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Department of Economics
University of Munich

Volkswirtschaftliche Fakultät
Ludwig-Maximilians-Universität München

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Income and Wealth Distributions in a Population of Heterogeneous Agents

PIETRO MULIERE

pietro.muliere@unibocconi.it

Bocconi University

DAVIDE SUVERATO

davide.suverato@econ.lmu.de

LMU University of Munich

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Abstract

This paper develops a simple framework to characterize the distribution of income and wealth in a real business cycle model. Agents are of two types depending on the human factor of production they own and they are located in separated markets, *cities*. In each city the two types of agent match to produce a composite factor, human service. We show that if the population is an *exchangeable* sequence of agents' types generated according to a *Pòlya urn* then (i) the share of agents' type follows a Beta distribution and (ii) the functional form of the matching function belongs to the family of the constant elasticity of substitution, with agent shares that depend on the composition of the population.

We nest this structure into a standard Bewley economy, in which the aggregate supply of human service is combined with physical capital to produce the homogeneous output. Given the results (i)-(ii) we perform the exact aggregation of income, consumption and asset holding across agents, leading to the solution of the real business cycle model with heterogeneous agents. Our framework predicts that the theoretical distributions of income and wealth are known real valued transformations of a Beta distribution. This result provides a simple way to characterize the equilibrium of macroeconomic models with heterogeneous agents.

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1 Introduction

The existing macroeconomic literature is rapidly growing in the direction of models able to account for the distribution of income, consumption and assets across heterogeneous agents. The quest for heterogeneity is motivated by the failure of the representative agent approach. For example, Heathcote et al. (2009) discuss the limitations of a representative agent framework when assessing the effect of macroeconomic policies on insurance, precautionary savings and welfare. Kirman (1992) shows how the assumption of a representative agent whose choices coincide with the aggregation over decisions of the heterogeneous individuals appears to be unjustified. Neglecting heterogeneity will in general lead to a trivial equilibrium in which agents do not trade. On the other hand, the aggregation over an heterogeneous population requires further assumptions on the similarity between agents (i.e. identical tastes or identical income) in order to guarantee the uniqueness of the equilibrium (Sonnenschein, Mantel and Debreu theorem).

We propose a new framework in which both (*i*) the distribution of agent types and (*ii*) the technology that matches agents of different types are endogenously determined. As a consequence, the measure for aggregating individual decisions across agents is derived in closed form and the macroeconomic outcome is the exact aggregation of the decentralized equilibrium with heterogeneous agents. In addition to aggregate resource allocation and prices, our approach allows to determine income, consumption and asset distributions across agents.

In this paper the economy is a collection of a agents located in separated cities. Agents are of two types differentiated by the human factor of production they own: labor for workers, managerial skills for managers. The match of the two factors gives a composite factor, human service. In each city competitive firms match workers and managers such that the supply of human service is maximized subject to the composition of the local population. The constrained optimization problem links the shape of the production function of human service to the composition of the population in each productive location. Under the assumption that the population in each city is a sequence of exchangeable binary random variables generated by a Pòlya urn, the share of agent types follows a Beta distribution; this result has been proved in Muliere et al. (2005). Under this framework, we show that the optimal matching function has to be of the C.E.S. form, with factor shares that depend on the composition of the population.

This methodology is new in the macroeconomic analysis, at least to the best of our knowledge. Nevertheless, we argue that the assumption of exchangeability is already common in the literature. As an example, consider the matching between agents of different types: the outcome of a matching process will in general depend on the number of agents of different types, whereas the order in which agents match is irrelevant. In this case agents are exchangeable across types, since the probability associated with a given permutation of the sequence of matches does not depend on the order in which types enter the sequence. Exchangeability is a very common property in Bayesian statistics and we choose this approach because it contains a broad class of experiments we are used to deal with in economics as pointed out by McCall (1991)[1]. The convenience of this setup is due to the fact the de Finetti representation theorem applies to a sequence of exchangeable random variables and, in our context, it guarantees the existence and uniqueness of a prior distribution for the share of workers over the total population across cities.

A well-known result in Bayesian statistics shows that Pòlya urns generate sequences of exchangeable random

[1]Notice that i.i.d. random variables are exchangeable, and any weighted average of i.i.d. sequences of random variables is exchangeable; moreover, exchangeable sequences are stationary. These example shows that most of the current literature on heterogeneous agents that makes use of idiosyncratic shocks is actually working with exchangeable sequences.

variables. This class of urns models with reinforcement is not the only data generating process with this property, but it has been extensively studied Johnson and Kotz (1997) and widely applied to social sciences Mahmoud (2008). Choosing a Pòlya urn we want to capture the (natural) idea that the composition of the population across productive locations and the adoption of a certain technology are two processes that mutually influenced each other over time. In fact, if this is the case, then in our framework: (i) the owners of the factor that is most intensively used in production should account for a larger share of the population; (ii) the distribution of the share of workers over total population across cities must be unimodal and the mode should be in a neighborhood of that particular factor intensity that maximizes production. Choosing a Pòlya urn as data generating process for the population of agent we obtain a theoretical framework that is consistent with this fundamental idea.

