

Non-associative deformations of geometry in double field theory

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ABSTRACT: Non-geometric string backgrounds were proposed to be related to a non-associative deformation of the space-time geometry. In the flux formulation of double field theory (DFT), the structure of mathematically possible non-associative deformations is analyzed in detail. It is argued that on-shell there should not be any violation of associativity in the effective DFT action. For imposing either the strong or the weaker closure constraint we discuss two possible non-associative deformations of DFT featuring two different ways how on-shell associativity can still be kept.

KEYWORDS: Flux compactifications, Non-Commutative Geometry

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1 Introduction

In string theory, a large scale geometric target space is rather an emergent phenomenon. The basic starting point is the two-dimensional field theory on the world-volume of the probe string equipped with the fundamental paradigm that on-shell solutions of string theory are provided by two-dimensional conformal field theories (CFTs) with the critical central charge. However, the generic left-right asymmetric CFT does not correspond to a fixed point of a non-linear sigma model with a geometric target space. Since string theory is strongly believed to provide a consistent theory of quantum gravity, one may wonder to which non-geometric generalizations of the target space-time the generic asymmetric CFT corresponds to. This could also enlighten relations to complementary target-space approaches to quantum gravity, like loop quantum gravity or non-commutative geometry.

During the last years some progress has been made towards a better understanding of this non-geometric regime of string theory. In fact, the recent developments go precisely in the direction of providing a quasi-geometric description of these asymmetric conformal field theories. T-duality is a left-right asymmetric transformation, so that it served as the main tool to shed some light into this mainly unexplored regime of the string theory landscape.

In [1] the simple closed string background of a flat space with constant H -flux and dilaton was considered. Successively applying the Buscher rules, one gets the well-known chain of T-dual configurations

$$H_{ijk} \xleftarrow{T_k} F_{ij}{}^k \xleftarrow{T_j} Q_i{}^{jk} \xleftarrow{T_i} R^{ijk} . \tag{1.1}$$

The last two were argued to be non-geometric. The Q -flux case is still geometric locally but the transition functions involve non-geometric T-duality transformations, whereas the R -flux case is considered to be even locally non-geometric.

A simple argument shows that this background does not allow the notion of a point [2]. Let us repeat it here to make clear that something drastic must happen for these backgrounds. Consider a D3-brane wrapping a three-torus carrying a constant three-form H -flux. In fact such a configuration is not allowed as it suffers from the Freed-Witten anomaly [3], i.e. it violates the Bianchi identity $d\mathcal{F} = H$ for the gauge flux on the brane. Now, by formally applying a T-duality along all three directions of the torus, one gets a D0-brane with transverse R -flux. Thus, placing a point-like object in an R -flux background is not allowed. This suggests that one has an uncertainty relation like $\Delta x \Delta y \Delta z \geq \ell_s^4 R^{xyz}$, pointing towards a relation to non-commutative geometry.

Indeed, it was abstractly argued that the R -flux involves a non-associativity of the coordinates [4]. More recently it was found [5–9] by explicit string and CFT computations that the string geometry indeed becomes non-commutative and non-associative for closed strings that are winding and moving in non-geometric backgrounds. Concretely, the equal-time cyclic double-commutator of three local coordinates was found to be

$$[x^i, x^j, x^k] = \begin{cases} 0 & H\text{-flux} \\ \ell_s^4 R^{ijk} & R\text{-flux} \end{cases} . \tag{1.2}$$

The same result arises from a commutator algebra

$$[x^i, x^j] = \frac{i}{3\hbar} \ell_s^4 R^{ijk} p_k, \quad [x^i, p_j] = i\hbar \delta^i_j \tag{1.3}$$

so that the Jacobiator gives precisely (1.2). If also Q -flux is present the commutator was argued to be generalized to

$$[x^i, x^j] = \frac{i}{3\hbar} \ell_s^4 \left(R^{ijk} p_k + Q_k^{ij} w^k \right), \tag{1.4}$$

where w^k is the winding operator. Analogous relations were also derived in the framework of matrix theory in [10].

In [7] this background was investigated using conformal perturbation theory and, analogous to the open string story [11], on-shell string scattering amplitudes of tachyons were computed. Actually, for both constant H -flux and R -flux the final scattering amplitude was associative, as expected from crossing symmetry of conformal correlation functions. However, prior to invoking momentum conservation, there was a difference between the H - and R -flux case, namely the appearance of world-sheet independent phase factors. For the H -flux the holomorphic and anti-holomorphic phases directly canceled each other while for the R -flux they added up. These phases could be encoded (at least at linear order in R^{ijk}) in the tri-product¹

$$(f \Delta g \Delta h)(x) = \exp\left(\frac{\ell_s^4}{6} R^{ijk} \partial_i^{x_1} \partial_j^{x_2} \partial_k^{x_3}\right) f(x_1) g(x_2) h(x_3) \Big|_x . \tag{1.6}$$

¹Choosing $f = \exp(ip_1 x)$ and similar for g, h the momentum conservation can be implemented by integrating the tri-product (1.6), so that the order ℓ_s^4 correction becomes

$$\int d^n x R^{ijk} p_i^1 p_j^2 p_k^3 e^{i(p^1 + p^2 + p^3) \cdot x} = R^{ijk} p_i^1 p_j^2 p_k^3 \delta(p^1 + p^2 + p^3) = 0 . \tag{1.5}$$

The aim of this paper is to generalize this result to non-constant fluxes on a curved space.

The three-bracket can then be defined as

$$[x^i, x^j, x^k] = \sum_{\sigma \in S_3} \text{sign}(\sigma) x^{\sigma(i)} \Delta x^{\sigma(j)} \Delta x^{\sigma(k)}, \quad (1.7)$$

where S_3 denotes the permutation group of three elements. Note that, formally one could also define such a tri-product with H^{ijk} instead of R^{ijk} . The tri-product (1.6) as well as an associated momentum dependent star-product was also derived in [12, 13] by starting with the non-associative commutator algebra (1.3). In addition the non-commutative and non-associative phase space structure of DFT as well as the magnetic field analogue of the string R -flux model was discussed in [13].

Besides these example based arguments, there was a successful approach to develop a manifestly T-duality, i.e. $O(D, D)$, covariant formulation of the dynamics of the massless modes of string theory. This was initiated in [14, 15] and pushed forward more recently in [16–19]. In this so-called double field theory (DFT) framework (see [20–22] for reviews) one doubles the number of target space coordinates by also introducing winding coordinates. It turned out that this is a constrained theory, where usually the weak and the strong constraint are imposed. Then, locally one ends up on a D -dimensional slice of the $2D$ -dimensional doubled geometry, which can be rotated to the supergravity frame via an $O(D, D)$ transformation.

DFT is related to generalized geometry [23–25] by setting the winding coordinates to zero while keeping the doubled tangent bundle $\text{TM} \oplus \text{TM}^*$. Moreover, it admits all the local symmetries, usual and winding diffeomorphisms, to allow for a global description of, for instance, the Q -flux and R -flux backgrounds. This is possible as T-duality exchanges ordinary and winding coordinates so that for these non-geometric backgrounds there appears a winding coordinate dependence either in the transition functions between two charts (Q -flux) or in the definition of the flux itself (R -flux). Thus, non-geometry just means explicit winding coordinate dependence in the background fluxes or in the transition functions.

There exist essentially two formulations of DFT. First, there is the generalized metric formulation, which was developed in a series of papers [16–19]. Here one invokes the so-called strong constraint to guarantee e.g. closure of the symmetry algebra (the C-bracket). Based on the previous work [14, 15, 26] and [27–30], in [31] a second formulation of DFT has been provided which from the onset incorporates the relation to gauged supergravity theories. This is the so-called flux formulation of DFT, which was shown to be equivalent to the generalized metric formulation, up to boundary terms and terms vanishing by the strong constraint. However, as will also be essential for our investigation, it allows to move away from the strong constraint and admit truly non-geometric duality orbits of fluxes in the sense of [32]. In fact, it makes use of the observation that requiring only closure of the symmetry algebra provides a weaker constraint than the strong constraint. A weakening of the strong constraint was first discussed in [33]. Maybe the simplest examples are given by Scherk-Schwarz reductions [34, 35] of DFT [28–31] (see also [36, 37]). Note that in [38] concrete examples of asymmetric orbifold CFTs were presented for which evidence was provided that they do correspond to such non-geometric duality orbits.

