



On the power counting in effective field theories

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ABSTRACT

We discuss the systematics of power counting in general effective field theories, focusing on those that are nonrenormalizable at leading order. As an illuminating example we consider chiral perturbation theory gauged under the electromagnetic $U(1)$ symmetry. This theory describes the low-energy interactions of the octet of pseudo-Goldstone bosons in QCD with photons and has been discussed extensively in the literature. Peculiarities of the standard approach are pointed out and it is shown how these are resolved within our scheme. The presentation follows closely our recent discussion of power counting for the electroweak chiral Lagrangian. The systematics of the latter is reviewed and shown to be consistent with the concept of chiral dimensions. The results imply that naive dimensional analysis (NDA) is incomplete in general effective field theories, while still reproducing the correct counting in special cases.

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1. Introduction

Effective field theories (EFTs) are the most efficient way of describing physics at a certain energy scale, provided there is a mass gap and the dynamical field content as well as the symmetries at that scale are known. What makes EFTs especially useful is that the operators one can build out of the fields can be organized according to their importance in a systematic expansion. The organizing principle is based on a power-counting argument. In weakly-coupled scenarios the power counting reduces to a dimensional expansion, where fields have canonical dimensions and higher-dimension operators are weighted with inverse powers of a cutoff scale Λ , whose value indicates the scale of new physics. In this case Λ can be arbitrarily large.

The situation is different in spontaneously broken strongly-coupled scenarios. Such theories are nonrenormalizable even at leading order. As a result, they are non-decoupling, *i.e.*, the scale of new physics is no longer arbitrary but required to be at $\Lambda \approx 4\pi f$, where f is the Goldstone-boson decay constant of the strong sector. Correspondingly, strongly-coupled effective theories can only be consistent if based on a loop expansion, where the loop divergences at a given order are renormalized by operators at the following order. The EFT is predictive if the size of the counterterms is of the same order as the loop contributions, to which they are related by renormalization [1]. Power counting is no longer based on the canonical dimension of fields and should be constructed

instead by analyzing the loop structure of a given (leading-order) Lagrangian. Knowledge of the effective Lagrangian at leading order is therefore necessary.

This strategy for the construction of EFTs with strongly-coupled dynamics is notably simplified in specific cases. The paradigm of simplicity is chiral perturbation theory (χ PT) [2,3] for massless pions, where the power counting reduces to an expansion in derivatives. When external sources are added and pion masses switched on [4,5], chiral symmetry is explicitly broken. It is common to extend the derivative counting to these new objects too. This formal assignment of derivative counting to couplings and fields goes under the name of chiral dimensional counting (χ DC). It is constrained by the requirement that the terms in the leading-order Lagrangian must have the same chiral dimension. Following this method, extensions of χ PT to include dynamical photons [6] and leptons [7] have been formulated.

Defined in this way, the assignment of chiral dimensions seems unsatisfactory in some respects. First, chiral dimensions may suggest a misleading interpretation of the physics of strongly-coupled dynamics. For instance, the electromagnetic coupling e is a parameter independent of chiral symmetry breaking, yet it has an assigned momentum scaling. χ DC should thus be rather understood as a formal tool. However, to the best of our knowledge, χ DC has never been justified in terms of a diagrammatic power counting. Second, χ DC alone does not yet allow one to construct an operator basis. In other words, it is no substitute for a full-fledged power-counting formula. These points have led to some confusion in the literature, especially in studies of electroweak effective theories [8,9].

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An alternative approach is naive dimensional analysis (NDA) [1]. Using the chiral quark model as a paradigmatic example, a simple set of rules has been inferred to build power-counting formulas for generic EFTs. In a nutshell, fundamental and composite fields are simply associated with different scales, $1/\Lambda$ for the former and $1/f$ for the latter [10]. This prescription is in contrast to χ DC.

Both χ DC and NDA have the common objective to describe the systematics of EFTs in which strongly-coupled and weakly-coupled sectors mix. Since both methods rely on some sort of dimensional expansion encoded in a set of rules, it would be interesting to explore the relation between both approaches and understand whether they are mutually consistent. In this Letter we will clarify these issues by reassessing the χ DC and NDA prescriptions in the light of a general power-counting formula for strongly-coupled theories with fermions, gauge bosons and scalars, initially derived in [11,12]. We will specialize it to the strong and electroweak interactions and compare it with the predictions of χ DC and NDA.