Our approach is related to several lines of research. First, the microfoundation of the production function as the outcome of a stochastic process of factors (or idea) has been studied in growth theory; from the seminal contribution of Houthakker (1955) to the more recent frameworks developed by Kortum (1997) and Jones (2005). Moreover, the distribution of agents we derive in this paper is a special case of Hill (1970). With respect to this literature, we endogenously characterize the distribution of the stochastic process and we apply its convenient aggregation properties to the solution of macroeconomic models with heterogeneous agents. Second, our approach applies to a class of macroeconomic models with heterogeneous agents, incomplete financial markets and idiosyncratic income shocks; as in Huggett (1993) and Aiyagari (1994). The advantage of our methodology is that we solve in closed form for the cross sectional distribution of types across agents. Therefore, our approach allows to determine the cross sectional distribution of income, consumption and asset each point in time. Third, the methodology we propose is alternative to several approaches to the macroeconomics of heterogeneous agents that focus on complexity of interacting agents Durlauf and Ioannides (2010) and Durlauf (2012), agent based computational economics, Gaffeo et al. (2008) and Lengnick (2013), and the combinatorial approach introduced by Aoki (1998) and discussed in Aoki and Yoshikawa (2007). This literature departs from recursive analytical solutions, therefore we could hardly reconcile our approach to these methodologies. Nevertheless the Bayesian structure that is behind our framework is a point of contact between the different techniques. In this respect our methodology establishes a possible bridge between the analytical solutions of the mainstream literature and these computational approaches.

The remainder of the paper is structured as follows. In the next section we discuss the economic framework. In the third section we introduce exchangeability and Pòlya urns and we determine the distribution of agent types in the population. In the fourth section we define and characterize the dynamic stochastic general equilibrium. In section five we illustrate the application of our methodology to the solution of a simple DSGE model with heterogeneous agents. The last section concludes and discusses several developments of this project.

2 A Population Based Production Economy

Our aim is to characterize an economy segmented in a continuum of productive locations. The composition of local population determines factor supply in each productive location.

2.1 Endowments and market structure

The economy is populated by a continuum of agents and a continuum of firms. Agents and firms are located in spatially separated markets. We call each of these markets *city* and we index those with the letter j. Agents are of two types, according to the factor of production they own. In a given city j a number of W_j workers are endowed with one unit of labor each and a number of M_j managers are endowed with one unit of human capital. Each city is populated by agents of both types. The share of workers out of total population in a city j is given by:

$$X_j = \frac{W_j}{W_j + M_j} \tag{1}$$

where X_j is a random variable that follows a distribution $F(x_j)$. In the third section we provide the sufficient assumptions to determine the distribution endogenously.

Firms are competitive and the same technological solutions are available to all firms in the economy. Output is homogeneous and tradable across cities. Factor markets are segmented, such that firms hire workers and managers who belong to the same city. It follows that firms maximize profit taking the output price and the factor prices as given.

2.2 Preferences

An homogeneous good is consumed in the economy. Agents are infinitely living and in every period they order their preferences according to a period utility function $u : \mathbb{R}_+ \to \mathbb{R}$ that is increasing and concave in consumption of the homogeneous good and it satisfies the Inada conditions. Since, there is not a a loss of utility to supply of labor and human capital, each agent supplies one unit of the factor she is endowed with. The total supply of labor and human capital in a city j are simply:

$$L_j \equiv W_j \quad , \quad H_j \equiv M_j \tag{2}$$

2.3 Technology

Given the market structure a representative producer exists in each city, indeed without loss of generality we assume that one and only one representative firm runs production in every city. The representative firm j rents labor l and human capital h in the city j to produce service according to a technology $T(l,h) : \mathbb{R}_+^2 \to \mathbb{R}_+$ that is homogeneous of degree one, it is increasing and concave in both arguments, it satisfies Inada conditions and it obeys to the normalization $T(1,1) = 1$. The total amount of service in the economy is $S = \int_0^1 S_j(x_j)\,dx_j$; which over a measure one of cities yields $S = \int_0^1 S_j(x_j)\,dF'(x_j)$. Let $q_j \in (0,\infty)$ be the value of the density $F'(x_j)$ evaluated at the realization $X_j = x_j$. The technology that supplies service in every city j attains the maximum value of service S_j that can be produced by choosing labor and human capital subject to the composition of the population:

$$\begin{aligned} S_j &= \max_{l,h>0} T(l,h) \\ s.t. &: F'(x_j) = q_j \end{aligned} \tag{3}$$

The distribution F is unknown and the functional form of T is determined such that $x_j = inv\left(F'\left(q_j\right)\right)$, for every given value q_j. Notice that cities that are associated with the same density q_j will adopt the same technology S_j. Nevertheless, cities with the same q_j might be characterized by a different worker share. Therefore, the existence of the inverse function is restored with the additional constraint that $F\left(x_j\right) \leq Q_j$, where Q_j is the value of the cumulative density function associated to $X_j \leq x_j$ for a given distribution $F\left(x_j\right)$.