It was observed that, in DFT, which is a priori a background independent formalism, generalized coordinate transformations compose in a non-standard manner, such that the composition is non-associative [39]. However this non-associativity vanishes after imposing the strong constraint on arbitrary fields. Besides that, in DFT no notion of a non-associative, background dependent deformation of the geometry is visible. Hence it is puzzling how DFT can be reconciled with the aforementioned claim that the R -flux is related to such a non-associative deformation, as described for constant flux via the tri-product (1.6). The resolution of this paradox is the purpose of this paper. To this end, we identify two important aspects:

- First, as is apparent from (1.6), the non-associativity is claimed to arise for an R -flux background contracted with ordinary partial derivatives $\partial/\partial x^i$. Note that, in this sense, the DFT T-dual of the H -flux background on ordinary space is an R -flux background on winding space.
- Second, in quantum theories, where observables are operators acting on some Hilbert space, one can get non-commutativity, but the product of operators is always associative. Since conformal field theories are ordinary (2-dimensional) quantum theories, on-shell, i.e. if the string equations of motion are satisfied, there should better not be any violation of associativity in CFT on-shell scattering amplitudes.

Indeed, in conformal field theory one requires crossing symmetry of the operator product expansion, which is related to the Jacobi identities for the algebra of the modes of the conformal fields. In string theory, from on-shell scattering amplitudes, one can determine an effective theory for the massless modes, which by construction does not show any on-shell sign of non-associativity. Therefore, we conclude that any admissible non-associative deformation given by a non-associative tri-product like (1.6) should have a trivial effect on the effective field theory, when going on-shell. However it is a priori not clear whether the off-shell effective string action is sensitive against non-associative deformations of the underlying geometry.

As we will discuss, the main result of this paper is that, on the level of the effective action, a non-associative deformation of the DFT generalization of both the H -flux and the R -flux only leads at most to boundary terms. For the first one has to invoke the DFT equations of motion, whereas the second deformation turns out to be trivial once one imposes either the strong or even the closure constraint.

A similar reasoning also applies to the case of open strings ending on D-branes supporting a non-trivial, in general non-constant gauge flux. The case when this product becomes non-associative was analyzed in a series of papers [40–42]. Thus, before we move on to briefly review the flux formulation of DFT in section 3, we review in section 2 two known examples of non-associativity, namely the system of an electric charge moving in a magnetic monopole field and a D-brane carrying non-constant gauge flux. In section 4 we will analyze possible tri-products for DFT. As we will see, a priori there are two candidates, one related to the tri-product (1.6) with H -flux and one to the tri-product with R -flux. Both cases will be discussed in detail.

2 Non-associativity in physics

In this section we review two instances where a non-associative structure has appeared in physics. First, we recall the story of quantizing the motion of an electrically charged particle in a magnetic field. Second, the effective theory on a D-brane with non-constant magnetic background field turned on is considered. This gives a non-vanishing $H = dB$ flux, which in general leads to a non-associative star-product.

2.1 Non-associativity for magnetic monopoles

As it is known for some time [43–47], non-associativity emerges when considering the quantization of a charged particle in the background of a magnetic monopole. Hence in this context immediately the question arises how the apparent emergence of non-associativity can be reconciled with the basic principles of quantum mechanics, where associativity of all operators is mandatory. This issue was recently addressed in [13], where also some remaining puzzles of the earlier work were resolved.

Here, let us just recall a few facts about this system following essentially [43, 44]. The commutator algebra between position and momentum of a particle in a background magnetic field \vec{B} in three space-dimensions takes the following form

$$[x^i, p_j] = i\hbar\delta_j^i, \quad [x^i, x^j] = 0, \quad [p_i, p_j] = i\hbar e \epsilon^{ijk} B_k(\vec{x}). \quad (2.1)$$

In turn, the Jacobiator becomes

$$[p_i, p_j, p_k] = -e\hbar^2 \epsilon^{ijk} \vec{\nabla} \cdot \vec{B} \quad (2.2)$$

with $\vec{\nabla} \cdot \vec{B} = 4\pi\rho_m$ in Gaussian-cgs units. These relations have the analogous form as the commutators (1.3) and three-bracket (1.2) after exchanging the role of momentum and position variables in these equations. Now, consider the finite translation operators $U(a) = \exp(\frac{i}{\hbar}a \cdot p)$. Using the Baker-Campbell-Hausdorff formula one obtains

$$U(a)U(b) = \exp\left(-\frac{ie}{\hbar}\Phi_{(a,b)}\right) U(a+b) \quad (2.3)$$

where $\Phi_{(a,b)} = \frac{1}{2}(a \times b)^k B_k$ denotes the magnetic flux through the (infinitesimally small) triangle spanned by the two vectors (a, b) . Similarly, one can compute the associator of three U s

$$(U(a)U(b))U(c) = \exp\left(-\frac{ie}{\hbar}\Phi_{(a,b,c)}\right) U(a)(U(b)U(c)) \quad (2.4)$$

where $\Phi_{(a,b,c)} = \frac{1}{6}[(a \times b) \cdot c]\vec{\nabla} \cdot \vec{B}$ denotes the magnetic flux through the tetrahedron spanned by the three vectors (a, b, c) . Due to Gauss law this is nothing else than the magnetic charge $4\pi m$ sitting inside the tetrahedron. Therefore, the non-associativity (2.4) vanishes if the phase is trivial, i.e.

$$\frac{em}{\hbar} = \frac{N}{2} \quad (2.5)$$

with an integer N . This is Dirac’s quantization rule for the magnetic charge. Thus we can cite from the abstract of [43] ‘Insisting that finite translations be associative leads to Dirac’s monopole quantization condition’.

As discussed in [13], only for the case of the magnetic monopole the classical equations of motion of a charged particle are still integrable. In this case, the so-called Poincaré vector provides an integral of motion, and angular momentum is still preserved. For a continuous magnetic charge distribution $\rho(x)$ angular symmetry gets broken.

In this paper, we are essentially generalizing the above mentioned logic by clarifying how the non-associative tri-product deformation of the DFT action can be made consistent with the requirements from CFT scattering amplitudes. The only main difference is that we are not considering quantized fluxes and momenta but the case where these are in general non-rational and space-time dependent. However, the main message still is that from the requirement of absence of non-associativity we can learn something very essential about the system.

2.2 Open string with non-associative star product

Let us recall that the conformal field theory of an open string ending on a D-brane supporting a non-trivial gauge flux $\mathcal{F} = B + 2\pi\alpha'F$ features a non-commutative geometry.

Indeed, by computing the disc level scattering amplitude of N -tachyons, certain relative phases appear which for constant gauge flux can be described by the Moyal-Weyl star-product

$$(f \star g)(x) = \exp\left(i \frac{\ell_s^2}{2} \theta^{ij} \partial_i^{x_1} \partial_j^{x_2}\right) f(x_1) g(x_2) \Big|_x, \tag{2.6}$$

where the relation of the open and closed string quantities is

$$G^{-1} + \theta = (g + \mathcal{F})^{-1}. \tag{2.7}$$

In the Seiberg-Witten limit the OPE exactly becomes the Moyal-Weyl star-product. This non-trivial product of functions lead to the non-commutative Moyal-Weyl plane with $[x^i, x^j] = i \ell_s^2 \theta^{ij}$. That in the on-shell string scattering amplitudes such a non-commutativity can show up, is possible because the conformal $SL(2, \mathbb{R})$ symmetry group only leaves the cyclic order of the inserted vertex operators invariant. By the same reason, the non-commutativity must be such that, on-shell, it preserves cyclicity.