We show that χ DC is a consistent prescription and can be rephrased in terms of systematic power-counting arguments. Its formal and, strictly speaking, unphysical scaling rules turn out to be the price to pay in order to force a simple dimensional counting onto a strongly-coupled EFT. We also show that the rules of NDA are not valid in general and lead to contradictions, for instance in the electroweak interactions. We point out how they should be modified. In particular, we will find out that power counting is insensitive to the fundamental or composite nature of fermions, yet very sensitive to their couplings with the Goldstone modes.

This Letter is organized as follows. We revisit χ PT with dynamical photons in Section 2 and derive the relevant power-counting formula. The latter is a new result, in spite of the fact that this EFT has been widely used. The formula allows us to prove that the definition of chiral dimensions employed in [6] is both consistent and unique. In Section 3 we turn to the electroweak interactions and discuss the power counting that applies when the spontaneous symmetry breaking is induced by strongly-coupled dynamics. We show how the general results derived in [11,12] can be reinterpreted in the language of chiral dimensions. Section 4 comments on the implications for the NDA prescription. We conclude in Section 5.

2. Chiral perturbation theory with photons

2.1. Lagrangian at leading order

Many of the essential features in the power counting of strongly-coupled effective field theories are already present in the case of chiral perturbation theory of pions and kaons coupled to electromagnetism. Due to its relative simplicity this case will serve as an instructive example for our discussion.

Under $SU(3)_L \times SU(3)_R$ the Goldstone boson matrix U transforms as

$$U \rightarrow g_L U g_R^\dagger, \quad g_{L,R} \in SU(3)_{L,R} \quad (1)$$

The explicit relation between the matrix U and the Goldstone fields φ^a is

$$U = \exp(2i\Phi/f), \quad \Phi = \varphi^a T^a \quad (2)$$

where $T^a = T_a = \lambda^a/2$ are the generators of $SU(3)$ and $f \approx 93$ MeV is the Goldstone-boson decay constant.

The vectorial subgroup of $SU(3)_L \times SU(3)_R$ is gauged under the electromagnetic $U(1)$, so that the covariant derivative is given by

$$D_\mu U = \partial_\mu U + ieA_\mu [Q, U] \quad (3)$$

where $Q = \text{diag}(2/3, -1/3, -1/3)$.

The full chiral symmetry $SU(3)_L \times SU(3)_R$ is broken by the quark-mass term (χ) and by electromagnetism (Q). This can be implemented in the standard way through the corresponding spurions transforming as

$$\chi \rightarrow g_L \chi g_R^\dagger, \quad Q_L \rightarrow g_L Q_L g_L^\dagger, \quad Q_R \rightarrow g_R Q_R g_R^\dagger \quad (4)$$

with the identification $Q_L = Q_R = Q$. Similarly, $\chi = 2B\mathcal{M}$ with $\mathcal{M} = \text{diag}(m_u, m_d, m_s)$.

The leading-order Lagrangian can then be written as [6,7,13–16]

$$\begin{aligned} \mathcal{L}_{\text{LO}} = & \frac{f^2}{4} \langle D_\mu U^\dagger D^\mu U \rangle + \frac{f^2}{4} \langle U^\dagger \chi + \chi^\dagger U \rangle \\ & - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + e^2 \Delta \langle U^\dagger Q U Q \rangle \end{aligned} \quad (5)$$

where $\langle \dots \rangle$ denotes the trace.

Eq. (5) is the lowest-order approximation to the theory of pseudo-Goldstone bosons and photons with typical energies of the order of f . The expansion parameter governing higher-order corrections is f^2/Λ^2 , with $\Lambda = 4\pi f$ the scale of chiral symmetry breaking. All terms in (5) are indeed of leading order ($\sim f^4$) in this expansion. This follows from the fact that in a scattering process involving Goldstone bosons and photons at a typical energy $\sim f \ll \Lambda$, the relevant quantities scale as

$$\partial_\mu \sim f, \quad \varphi^a \sim f, \quad \chi \sim f^2, \quad A_\mu \sim f, \quad e \sim 1 \quad (6)$$

The spurion χ is proportional to the pseudo-Goldstone masses squared, which are counted in the standard way as $\sim f^2$, consistent with the homogeneous scaling of the meson propagator. The coupling e is an independent parameter, which can be viewed as a quantity of order one. A further expansion for $e^2 \ll 1$ can always be performed if desired.

