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Renormalization of the classical massless scalar field theory with quartic self-interaction

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Renormierung der klassischen masselosen Skalarfeldtheorie mit quartischer Selbstwechselwirkung

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Abstract

The classical theory of the massless scalar field with negative quartic self-interaction shows asymptotic freedom and confinement. These findings might help in understanding evidence, gathered from experiment and lattice simulations of quantum chromodynamics, that quarks can't be found alone. To improve the understanding of these results the renormalization of the classical massless scalar field theory with quartic self-interaction is studied in detail. The renormalized effective coupling is obtained twice, using different regularization schemes: Cutoff regularization and dimensional regularization. The renormalization group is explained. The perturbative expansion of the effective coupling is improved by enforcing the renormalization group law. A connection to renormalization and the renormalization group found in other areas of physics is made.

Contents

1 Introduction

1.1 Review

Quantum field theories like quantum electrodynamics and the standard model can predict transition probabilities between initial and final states. If the initial state is a single particle state then a decay rate can be computed and if the initial state is a two particle state then a cross section can be computed. These transition probabilities are proportional to a weighted sum and integral over all possible intermediate states. For some quantum field theories some of the momentum integrals over intermediate states diverge. That leads to an infinite relation between the bare parameters of the theory and the transition probabilities. In renormalizable theories renormalization can be used to rewrite the transition probabilities in terms of physical parameters. Then the resulting relations become finite. For general references on renormalization in quantum field theory see [30, 26, 18, 21, 11, 23] .

As the author of [4] explains the procedure of renormalization is not specific to quantum field theory. It is a procedure to rewrite physical quantities given as a power series in a bare parameter with divergent coefficients as a power series in a physical parameter with finite coefficients. In [7] the authors confirm the renormalizability of the classical massless scalar field theory with quartic self-interaction by showing that the regulated effective coupling can be made independent of the regulator by letting the bare coupling become a function of the regulator. In [29] a similar analysis in done for the massive case.

Associated with renormalizable theories is the renormalization group. The renormalization group is used to relate physical couplings measured at different scales. For instance the coupling of quantum electrodynamics increases with the energy at which it is measured while the coupling of quantum chromodynamics decreases with the energy. The authors of [7] use the renormalization group to resum the effective coupling. By that they obtain expressions for all the leading, subleading, ... logarithms. The general theory of resummation using the renormalization group is explained in [4].

The renormalization group is also used in statistical mechanics. Starting from a microscopic description microscopic details are integrated out to yield an effective description of the macroscopic behavior of the system. One is interested in fixed points of the renormalization group that indicate scale invariance of the system. By comparing different systems after the microscopic details have been integrated out they can be grouped into universality classes because theories with different microscopic details can show the same macroscopic behavior.

Effective field theories are quantum field theories with a built-in momentum cutoff. After transition probabilities have been computed the cutoff is not removed, that is, it is not sent to infinity. It might not be possible to remove the cutoff because the theory might not be a renormalizable quantum field theory. The interpretation is that the effective field theory is obtained from a more fundamental theory. This more fundamental theory can itself be another quantum field theory with more degrees of freedom. The high-energy degrees of freedom are integrated out down to the momentum cutoff, yielding the effective field theory.

In general the integrating out of high-energy degrees of freedom can produce infinitely man interactions, most of which are non-renormalizable. By taking only some of these interactions into account an error is introduced that is of the order of some power of the ratio of the energy of the process to the cutoff. Therefore this error is small if the energy of the process is small compared to the cutoff.

Like in statistical mechanics the process of integrating out high-energy degrees of freedom constitutes the operation of a group, called the functional or exact renormalization group. See [5, 17, 19, 12, 27, 20, 15, 14] for the renormalization group in statistical mechanics and effective field theory.