There is no need to only consider a constant antisymmetric two-vector θ^{ij} . Indeed, in [48] it has been shown that for every Poisson structure θ^{ij} one can define a corresponding associative star-product, which will also involve derivatives of the Poisson structure. The same product can also be considered for a quasi Poisson structure, but then leads to a non-associative star-product. This is related to the physical situation of an open string ending on a D-brane with generic non-constant B -field, i.e. non-vanishing field strength H . At leading order in derivatives this leads to a non-commutative product

$$\begin{aligned} f \circ g = f \cdot g + & i \frac{\ell_s^2}{2} \theta^{ij} \partial_i f \partial_j g - \frac{\ell_s^4}{8} \theta^{ij} \theta^{kl} \partial_i \partial_k f \partial_j \partial_l g \\ & - \frac{\ell_s^4}{12} (\theta^{im} \partial_m \theta^{jk}) (\partial_i \partial_j f \partial_k g + \partial_i \partial_j g \partial_k f) \dots \end{aligned} \tag{2.8}$$

The associator for this product becomes

$$(f \circ g) \circ h - f \circ (g \circ h) = \frac{\ell_s^4}{6} \theta^{ijk} \partial_i f \partial_j g \partial_k h + \dots \quad (2.9)$$

with $\theta^{ijk} = 3 \theta^{[im} \partial_m \theta^{jk]}$, which precisely vanishes for a Poisson tensor. But now the puzzle arises that in the open string CFT we should *not* see the effect of such a non-associative deformation of the underlying space-time. Indeed this question was analyzed in some detail in [41, 42] and we briefly repeat their essential observation here.

From the open string scattering amplitudes one can determine the low-energy effective action so that also the effect of the non-associativity in its quantum deformation should be trivial. Indeed, consider the DBI action

$$S_{\text{DBI}} = \int d^n x \sqrt{g + \mathcal{F}} \quad (2.10)$$

and vary it with respect to the gauge potential A in $\mathcal{F} = B + dA$. One gets

$$\partial_i \left(\sqrt{g + \mathcal{F}} [(g + \mathcal{F})^{-1}]^{[ij]} \right) = \partial_i \left(\sqrt{g + \mathcal{F}} \theta^{ij} \right) = 0 \quad (2.11)$$

where we have used (2.7). Then, it directly follows that up to leading order in $\partial\theta$ the \star -product satisfies the property

$$\int d^n x \sqrt{g + \mathcal{F}} f \circ g = \int d^n x \sqrt{g + \mathcal{F}} f \cdot g. \quad (2.12)$$

Indeed, e.g. at order $O(\ell_s^2)$ the difference between the left and the right hand side is a total derivative on-shell

$$i \frac{\ell_s^2}{2} \int d^n x \sqrt{g + \mathcal{F}} \theta^{ij} \partial_i f \partial_j g = i \frac{\ell_s^2}{2} \int d^n x \partial_i \left(\sqrt{g + \mathcal{F}} \theta^{ij} f \partial_j g \right) = 0 \quad (2.13)$$

where here and in the following sections we always assume that the functions f, g are sufficiently well behaving so that integrals over total derivatives vanish. Thus, as expected from CFT, in the effective action the product of two functions is commutative (cyclic), once the background satisfies the string equations of motion.

Similarly, the associator below the integral also gives a total derivative at leading order in $\partial\theta$. E.g. at order $O(\ell_s^4)$ we find

$$\int d^n x \sqrt{g + \mathcal{F}} \left((f \circ g) \circ h - f \circ (g \circ h) \right) = \frac{\ell_s^4}{6} \int d^n x \partial_i \left(\sqrt{g + \mathcal{F}} \theta^{ijk} f \partial_j g \partial_k h \right) = 0, \quad (2.14)$$

where we have used

$$\partial_i \left(\sqrt{g + \mathcal{F}} \theta^{ijk} \right) = 0, \quad (2.15)$$

which can be seen by expanding θ^{ijk} and successively employing the equation of motion (2.11) and the anti-symmetry of θ^{ij} . The two relations (2.12) and (2.14) also hold for higher orders in derivatives of θ^{ij} [42]. Note, that as here one is using the DBI action, the

star-product is exact in α' at leading order in $\partial\theta$. Thus, we conclude that, as expected from the open string conformal field theory, on-shell the non-associativity of the \circ -product is not visible.

In the following we will generalize this kind of analysis to the closed string case. Since there we are dealing with non-geometric fluxes, the appropriate framework to discuss it is double field theory. Therefore, let us recall those aspects of DFT which will be used in the main section 4.

3 Flux formulation of DFT

In this section we summarize the main features of the flux formulation of DFT, as it has been described in [21, 31], based on the earlier work [14, 15] and [28–30]. For more details we refer to these papers.

3.1 Basics of DFT

The main new feature of DFT is that one doubles the number of coordinates by introducing winding coordinates \tilde{x}_m and arranges them into a doubled vector $X^M = (\tilde{x}_m, x^m)$. One defines an $O(D, D)$ invariant metric

$$\eta_{MN} = \begin{pmatrix} 0 & \delta^m_n \\ \delta_m^n & 0 \end{pmatrix} \tag{3.1}$$

and introduces a generalized bein E^A_M with metric

$$S_{AB} = \begin{pmatrix} s^{ab} & 0 \\ 0 & s_{ab} \end{pmatrix} \tag{3.2}$$

with s_{ab} being the flat D -dimensional Minkowski metric. The most generic parameterization of this generalized bein reads

$$E^A_M = \begin{pmatrix} e_a^m & e_a^k B_{km} \\ e^a_k \beta^{km} & e^a_m + e^a_k \beta^{kl} B_{lm} \end{pmatrix}, \tag{3.3}$$

with the ordinary bein $e_a^m s^{ab} e_b^n = G^{mn}$. Note that (3.3) contains both a two form B_{mn} and a two-vector β^{mn} . The flat derivative is defined as

$$\mathcal{D}^A = E^A_M \partial^M. \tag{3.4}$$

Using these beins, one defines the generalized fluxes F_{ABC} as

$$\mathcal{F}_{ABC} = 3\Omega_{[ABC]} \tag{3.5}$$

in terms of the generalized Weitzenböck connection²

$$\Omega_{ABC} = \mathcal{D}_A E_B^M E_{CM}. \tag{3.6}$$

²For a recent discussion of the role of a Weitzenböck connection in DFT see [49].

The components of these DFT fluxes \mathcal{F}_{ABC} are precisely the geometric and non-geometric fluxes H, F, Q and R

$$\mathcal{F}_{abc} = H_{abc}, \quad \mathcal{F}^a{}_{bc} = F^a{}_{bc}, \quad \mathcal{F}_c{}^{ab} = Q_c{}^{ab}, \quad \mathcal{F}^{abc} = R^{abc}. \quad (3.7)$$

The explicit form of these fluxes in terms of B and β can be found in [28, 31, 50] (see also [51]). For later use we just list the fluxes for the choice $B_{mn} = 0$ in (3.3). Defining

$$f^c{}_{ab} = e_i{}^c (\partial_a e_b{}^i - \partial_b e_a{}^i), \quad \tilde{f}_a{}^{bc} = e_a{}^i (\tilde{\partial}^b e_i{}^c - \tilde{\partial}^c e_i{}^b), \quad (3.8)$$

one finds $H_{abc} = 0$ and the geometric flux $F^c{}_{ab} = f^c{}_{ab}$. The non-geometric fluxes are

$$Q_c{}^{ab} = \tilde{f}_c{}^{ab} + \partial_c \beta^{ab} + f^a{}_{cm} \beta^{mb} + f^b{}_{cm} \beta^{am} \quad (3.9)$$

and

$$R^{abc} = 3 \left(\tilde{\partial}^{[a} \beta^{bc]} + \tilde{f}_m{}^{[ab} \beta^{c]m} \right) + 3 \left(\beta^{[am} \partial_m \beta^{bc]} + \beta^{[am} \beta^{bn} f^c{}_{mn}] \right). \quad (3.10)$$

Similar to the open string case (2.9), the contribution $R_{cl}^{abc} = 3(\beta^{[am} \partial_m \beta^{bc]} + \dots)$ can be considered as the defect for associativity, when we consider β^{ab} as a classical (quasi-)Poisson tensor.