The scaling $\sim f^4$ follows immediately for the first three terms in (5). The last term has no derivatives and amounts to a potential for the Goldstone bosons, induced by virtual photons. It is proportional to the cut-off squared, but carries a loop suppression. It scales as [17]

$$\Delta \sim f^2 \frac{\Lambda^2}{16\pi^2} \sim f^4 \quad (7)$$

and is consistently included in \mathcal{L}_{LO} . We note that the leading-order Lagrangian has terms with canonical dimension zero (second and fourth term), two (first term) and four (third term). As is well known, the Lagrangian is not ordered by canonical dimension of operators in the case of a strongly-interacting system. This is hardly surprising, since already the first term in \mathcal{L}_{LO} contains operators of arbitrarily large canonical dimension when expanded out in powers of the field Φ .

Rather than by dimensional counting, the higher-order terms correcting the Lagrangian in (5) are governed by a loop expansion, which corresponds to a series in powers of $1/(16\pi^2) = f^2/\Lambda^2$. The systematics of this construction can be described by a power-counting formula, which we discuss in the following section.

2.2. Power counting and the Lagrangian at NLO

The leading-order Lagrangian (5) is nonrenormalizable. Corrections can be organized in the form of a loop expansion. The relevant power counting has been discussed in [11,12] for the electroweak chiral Lagrangian. It makes use of the assumption that the loop effects in the strong sector $\sim 1/(16\pi^2)$ are actually of the same order of magnitude as the corresponding coefficients of NLO operators $\sim f^2/\Lambda^2$ [1,10]. This implies the identification $\Lambda = 4\pi f$.

A priori, the coefficients $\sim f^2/\Lambda^2$ need only be at least of the size of the loop contribution $\sim 1/(16\pi^2)$, giving $\Lambda \lesssim 4\pi f$. The approximate equality is a natural assumption for QCD. In the case of electroweak symmetry breaking the assumption is also justified as long as new-physics states appear only at the (few) TeV scale or above.

Adapted to the present case, the power counting for a diagram \mathcal{D} with L loops, built from the vertices of (5), can be summarized by the formula

$$\mathcal{D} \sim \frac{p^d}{\Lambda^{2L}} \left(\frac{F_{\mu\nu}}{f} \right)^V \left(\frac{\varphi}{f} \right)^B \quad (8)$$

where the power of external momenta p is

$$d \equiv 2L + 2 - V - m - 2r - 2\omega_\chi - 2\omega_Q \quad (9)$$

Here V is the number of external factors of photon field strength $F_{\mu\nu}$, m (r) is the number of photon-meson vertices of the form $A_\mu\phi^l$ ($A_\mu^2\phi^s$), and ω_χ (ω_Q) is the number of meson vertices from the term with χ (Q) in (5). The number B of external meson lines does not enter d in (9).

Assuming dimensional regularization, an exponent $d \geq 0$ in (8) indicates a divergence by power counting, as well as the number of derivatives in the corresponding counterterm (not counting those in the factors of $F_{\mu\nu}$). Using (9), one finds that the classes of counterterms at next-to-leading order ($L = 1$) are exhausted by the six cases

$$(V, \omega_\chi + \omega_Q; d): \quad (0, 0; 4), \quad (0, 1; 2), \quad (0, 2; 0), \\ (1, 0; 2), \quad (1, 1; 0), \quad (2, 0; 0) \quad (10)$$

The explicit operators in each of these classes that are compatible with chiral symmetry and its breaking by spurions, and even under C and P , can be listed as follows:

(0,0;4):

$$\langle D_\mu U^\dagger D^\mu U \rangle^2, \quad \langle D_\mu U^\dagger D_\nu U \rangle \langle D^\mu U^\dagger D^\nu U \rangle, \\ \langle D_\mu U^\dagger D^\mu U D_\nu U^\dagger D^\nu U \rangle \quad (11)$$

(0,1;2):

$$\langle D_\mu U^\dagger D^\mu U \rangle \langle \chi U^\dagger + U \chi^\dagger \rangle, \quad \langle D_\mu U^\dagger D^\mu U (U^\dagger \chi + \chi^\dagger U) \rangle, \\ e^2 \langle D_\mu U^\dagger D^\mu U \rangle \langle U^\dagger Q U Q \rangle, \quad e^2 \langle D_\mu U^\dagger (U Q \pm Q U) \rangle^2, \\ e^2 \langle (D_\mu U^\dagger D^\mu U + D_\mu U D^\mu U^\dagger) Q \rangle, \\ e^2 \langle D_\mu U^\dagger D^\mu U (U^\dagger Q U Q + Q U^\dagger Q U) \rangle \quad (12)$$