Renormalization has also been discussed in semi-classical, classical and electrostatic systems: The Casimir effect as explained in [30], where the energy of the vacuum fluctuations of the electric field between and outside of two metal plates diverges. In [3], where the electric fields around some charge configurations are obtained from divergent electric potentials. In [25], where the ionization energy is computed from the difference of two infinite energies, the energies of the electric field of the charged and the neutral atom. They have in common that a physical quantity is computed from another intermediate quantity that is infinite. To obtain a result this intermediate quantity is regulated and after the physical quantity is obtained, the regulator is removed.

In [2] it is shown that that radiative corrections can introduce a mass in an initially massless theory. The theory trades a dimensionless parameter for a dimensionful parameter. This is called dimensional transmutation. A physical quantity that might depend in a complicated way on the dimensionless parameter will have a trivial dependence on the dimensionful parameter due to dimensional analysis.

In [1] the concept of dimensional transmutation is studied in nonrelativistic quantum mechanics. In quantum mechanics scale invariant potentials lead to a singular spectrum. That is, there are either no bound states, or the energy of the bound states is not bounded from below, or there is only one bound state with negative infinite energy. Regularization is used to break the scale invariance and by that a non-singular spectrum is obtained. Scale invariance is broken by the regularization scheme because it always introduces a dimensionful parameter: In cutoff regularization the cutoff is dimensionful while in dimensional regularization an arbitrary dimensionful parameter is introduced to leave dimensionless the dimensionless parameter in any spatial dimension. After renormalization these dimensionful parameters disappear but a new dimensionful parameter appear: The scale, also called the renormalization point or subtraction point.

Non-Abelian gauge theories become weakly coupled at short distance or high momentum: They are asymptotically free [9, 16]. Quantum chromodynamics, the Yang-Mills theory [28] of gauge group $SU(3)$, is supposed to be the theory of the strong interactions. Apart from asymptotic freedom, it is supposed to have another feature: Confinement.

It has been possible to compute the large momentum behavior of the quantum scalar field theory with negative quartic self-interaction because the corrections to the leading terms become smaller the larger the momentum becomes [22, 13]. The classical counterpart has been analyzed in [7]. By letting the bare coupling become a function of the cutoff they make the effective coupling independent of the cutoff. With the cutoff-dependent bare coupling they construct a scale which is independent of the cutoff. They show that the theory is asymptotically free and that it is confining. Confinement is shown by demonstrating that the energy of the field generated by a point source is infinite while the energy of a dipole is finite.

The appearance of a scale in the classical theory supports the statement of [1] about dimensional transmutation being a general mechanism, not specific to quantum field theory. It is crucial for the classical theory to develop a scale in order to be both asymptotically free and confining. Without a dimensionful scale there would be no distance or momentum range in which the theory transitions from weak to strong coupling. The scale appears in the expression for the energy of a dipole in the form of a power of the ratio of the distance of the sources to the scale. Without a scale such a power law could not exist because of dimensional analysis.

Quarks have not been observed alone. Even though lattice simulations of quantum chromodynamics indicate that the theory is confining a mathematical proof is not available yet [10, 6]. Being able to study confinement in classical field theories might shed some light on its mechanism. In addition numerical methods can be used to obtain non-perturbative results more easily in the classical theory than in a quantum field theory [8].

There might be different concepts involved in explaining confinement of quantum chromodynamics. Some of them might turn out not to be related while others will. To decide on these matters a precise understanding of each concept will be necessary. This text focusses on one of the possibly involved concepts: Renormalization and the renormalization group.

1.2 Theory

A physical theory relates physical quantities. Given some of these related physical quantities others can be predicted for a universe that follows the laws of that physical theory. For instance in a classical field theory knowledge about field values and derivatives at some spacetime points or regions allows the prediction of field values at other spacetime points. This means that the predicted field values are functions of the measured field values. Intermediate variables are typically introduced and called parameters of the theory.

If the theory imposes a relation between these parameters some of them can be eliminated so that only independent parameters remain. To fully specify the theory all the parameters have to be determined by doing as many independent measurements as there are independent parameters.

