing the prices vector v and being satisfied.

For example, suppose a vector $v = \{8, 5, 9\} \in BS_3$, has $v_m = 5$. v implies the following: the searcher visited 3 stores. In the first observed the unsatisfactory price of 8. In the second the price of 5 was observed, and still was unsatisfactory. After a third search the price of 9 was observed, making the price 5 satisfactory. Such an example can arise when the last visited seller mixes between 9 and some extremely low prices. After seeing that the realization was 9, a previously unsatisfactory price becomes attractive.

Let the supremum element of BS_1 (the first satisfaction set) be denoted as P_M. Let P_S denote the maximal (supremum) price which is in the support of a seller in a given NE. Implying - in NE no one offers a price above P_S. From an assumption no price above M can be set, and therefore, P_S and P_M are well defined and finite.

Note the following property. Let P_{min} be the lowest (infimum) price that is in the support of a strategy in NE. Any price below P_{min} is in BS_1. Namely, any price below the lowest price offered in NE is satisfactory. This follows directly from the Bayesian structure of the beliefs: such price is satisfactory as any additional search will end up with a weakly higher price.

2.3 Game Structure

Now after discussing the knowledge and behavior of searchers, we can turn to the structure of the game.

The game is played between sellers, searchers and shoppers. The time line of the game is as follows:

At the first stage, the sellers set their pricing strategies and simultaneously searchers set their common satisfaction sets. Then, if some sellers used a mixed strategy, realization of the mixed strategy is taking place. At the second stage the shoppers observe all the realized prices and purchase the item at the cheapest store. At the third stage, searchers sample randomly (uniformly) selected price. If a searcher is satisfied she purchases the item. If not - she pays c and observes another price, and so forth until ether she is satisfied or sampled all prices. When the searcher observed all stores and observed only unsatisfactory prices she would buy at the cheapest store encountered.

Additionally, when setting the prices and satisfaction sets the following beliefs are taken into consideration: Searchers have beliefs about which strategies were actually played by the sellers and about sellers' sizes. Sellers have beliefs regarding their competitors' strategies and regarding the satisfaction sets. Shoppers will know the real price in each store in the moment it is realized.

The probability that seller i sells to the shoppers when offering price p is denoted $\alpha_i(p)$. Let q be defined as the expected quantity that seller i sells when offering price p. It consists of the expected share of searchers that will purchase at her store, plus the probability she is the cheapest store times μ. It is also the expected market share of the seller.

The seller utility is the price charged multiplied by the expected quantity sold (equals to pq). Since she is setting price at the early stage of the game, she can only has ex ante utility, reflecting the expected income. The consumer utility is a large constant M, from which the price paid for the item and the search costs are subtracted.

The NE of the game has a Bayesian form and is as follows:

- The searchers beliefs coincide with the actual strategies played by sellers.
- The satisfaction sets are rational for the searchers, and they cannot profitably deviate to different satisfaction sets profitably in expected terms.
- No seller can unilaterally adjust the pricing strategy and gain higher utility in expected terms.
- Searchers beliefs coincide with actual strategies and sizes of sellers
- As usually in NE, there is a common knowledge of rationality.

Remark 2.2 *As the sum of the searcher and seller utilities may differ only in the search cost, any strategy profile where the searchers always purchase the item at the first store visited is socially optimal.*

2.4 Undercut Proof

Lastly, an undercutting condition is defined, which will lead to existence of a reserve price.