Next, one introduces the T-duality invariant dilaton

$$e^{-2d} = e^{-2\phi} \sqrt{g} \quad (3.11)$$

which is used to also define

$$\mathcal{F}_A = \Omega^B{}_{BA} + 2E_A{}^M \partial_M d. \quad (3.12)$$

DFT is required to be invariant under a large symmetry group. First it is invariant under global $G = O(D, D)$ transformation and second it is invariant under a local $H \subset G$ symmetry with $H = O(D) \times O(D)$. This local symmetry acts on the bein as

$$\delta_\Lambda E_A{}^M = \Lambda_A{}^B E_B{}^M \quad \text{with} \quad \Lambda_A{}^C S_{CD} \Lambda_B{}^D = S_{AB} \quad (3.13)$$

so that they can be viewed as local double Lorentz transformations. Besides that, the usual diffeomorphism symmetry is enhanced to so-called generalized diffeomorphism with infinitesimal parameter $\xi^M = (\tilde{\lambda}_m, \lambda^m)$ and generalized Lie-derivative, acting e.g. on a doubled vector V as

$$\mathcal{L}_\xi V^M = \xi^N \partial_N V^M + (\partial^M \xi_N - \partial_N \xi^M) V^N. \quad (3.14)$$

For instance the beins E_A transform as a doubled vector, whereas the dilaton d transforms as a scalar density

$$\delta_\xi d = \mathcal{L}_\xi d = \xi^M \partial_M d - \frac{1}{2} \partial_M \xi^M. \quad (3.15)$$

This allows to define a generalized tensor calculus by defining that the variation of a tensor with respect to generalized diffeomorphisms is

$$\delta_\xi T^{M_1 \dots M_k} = \mathcal{L}_\xi T^{M_1 \dots M_k} . \tag{3.16}$$

In contrast to the usual Lie-derivative, the Lie-derivative of a generalized tensor is not automatically again a generalized tensor. To ensure this, one has to impose the so-called *closure constraint*

$$\Delta_{\xi_1} (\mathcal{L}_{\xi_2} T^{M_1 \dots M_k}) = 0 \tag{3.17}$$

with the anomalous variation $\Delta(\cdot) = \delta_\xi(\cdot) - \mathcal{L}_\xi(\cdot)$.

The invariant action of the flux formulation of DFT reads

$$S_{\text{DFT}} = \int dX e^{-2d} \left[\mathcal{F}_A \mathcal{F}_{A'} S^{AA'} + \mathcal{F}_{ABC} \mathcal{F}_{A'B'C'} \left(\frac{1}{4} S^{AA'} \eta^{BB'} \eta^{CC'} - \frac{1}{12} S^{AA'} S^{BB'} S^{CC'} \right) - \frac{1}{6} \mathcal{F}_{ABC} \mathcal{F}^{ABC} - \mathcal{F}_A \mathcal{F}^A \right] . \tag{3.18}$$

Note that in CFT we can assign a world-sheet parity Ω to every field (see e.g. [50]). Then, the terms in the first two lines are Ω -even and the term in the last line are Ω -odd. The DFT action has to be supplemented by one of the following constraints.

- *Strong constraint*: in this case one requires the so-called weak and strong constraint

$$\partial_M \partial^M = 0, \quad \partial_M f \partial^M g = \mathcal{D}_A f \mathcal{D}^A g = 0 \tag{3.19}$$

with f, g being the fundamental objects like E^A_M and ξ^M . Locally, up to an $O(D, D)$ transformation these constraints remove the winding dependence. In particular, the constraints guarantee the closure constraint. In the following, we always implement the weak and strong constraint for the uncompactified directions.

- *Closure constraint*: for compact spaces one can weaken the strong constraint and only require that the symmetry algebra closes [30], i.e. that a Lie-derivative of a generalized tensor is again a generalized tensor (3.17). Scherk-Schwarz reductions are prototype examples, whose reduced action is closely related to gauged supergravity and whose internal spaces are truly non-geometric in the sense that fields depend on doubled coordinates (y^m, \tilde{y}_m) .

Let us analyze some of the consequences of just imposing the closure constraint. First, if f is a generalized scalar, then we can write

$$\mathcal{D}_A f = E_A^M \partial_M f = \mathcal{L}_{E_A}(f) \tag{3.20}$$

which by the closure constraint implies that $\Delta_\xi(\mathcal{L}_{E_A}f) = 0$. Therefore, $\mathcal{D}_A f$ is also generalized scalar. Now, by direct computation one obtains

$$\begin{aligned}\Delta_\xi(\mathcal{D}_B f) &= \delta_\xi(\mathcal{D}_B f) - \mathcal{L}_\xi(\mathcal{D}_B f) \\ &= (\mathcal{D}^C \xi^M) E_{BM} \mathcal{D}_C f = 0.\end{aligned}\tag{3.21}$$

Thus, choosing $\xi = E_A$ we can conclude

$$(\mathcal{D}^C E_A^M) E_{BM} \mathcal{D}_C f = \Omega^C{}_{AB} \mathcal{D}_C f = 0.\tag{3.22}$$

For a generalized scalar g , we can also choose $\xi = E_B g$ in (3.21) and, using the relation (3.22), obtain

$$\delta_{AB} \mathcal{D}^C g \mathcal{D}_C f = 0.\tag{3.23}$$

Thus, we conclude that the closure constraint implies that for scalars f and g the strong constraint still has to hold. A particular example which we will use later is

$$(\mathcal{D}_C \mathcal{F}_A) \mathcal{D}^C f = 0.\tag{3.24}$$

Similarly, the fluxes $\mathcal{F}_{ABC} = E_{CM} (\mathcal{L}_{E_A} E_B^M)$ and $\mathcal{F}_A = -e^{2d} (\mathcal{L}_{E_A} e^{-2d})$ with flat indices transform as scalars with respect to generalized diffeomorphisms, i.e.

$$\delta_\xi \mathcal{F}_{ABC} = \xi^M \partial_M \mathcal{F}_{ABC}, \quad \delta_\xi \mathcal{F}_A = \xi^M \partial_M \mathcal{F}_A.\tag{3.25}$$

However, under a local double Lorentz transformation one gets as

$$\delta_\Lambda \mathcal{F}_{ABC} = 3 \left[\mathcal{D}_{[A} \Lambda_{BC]} + \Lambda_{[A}{}^D \mathcal{F}_{BC]D} \right], \quad \delta_\Lambda \mathcal{F}_A = \mathcal{D}^B \Lambda_{BA} + \Lambda_A{}^B \mathcal{F}_B,\tag{3.26}$$

where the first terms are anomalous. We also write e.g. $\Delta_\Lambda \mathcal{F}_{ABC} = 3 \mathcal{D}_{[A} \Lambda_{BC]}$. For the relation (3.22) to be well defined we also require

$$0 = \Delta_\Lambda (\Omega^C{}_{AB} \mathcal{D}_C f) = (\mathcal{D}^C \Lambda_{AB}) \mathcal{D}_C f,\tag{3.27}$$

which could also be read off from (3.23)

Moreover, the fluxes satisfy the generalized Bianchi identities

$$\mathcal{D}_{[A} \mathcal{F}_{BCD]} - \frac{3}{4} \mathcal{F}_{[AB}{}^M \mathcal{F}_{CD]M} = \mathcal{Z}_{ABCD}\tag{3.28}$$

and

$$\mathcal{D}^M \mathcal{F}_{MAB} + 2 \mathcal{D}_{[A} \mathcal{F}_{B]} - \mathcal{F}^M \mathcal{F}_{MAB} = \mathcal{Z}_{AB},\tag{3.29}$$

where the right hand sides are given by

$$\begin{aligned}\mathcal{Z}_{ABCD} &= -\frac{3}{4} \Omega_{E[AB} \Omega^E{}_{CD]} \\ \mathcal{Z}_{AB} &= \left(\partial^M \partial_M E_{[A}{}^N \right) E_{B]N} - 2 \Omega^C{}_{AB} \mathcal{D}_C d.\end{aligned}\tag{3.30}$$

Both quantities vanish by the strong constraint. As shown in [31], realizing that $\Delta_{E_A} \mathcal{F}_B = \mathcal{Z}_{AB}$ and $\Delta_{E_A} \mathcal{F}_{BCD} = \mathcal{Z}_{ABCD}$ this also holds for the closure constraint.