The choice of independent parameters is arbitrary as any invertible function of them does equally well specify the same theory. Particular choices can be made, for instance, observables in a quantum field theory can be parametrized by the bare parameters appearing in the Hamilton operator, or classical fields can be parametrized by the bare parameters appearing in the Lagrangian. Sometimes this choice of parameters is not available because the relation between physical quantities and these parameters turns out to be ill-defined, that is, a finite value for some physical quantity requires an infinite value for at least one of the parameters. The theory is still useful if it can be rewritten in terms of a different set of parameters that contains only finite parameters.

The infinite relations between physical quantities and bare parameters appear in the form of divergent integrals. To reparametrize the theory these integrals are regulated. After the theory has been rewritten in terms of physical parameters the regulator is removed.

The integrals are regulated because otherwise every time infinity is multiplied or added to information were lost. The results would be ambiguous.

As a regulator different functions can be chosen to make certain terms finite. As the regulator is removed after the reparametrization no trace remains of which regulator has been used. Therefore the renormalization procedure is independent of the regulator.

Changing from one parametrization to a second parametrization and from the second one to a third one has the same effect as changing from the first to the third parametrization directly. Therefore the operation of changing the parametrization of a theory forms the operation of a group, the renormalization group [4].

If a physical theory is known only in terms of a power series in some parameter up to a finite order then the renormalization group law can be violated. By enforcing the group law additional terms can be obtained that are not present in the initial perturbative expansion.

The beta function β captures the rate of change of a physical quantity with the change of scale at which it is measured. Integrating this beta function yields an expression that automatically satisfies the group law. This expression can contain additional terms that are not present in the perturbative expression from which the beta function was obtained.

1.3 Overview

In this text renormalization of a classical field theory is explained at the example of the massless scalar field theory with quartic self-interaction. From an expression of the effective coupling α, given as a power series in the bare coupling α_0 with divergent coefficients, a power series with finite coefficients in a physical coupling α_p, which is measured at a scale r_5, is obtained with the help of renormalization. First cutoff regularization is used, then dimensional regularization. Both results coincide. Finally, using the renormalization group, the effective coupling is resummed, giving exact expressions for the first four series of leading and lower order logarithms.

In particular, in chapter 3 a static, spherically symmetric ansatz for the equations of motion is chosen, leading to a second-order ordinary differential equation. An equivalent integral equation is obtained. The meaning on the constants of integration, the free parameters and the relation to measurement is explained and the effective coupling α is introduced. The general process of renormalization is explained. In chapter 4 the integral equation is regulated using a cutoff and solved iteratively. After imposing the renormalization condition and expressing the bare coupling α_0 as a power series in the physical coupling α_p a finite, cutoff-independent expression for the effective coupling is obtained. In chapter 5 dimensional regularization is used. After imposing the renormalization condition a regulator-dependent expression for the effective coupling is obtained. After removing the regulator the result coincides with the effective coupling obtained using cutoff regularization. In chapter 6 the renormalization group is explained. Terms in the renormalized effective coupling are grouped into leading, subleading, ... logarithms. It is demonstrated that by enforcing the renormalization group law the second term in the series of leading logarithms can be obtained from the knowledge of only the first term. Then the beta function is introduced and an expression is obtained that contains all the leading logarithms. The integration of the beta function is also demonstrated for the subleading and the subsubleading logarithms. The result for the subsubsubleading logarithms is given in the appendix.

All computations have been carried out to an order necessary to obtain the effective coupling to order twelve in the physical coupling α_p. The appendix contains the effective coupling and intermediate results for both regularization schemes to some lower order for reference. Higher orders are not shown because the expressions are huge and they are straightforward to obtain using computer algebra.

2 Scalar field theory

In this chapter the classical massless scalar field theory with quartic self-interaction is studied in $d+1$ spacetime dimensions. A static, spherically symmetric ansatz is chosen, yielding a second-order ordinary differential equation. An equivalent integral equation is obtained. The meaning of the constants of integration and free parameters is explained and the effective coupling α is introduced. Then renormalization is explained.