The goal of the next section is to determine the distribution of worker shares across cities $F\left(x_j\right)$.

3 Characterizing the population in a city

We investigate the sequence of agents that describes the population in a city. We proceed building a statistical model, that is a triplet $(\mathcal{Z}, \mathcal{F}, \mathcal{P})$ where $\mathcal{Z}$ is the space of possible realizations of the experiment, $\mathcal{F}$ is the $\sigma-algebra$ associated to $\mathcal{Z}$, and $\mathcal{P}$ is the family of probability measures over the measurable space $(\mathcal{Z}, \mathcal{F})$. The family $\mathcal{P}$ is parametrized by the random variable $\theta \in \Theta$, where Θ is the parameter space.

The object of observation is the population of a city. The single object in a population is an agent, which is a random variable because it takes one out of two possible types. Define the binary random variable A_i that takes values $a_i = 1$ if the agent i is a worker and $a_i = 0$ if the agent i is a manager. The statistical model that characterizes an agent is therefore a Bernoulli; we refer to this as the baseline model. Assume in each city we observe n agents, then the population in a city is a sequence of random variables $\mathcal{A}^{(n)} = (A_1, ..., A_n)$. One possible realization of the experiment is then the sequence of values $z^{(n)} = (a_1, ..., a_n)$ and $\mathcal{Z}$ is the set of all sequences $z^{(n)}$. Under Bernoullian sampling, the joint density of n observations of the baseline model is equal to n times the marginal densities.[2] Since the baseline model is Bernoulli, the marginal density is $\theta^{a_i}\left(1-\theta\right)^{1-a_i}$, with success probability $0 \leq \theta \leq 1$. The family of probability measure takes the form $\mathcal{P} = \prod_{i=1}^{n} \theta^{a_i}\left(1-\theta\right)^{1-a_i}$ where θ is a random variable defined on the parameter space $\Theta = [0, 1]$.

Our goal is to characterize the composition of the population in a city. Given the structure we frame the problem into, our task is now to find the joint density of the event $(A_1 = a_1,\ A_2 = a_2,\ ...,\ A_n = a_n)$. By construction, conditionally on the value of θ the random variables $(A_1, A_2, ..., A_n)$ are independent and identically distributed. Nevertheless, modeling the population in a city without knowledge about the value of θ requires to choose the type of dependence that links the elements A_i within the sequence $\{A_i\}_{i=1}^{n}$. There is a number of alternatives, we opt for a very simple form that describes a very large class of experiments in human sciences: *exchangeability*; that we discuss in the next section.

3.1 Exchangeability and the de Finetti representation theorem

Exchangeability is a powerful and elegant theoretical framework that applies to experiments that takes the form of sequence of events in which the order of the realization of a single event does not matter to determine the outcome of the experiment itself. In our framework, the random variable X_j characterizes the composition of the population in city j. We apply exchangeability to model the probability associated with a particular composition of the population

[2] The practice to assume a Bernoullian sampling is very common in statistics and in this paper is due to technical convenience.

$X_j = x_j$ because the order in which we observe managers and workers, as realizations a_i of the variable A_i, does not affect the probability associated to the composition of the population ($A_1 = a_1$, $A_2 = a_2$, ..., $A_n = a_n$).

We now introduce the definitions and theorems that are strictly necessary to perform the following analysis. Proofs and a more general discussion of the results can be found in Savage (1956).

Definition (1). *Random variables* $(A_1, A_2, ..., A_n)$ *are called exchangeable if the cumulative distribution function of each* $(A_{\rho_1}, A_{\rho 2}, ..., A_{\rho_n})$*, where* $(\rho_1, \rho_2, ..., \rho_n)$ *is any permutation of* $(1, 2, ..., n)$*, coincides with the cumulative distribution of* $(A_1, A_2, ..., A_n)$*.*

Definition (2). *The sequence* $\{A_i\}_{i=1}^n$ *is exchangeable if each finite subsequence is exchangeable.*