Definition 2.1 *Let σ be a strategy profile of the sellers. Let MP denote the set of prices where some sellers have set a mass point.*

Definition 2.2 *Let v be a price vector in BS_k with $v_m = p$. Suppose exists $\varepsilon > 0$ such that when subtracting any number smaller than ε from any single coordinate in $Min(v)$ will keep the resulting vector in BS_k. Then v is denoted ε-undercut proof at price p.*

Definition 2.3 *Fix a natural number k. Suppose that for any price p in MP exists $\varepsilon_p > 0$ such that for any vector $v \in BS_k$ with $v_m = p$, the vector v is ε_p-undercut proof. Then BS_k will be denoted as undercut proof.*

The condition in the last definition says the following: if a certain vector is satisfactory, then reducing one of the lowest elements in it by a small amount will not affect the decision of the searchers. The main implication of this assumption is simple - for any price p, with a mass point on it, it is possible to undercut it without losing any searchers. This condition can be interpreted as a trebling hand perfection condition. A seller sets a certain price with a mass point, but due to a trembling hand the price is set slightly lower. Consumers, which expect such a possibility, believe that for some small ε, the price is selected by someone who trembled.

Trembling is important only on mass points, as there is a mass of probability. Trembling on prices with regular continuous distributions have zero probability to occur, and therefore, are far less significant than trembling on mass points. A good comparison would be first order and second order 'trembling', where the first order is on mass points and the second order over continuous distributions, where any given price is selected with prob. zero (in comparison to mass points).

Note that only one side trembling is accepted. This is the one which increases the utility of the consumer due to seller trembling. Trembling and offering slightly higher prices is not accepted by consumers, as it may have some strategic benefit for sellers.

As the number of mass points with mass above any positive number is finite, it is possible to take an interval without any significant mass points. The searcher observing such a price will think that it is much more probable that a seller trembled rather than such a price was selected by a distribution without mass points. To sum it up, the condition states that for a vector v and a small positive $\delta < \varepsilon_p$:

$$\begin{array}{rcl} & Min(v) = p \in MP & \\ v = & (v_1, v_2, \ldots v_{i-1}, p, v_{i+1}, \ldots v_k) & \in BS^k \\ & \Downarrow & \\ v - \delta e_i & = (v_1, v_2, \ldots v_{i-1}, p - \delta, v_{i+1}, \ldots v_k) & \in BS^k \end{array}$$

Remark 2.3 *Reserve price condition is a specific case of undercut proof, if a price p is satisfactory it lays below the reserve price. Then also any price below p is satisfactory.*

3 No Reserve Price Examples

Firstly, consider a case where, in a constellation which is not an equilibrium a situation without a reserve price may arise. It will additionally serve as an example to the relevant calculations.

Example 3.1 *Consider a case where three sellers each own a single store. Suppose the search cost c is 0.9 and pricing strategies, equally probable from searchers beliefs, are as follows:*

1. *Uniform in [1, 9], exp. value of 5*
2. *Uniform in [5, 9], exp. value of 7*
3. *Pure strategy of 7.*

After observing the price of 7 the searcher is certain with prob. 1 that she had encountered the third strategy seller. An additional search will yield the average between the expected values of the two strategies: 6, making an additional search worthy.

After observing the price of 7.1 the searcher knows that she had encountered one of the mixed strategies, and due to a distribution likelihood ratio - twice more probable that it is the second strategy. Therefore, with probability 1/3 it is the first str. and probability 2/3 the second str.

If the first strategy was encountered, then an additional search will end up in ether second or third strategy - both with expected price of 7.

If it is the second strategy, then an additional search will end up with expected price of 5 or of 7, as both can occur with equal probability (due to the beliefs) expected price in an additional search in this case is 6.

Combining the two possibilities, when taking into account that the second case is twice more probable, the expected price in an additional search is $(2 \cdot 6 + 7)/3 = 6.333$, making another search not profitable.

Example 3.2 *An additional example is provided to elaborate on the possibility of equilibrium without a reserve price.*

Consider a Stahl model with two sellers, but the following changes:

- There are only three prices available for a seller to offer - L, M and H (low, medium and high).
- In the case both sellers select the high price a legislator comes in and fines both sellers. Thus, (H, H) cannot be an equilibrium.
- The sellers choose strategies sequentially, where the latter seller (seller 2) can observe the realization of the former seller (seller 1).

Set the following properties of the model:

- The three prices are $L = 24.99, M = 49$ and $H = 51$.
- The search cost c is equal to 2.
- The share of informed consumers is 2%, making $\mu = 0.02$.
- The fine is 15 units.