2.1 Integral equation

A massless scalar field theory with quartic self-interaction in $d+1$ spacetime dimensions is given by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - \frac{1}{4}\lambda_0\phi^4(x). \tag{1}$$

It satisfies the equation of motion

$$\partial_\mu\partial^\mu\phi(x) = -\lambda_0\phi^3(x), \tag{2}$$

where the metric is $\eta = diag(1, -1, \ldots, -1)$. Plugging in a static, spherically symmetric ansatz

$$\phi(x) = \phi(|\vec{x}|) \tag{3}$$

and using hyperspherical coordinates

$$\begin{aligned}
x_1 &= r\cos\theta_1 \\
x_2 &= r\sin\theta_1\cos\theta_2 \\
x_3 &= r\sin\theta_1\sin\theta_2\cos\theta_3 \\
&\cdots \\
x_{d-2} &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_{d-3}\cos\theta_{d-2} \\
x_{d-1} &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_{d-2}\cos\phi \\
x_d &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_{d-2}\sin\phi
\end{aligned} \tag{4}$$

the following second-order ordinary differential equation is obtained:

$$\frac{1}{r^{d-1}}\frac{\partial}{\partial r}\left(r^{d-1}\phi'(r)\right) = \lambda_0\phi^3(r) \tag{5}$$

An equivalent integral equation can be obtained by integrating once,

$$\phi'(r) = \lambda_0\frac{1}{r^{d-1}}\int_0^r r_1^{d-1}\phi^3(r_1)\,dr_1 + \frac{a(\lambda_0)}{r^{d-1}}, \tag{6}$$

and once more:

$$\phi(r) = -\lambda_0\int_r^\infty \frac{1}{r_2^{d-1}}\left(\int_0^{r_2} r_1^{d-1}\phi^3(r_1)\,dr_1\right)dr_2 - \frac{1}{d-2}\frac{a(\lambda_0)}{r^{d-2}} + b(\lambda_0), \tag{7}$$

where a and b are constants of integration. Using the ansatz

$$\phi(r) = -\frac{1}{d-2}\frac{a(\lambda_0)f(r)}{r^{d-2}}, \tag{8}$$

motivated in [7], the integral equation can be rewritten as

$$f(r) = 1-\lambda_0\frac{a^2(\lambda_0)}{(d-2)^2}r^{d-2}\int_r^\infty \frac{1}{r_2^{d-1}}\left(\int_0^{r_2}\frac{f^3(r_1)}{r_1^{2d-5}}\,dr_1\right)dr_2 - r^{d-2}\frac{b(\lambda_0)}{a(\lambda_0)}(d-2). \tag{9}$$

Introducing the bare coupling α_0,

$$\alpha_0 = -\lambda_0\frac{a^2(\lambda_0)}{(d-2)^2}, \tag{10}$$

the integral equation takes the form

$$f(r) = 1+\alpha_0 r^{d-2}\int_r^\infty \frac{1}{r_2^{d-1}}\left(\int_0^{r_2}\frac{f^3(r_1)}{r_1^{2d-5}}\,dr_1\right)dr_2 - r^{d-2}\frac{b(\lambda_0)}{a(\lambda_0)}(d-2). \tag{11}$$

2.2 Free parameters

The solution of an n-th order ordinary differential equation contains n constants of integration. These can be fixed by imposing n initial conditions. If the differential equation depends on one parameter then one additional condition is necessary to uniquely specify the solution. If the differential equation describes a physical system then a choice of constants of integration corresponds to the choice of a state of the system at a certain value of the independent variable, say, the time. By observing the system at a different time the parameter can be determined. This is called a measurement. After the parameter has been determined the solution of the differential equation can be used to predict the state of the system at any time given the state at one particular time. If there are multiple independent parameters then the same number of independent measurements is necessary to determine all of them.