(0,2;0):

$$\langle \chi U^\dagger \pm U \chi^\dagger \rangle^2, \quad \langle \chi U^\dagger \chi U^\dagger + U \chi^\dagger U \chi^\dagger \rangle, \\ e^2 \langle \chi U^\dagger + U \chi^\dagger \rangle \langle U^\dagger Q U Q \rangle, \quad e^2 \langle (\chi U^\dagger + U \chi^\dagger + U^\dagger \chi + \chi^\dagger U) Q \rangle, \\ e^2 \langle (\chi U^\dagger \pm U \chi^\dagger) Q U Q U^\dagger + (\chi^\dagger U \pm U^\dagger \chi) Q U^\dagger Q U \rangle, \\ e^4 \langle U^\dagger Q U Q \rangle^2 \quad (13)$$

(2,0;0):

$$e^2 F_{\mu\nu} F^{\mu\nu} \langle U^\dagger Q U Q \rangle \quad (14)$$

Classes **(1,0;2)** and **(1,1;0)** contain no independent operators.

The complete list of NLO operators in (11)–(14) is consistent with the results originally obtained in [6]. NLO counterterms that reduce to LO structures when $Q_{L,R} \rightarrow Q = \text{const.}$ are not included in our list above.

2.3. Alternative scheme based on chiral dimensions

It is possible to interpret the results of the previous sections in a somewhat different, but equivalent, way. Since the chiral Lagrangian is organized as a loop expansion, the different orders are just given by L , the number of loops. For convenience we may define $2L + 2$ as the *chiral order* of the corresponding terms in the Lagrangian. Using (9) the chiral order can be written as

$$2L + 2 = d + V + m + 2r + 2\omega_Q + 2\omega_\chi \quad (15)$$

This implies that the chiral order $2L + 2$ is obtained for any term by adding the number d of derivatives it contains (not counting those in $F_{\mu\nu}$), the number V of photon field-strength factors $F_{\mu\nu}$, the total number $m + 2r + 2\omega_Q$ of couplings e , and twice the number ω_χ of factors of χ . A *chiral dimension* $[x]_c$ can therefore be assigned to any quantity x in the chiral Lagrangian according to the number it contributes to the chiral order of an operator. It follows from (15) that

$$[\partial_\mu]_c = 1, \quad [F_{\mu\nu}]_c = 1, \quad [e]_c = 1, \quad [\chi]_c = 2, \\ [U]_c = 0 \quad (16)$$

and consequently

$$[A_\mu]_c = 0, \quad [D_\mu]_c = 1, \quad [\Phi]_c = 0 \quad (17)$$

Applying these rules, we see that all terms in the leading-order Lagrangian (5) have chiral order 2, irrespective of their canonical dimension. All the terms at NLO at the end of Section 2.2 have chiral order 4. We also note the well-known result that in the case of pure chiral perturbation theory (without photons and χ term) the chiral order is simply given by the number of derivatives, $2L + 2 = d$, which follows from (15) as a special case. Based on this result chiral perturbation theory is often viewed as an expansion in the number of derivatives. While this is correct in the simplest case, it should not be misinterpreted as being equivalent to an expansion in (canonical) dimension.

The assignment in (16) and (17) is the counting introduced for chiral perturbation theory with photons by Urech in [6]. The justification given there was somewhat different from ours, although essentially equivalent in the end. It was noted in [6] that the formal counting in (16) and (17) leads to a homogeneous chiral order of 2 for the Lagrangian (5). The one-loop counterterms were then computed by the heat-kernel method and shown to correspond to the terms of chiral order 4. We emphasize, however, that the assignment for e and A_μ in (16) and (17) is purely formal and has a consistent basis in the power-counting formulas of (9) and (15). Those can be derived from the physical scaling of parameters and fields and a priori without recourse to the concept of chiral dimension.

3. Electroweak chiral Lagrangian

3.1. Basic structure

The leading-order electroweak chiral Lagrangian including a light Higgs boson can be schematically written in the form [18–20]

$$\mathcal{L}_{\text{EW,LO}} = -\frac{1}{4} X_{\mu\nu} X^{\mu\nu} + \bar{\psi} i \not{D} \psi + \frac{v^2}{4} \langle D_\mu U^\dagger D^\mu U \rangle (1 + F(h)) \\ - yv [\bar{\psi}_L F_Y(h) U \psi_R + \text{h.c.}] \\ + \frac{1}{2} \partial_\mu h \partial^\mu h - V(h) \quad (18)$$

The construction of next-to-leading order terms (in the higgsless case) dates back to the work of [21–23]. Including a light Higgs,

a subset of NLO operators has been given in [24], the power counting and the complete list has been worked out in [11,12]. Beyond power counting, the actual size of the operator coefficients may be further suppressed through additional symmetries (such as flavor or CP).