Due to (3.25) the DFT action (3.18) is apparently invariant under generalized diffeomorphisms. Taking the anomalous terms in (3.26) into account, under local double Lorentz transformations, the action transforms into a boundary term plus

$$\delta_\Lambda S_{\text{DFT}} = \int dX e^{-2d} \Lambda_A{}^C (\eta^{AB} - S^{AB}) \mathcal{Z}_{BC} \quad (3.31)$$

which indeed vanishes for all possible constraints.

The derivative (3.4) satisfies the commutation relations

$$[\mathcal{D}_A, \mathcal{D}_B] = \mathcal{F}^C{}_{AB} \mathcal{D}_C - \Omega^C{}_{AB} \mathcal{D}_C = \mathcal{F}^C{}_{AB} \mathcal{D}_C, \quad (3.32)$$

where $\Omega^C{}_{AB} \mathcal{D}_C$ vanishes after invoking either the strong or the closure constraint (3.22).

Now, varying the action with respect to the beins, one obtains the equations of motion

$$\mathcal{G}^{[AB]} = \mathcal{Z}^{AB} + 2S^{C[A} \mathcal{D}^{B]} \mathcal{F}_C + (\mathcal{F}_C - \mathcal{D}_C) \check{\mathcal{F}}^{C[AB]} + \check{\mathcal{F}}^{CD[A} \mathcal{F}_{CD}{}^{B]} = 0 \quad (3.33)$$

with

$$\check{\mathcal{F}}^{ABC} = \check{\mathcal{S}}^{ABCDEF} \mathcal{F}_{DEF} \quad (3.34)$$

and

$$\check{\mathcal{S}}^{ABCDEF} = \frac{1}{2} S^{AD} \eta^{BE} \eta^{CF} + \frac{1}{2} \eta^{AD} S^{BE} \eta^{CF} + \frac{1}{2} \eta^{AD} \eta^{BE} S^{CF} - \frac{1}{2} S^{AD} S^{BE} S^{CF}. \quad (3.35)$$

Note that the Ω -odd terms in (3.18) do not contribute to these equations of motion. The dilaton equation of motion is that the integrand of the action (3.18) vanishes. It is remarkable that it is possible to express the equations of motions, including the gravity part, in this unified way just in terms of doubled fluxes \mathcal{F}_{ABC} and \mathcal{F}_A .

Finally, let us mention that, by analyzing a Scherk-Schwarz reduction of DFT, it was pointed out in [28, 29] that the quadratic constraints of gauged supergravity are satisfied even though the strong constraint is not. Additionally, in [30, 31] it was shown that for such Scherk-Schwarz reductions the closure constraint of DFT is satisfied. Thus, in a compactified DFT the strong constraint seems only to be a sufficient but not a necessary requirement. These Scherk-Schwarz reductions provide explicit examples of truly doubled geometries [32]. Whether all such truly non-geometric backgrounds are honest solutions of string theory is still under debate.

4 Non-associative deformations of DFT

In this section we investigate the generalization of the open string analysis from section 2.2 to the closed string, which we describe by DFT. As we argued, (on-shell) closed string scattering amplitudes are not expected to show any sign of non-associativity. The latter is due to the fact that CFT amplitudes are crossing symmetric, which correspond to satisfied

Jacobi-identities in an operator formalism. Therefore, we again expect that the deformation of the effective action by a (non-associative) tri-product should better be trivial (at least) on-shell. However, let us stress that, if one can identify such a specific non-trivial tri-product, one definitely has made a big change of the underlying geometry. We will show that, under certain conditions, it remarkably has no effect for the DFT action. In a similar vein, the conformal $SL(2, \mathbb{C})$ symmetry does not preserve the (radial) ordering of points on the sphere. Therefore, on-shell one also does not expect to see any imprint of non-commutativity.

In DFT, there exist two possible tri-products. First, there is the tri-product

$$f \Delta g \Delta h = f g h + \frac{\ell_s^4}{6} \check{\mathcal{F}}^{ABC} \mathcal{D}_A f \mathcal{D}_B g \mathcal{D}_C h + O(\ell_s^8). \tag{4.1}$$

Since (4.1) contains the component $H^{abc} \partial_{af} \partial_{bg} \partial_{ch}$, with $H^{ijk} = g^{ii'} g^{jj'} g^{kk'} H_{i'j'k'}$, it can be considered as the DFT generalization of the three-product (1.6) with H -flux deformation. Even though there does not exist evidence for the presence of some non-associativity for H -flux, we study it here, as it is the direct generalization of the open string story and it still shows some remarkable properties.

The second possibility is the generalization of the tri-product with R^{ijk} deformation

$$f \Delta g \Delta h = f g h + \frac{\ell_s^4}{6} \mathcal{F}_{ABC} \mathcal{D}^A f \mathcal{D}^B g \mathcal{D}^C h + O(\ell_s^8). \tag{4.2}$$

As mentioned in the introduction, for this case the CFT analysis showed some signs of non-associativity.

In this section we will see that both of these in principle possible non-associative deformations do not lead to any physical effect in on-shell DFT, though the mechanisms turn out to be different for the two cases.

4.1 A tri-product for $\check{\mathcal{F}}^{ABC}$

In analogy to the non-associative product for the open string, we consider the DFT tri-product

$$f \Delta g \Delta h = f g h + \frac{\ell_s^4}{6} \check{\mathcal{F}}^{ABC} \mathcal{D}_A f \mathcal{D}_B g \mathcal{D}_C h + O(\ell_s^8). \tag{4.3}$$

We assume that f, g, h are scalars under generalized diffeomorphisms and are invariant under doubled local Lorentz transformations.

Invoking the strong or closure constraint, $\check{\mathcal{F}}^{ABC}$ and $\mathcal{D}_A f$ transform as scalars under generalized diffeomorphisms so that the tri-product is invariant under the latter. The anomalous transformation behavior of the tri-product under doubled local Lorentz transformations is

$$\Delta_\Lambda \left(\check{\mathcal{F}}^{ABC} \mathcal{D}_A f \mathcal{D}_B g \mathcal{D}_C h \right) = 3S^{[AD} \mathcal{D}^B \Lambda^C]_D \mathcal{D}_A f \mathcal{D}_B g \mathcal{D}_C h \tag{4.4}$$

which vanishes directly for the strong constraint and due to (3.27) for the closure constraint.

Now consider the effect of the order ℓ_s^4 term under the integral. Performing an integration by parts and using that for both constraints we have $[\mathcal{D}_A, \mathcal{D}_B] = \mathcal{F}^C{}_{AB} \mathcal{D}_C$, we find

$$\int dX e^{-2d} \check{\mathcal{F}}^{ABC} \mathcal{D}_A f \mathcal{D}_B g \mathcal{D}_C h = \int dX \partial_M (e^{-2d} V^M) + \int dX e^{-2d} [(\mathcal{F}_C - \mathcal{D}_C) \check{\mathcal{F}}^{C[AB]} + \check{\mathcal{F}}^{CD[A} \mathcal{F}_{CD}{}^{B]}] f \mathcal{D}_A g \mathcal{D}_B h. \quad (4.5)$$

with

$$V^M = E_A{}^M \check{\mathcal{F}}^{ABC} f \mathcal{D}_B g \mathcal{D}_C h \quad (4.6)$$

transforming as a vector under generalized diffeomorphisms. Thus, invoking Stokes theorem this gives a boundary term, which vanishes on well defined compact doubled geometries patched by generalized diffeomorphisms and double Lorentz transformations. Here we have also used the relation

$$\partial_M (E_A{}^M e^{-2d}) = -e^{-2d} \mathcal{F}_A. \quad (4.7)$$

The second term can be written as

$$\int dX e^{-2d} [\mathcal{G}^{[AB]} - 2S^{M[A} \mathcal{D}^{B]} \mathcal{F}_M] f \mathcal{D}_A g \mathcal{D}_B h = 0 \quad (4.8)$$

where, due to (3.33), $\mathcal{G}^{[AB]}$ vanishes on-shell and the second term vanishes for both the strong and, due to (3.24), also for the closure constraint. Thus, we conclude that the order ℓ_s^4 term in the tri-product is a surface term on-shell. In this respect this tri-product is very similar to the open string story.