Note that if any seller selects L, searchers would search in the second store if L was not observed. If no seller is believed to offer L, searchers visit only one store. Based on this, the seller utilities for pure strategies are as follows [2]:

- If seller i offers L and seller j offers M or H, seller i has utility of L and seller j has zero utility.
- If seller i and seller j both offer M, both get utility $M/2$.
- If seller i selects M and seller j selects H, seller i gets $M(1+\mu)/2$ and seller j gets $H(1-\mu)/2$.
- If both sellers select H, both get $H/2-F$, where F is the fine imposed by a regulator.

When we put payoffs in a table, in a similar way to a normal form (ignoring the parts of seller 2 strategy which did not occur), the possible outcomes and payoffs are:

Strategy	Low	Medium	High
Low	12.995, 12.995	24.99, 0	24.99, 0
Medium	0, 24.99	24.5, 24.5	24.99, 24.99
High	0, 24.99	24.99, 24.99	10.50, 10.50

Due to the sequential form of the game, consider that first seller 1 selects the row, and after seller 2 knows the row selected, she selects the column.

Note that $(H, [L, L, M]), (H, [L, H, M])$ and $(M, [L, H, L])$[3] are pure strategy equilibria where searchers always buy at the first store visited. However, exists an additional equilibrium without a reserve price. Consider the situation where seller 1 selects M, and seller

[2] Due to symmetry in payoffs it does not matter which of the sellers is i or j.

[3] The strategy of seller 2 is a combination of three actions, what she does when seller 1 chose L, M or H.

2 mixes between L (prob. 2/13)and H with prob. 11/13. If seller 1 does not select M, seller 2 selects L.

Note that, in the case seller 1 selects M, the expected price offered by seller 2 is:

$$E(P_i) = 24.99\frac{2}{13} + 51\frac{11}{13} = 46.998 \tag{3}$$

Note that in such case a searcher after observing $M = 49$ knows it is seller 2. In the other store awaits expected price of 46.998, and with search costs of 2 an additional search is worthy. If L was observed no improvement is possible and, therefore, no further search. Now, if $H = 51$, it is seller 2 who was visited. An additional search would bring price of 49, and is not worthy.

Note that seller 1 has no better alternative, as selecting H would bring him utility of zero, and L utility of $P_L/2 = 12.495$. In the current situation seller i has utility of $11/13P_M(1+\mu)/2 \approx 21.15$. Seller 2 selects in all cases a best response to seller 1 strategy, gaining utility of 24.99 if M or H was chosen, and 12.495 if L was chosen. Therefore, it is an equilibrium without a reserve price.

So, regarding reserve price - it might be the case that it does not exist. However, the literature concentrates on NE with a reserve price. Additional equilibria could exist to the ones commonly known. This paper will provide a basic condition which ensures existence of a reserve price in equilibrium, which will involve a specific type of trembling hand perfection.

4 Results

The main result of the paper is as follows:

Theorem 1 *Suppose that all BS_k are undercut proof. Then all NE of the model have reserve price of P_M, up to zero measure adjustments. Additionally, P_M is the only price which may have mass points in NE.*

The main part of the proof shifted to the appendix.

Here, let me address the zero measure adjustments, for mathematic completeness. It might be the case that all prices below some P_M are accepted, except several prices of measure zero. It is clear that such prices will affect the equilibrium with probability zero, as the probability for such 'unsupported' price to be selected is zero, as mass points may exist only at P_M. For example, consider a continuous distribution on $[a, b]$ with positive density everywhere in the interval. Now, if $c \in [a, b]$ is excluded from the distribution, the probability for the change to make a difference is zero. Therefore, the zero measure adjustments do not have any effect on the equilibrium, with certainty (prob. 1).

5 Summary

The result presented here allows extending the consumer search models and allowing dealing with a finite number of heterogeneous sellers. There some NE without a reserve price may exist, but in very limited setting. The condition presented here allows eliminating the possibility of NE without a reserve price, allowing a simple analysis of many search models.