The population in each city is an exchangeable sequence of Bernoulli random variables. It follows by definition that all couples, triplets and tuple of variables A_i are identically distributed. It is possible to show that given an exchangeable sequence with n elements, finite and positive variance, then the linear correlation between any given couple of random variables is $corr\,(A_i, A_k) \geq -\frac{1}{n-1}$ for $i \neq k$ and if the sequence is infinite, $n \to \infty$, then $corr \geq 0$. Working with infinite exchangeable sequence we can apply de Finetti representation theorem (1937)[3]:

Theorem (1). *Let* $\{A_i\}_{i=1}^n$ *be an infinite sequence of exchangeable random variables such that each random variable takes on values* 0 *and* 1 *then the probability associated to each of the possible realizations* $(a_1, ..., a_n)$ *takes the form:*

$$Prob\,(A_1 = a_1, A_2 = a_2, ..., A_n = a_n) = \int_0^1 \theta^{\sum_{i=1}^n a_i} (1-\theta)^{n-\sum_{i=1}^n a_i}\, dF(\theta) \tag{4}$$

where $F(\theta)$ *is the almost sure limit of* $\frac{1}{n}\sum_{i=1}^n A_i$*:*

$$F(\theta) = \lim_{n\to\infty} Prob\left\{\frac{1}{n}\sum_{i=1}^n A_i \leq \theta\right\} \tag{5}$$

and its distribution is uniquely determined by the sequence $\{A_i\}_{i=1}^n$.

The probability distribution F is the *de Finetti measure.* Let $\{A_{ij}\}_{i=1}^n$ be the sequence that describes the population of agents i in city j, then notice that the worker share in city j is $X_j \equiv \frac{1}{n}\sum_{i=1}^n A_{ij}$. We assume that in each city the population is large enough that the asymptotic case $n \to \infty$ applies then F is the distribution of the worker share across cities.

The distribution F is a *prior*, because it adds an à priori information to the data, a subjective belief of the researcher on the unknown data generating process that does not originate from the observation of the data. The selection of a prior allows us to characterize the composition of the population across cities.

3.2 Select the prior with a Pòlya urn

One simple way to generate sequences of exchangeable random variables is the *Pòlya urn.* According to this mental exercise, the data generating process is described as a sequence of stages that starts with an urn containing an initial

[3]We state the theorem in its origenal version, de Finetti (1937), that applies to the Bernoulli distribution, because this our case. But the theorem has been proved in general and it became a pillar of the subjective probability approach. The interest reader could refer to Savage (1956).

number of balls of different colors. At each stage of the experiment a ball is randomly drawn from the urn, the color is observed and the ball is put back in the urn with in addition a given number of balls of the same color; that number of additional balls is called reinforcement. The distribution of the proportion of balls of one color in the urn is the outcome of repeating draws and reinforcements for a large number of stages.

Applying the Pòlya urn scheme to our framework, A_k is the random variable that records the type of the agent we observe at the $k-th$ stage in a given city. Workers and managers in the city before the $k+1$ stage are W_k and M_k. For a given initial allocation (W_0, M_0), the dynamics of processes A_{k+1}, W_{k+1} and M_{k+1} conditionally on the information at time k are described by:

$$A_{k+1} = \begin{cases} 1 \text{ with probability } \frac{W_k}{W_k+M_k} \\ 0 \text{ with probability } \frac{M_k}{W_k+M_k} \end{cases} \tag{6}$$

$$(W_{k+1}, M_{k+1}) = \begin{cases} (W_k + r, M_k) \text{ with probability } \frac{W_k}{W_k+M_k} \\ (W_k, M_k + r) \text{ with probability } \frac{M_k}{W_k+M_k} \end{cases}$$

where $r > 0$ is the reinforcement. The following theorem guarantees that the population in a city is an exchangeable sequence of agents of two types.

THEOREM (2). *The sequence $\{A_k\}_{k=0}^n$ generated by the Polya urn $\{W_k, M_k\}_{k=0}^n$ is exchangeable and its de Finetti measure is a Beta Distribution with parameters $\left(\frac{W_0}{r}, \frac{M_0}{r}\right)$.*

The limit behavior of the proportion of workers conditional on the previous information about the city is known.

THEOREM (3). *Given the Polya urn $\{W_k, M_k\}_{k=0}^n$, as n grows to infinity, the worker share at the n-th stage $X_n = \frac{W_n}{W_n+M_n}$ converges almost surely to a random limit. Moreover, the distribution of the limit is a Beta with parameters $\left(\frac{W_0}{r}, \frac{M_0}{r}\right)$.*

If each city is populated by an infinite numerable set of agents, then according to Theorem (3) the prior distribution of the worker share across cities is a Beta with parameters $\alpha_0 = \frac{W_0}{r} > 0$, $\beta_0 = \frac{M_0}{r} > 0$:

$$F(\theta) = \int_0^\theta \frac{\theta^{\alpha_0 - 1}(1-\theta)^{\beta_0 - 1}}{B(\alpha_0, \beta_0)} d\theta \tag{7}$$

where $B(\alpha_0, \beta_0) = \int_0^1 v^{\alpha_0 - 1}(1-v)^{\beta_0 - 1} dv$ is the beta function, a constant given the couple (α_0, β_0).