In [11,12] it has been shown that the result of power counting for a generic L -loop diagram \mathcal{D} built from vertices of the leading-order Lagrangian (18) can be expressed as

$$\mathcal{D} \sim v^{2+2\omega} (y\nu)^\nu (g\nu)^\gamma \frac{p^d}{\Lambda^{2L}} \left(\frac{\psi}{v}\right)^F \left(\frac{X_{\mu\nu}}{v}\right)^V \left(\frac{\varphi}{v}\right)^B \left(\frac{h}{v}\right)^H \quad (19)$$

where the power of external momenta p is

$$d \equiv 2L + 2 - \frac{F}{2} - V - \nu - \gamma - 2\omega \quad (20)$$

Here F and V are the number of external fermions and gauge-boson field-strength factors, respectively. ν is the number of Yukawa couplings, γ the number of gauge couplings, and ω the number of (non-derivative) Higgs-boson self interactions.

3.2. Chiral dimensions

We note that this general result, which includes chiral fermions, can also be interpreted using the concept of chiral dimensions. This leads to an immediate generalization of the case discussed in Section 2.3. Indeed, the chiral order may now be written as

$$2L + 2 = d + \frac{F}{2} + V + \nu + \gamma + 2\omega \quad (21)$$

which corresponds to the following assignment of chiral dimension to derivatives ∂_μ , Goldstone fields φ , Higgs fields h , gauge-field strengths $X_{\mu\nu}$, fermions $\psi_{L,R}$, gauge couplings g and Yukawa couplings y :

$$\begin{aligned} [\partial_\mu]_c = 1, \quad [\varphi]_c = [h]_c = 0, \quad [X_{\mu\nu}]_c = 1, \quad [\psi_{L,R}]_c = \frac{1}{2}, \\ [g]_c = [y]_c = 1 \end{aligned} \quad (22)$$

With this counting all the terms in the leading-order Lagrangian (18) have chiral dimension 2. The potential is assumed to be radiatively generated and thus implicitly includes a factor of g^2 or y^2 .

The classes of next-to-leading order operators worked out in [11,12]

$$\begin{aligned} UhD^4, \quad g^2 X^2 Uh, \quad gXUhD^2, \quad y^2 \psi^2 UhD, \\ y\psi^2 UhD^2, \quad y^2 \psi^4 Uh \end{aligned} \quad (23)$$

have chiral dimension 4. Note that the counting based on chiral dimension necessitates the inclusion of the appropriate powers of couplings in the NLO counterterms.

The chiral dimensions in (22) have been discussed previously in the work of Nyffeler and Schenk [8] in the context of the (higgsless) electroweak chiral Lagrangian. However, the assignment of chiral dimensions in [8] was not based on an explicit power-counting analysis of loop corrections and counterterms. Instead, [8] defined the chiral dimensions of couplings and fields such as to ensure a homogeneous scaling of all the kinetic terms, following [6]. For instance, with $[D_\mu U^\dagger D^\mu U]_c = 2$, the same scaling of the fermion kinetic term, $[\bar{\psi} i \not{D} \psi]_c = 2$, is obtained for $[\psi]_c = 1/2$, while $[y \bar{\psi}_L U \psi_R]_c = 2$ then requires $[y]_c = 1$, etc. This leads to a systematic assignment of chiral dimension $2L + 2$ to terms of order L in the effective-theory series, in agreement with our derivation above.

Nevertheless, the mere assignment of chiral dimensions as in (22) does not by itself suffice to specify the full systematics of the effective-theory power counting. A clear example is given by the 4-fermion operators $\bar{\psi} \psi \bar{\psi} \psi$. These terms have a chiral dimension of 2 and would seem to be part of the leading-order Lagrangian. However, such an assignment would not be consistent for the electroweak chiral Lagrangian. A 4-fermion operator would arise at leading-order for instance from the exchange of a heavy resonance of mass $\Lambda = 4\pi v$ with a strong coupling $\sim 4\pi$ to the fermionic current $\bar{\psi} \psi$, giving an unsuppressed coefficient $\sim (4\pi)^2 / \Lambda^2 = 1/v^2$ to the 4-fermion term. Even for the top quark, which has the strongest coupling to the symmetry-breaking sector of any standard-model fermion, this coupling is only of order 1, giving a suppression of the coefficient. Assuming therefore a weak coupling of fermions to the strong sector (of order unity or less), eliminates 4-fermion terms from the leading-order Lagrangian. We also note that in the case of a strong coupling of a fermion to the symmetry-breaking sector, its mass would be $\sim 4\pi v = \Lambda$. Such a fermion would not be part of the spectrum at low energies and therefore not included as a field in the effective Lagrangian.