Matter corrections. However, these equations of motion receive stringy higher derivative corrections, so that the tri-product, i.e. the coefficient $\check{\mathcal{F}}^{ABC}$, needs to be adjusted accordingly. Moreover, coupling DFT to extra matter sources, which, in particular, means *any* additional field contributing to the energy-momentum tensor, the equations of motion change to

$$2S^{C[A} \mathcal{D}^{B]} \mathcal{F}_C + (\mathcal{F}_C - \mathcal{D}_C) \check{\mathcal{F}}^{C[AB]} + \check{\mathcal{F}}^{CD[A} \mathcal{F}_{CD}{}^{B]} = \mathcal{T}^{AB}. \quad (4.9)$$

For instance, including the R-R sector [52, 53], one can put all R-R fields in the spinor representation of $O(D, D)$

$$\mathcal{G} = \sum_n \frac{e^\phi}{n!} G_{i_1 \dots i_n}^{(n)} e_{a_1}{}^{i_1} \dots e_{a_n}{}^{i_n} \Gamma^{a_1 \dots a_n} |0\rangle, \quad (4.10)$$

where $\Gamma^{a_1 \dots a_n}$ defines the totally anti-symmetrized product of n Γ -matrices. Then, the R-R contribution to the DFT equation of motion is

$$\mathcal{T}^{AB} = \frac{1}{4} \bar{\mathcal{G}} \Gamma^{AB} \mathcal{G}. \quad (4.11)$$

In order to still keep the total derivative property, the only thing one can do is to adjust the tri-product (4.3) as

$$f \Delta g \Delta h = \dots + \frac{\ell_s^4}{18} \mathcal{T}^{AB} \left(f \mathcal{D}_A g \mathcal{D}_B h + \mathcal{D}_A f \mathcal{D}_B g h + \mathcal{D}_B f g \mathcal{D}_A h \right). \quad (4.12)$$

This means that one already has to introduce a non-trivial two-product as

$$f \Delta_2 g = f \cdot g + \frac{\ell_s^4}{18} \mathcal{T}^{AB} \mathcal{D}_A f \mathcal{D}_B g + O(\ell_s^8). \quad (4.13)$$

Let us discuss its effect for the case that one imposes the strong constraint. Below the integral the order ℓ_s^4 correction to this two-product can be written as

$$\begin{aligned} \int dX e^{-2d} \mathcal{T}^{AB} \mathcal{D}_A f \mathcal{D}_B g &= \int dX \partial_M (\dots)^M \\ &+ \int dX e^{-2d} \left[(\mathcal{F}_A - \mathcal{D}_A) \mathcal{T}^{AB} - \frac{1}{2} \mathcal{T}^{CD} \mathcal{F}_{CD}{}^B \right] f \mathcal{D}_B g. \end{aligned} \quad (4.14)$$

Employing the Bianchi identities (3.28) and (3.29) and the strong or the closure constraint, from (4.9) we derive the continuity equation for the energy-momentum tensor

$$(\mathcal{D}_A - \mathcal{F}_A) \mathcal{T}^{AB} + \frac{1}{2} \mathcal{F}_{CD}{}^B \mathcal{T}^{CD} = S^{CA} \mathcal{D}^B \left(\mathcal{D}_A \mathcal{F}_C - \frac{1}{2} \mathcal{F}_A \mathcal{F}_C \right). \quad (4.15)$$

Thus, due to the strong constraint the second line in (4.14) vanishes and the order ℓ_s^4 correction to the two-product gives a total derivative below the integral. Note that such a two-product implies a two-bracket

$$[x^i, x^j] = \frac{\ell_s^4}{9} \mathcal{T}^{ij}. \quad (4.16)$$

Thus, we conclude that, due to higher order and source term corrections to the equations of motion, the tri-product needs to be adjusted accordingly. For the matter source term, we showed explicitly that at order ℓ_s^4 this is indeed possible. We find it compelling that the definition of a tri-product and the DFT/string equations of motion are related in this intricate manner. Deforming the underlying geometry in this non-associative way does not effect the on-shell DFT.

4.2 A tri-product for \mathcal{F}_{ABC}

Now consider the DFT generalization of the tri-product (1.6)

$$f \Delta g \Delta h = f g h + \frac{\ell_s^4}{6} \mathcal{F}_{ABC} \mathcal{D}^A f \mathcal{D}^B g \mathcal{D}^C h + O(\ell_s^8). \quad (4.17)$$

Note that, once the strong or closure constraint is imposed, the order ℓ_s^4 term in (4.17) transforms as a scalar under generalized diffeomorphisms if f, g, h are scalars. In addition this tri-product is also invariant under local double Lorentz transformations. However, a second look reveals that this is trivial as, imposing either constraint, one immediately realizes that due to (3.22) the whole order ℓ_s^4 term actually *vanishes*. Thus, in this constrained DFT framework this tri-product is actually trivial.

For illustrative purposes, nevertheless let us apply a partial integration to the tri-product (4.17) written below an integral. The order ℓ_s^4 term can be written as

$$\int dX e^{-2d} \mathcal{F}_{ABC} \mathcal{D}^A f \mathcal{D}^B g \mathcal{D}^C h = \int dX \partial^M (\dots)_M - \int dX e^{-2d} [(\mathcal{D}^C - \mathcal{F}^C) \mathcal{F}_{CAB}] \mathcal{D}^A f \mathcal{D}^B g h \quad (4.18)$$

where the term in the last line can be written as

$$\int dX e^{-2d} [\mathcal{Z}_{AB} - 2\mathcal{D}_{[A} \mathcal{F}_{B]}] \mathcal{D}^A f \mathcal{D}^B g h. \quad (4.19)$$

Here we have used $\mathcal{F}_{MN[A} \mathcal{F}^{MN}_{B]} = 0$. Consistently, due to the Bianchi-identity (3.29) and the relation (3.24) this expression vanishes for both constraints. Since the terms appearing in this computation are related to the ones appearing in a topological Bianchi identity and not a dynamical equation of motion, one might expect that there are no stringy higher order derivative corrections to the, in general, non-constant tri-product parameter \mathcal{F}_{ABC} .

Comments on relaxing the closure constraint. Relaxing even the closure constraint is the only option to get a non-trivial tri-product (4.17). For compact configurations it is clear that string theory contains momentum and winding modes not subject to the weak and consequently the strong constraint. For instance, for a toroidal compactification, the level matching condition becomes

$$L_0 - \bar{L}_0 = \alpha' p \cdot w + N - \bar{N} = 0 \quad (4.20)$$

where N and \bar{N} denote the number of left and right-moving oscillator excitations. Including these modes is expected to go beyond the realm of DFT.

Another way of relaxing the closure constraint could be by splitting the fluxes into backgrounds and fluctuations and relaxing the strong and closure constraint between the two. Whether this is an allowed relaxation in DFT remains to be seen and is beyond the scope of this paper. Here we just discuss its consequences for the tri-product.

Independent of how actually the constraints are relaxed, let us now discuss the consequences for the tri-product. Up to boundary terms, after partially integrating the order ℓ_s^4 term under the integral we get

$$\int dX e^{-2d} [(\mathcal{D}^C - \mathcal{F}^C) \mathcal{F}_{C[AB]} + 2\Omega_{CD[A} \mathcal{F}_{B]}{}^{CD}] (\mathcal{D}^A f) (\mathcal{D}^B g) h. \quad (4.21)$$

The additional term compared to (4.18) arises from the Ω term in the commutator (3.32) when violating closure. Taking into account that, in string theory, non-associativity should still be vanishing at least on shell, we can imagine two ways to proceed from here.