4 The Shape of the Matching Function

The properties of the technology T imply that in every city j the efficient production of human service S_j employs the total endowment of human factors L_j, H_j. The disposal of factors L_j, H_j is fixed by the factor supply (2). Indeed, a solution to the problem (3) has to satisfy $S_j(q_j) = T(W_j, M_j)$. From equation (7) the density of the distribution of worker shares is $F'(x_j) = x_j^{\alpha_0 - 1}(1-x_j)^{\beta_0 - 1} B(\alpha_0, \beta_0)^{-1}$. Let $n > 0$ be the arbitrary size of the population in a city. Workers share is x_j, workers and managers are respectively: $W_j = x_j n$ and $M_j = (1-x_j)n$. For a given value of the density $F'(x_j) = q_j$, the problem (3) is equivalent to:

$$\begin{aligned} S_j &= \max_{W_j, M_j > 0} T(W_j, M_j) \\ s.t. &: \quad W_j^{(\alpha_0 - 1)} M_j^{(\beta_0 - 1)} = \tilde{q}_j \end{aligned} \tag{8}$$

where $\tilde{q}_j = q_j B(\alpha_0, \beta_0) n^{\alpha_0 + \beta_0 - 2} > 0$. The first order conditions of this problem imply that the production function S_j belongs to the class of technologies with constant elasticities of substitution between factors:

$$\epsilon = \frac{\frac{\partial S_j}{\partial L_j}\frac{L_j}{S_j}}{\frac{\partial S_j}{\partial H_j}\frac{H_j}{S_j}} = \frac{\frac{\partial S_j}{\partial W_j}\frac{W_j}{S_j}}{\frac{\partial S_j}{\partial M_j}\frac{M_j}{S_j}} = \frac{\alpha_0 - 1}{\beta_0 - 1} \tag{9}$$

A functional form that satisfies condition (9) is given by $S_j = \left[\gamma L_j^{\frac{\epsilon-1}{\epsilon}} + (1-\gamma) H_j^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{\epsilon}{\epsilon-1}}$, for $\gamma > 0$ that has to be determined. A unique solution S_j satisfies the properties of the technology T: homogeneity of degree one and the normalization of units $T(1,1) = 1$. The first feature yields $\frac{S_j}{H_j} = T\left(\frac{L_j}{H_j}, 1\right)$; the second feature implies that if $L_j = H_j$ then $S_j = H_j = L_j$. The elasticity of substitution in the point $L_j = H_j = 1$ is equivalent to the marginal rate of technical substitution $\frac{\partial S_j}{\partial L_j} / \frac{\partial S_j}{\partial H_j} = \frac{\alpha_0 - 1}{\beta_0 - 1}$. The function S_j passes through a point that satisfies $S_j = H_j = L_j = 1$ and the condition $\frac{\gamma}{1-\gamma} = \frac{\alpha_0 - 1}{\beta_0 - 1}$. The constraint fixes the labor share γ:

$$\gamma = \frac{\alpha_0 - 1}{\alpha_0 + \beta_0 - 2} \tag{10}$$

From an economic point of view we want positive factors shares, then either $\alpha_0 > 1$ and $\beta_0 > 1 > 2 - \alpha_0$ or $\alpha_0 < 1$ and $\beta_0 < 1 < 2 - \alpha_0$. Remarkably notice that if $\alpha_0, \beta_0 > 1$ then the labor share is the mode of the Beta distribution describing the composition of the population. The technology that combines labor and human capital to produce service takes the form:

$$S(L_j, H_j) = \left[\left(\frac{\alpha_0 - 1}{\alpha_0 + \beta_0 - 2}\right) L_j^{\frac{\epsilon-1}{\epsilon}} + \left(\frac{\beta_0 - 1}{\alpha_0 + \beta_0 - 2}\right) H_j^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{\epsilon}{\epsilon-1}} \tag{11}$$

It is important to notice that the elasticity of substitution between factors and the distribution of agent types are linked. Consider the limit case of a Cobb-Douglas. Notice that $\epsilon \to 1$ if and only if $\alpha_0 = \beta_0$, then the distribution of worker share is symmetric around $\frac{1}{2}$. When $\alpha_0 > \beta_0$ the elasticity of substitution between factors is larger than one. The opposite is true for $\beta_0 > \alpha_0$.