Similar arguments suggest that any operator of the form $X^2 Uh$, which also has a chiral dimension of 2, cannot appear at leading order since the gauge fields X are again weakly coupled to the strong sector, that is with couplings of order 1 instead of 4π . Note that an operator $X^2 Uh$ appearing at leading order could induce, for instance, $h \rightarrow \gamma\gamma$ decays with an amplitude larger than in the standard model by a factor of $16\pi^2$, which is excluded by experiment.

Rather than at leading order, terms such as $\bar{\psi} \psi \bar{\psi} \psi$ or $X^2 Uh$ arise at next-to-leading order where they come with explicit factors of couplings y^2 or g^2 , which shifts their chiral dimension to 4.

In [8] 4-fermion operators have been listed as leading-order terms, on the formal ground that they have chiral dimension 2. Even though they seem to have been recognized as phenomenologically undesirable at this order, the physical implications were not clearly spelled out. Similar issues apparently prompted [9] to question the standard assignment of chiral dimensions and to attempt a modification that employed spurions to eliminate unwanted terms. This approach remained largely inconclusive.

The analysis discussed in the present article addresses both the physical content of the electroweak chiral Lagrangian and the justification of chiral dimensions in terms of standard methods of power counting. This clarifies the systematics of the effective Lagrangian and resolves the peculiarities encountered in [8,9], and more recently in the discussion of derivative counting in [24].

3.3. Counting of chiral dimensions to all orders

In the previous sections we have discussed the power counting to all orders in the loop expansion, but only with vertices of the leading-order Lagrangian. This is sufficient to construct the counterterms at all orders. Here we show how the counting of chiral dimensions is extended to the fully general case, including any loop order, as well as vertices from any order in the effective Lagrangian.

A general term in the effective Lagrangian can be denoted by

$$\kappa^{k_i} \psi^{F_i} X_{\mu\nu}^{V_i} D^{d_i} Uh \quad (24)$$

with a fixed number k_i of couplings κ (gauge or Yukawa couplings), F_i fermion fields ψ , V_i field-strength factors $X_{\mu\nu}$, d_i covariant derivatives D , and an arbitrary number of (pseudo-) Goldstone bosons (Uh). This term defines vertices of type ij , in general with B_{ij} Goldstone lines. The total chiral dimension of (24) determines the loop order L_i , that is the order of the term in the effective theory, from (21) as

$$d_i + k_i + \frac{F_i}{2} + V_i = 2L_i + 2 \quad (25)$$

We consider next an arbitrary diagram \mathcal{D} with L loops and any number of type- ij vertex insertions. Denoting the number of external (internal) fermion, gauge-field and Goldstone lines by F , V , B (\mathcal{F} , \mathcal{V} , \mathcal{B}), respectively, the familiar topological identities give

$$\begin{aligned} F + 2\mathcal{F} &= \sum_i n_i F_i \\ V + 2\mathcal{V} &= \sum_i n_i V_i \\ B + 2\mathcal{B} &= \sum_{i,j} n_{ij} B_{ij} \\ L &= \mathcal{F} + \mathcal{V} + \mathcal{B} - \sum_{i,j} n_{ij} + 1 \end{aligned} \quad (26)$$

where n_{ij} is the number of vertices of type ij and $n_i = \sum_j n_{ij}$.

Defining the total number of couplings in diagram \mathcal{D} by $k = \sum_i n_i k_i$, one finds from (26)

$$d + k + \frac{F}{2} + V = 2L + 2 + \sum_i n_i 2L_i \quad (27)$$

This relates the chiral dimension of diagram \mathcal{D} on the left to the number of loops L and the loop order of the vertex insertions L_i . Put differently, the chiral dimension $[\mathcal{D}]_c = d + k + F/2 + V$ gives the total loop order as

$$\frac{[\mathcal{D}]_c - 2}{2} = L + \sum_i n_i L_i \quad (28)$$

This implies that, in general, the chiral dimension of a diagram simply counts the number of loops. More precisely, it gives the order of a diagram, or an operator, in the loop expansion, on which the effective theory construction is based.