First, we can require a new constraint

$$\zeta_{AB} \mathcal{D}^A f \mathcal{D}^B g = 0 \quad (4.22)$$

with

$$\zeta_{AB} = (\mathcal{D}^C - \mathcal{F}^C)\mathcal{F}_{C[AB]} + 2\Omega_{CD[A}\mathcal{F}_{B]}{}^{CD} \quad (4.23)$$

that is weaker than the closure constraint. The second possibility is to cancel these terms by an appropriately adjusted tri-product

$$\begin{aligned} f \Delta g \Delta h &= f g h + \frac{\ell_s^4}{6} \mathcal{F}_{ABC} \mathcal{D}^A f \mathcal{D}^B g \mathcal{D}^C h \\ &+ \frac{\ell_s^4}{18} \zeta_{AB} \left(f \mathcal{D}^A g \mathcal{D}^B h + \mathcal{D}^A f \mathcal{D}^B g h + \mathcal{D}^B f g \mathcal{D}^A h \right). \end{aligned} \quad (4.24)$$

Note that one can rewrite the adjusted tri-product (4.24) as

$$f \Delta g \Delta h = f g h + e^{2d} \partial_M \left(\frac{\ell_s^4}{6} E_A{}^M e^{-2d} \mathcal{F}_{ABC} f \mathcal{D}^B g \mathcal{D}^C h + \text{cycl}_{f,g,h} \right) \quad (4.25)$$

showing that it is really designed to give a boundary term below the integral. One can show that also the induced two-product gives a boundary term if written under an integral.

Summarizing, relaxing the closure constraint, one can either impose (4.22) or define the tri-product deformation trivially as a total derivative. In both cases one formally has non-vanishing brackets (1.3) and (1.4) that leave no trace under an action integral.

Holonomic basis. In order to see more concretely what is happening here, let us consider as an example a holonomic basis with $B_{ab} = 0$, $f_{ab}{}^c = 0$ and $\tilde{f}^{ab}{}_c = 0$. In this case one finds

$$\begin{aligned} \mathcal{F}_{ABC} \mathcal{D}^A f \mathcal{D}^B g \mathcal{D}^C h &= R^{ijk} \partial_i f \partial_j g \partial_k h + \\ &Q_k{}^{ij} \left(\partial_i f \partial_j g (\tilde{\partial}^k + \beta^{kl} \partial_l) h + \text{cycl}_{f,g,h} \right) \\ &= 3 \left(\tilde{\partial}^{[i} \beta^{jk]} + \beta^{[im} \partial_m \beta^{jk]} \right) \partial_i f \partial_j g \partial_k h \\ &\quad - 3 \left(\beta^{[im} \partial_m \beta^{jk]} \right) \partial_i f \partial_j g \partial_k h + \partial_k \beta^{ij} \left(\partial_i f \partial_j g \tilde{\partial}^k h + \text{cycl}_{f,g,h} \right) \end{aligned} \quad (4.26)$$

where we have split the R -flux as

$$R^{ijk} = \hat{R}^{ijk} + R_{\text{cl}}^{ijk} = 3 \left(\tilde{\partial}^{[i} \beta^{jk]} + \beta^{[im} \partial_m \beta^{jk]} \right). \quad (4.27)$$

Therefore, the second and third term cancel and the sum of the first and fourth vanish by the constraint. In particular, this means that in DFT the classical part $R_{\text{cl}}^{ijk} = \beta^{[im} \partial_m \beta^{jk]}$ does not contribute to the tri-product.

In order to derive the tri-bracket among three coordinates, let us choose for the three functions $f = x^i$, $g = x^j$ and $h = x^k$. Without imposing neither the strong nor the closure constraint³ the resulting tri-bracket is then given by

$$[x^i, x^j, x^k] = \ell_s^4 \hat{R}^{ijk}, \quad (4.28)$$

and, in particular, only contains the R -flux \hat{R}^{ijk} .

³The CFT computations performed in [5–9] were not imposing any constraints so that they can be considered to be reliable for the compact torus case for which the level matching condition is (4.20).

Let us also consider the general commutator (1.4) for the case that both Q - and R -flux is present in more detail. Our DFT analysis suggests that the commutator for general functions should be defined as

$$\begin{aligned}
 -\frac{3i\hbar}{\ell_s^4} [f, g] &= R^{ijk} \partial_i f \partial_j g \partial_k + Q_k^{ij} \left(\partial_i f \partial_j g (\tilde{\partial}^k + \beta^{kl} \partial_l) \right. \\
 &\quad \left. + (\tilde{\partial}^k + \beta^{kl} \partial_l) f \partial_i g \partial_j + \partial_j f (\tilde{\partial}^k + \beta^{kl} \partial_l) g \partial_i \right). \tag{4.29}
 \end{aligned}$$

Inserting the definition of the R -flux (4.27), again the term R_{cl}^{ijk} completely cancels against terms appearing in the Q -flux contribution and we are left with

$$\begin{aligned}
 -\frac{3i\hbar}{\ell_s^4} [f, g] &= \hat{R}^{ijk} \partial_i f \partial_j g \partial_k \\
 &\quad + Q_k^{ij} \left(\partial_i f \partial_j g \tilde{\partial}^k + \tilde{\partial}^k f \partial_i g \partial_j + \partial_j f \tilde{\partial}^k g \partial_i \right). \tag{4.30}
 \end{aligned}$$

Note that, invoking the constraint, the commutator vanishes. Computing the commutation relations for the coordinate functions, without imposing any constraint, one finds

$$\begin{aligned}
 [x^i, x^j] &= i \frac{\ell_s^4}{3\hbar} \left(\hat{R}^{ijk} \partial_k + Q_k^{ij} \tilde{\partial}^k \right), \\
 [x^i, \tilde{x}_k] &= -i \frac{\ell_s^4}{3\hbar} Q_k^{ij} \partial_j. \tag{4.31}
 \end{aligned}$$

Thus, DFT suggests that the interpretation of the commutation relation (1.4) in terms of derivatives is (4.31). In particular, the contribution R_{cl}^{ijk} drops out and all commutators vanish after imposing any constraint.

Higher order corrections. At leading order in derivatives of \mathcal{F}_{ABC} there is a natural candidate for the all order in ℓ_s^4 tri-product, namely

$$(f \triangle g \triangle h)(X) = \exp \left(\frac{\ell_s^4}{6} \mathcal{F}_{ABC} \mathcal{D}_{X_1}^A \mathcal{D}_{X_2}^B \mathcal{D}_{X_3}^C \right) f(X_1) g(X_2) h(X_3) \Big|_X. \tag{4.32}$$

At leading order in $(\mathcal{D}\mathcal{F}_{ABC})$, except fgh , all terms give a total derivative below the integral. The appearing derivatives can be canceled by defining the overall tri-product as

$$\begin{aligned}
 f \blacktriangle g \blacktriangle h &= f \triangle g \triangle h + \sum_{k=2}^{\infty} \frac{\ell_s^{4k}}{3 \cdot 6^k k!} \left\{ \mathcal{F}_{A_1 B_1 D} \mathcal{D}^D (\mathcal{F}_{A_2 B_2 C_2} \dots \mathcal{F}_{A_k B_k C_k}) \right. \\
 &\quad \left. \left((\mathcal{D}^{A_1} \dots \mathcal{D}^{A_k} f) (\mathcal{D}^{B_1} \dots \mathcal{D}^{B_k} g) (\mathcal{D}^{C_2} \dots \mathcal{D}^{C_k} h) + \text{cycl}_{\{f, g, h\}} \right) \right\}. \tag{4.33}
 \end{aligned}$$

This product is designed to satisfy

$$\int dX e^{-2d} f \blacktriangle g \blacktriangle h = \int dX e^{-2d} f g h. \tag{4.34}$$

A possible generalization of the tri-product to the product of K functions is presented in the appendix.

5 Conclusions

Using the flux formulation of DFT, we have analyzed the consequences of introducing non-associativity via a non-trivial tri-product for the functions on the manifold. We analyzed two different such non-associative deformations. For the first the deforming flux was given by $\check{\mathcal{F}}^{ABC}$ and for the second by \mathcal{F}_{ABC} . The first case is the DFT generalization of the H^{ijk} -flux deformation and the second one the generalization of the R^{ijk} -flux deformation.