4.1 Service and income across cities

Since $L_j = W_j = x_j n$ and $H_j = M_j = (1 - x_j) n$ then, the production function of service can be written in terms of the worker share x_j, given the population size n:

$$S(x_j; n) = n\left[\left(\frac{\alpha_0 - 1}{\alpha_0 + \beta_0 - 2}\right) x_j^{\frac{\epsilon-1}{\epsilon}} + \left(\frac{\beta_0 - 1}{\alpha_0 + \beta_0 - 2}\right) (1 - x_j)^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{\epsilon}{\epsilon-1}} \tag{12}$$

Service is traded in the economy and the production of service within a city is competitive. Let $p^s > 0$ be the price of human service in the economy, then worker's wage w and manager's reward m are equal to the value of marginal productivities: respectively $\frac{w_j}{p^s} = \left(\frac{\alpha_0 - 1}{\alpha_0 + \beta_0 - 2}\right)\frac{S_j}{L_j}$ and $\frac{m_j}{p^s} = \left(\frac{\beta_0 - 1}{\alpha_0 + \beta_0 - 2}\right)\frac{S_j}{H_j}$. As in the case of service output, competitive factor rewards are given as a function of worker share:

$$w_j = w\left(x_j; p^s\right) = p^s \left[\left(\frac{\alpha_0 - 1}{\alpha_0 + \beta_0 - 2}\right) + \left(\frac{\beta_0 - 1}{\alpha_0 + \beta_0 - 2}\right)\left(\frac{1 - x_j}{x_j}\right)^{\frac{\epsilon - 1}{\epsilon}}\right]^{\frac{\epsilon}{\epsilon - 1}} \tag{13}$$

$$m_j = m\left(x_j; p^s\right) = p^s \left[\left(\frac{\alpha_0 - 1}{\alpha_0 + \beta_0 - 2}\right)\left(\frac{x_j}{1 - x_j}\right)^{\frac{\epsilon - 1}{\epsilon}} + \left(\frac{\beta_0 - 1}{\alpha_0 + \beta_0 - 2}\right)\right]^{\frac{\epsilon}{\epsilon - 1}} \tag{14}$$

Service $S\left(x_j; n\right)$, wage $w\left(x_j; p^s\right)$ and managerial income $m\left(x_j; p^s\right)$ are continuous, differentiable and real valued functions of the random variable X_j. Therefore, the distributions of service, wage and managerial income across cities are derived from the distribution of worker share (7).

Service per capita $s_j = s\left(x_j\right) = \frac{S(x_j;n)}{n}$, real wage $\tilde{w}_j = \tilde{w}\left(x_j\right) = \frac{w(x_j;n)}{p^s}$ and real managerial income $\tilde{m}_j = \tilde{m}\left(x_j\right) = \frac{m(x_j;n)}{p^s}$ depend on the worker share only and not on city size n and price of service p^s. Average service $\mathbb{E}\left[S\right]$, wage $\mathbb{E}\left[w\right]$ and managerial income $\mathbb{E}\left[m\right]$ are the product of two components:

$$\mathbb{E}\left[S\right] = n\ \mathbb{E}\left[s\left(x_j\right)\right] \quad , \quad \mathbb{E}\left[w\right] = p^s\ \mathbb{E}\left[\tilde{w}\left(x_j\right)\right] \quad , \quad \mathbb{E}\left[m\right] = p^s\ \mathbb{E}\left[\tilde{m}\left(x_j\right)\right] \tag{15}$$

one that does not depend on the composition of the population (n and p^s); the second one that is fixed uniquely by the distribution of worker shares. The dichotomy emerges endogenously, since it is due to the functional form of the production function S that is induced by the statistical process behind the distribution of worker shares. This feature is desirable while addressing the aggregation problem. The aggregate the supply of service and agent incomes across cities is obtained in exact form, as the distribution of the worker share X_j is known. Service and income are scaled up by city size n and service price p^s which are independent on the aggregation problem.

5 A Simple DSGE Model with Income Distribution

In this section we outline a simple dynamic stochastic general equilibrium model in which agents face idiosyncratic uncertainty. No contingent assets are traded, indeed, agents cannot insure against income uncertainty.

5.1 Decentralized equilibrium with heterogeneous agents

At the aggregate level, human service and physical capital are combined to produce an homogeneous output, that is taken as the numeraire. The relative price of service in terms of the homogeneous good is $p > 0$, the (real) rental rate of capital is $r > 0$. Period consumption and (non-contingent) asset holding of an agent of type $k = \{w, m\}$ are respectively $c_k \geq 0$ and $a_k \geq -b$, where the value $b \geq 0$ is a borrowing constraint. The period income of workers and managers are respectively: $i_w = p\tilde{w}\left(x_j\right) + \left(1 - d + r\right)a_w$ and $i_m = p\tilde{m}\left(x_j\right) + \left(1 - d + r\right)a_m$; where where $0 \leq d \leq 1$ is the depreciation rate of capital in a period.