The integral equation contains two constants of integration, a and b, and one parameter λ_0. After making a choice for the constants of integration a and b one measurement is necessary to determine the parameter λ_0. One convenient choice is

$$b(\lambda_0) = 0, \tag{12}$$

while a can be chosen to take any value other than 0 because a different choice can be compensated by a redefinition of the parameter λ_0. In the following the bare coupling α_0 will be used instead of the bare parameter λ_0. With this choice of initial conditions the integral equation takes the form

$$f(r) = 1 + \alpha_0 r^{d-2} \int_r^\infty \frac{1}{r_2^{d-1}} \left(\int_0^{r_2} \frac{f^3(r_1)}{r_1^{2d-5}} \, dr_1 \right) dr_2. \tag{13}$$

A measurement gives a relation of the form

$$f(r_5) = f_p, \tag{14}$$

where f_p is the value of f measured at the scale r_5. From this relation the bare coupling α_0 can be obtained by solving the integral equation and then solving for α_0. Alternatively an effective coupling α can be defined:

$$\alpha(r) = \alpha_0 f^2(r) \tag{15}$$

A corresponding experiment will give a relation

$$\alpha(r_5) = \alpha_p, \tag{16}$$

where α_p is called the physical [30] coupling. Working with the effective coupling α instead of directly working with the solution f can be motivated with an analogy to quantum electrodynamics where an effective coupling is introduced to describe the deviation of the fine structure constant due to quantum effects from its classical value [24, 7]. In this interpretation the constant of integration a corresponds to the charge of a point source located at the origin.

2.3 Renormalization

After the bare coupling α_0 has been determined from experiment the effective coupling α can in principle be computed for every radius r. The dependence of the effective coupling α on the bare coupling α_0 however turns out not to be well defined in four spacetime dimensions: A finite value for the effective coupling α requires an infinite value for the bare coupling α_0.

The relation

$$\alpha(r_5) = \alpha_p, \tag{17}$$

called the renormalization condition [4], can be used to express the bare coupling α_0 in terms of the physical coupling α_p, a relation that is not well defined neither. By eliminating the bare coupling α_0 from the effective coupling α in favor of the physical coupling α_p a finite relation between the effective coupling α and the physical coupling α_p is obtained.

In intermediate steps of the calculation infinite expressions are regulated. Once the effective coupling α is expressed in terms of the physical coupling α_p the regulator is removed. The result is independent of which regulator has been used in intermediate calculations.

More concretely, the regulated effective coupling α_Λ is first computed as a power series in the bare coupling α_0,

$$\alpha_\Lambda(r) = \alpha_0 + c_{2,\Lambda}\alpha_0^2 + c_{3,\Lambda}\alpha_0^3 + \cdots , \tag{18}$$

where the coefficients have been regulated with some regulator. The regulator depends on some parameter Λ. Then the bare coupling α_0 is expressed as a power series in the physical coupling α_p:

$$\alpha_0 = \alpha_p + \delta_{2,\Lambda}\alpha_p^2 + \delta_{3,\Lambda}\alpha_p^3 + \cdots \tag{19}$$

The renormalization condition is used to determine the coefficients δ order by order in the physical coupling α_p. Imposing the renormalization condition at the lowest relevant order,

$$\begin{aligned}\alpha_p &= \alpha_\Lambda(r_5)\\ &= \alpha_0 + c_{2,\Lambda}\alpha_0^2 + O(\alpha_0^3)\\ &= \alpha_p + (\delta_{2,\Lambda} + c_{2,\Lambda}(r_5))\alpha_p^2 + O(\alpha_p^3),\end{aligned} \tag{20}$$

where in the last step the bare coupling α_0 has been replaced with its power series expansion in the physical coupling α_p, implies

$$\delta_{2,\Lambda} = -c_{2,\Lambda}(r_5). \tag{21}$$

Order α_p^3 terms have been neglected. Imposing the renormalization condition at the next higher order,