The condition is a natural introduction of trembling hand perfection. When sellers can tremble, or even just believed to be, it is sufficient. Sellers, by such trembling mistake, offer prices just below what they intended, when selecting prices with mass points, as those are more significant than continuous distribution. When one adds to beliefs such trembling hand perfection, the only equilibria possible in the Stahl model are ones with reserve price. Note that a trembling hand upwards is less plausible, and can be a strategic behavior of a seller to gain a bit higher utility. Note that this is a one direction trembling, which gives a slightly better price offer to consumers. Therefore, consumers are inclined to accept this type of trembling. Trembling to a higher price would be less accepted, as then consumers are the ones paying the cost of the trembling, and giving a slightly higher benefit for sellers.

Further research can deal with additional search models and extensions thereof in various fields. Testing whether such a condition is a sufficient one also in other areas. This will allow making the search models more extended and realistic. An additional further research is to identify NE without a reserve price and find a characterizing condition on when those are possible.

A Omitted Proofs

In this appendix Theorem 1 is proven. The theorem will be proved in a sequence of lemmas. Firstly, it is shown that no price above P_M is used and P_M cannot be an isolated point of BS_1. Following step is showing that no mass point exist, except possibly at P_M (supremum element of BS^1), and then that no interval holes can exist in equilibrium. All lemmas refer to a NE strategy profile of the game.

Let P_S be the supremum price in the support union of all sellers.

Lemma A.1 *In NE no seller has prices above P_M in support: $P_S \leq P_M$*

Suppose $P_S > P_M$. Let us distinguish between several cases:

Case 1: All sellers have mass points on P_S. Due to undercutting this cannot be an equilibrium - with positive prob. get all of the market instead of only $1/n$ of it.
Case 2: Some sellers have mass points on P_S. Undercutting here is also a profitable deviation. The only case when a seller has non zero utility is when some searchers, after observing P_S by $k > 1$ sellers are satisfied with P_S. Due to undercutting proof condition it is possible to undercut P_S and get all of these searchers, and not just $1/k$ of them.
Case 3: No seller has a mass points on P_S. Then, P_S has zero utility, and prices arbitrarily close to P_S have utility arbitrarily close to zero. This is since all sellers offer price bellow P_S w.p. 1, no shoppers would purchase at this price, and searchers would search on and find a cheaper price w.p. 1. Similarly, with prices arbitrarily close below P_S have prob. arbitrarily close to one and utility arbitrarily close to zero. Therefore, a deviation to c would be profitable, as it ensures positive utility. □

Corollary A.1 *In NE, for any price p with positive prob. to attract shoppers, at most a single seller has a mass point at p.*

If several sellers have a mass point at some price p, there is a positive prob. for multiple sellers to select p. Thus, undercutting p is possible and the shoppers share will increase discontinuously. Due to the undercutting condition no searchers will be lost. The prob. to satisfy shoppers is strictly higher. The result is that undercutting is a profitable deviation.

Lemma A.2 *If $P_M \in BS_1$ and is selected by some sellers it is not an isolated point of BS_1.*

If it is selected with a mass point by some sellers it cannot be an isolated point of BS^1 due to undercut proof. If it is selected without mass points (continuously), it must be a part of an interval, and cannot be isolated. □

Let the maximal (supremum) price in the sellers support which is not in BS_1 (if exists) be denoted as P_{src}. Additionally, let the maximal (supremum) price which has a positive probability to attract shoppers be denoted as P_{shop}.

Lemma A.3 $P_{shop} = P_M$ *In words: P_M is the supremum point of strategy support for each seller.*

From lemma A.1 it cannot be higher than P_M. Suppose that the supremum price of seller i is $p < P_M$, an is the lowest supremum among all sellers. Formally, $P_{shop} = p < P_M$. For any price above p and below P_M the probability to sell to shoppers is zero.