Workers and managers have the same preferences, which are represented by a continuous, increasing and concave utility function $u : \mathbb{R}^+ \to \mathbb{R}$ that satisfies the Inada conditions. Agents of type k conditionally on being in city j solve the following maximization problem in recursive form:

$$v\left(a_k, x_j; p, r\right) = \sup_{a'_k \in [-b, i_k]} \left\{u\left(i_k - a'_k\right) + \delta \mathbb{E}\left[v\left(a'_k, x_j; p', r'\right) | \varphi\right]\right\} \tag{16}$$

where the variables with prime superscript refer to the next period and $\delta \in (0,1)$. The pair of values $\{a, x_j\}$ is the individual state of the dynamic problem, while the aggregate state is given by $\{p, r\}$. The expectation operator $\mathbb{E}\left[v\left(\cdot; p', r'\right)|\varphi\right]$ applies to the next period realization of the aggregate state.

The total endowment of capital in the economy in a given period K consists of the aggregation asset holdings across agents and cities. The total amount of human service in an economy is the aggregation of service across cities. Split the economy in a measure one of cities, each city is populated by a measure one of agents of two types, then aggregate consumption, capital and service are:

$$\begin{aligned} C &= \frac{1}{2}\mathbb{E}\left[x_j c_w\left(x_j\right) + \left(1 - x_j\right) c_m\left(x_j\right)\right] \\ K &= \frac{1}{2}\mathbb{E}\left[x_j a_w\left(x_j\right) + \left(1 - x_j\right) a_m\left(x_j\right)\right] \\ S &= \mathbb{E}\left[s\left(x_j\right)\right] \end{aligned} \tag{17}$$

Notice that the service is constant over time, as it only depends on the population of agents.

A representative firm produces the homogeneous output according to a Cobb-Douglas technology $Y = \varphi K^{\sigma} S^{(1-\sigma)}$, with total factor productivity $\varphi > 0$ and capital share $0 < \sigma < 1$. Express the aggregate variables per unit of service: $c = \frac{C}{S}$, $y = \frac{Y}{S}$ and $k = \frac{K}{S}$. Production reads:

$$y = \varphi k^{\sigma} \tag{18}$$

Let investment be $I = Y - C$ and next period capital $K' = (1-d) K + I$. The aggregate budget constraint implies:

$$c + k' - (1-d) k = y \tag{19}$$

The competitive factor rewards for capital and service are respectively:

$$\begin{aligned} r &= \sigma \varphi k^{\sigma - 1} \\ p &= (1 - \sigma) \varphi k^{\sigma} \end{aligned} \tag{20}$$

When the aggregate conditions (17)-(20) are understood, a solution to the agent's problem (16) is a policy function,

$$a'_k\left(x_j\right) = \pi\left(a_k\left(x_j\right); \varphi, k\right) \tag{21}$$

that prescribes the next period asset holding as a function of the current individual state $\{a_k, x_j\}$ and aggregate state $\{\varphi, k\}$.

5.2 Income distribution

The aim of this section is to present the simplest specification of the model outlined in equations (16)-(21), such that we can obtain a closed form solution and discuss the distribution of income across agents.

We restrict the discussion in three directions: (*i*) utility takes the log form $u = \ln(c)$; (*ii*) there is full capital depreciation $d = 1$; (*iii*) the share of total capital $f_k(x_j)$ hold by an agent of type $k = \{w, m\}$ in city j does not depend on the aggregate state, such that $a_k(x_j) = f_k(x_j)\,k$. Under these restrictions, the model is just an extension of the classical Brock and Mirman (1972). The income of an agent reads $i_k = \iota_k(x_j)\,\varphi k^{\sigma}$ where $\iota_w = [(1-\sigma)\,\tilde{w}(x_j) + \sigma f_w(x_j)]$ and $\iota_m = [(1-\sigma)\,\tilde{m}(x_j) + \sigma f_m(x_j)]$. In order to derive the solution, guess that an optimal path for asset holding satisfies: $a'_k(x_j) = g_k(x_j)\,\varphi k^{\sigma}$. The first order condition for an interior solution of the inter-temporal problem (16) implies:

$$\frac{1}{[\iota_k(x_j) - g_k(x_j)]\,\varphi k^{\sigma}} = \delta \mathbb{E}\left\{\frac{\sigma \varphi' k'^{\sigma-1}}{[\iota_k(x_j) - g_k(x_j)]\,\varphi' k'^{\sigma}} |\varphi\right\}$$

Therefore, the stock of capital in the economy evolves according to the policy:

$$k' = \sigma \delta \varphi k^{\sigma} \tag{22}$$

Substituting for $k' = \frac{a'_k(x_j)}{f_k(x_j)}$ yields $g_k(x_j) = f_k(x_j)\,\sigma\delta$ and the policy for asset holding is determined:

$$\begin{aligned} a'_w(x_j) &= a_w(x_j)\,\delta\sigma\varphi k^{\sigma-1} \\ a'_m(x_j) &= a_m(x_j)\,\delta\sigma\varphi k^{\sigma-1} \end{aligned} \tag{23}$$