$$\begin{aligned}\alpha_p &= \alpha_\Lambda(r_5)\\ &= \alpha_0 + c_{2,\Lambda}(r_5)\alpha_0^2 + c_{3,\Lambda}(r_5)\alpha_0^3 + O(\alpha_0^4)\\ &= \alpha_p + (\delta_{2,\Lambda} + c_{2,\Lambda}(r_5))\alpha_p^2 + (\delta_{3,\Lambda} + 2\delta_{2,\Lambda}c_{2,\Lambda}(r_5) + c_{3,\Lambda}(r_5))\alpha_p^3 + O(\alpha_p^4),\end{aligned} \tag{22}$$

implies

$$\begin{aligned}\delta_{3,\Lambda} &= -2\delta_{2,\Lambda}c_{2,\Lambda}(r_5) - c_{3,\Lambda}(r_5)\\ &= 2c_{2,\Lambda}^2(r_5) - c_{3,\Lambda}(r_5),\end{aligned} \tag{23}$$

and so on. Substituting the expression for the bare coupling α_0 into the expression for the effective coupling α yields

$$\begin{aligned}\alpha_\Lambda(r) &= \alpha_p + (\delta_{2,\Lambda} + c_{2,\Lambda}(r))\alpha_p^2\\ &\quad + (\delta_{3,\Lambda} + 2\delta_{2,\Lambda}c_{2,\Lambda}(r) + c_{3,\Lambda}(r))\alpha_p^3 + O(\alpha_p^4)\\ &= \alpha_p + (c_{2,\Lambda}(r) - c_{2,\Lambda}(r_5))\alpha_p^2\\ &\quad + (c_{3,\Lambda}(r) - c_{3,\Lambda}(r_5) - 2c_{2,\Lambda}(r)c_{2,\Lambda}(r_5) + 2c_{2,\Lambda}^2(r_5))\alpha_p^3 + O(\alpha_p^4).\end{aligned} \tag{24}$$

After the bare coupling α_0 has been eliminated in favor of the physical coupling α_p the regulator can be removed, yielding a finite result for the effective coupling α. The theory is called renormalizable.

3 Renormalization using cutoff regularization

In this chapter the integral equation for $d = 3$ spatial dimensions is regulated using a cutoff r_0 and solved iteratively for f, yielding the regulated effective coupling α_{r_0},

$$\alpha_{r_0}(r) = \alpha_0 f^2(r), \tag{25}$$

as a power series in the bare coupling α_0. The bare coupling α_0 is then expressed as a power series in the physical coupling α_p,

$$\alpha_0 = \alpha_p + \delta_{2,r_0}\alpha_p^2 + \delta_{2,r_0}\alpha_p^3 + \cdots . \tag{26}$$

The coefficients δ are determined by imposing the renormalization condition

$$\alpha_{r_0}(r_5) = \alpha_p \tag{27}$$

at the scale r_5, yielding a power series of the regulated effective coupling α_{r_0} in the physical coupling α_p with cutoff independent and finite coefficients.

3.1 Regularization

The integral equation will be solved iteratively. In three spatial dimensions, when starting with $f_0(r) = 1$, the inner integral diverges at the lower bound, that is, for small radius r. It can be regulated by introducing a cutoff r_0:

$$f(r) = 1 + \alpha_0 r \int_r^\infty \frac{1}{r_2^2} \int_{r_0}^{r_2} \frac{f^3(r_1)}{r_1} \, dr_1 \, dr_2 \tag{28}$$

The regulator is removed in the limit $r_0 \to 0$.