We argued from conformal field theory that on-shell any non-associative deformation should not lead to any physical effect. Note that in the open string case, the situation is different. There the DBI action can be expressed in the Seiberg-Witten limit as a non-commutative gauge theory and the higher orders in the star-product really contribute physical terms to the deformed action. However, also here cyclicity and associativity are preserved on-shell.

The $\check{\mathcal{F}}^{ABC}$ flux case is conceptually very close to its open string analogue. Similarly, we found that, at leading order in ℓ_s^4 , the deformation gives a boundary term under the integral if the DFT equations of motion are satisfied and the strong or closure constraint is employed. We showed that, for additional matter contributions, the tri-product can be adjusted accordingly. This led to a new deformation of the two-product, whose on-shell triviality was guaranteed by the continuity equation of the energy momentum tensor. This means that on-shell DFT or string theory cannot distinguish between ordinary smooth geometry and a fuzzy one with fundamental tri-bracket

$$[x^i, x^j, x^k] = \ell_s^4 H^{ijk}. \tag{5.1}$$

Even though from [5–9] we do not have any evidence for such a non-associative behavior of the coordinates, we find this a remarkable property of DFT. Turning the logic around, up to the dilaton sector, one can derive the DFT equations of motion from the concept of the *absence of on-shell non-associativity*. We emphasize, that in the flux formulation of DFT also the gravity part is fully encoded in the generalized three-form flux. At least in spirit, this is very similar to the familiar magnetic monopole example discussed in the first section.

The \mathcal{F}_{ABC} flux case is the one where non-associativity was expected. We realized that in the DFT framework this tri-product actually vanishes after imposing either the strong or the closure constraint. Therefore, in order to get something non-trivial even the closure constraint need to be weakened. Only then one could obtain a non-associative deformation of the target space action with the three-bracket for the internal coordinates x^i being

$$[x^i, x^j, x^k] = \ell_s^4 \hat{R}^{ijk}. \tag{5.2}$$

Again note that the \hat{R}^{ijk} only contain the winding part of the full R -flux, the classical part has canceled out.

On a more speculative level, we also proposed a generalization of the tri-product to higher orders in ℓ_s^4 and for products of K -terms.

Summarizing, the resolution to the initially raised paradox is that one can have a non-associative deformation of the target space, while nothing of it is immediately apparent

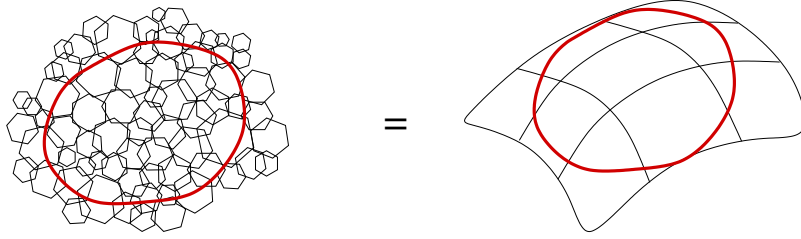


Figure 1. Stringy equivalence between fuzzy non-associative geometry and smooth Riemannian geometry.

in the effective string and DFT actions for the massless modes. Deforming the product to a tri-product we have found two different ways how such a deformation can become trivial (on-shell).

One could imagine that, due to the finite size and resolution of the string, there exists a certain non-associative deformation of the target space that is “under the radar” of the string. Therefore, string theory can very well admit such non-geometric space as honest backgrounds. An artist’s impression of this picture is presented in figure 1.

It would be interesting to carry out a similar analysis for the (precursor) non-commutative closed string star product defined on phase space, which was introduced and discussed in [12, 13]. Moreover, one could contemplate what other deeper conceptual consequences the existence of such a non-geometric regime of string theory might have. Including also the massive string states, can it be generalized to string field theory? Does there exist an analogous structure for M-theory?

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A K tri-product

In this appendix we discuss how to treat terms which involve for instance a product of K functions. Clearly, e.g. for $K = 4$ this is not defined by an iteration of the tri-product (4.32). From the analysis of multiple tachyon scattering amplitudes in CFT, in [7] a proposal was made, how to deform the product of K functions. Analogously, at leading order in $(\mathcal{D}\mathcal{F}_{ABC})$ (or $(\check{\mathcal{D}}\mathcal{F}_{ABC})$) we now define the K -fold tri-product as

$$(f_1 \Delta_K f_2 \Delta_K \dots \Delta_K f_K)(X) \stackrel{\text{def}}{=} \exp\left(\frac{\ell_s^4}{6} \mathcal{F}_{ABC} \sum_{1 \leq a < b < c \leq K} \mathcal{D}_{X_a}^A \mathcal{D}_{X_b}^B \mathcal{D}_{X_c}^C\right) f_1(X_1) f_2(X_2) \dots f_K(X_K) \Big|_X. \quad (\text{A.1})$$

Below we prove the remarkable feature that for each K all terms beyond leading order give a total derivative under the internal integral, i.e.

$$\int dX e^{-2d} f_1 \Delta_K f_2 \Delta_K \dots \Delta_K f_K = \int dX e^{-2d} f_1 f_2 \dots f_K. \quad (\text{A.2})$$

Moreover, this K tri-product has the property

$$f_1 \Delta_K \dots \Delta_K 1 = f_1 \Delta_{K-1} \dots \Delta_{K-1} f_{K-1} \quad (\text{A.3})$$

which suggests to define $f_1 \Delta_2 f_2 = f_1 \cdot f_2$, i.e. the two tri-product is the ordinary multiplication of functions.

Note that the total derivative property does *not* hold for a similar definition of an K star-product

$$(f_1 \star_K f_2 \star_K \dots \star_K f_K)(X) \stackrel{\text{def}}{=} \exp\left(i \frac{\ell_s^2}{2} \theta^{IJ} \sum_{1 \leq a < b \leq K} \partial_I^{X_a} \partial_J^{X_b}\right) f_1(X_1) f_2(X_2) \dots f_K(X_K) \Big|_X, \quad (\text{A.4})$$

This is why for the open string case, the non-commutativity of the underlying space-time has a non-trivial effect on the action.

Proof. Here we present the proof that at leading order in \mathcal{DF}_{ABC} the K tri-product (A.1) gives a total derivative under the integral, i.e.

$$\int dX e^{-2d} f_1 \Delta_K \dots \Delta_K f_K = \int dX e^{-2d} f_1 \dots f_K. \quad (\text{A.5})$$

We first consider just the order ℓ_s^4 term, which is given by

$$\frac{\ell_s^4}{6} \mathcal{F}_{ABC} \sum_{1 \leq a < b < c \leq K} \mathcal{D}_{X_a}^A \mathcal{D}_{X_b}^B \mathcal{D}_{X_c}^C \left(f_1(X_1) f_2(X_2) \dots f_K(X_K) \right) \Big|_X.$$

Inspection reveals, that the $\binom{K}{3}$ terms can be grouped together as

$$\begin{aligned} & \mathcal{D}^A(f_1) \mathcal{D}^B f_2 \mathcal{D}^C(f_3 \dots f_K) \\ & + \mathcal{D}^A(f_1 f_2) \mathcal{D}^B f_3 \mathcal{D}^C(f_4 \dots f_K) \\ & + \mathcal{D}^A(f_1 f_2 f_3) \mathcal{D}^B f_4 \mathcal{D}^C(f_5 \dots f_K) \\ & + \dots \\ & + \mathcal{D}^A(f_1 \dots f_{K-2}) \mathcal{D}^B f_{K-1} \mathcal{D}^C(f_K). \end{aligned} \quad (\text{A.6})$$

Note that the sum fixes the order of the derivatives and the number of terms is correct, since

$$\binom{K}{3} = 1 \cdot (K-2) + 2 \cdot (K-3) + \dots + (K-2) \cdot 1. \quad (\text{A.7})$$

As one can see, the K tri-product splits into $K-2$ three tri-products and therefore shares its properties under an integral. The higher order terms follow immediately by iteration. This is owed to the fact that, in the derivation of the total derivative property, only first three derivatives are relevant.

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