It is interesting to compare the systematics of chiral dimensions with the standard counting by canonical dimension that governs the low-energy description of weakly-coupled theories. In the latter case, the leading-order, dimension-4 Lagrangian is renormalizable and higher-dimensional terms can be added as corrections. They are increasingly suppressed by inverse powers $1/M^{d-4}$ of the new-physics scale M with increasing canonical dimension d of the operators. As is well known, in a general diagram, with an arbitrary number of operator insertions, the corresponding powers of $1/M$ simply add up to the total power of $1/M$ for the entire diagram, independently of the number of loops. Eq. (28) implies that a formally similar rule holds for the counting of chiral dimensions. However, as discussed in the previous section, collecting the terms of a given chiral dimension is not sufficient to establish the operators at a given order in the strongly-coupled case. A consistent definition of the leading-order Lagrangian and an analysis of counterterms is also needed. This important difference to the dimensional case is a consequence of the fact that the theory is organized as a loop expansion.

4. Naive dimensional analysis

The order at which a given operator appears in a general effective field theory is often determined using naive dimensional analysis (NDA). This procedure has been introduced in [1] in the context of the chiral quark model, with the understanding that its validity is more general. In the example of the chiral quark model with quarks ψ , gluons G_μ , gauge coupling g , Goldstone bosons φ

and derivatives p , the coefficient of a general term in the effective Lagrangian has been given in [1] as

$$\left(\frac{\varphi}{f}\right)^A \left(\frac{\psi}{f\sqrt{\Lambda}}\right)^B \left(\frac{gG_\mu}{\Lambda}\right)^C \left(\frac{p}{\Lambda}\right)^D f^2 \Lambda^2 \quad (29)$$

Eq. (29) can be easily translated to the language of chiral dimensions. The order of suppression of the different terms in the effective theory is determined by the inverse powers of Λ it contains. Since $(f^2/\Lambda^2)^L$ corresponds to the loop order, (29) implies

$$2L + 2 = D + \frac{B}{2} + C \quad (30)$$

or, in the notation of (21),

$$2L + 2 = d + \frac{F}{2} + V + \gamma \quad (31)$$

with the definitions $d \equiv D - C$, $F \equiv B$, $V \equiv C$, and with the additional assumptions $\gamma = C = V$, $v = \omega = 0$. Differences arise since in (29) the number of gauge fields (V) and of gauge couplings (γ) are identified (because internal gauge-boson lines are neglected), and Yukawa terms (v) and Goldstone-boson non-derivative couplings are not included.

The formula (31) agrees with the result (21) in assigning the correct chiral dimension to φ (chiral dimension 0), ψ (1/2), gG_μ (1) and p (1). Note in particular that the chiral dimension of the vector-like fermion ψ here is identical to the chiral dimension of the chiral fermions in (22). However, the NDA prescription does not specify the separate counting of g and G_μ . As we have seen, the assignment of chiral dimension has to be $[g]_c = 1$ and $[G_\mu]_c = 0$ to achieve a universal counting for parameters and fields. Such a universal counting is an essential objective of NDA.

The counting of NDA has been summarized in a compact way in [10]. This paper states that the size of the coefficient for any term in the effective Lagrangian is obtained by including an overall factor of $f^2 \Lambda^2$, a factor of $1/f$ for each strongly-interacting field, and factors of Λ to get the dimension to 4. Weakly interacting fields enter with a suppression by inverse powers of Λ according to their canonical dimension. Implicit is the assumption that gauge couplings g are to be treated as factors of order one. These rules have been abstracted from (29) and postulated to be of general validity for effective theories with Goldstone bosons from a strongly-interacting sector.

We point out that the rules of NDA used in this form are incomplete and can lead to incorrect results. As an illustration let us consider the following examples:

- The coefficient of the photon kinetic term in (5) is obtained in NDA as $f^2 \Lambda^2 / \Lambda^4 = f^2 / \Lambda^2$ instead of the correct size of order 1. This problem had already been noted in [1], but the consequences for NDA were not fully explored.
- Applying the NDA rules to the electromagnetic mass term in (5), one finds a coefficient of $f^2 \Lambda^2$ instead of f^4 . In this case the coefficient is too large by a factor of Λ^2 / f^2 .
- In the electroweak chiral Lagrangian the operator classes $\psi^2 U h D$ and $\psi^2 U h D^2$ are both present at next-to-leading order. It is clear that no scaling of fields and derivatives can be devised that would yield coefficients of the same order for these two classes. The NDA scaling in (29) would suggest coefficients of order 1 for the first class and of order $1/\Lambda$ for the second, different from the correct scaling $\sim 1/\Lambda^2$ in both cases. As discussed above, an appropriate (formal) scaling of the associated Yukawa couplings will ensure the correct size of the coefficients.