3.2 Iteration

The regulated integral equation can be solved iteratively by repeatedly plugging the result of the previous integration back into the integral,

$$f_n(r) = 1 + \alpha_0 r \int_r^\infty \frac{1}{r_2^2} \int_{r_0}^{r_2} \frac{f_{n-1}^3(r_1)}{r_1} \, dr_1 \, dr_2, \tag{29}$$

starting with

$$f_0(r) = 1. \tag{30}$$

First iteration:

$$\begin{aligned} f_1(r) &= 1 + \alpha_0 r \int_r^\infty \frac{1}{r_2^2} \int_{r_0}^{r_2} \frac{1}{r_1} \, dr_1 \, dr_2 \\ &= 1 + \alpha_0 r \int_r^\infty \frac{1}{r_2^2} \log\left(\frac{r_2}{r_0}\right) \, dr_2 \\ &= 1 + \left(\log\left(\frac{r}{r_0}\right) + 1\right) \alpha_0 \end{aligned} \tag{31}$$

Second iteration, keeping only terms of order α_0^2:

$$
\begin{aligned}
f_2(r) &= 1 + \alpha_0 r \int_r^\infty \frac{1}{r_2^2} \int_{r_0}^{r_2} \frac{1}{r_1} \left(1 + \left(\log\left(\frac{r_1}{r_0}\right) + 1\right) \alpha_0\right)^3 dr_1\, dr_2 \\
&= 1 + \alpha_0 r \int_r^\infty \frac{1}{r_2^2} \int_{r_0}^{r_2} \frac{1}{r_1} \left(1 + 3\left(\log\left(\frac{r_1}{r_0}\right) + 1\right) \alpha_0\right) dr_1\, dr_2 + O(\alpha_0^3) \\
&= 1 + \left(\log\left(\frac{r}{r_0}\right) + 1\right) \alpha_0 \\
&\quad + 3\alpha_0^2 r \int_r^\infty \frac{1}{r_2^2} \int_{r_0}^{r_2} \frac{1}{r_1} \left(\log\left(\frac{r_1}{r_0}\right) + 1\right) dr_1\, dr_2 + O(\alpha_0^3) \\
&= 1 + \left(\log\left(\frac{r}{r_0}\right) + 1\right) \alpha_0 \\
&\quad + \left(\frac{3}{2}\log^2\left(\frac{r}{r_0}\right) + 6\log\left(\frac{r}{r_0}\right) + 6\right) \alpha_0^2 + O(\alpha_0^3)
\end{aligned} \tag{32}
$$

Third iteration, keeping only terms of order α_0^3:

$$
\begin{aligned}
f_3(r) &= 1 + \alpha_0 r \int_r^\infty \frac{1}{r_2^2} \int_{r_0}^{r_2} \frac{1}{r_1} \left(1 + \left(\log\left(\frac{r_1}{r_0}\right) + 1\right) \alpha_0 \right. \\
&\quad \left. + \left(\frac{3}{2}\log^2\left(\frac{r_1}{r_0}\right) + 6\log\left(\frac{r_1}{r_0}\right) + 6\right) \alpha_0^2\right)^3 dr_1\, dr_2 + O(\alpha_0^4) \\
&= 1 \\
&\quad + \left(\log\left(\frac{r}{r_0}\right) + 1\right) \alpha_0 \\
&\quad + \left(\frac{3}{2}\log^2\left(\frac{r}{r_0}\right) + 6\log\left(\frac{r}{r_0}\right) + 6\right) \alpha_0^2 \\
&\quad + \left(\frac{5}{2}\log^3\left(\frac{r}{r_0}\right) + \frac{39}{2}\log^2\left(\frac{r}{r_0}\right) + 60\log\left(\frac{r}{r_0}\right) + 60\right) \alpha_0^3 \\
&\quad + O(\alpha_0^4)
\end{aligned} \tag{33}
$$

It can be observed that successive iterations do not alter lower order coefficients that have been determined in a previous iteration. This allows determining at each step of iteration one higher order coefficient. In terms of the bare coupling α_0 the regulated effective coupling α_{r_0} is:

$$\begin{aligned}\alpha_{r_0}(r) &= \alpha_0 f^2(r) \\ &= \alpha_0 \\ &\quad + \left(2\log\left(\frac{r}{r_0}\right) + 2\right)\alpha_0^2 \\ &\quad + \left(4\log^2\left(\frac{r}{r_0}\right) + 14\log\left(\frac{r}{r_0}\right) + 13\right)\alpha_0^3 \\ &\quad + \left(8\log^3\left(\frac{r}{r_0}\right) + 54\log^2\left(\frac{r}{r_0}\right) + 144\log\left(\frac{r}{r_0}\right) + 132\right)\alpha_0^4 \\ &\quad + O(\alpha_0^5)\end{aligned} \tag{34}$$