The agent budget constraint and the policy for asset holding (23) imply the consumption decisions: $c_w(x_j) = p\tilde{w}(x_j) + (1-\delta)\,r a_w(x_j)$ and $c_m(x_j) = p\tilde{m}(x_j) + (1-\delta)\,r a_m(x_j)$. The solution of the model is given for an initial distribution of asset across agents and cities $\left\{a_w^0(x_j), a_m^0(x_j)\right\}$ that satisfies the aggregate conditions (17).

Steady state. The aggregate steady state is determined by an allocation of capital such that $k' = k$. The total amount of service only depends on the composition of the population, indeed it is constant over time. Let $\bar{\varphi} > 0$ be the steady state level of total factor productivity, that is exogenous to the model. From the law of motion for capital (22), the steady state level of capital per unit of service is $\bar{k} = (\sigma\delta\bar{\varphi})^{\frac{1}{1-\sigma}}$. The aggregate budget constraint (19) and the production function (18) yield the steady state level of aggregate consumption per unit of service $\bar{c} = \bar{\varphi}\bar{k}^{\sigma} - \bar{k} > 0$. Factor prices in steady state are $\bar{r} = \sigma\bar{\varphi}\bar{k}^{\sigma-1}$ and $\bar{p} = (1-\sigma)\,\bar{\varphi}\bar{k}^{\sigma}$, as it follows from (20).

Initial allocation. We determine the initial allocation by conjecture: if at the beginning of time agents settled freely across cities then they had to be indifferent among alternative locations. Under this assumption, consider the economy to be endowed with a steady state level of capital k^0. Let $c\left(k^0\right) = \mathbb{E}\left[s(x_j)\right]\left(\varphi^0 k^{0\sigma} - k^0\right)$ be the per capita consumption at the beginning of time in each city of the economy. Then combining the budget constraint and the optimal policy for asset holding yields the initial wealth allocation,

$$\begin{aligned} a'_w\left(x_j; k^0\right) &= a_w\left(x_j; k^0\right) = z \times \frac{p\left(k^0\right)\tilde{w}(x_j) - c\left(k^0\right)}{(1-\delta)\,r\left(k^0\right)} \\ a'_m\left(x_j; k^0\right) &= a_m\left(x_j; k^0\right) = z \times \frac{p\left(k^0\right)\tilde{m}(x_j) - c\left(k^0\right)}{(1-\delta)\,r\left(k^0\right)} \end{aligned} \tag{24}$$

for every value of the scaling factor $z > 0$. An initial endowment of asset across agents that satisfies the aggregate condition (17) must be a solution to the following equation:

$$\mathbb{E}\left[x_j a_w\left(x_j; k^0\right) + (1 - x_j)\, a_m\left(x_j; k^0\right)\right] = \frac{p\left(k^0\right) k^0 \mathbb{E}\left[x_j \tilde{w}\left(x_j\right) + (1 - x_j)\, \tilde{m}\left(x_j\right)\right]}{c\left(k^0\right) - k^0\left(1 - \delta\right) r\left(k^0\right)} \tag{25}$$

which fixes the first moment of asset per agent $x_j a_w\left(x_j; k^0\right) + (1 - x_j)\, a_m\left(x_j; k^0\right)$ across cities for a given capital k^0. Notice that the system of (24) and (25) yields the unique value for the scaling factor z. If k^0 is a steady state level of capital then $k^0 = \bar{k}$ gives the initial distribution of wealth.

6 Conclusion

This paper provides a simple framework to handle heterogeneity across agents in a real business cycle model. Agents are of two types differentiated by the human factor of production they own and they are located in separated cities. In each city competitive firms choose the technology that maximizes output subject to the composition of the local population. Assuming the population of agents is a collection of exchangeable random variables we microfound the distribution of agent types in the population and the matching function that combines human factors. Aggregation of individual decisions over the known distribution of agent types yields macroeconomic variable outcomes together with the distribution of income, consumption and asset holdings across agents.

The current framework has at least two major limitations from an empirical point of view. First, the composition of the population is independent across productive locations. A natural development of this paper is to introduce dependence across productive locations. One way to incorporate this is to allow for the population of agents being a sequence of partially exchangeable random variables in which one can control for the dependence across productive locations. The second limitation is the choice of two types of agents only. This issue can be overcome extending the present framework with a Bernoullian statistical model to a non parametric approach based on the Dirichelet process. We think that this direction of research will provide a tractable framework to fit income distributions and their pattern of change over the business cycle.

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