In summary, while we agree with the basic assumptions of NDA, which have been clearly explained in [1,10], we find that the specific counting rules given in these papers are incomplete. Complete and consistent formulations of the power counting are described in Sections 2 and 3. In particular, a direct comparison of (29) with the power counting in (19) shows that agreement is reached if the NDA formula is enlarged by a factor

$$R = \left(\frac{1}{4\pi}\right)^{2\omega} \left(\frac{y}{4\pi}\right)^{\nu} \left(\frac{g}{4\pi}\right)^{\gamma-\nu} \quad (32)$$

A similar generalization of the NDA formula has recently been considered in [25]. There it has been used in the context of weakly-coupled effective field theories with dimensional counting, rather than for strongly-coupled scenarios. The main result of [25] is an identity relating the perturbative order of a diagram (i.e. the total number $2N$ of weak couplings) to the number of loops L as

$$N = L + w - \sum_k w_k \quad (33)$$

where w and w_k are the NDA weights, defined in [25], of the diagram and the inserted operators, respectively. We find that in general the weight w can be written in terms of the canonical dimension $[\mathcal{O}]$ and the chiral dimension $[\mathcal{O}]_c$ of an operator as

$$w \equiv \frac{[\mathcal{O}] - [\mathcal{O}]_c}{2} - 1 \quad (34)$$

The identity (33) can be immediately obtained from (26) and is therefore seen to be entirely of topological origin. As such it does not by itself determine the degree of divergence or the order of a given operator in the EFT expansion. This is reflected in the fact that the identity holds both for the chiral Lagrangian and the Lagrangian with dimensional counting, which have a very different organization of the EFT counting.

5. Conclusions

We have presented a detailed discussion of the power counting for effective field theories valid at a scale v and with strong dynamics at their cut-off $\Lambda \gg v$. This counting is the key element for organizing the possible terms in the effective Lagrangian according to their order in powers of v^2/Λ^2 . The basic assumptions can be stated as follows:

- The degrees of freedom at the low scale v are, in general, (chiral) fermions $\psi_{L,R}$, gauge fields A_μ and (pseudo-) Goldstone bosons φ .
- A mass gap separates the scale v from the high scale Λ , which has been integrated out in the low-energy effective theory.
- At the scale Λ (part of) the dynamics is strongly coupled, the natural cut-off is then $\Lambda = 4\pi v \gg v$.
- The Goldstone sector is strongly coupled, with couplings $\sim 4\pi$, to the strong interactions at Λ .
- Chiral fermions and gauge fields are weakly coupled to the dynamics at Λ , that is with couplings of order unity (or smaller).

Important examples for such a scenario are the chiral perturbation theory for pions and kaons in the presence of electromagnetism, or the electroweak chiral Lagrangian with a light (pseudo-Goldstone) Higgs. In the latter case the actual cut-off may be at a scale $4\pi f$, with $f > v$. When the parameter $\xi \equiv v^2/f^2$ is taken to zero, the ordinary standard model is recovered. Expanded to first order in ξ , the electroweak chiral Lagrangian contains [12] the SILH framework [26]. The full chiral Lagrangian amounts to a resummation of

terms to all orders in ξ , which is parametrically viewed as a quantity of order one.

We have emphasized the importance of specifying the leading-order Lagrangian consistently with the assumptions above. The leading-order Lagrangian provides the basis for the power-counting analysis of loop corrections and their divergence structure. The latter determines the required classes of counterterms, which yield the higher-order operators in the effective Lagrangian.

Previous treatments of power counting appear to have followed one of two different lines of approach, the first employing naive dimensional analysis [1,10], the second using the concept of chiral dimensions [6,8,9]. There seems to have been little overlap between the parts of the literature applying one or the other framework. We have shown how the two methods are related. In particular, we have demonstrated how chiral dimensions follow from standard power counting and we have clarified the physical assumptions that are needed in addition to the chiral dimensions in order to construct effective Lagrangians. We have shown that chiral dimensions simply count the loop order of diagrams, and in that sense they have a topological nature. Our approach is also consistent with the basic philosophy of NDA and shows how the simple NDA rules need to be generalized.

The discussion of power counting presented here provides a general and unified framework for constructing low-energy effective theories of a strong sector. It encompasses chiral perturbation theory weakly coupled to gauge fields, the electroweak chiral Lagrangian and further theories of this kind with other patterns of symmetries and symmetry breaking.

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