Firstly, note that no seller would select prices in (p, P_M). The probability to sell to shoppers in this interval is zero, and only uninformed consumers would purchase for a price in (p, P_M). Each seller would maximize her utility when selecting P_M and not prices in (p, P_M).

Consider the case that P_M is not an isolated point of BS^1, and some prices just below P_M are in BS^1.

Let us distinguish between several cases regarding mass points on p:

Case 1: seller i has a mass point at p: Due to corollary A.1 no other seller has a mass point on p. Since no other seller has (p, P_M) in support i can increase her price from p into (p, P_M) and gain higher utility, without loosing any market share.
Case 2: Seller(s) that are not i have mass points at p: Similarly, one of those sellers can deviate profitably into (p, P_M) instead of offering p.
Case 3: No seller has mass points at p: If there are no mass points at p, seller i can deviate to a price just below P_M instead of p and prices just below p and get a higher utility due to a higher price.

If P_M is not an isolated point of BS^1 no seller selects it. Since no seller selects prices in (p, P_M), all sellers have p as the support supremum of their strategy. In this case some sellers have no mass point on p, and therefore, it has prob. zero to attract shoppers. Similarly to the cases above a deviation to P_M is profitable, ether from a mass point at p, or if none exists from some interval just below p to P_M.

Therefore, in all cases a profitable deviation exists. □

Corollary A.2 *Let p be a price below P_M. At most one seller has a mass point at p.*

Follows directly from Lemma A.3 and Corollary A.1.

From the construction, BS_1 is dense in some interval of the form $(a, P_M]$. The highest interval in BS_1 is defined as follows:

Definition A.1 *Let I be the interval of the form $(a, b]$ such that $b = P_M$ and a is the highest number below P_M such that exists ε satisfying: $(a - \varepsilon, a) \cap BS = \emptyset$.*

This allows I to cover some prices that are not covered in BS_1, as noted in the theorem.

Lemma A.4 *P_{src} is below I.*

Suppose $P_{src} \in I$ and in support of seller i. From construction P_{src} is not the infimum of I. Therefore, some prices below P_{src} are in I and BS^1. Thus there is a positive probability for some sellers to select prices below P_{src}, and therefore, some of the searchers initially visiting i may purchase in a different store. Thus, a deviation to a price just below p, which is in BS_1 from construction, would be a profitable deviation. The loss in price is arbitrarily small, and increase in searchers purchasing at seller i would overweight it. $\square$

Lemma A.5 *If there are mass points in I, these must be at P_M.*

Suppose that $p \neq P_M \in I$ and seller i has a mass point on it. From corollary A.2, No other seller has a mass point on I. Additionally, from lemma A.4 all unsatisfied searchers after first visit observe a price below I, implying that additional unsatisfied searchers will not purchase for p.

For seller $j \neq i$ the probability to attract shoppers decrease discontinuously at p. Therefore, in equilibrium seller j would not have prices just above p in support. As a result i can deviate and increase the price from p to a slightly higher price in BS^1, without losing any market share, reaching a contradiction. $\square$

Lemma A.6 *No seller has prices below the interval I in support.*

Let the maximal (supremum) price that can be offered by sellers below I denoted as P_I.

Note that no seller has prices above P_I and below I in support. From corollary A.2 at most a single seller has a mass point at P_I. Suppose seller i is a seller with P_I in support, and if there is a mass point on P_I she is the seller with the mass point.

Seller i can deviate into the interval I, arbitrarily close to the infimum of I, instead of selecting P_I or prices just below it. The loss of shoppers is arbitrarily small, as no other seller has any mass on any prices between P_I and P_M from lemma A.5. The price is higher, and the share of searchers will be weakly higher (or equal if P_I is in BS^1). The latter is due to the fact that all initial searchers now purchase at seller i, and previously some could observe a lower price and never return. Note that due to lemma A.4 all unsatisfied searchers after first visit observe a price of at most P_I, implying that additional unsatisfied searchers will not purchase at i. $\square$

Any price below I would be below the seller support union and in BS^1. This completes the proof of the theorem.

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