3.3 Renormalization

The renormalization condition

$$\alpha_{r_0}(r_5) = \alpha_p \tag{35}$$

relates the regulated effective coupling α_{r_0} to a physical coupling α_p, measured at the scale r_5. It is used to determine the coefficients δ of the power series expansion of the bare coupling α_0 in the physical coupling α_p,

$$\alpha_0 = \alpha_p + \delta_{2,r_0}\alpha_p^2 + \delta_{3,r_0}\alpha_p^3 + \cdots . \tag{36}$$

The coefficients δ are determined by imposing the renormalization condition at successively higher orders in the physical coupling α_p, starting at the second order:

$$\begin{aligned}\alpha_p &= \alpha_{r_0}(r_5) \\ &= \alpha_0 + \left(2\log\left(\frac{r_5}{r_0}\right) + 2\right)\alpha_0^2 + O(\alpha_0^3) \\ &= \alpha_p + \delta_{2,r_0}\alpha_p^2 + \left(2\log\left(\frac{r_5}{r_0}\right) + 2\right)\alpha_p^2 + O(\alpha_p^3)\end{aligned} \tag{37}$$

In the last line the expansion of the bare coupling α_0 has been plugged in. This implies, neglecting order α_p^3 terms:

$$\delta_{2,r_0} = -2\log\left(\frac{r_5}{r_0}\right) - 2 \tag{38}$$

Impose the renormalization condition at order α_p^3 to determine δ_3:

$$\begin{aligned}
\alpha_p &= \alpha_{r_0}(r_5) \\
&= \alpha_0 + \left(2\log\left(\frac{r_5}{r_0}\right) + 2\right)\alpha_0^2 \\
&\quad + \left(4\log^2\left(\frac{r_5}{r_0}\right) + 14\log\left(\frac{r_5}{r_0}\right) + 13\right)\alpha_0^3 + O(\alpha_0^4) \\
&= \alpha_p + \delta_{2,r_0}\alpha_p^2 + \delta_{3,r_0}\alpha_p^3 + \left(2\log\left(\frac{r_5}{r_0}\right) + 2\right)\left(\alpha_p^2 + 2\delta_2\left(r_0\right)\alpha_p^3\right) \\
&\quad + \left(4\log^2\left(\frac{r_5}{r_0}\right) + 14\log\left(\frac{r_5}{r_0}\right) + 13\right)\alpha_p^3 + O(\alpha_p^4)
\end{aligned} \tag{39}$$

This implies, neglecting order α_p^4 terms:

$$\delta_{3,r_0} = 4\log^2\left(\frac{r_5}{r_0}\right) + 2\log\left(\frac{r_5}{r_0}\right) - 5 \tag{40}$$

Impose the renormalization condition at order α_p^4 to determine δ_4:

$$\begin{aligned}
\alpha_p &= \alpha_{r_0}(r_5) \\
&= \alpha_0 \\
&\quad + \left(2\log\left(\frac{r_5}{r_0}\right) + 2\right)\alpha_0^2 \\
&\quad + \left(4\log^2\left(\frac{r_5}{r_0}\right) + 14\log\left(\frac{r_5}{r_0}\right) + 13\right)\alpha_0^3 \\
&\quad + \left(8\log^3\left(\frac{r_5}{r_0}\right) + 54\log^2\left(\frac{r_5}{r_0}\right) + 144\log\left(\frac{r_5}{r_0}\right) + 132\right)\alpha_0^4 \\
&\quad + O(\alpha_0^5) \\
&= \alpha_p + \delta_{2,r_0}\alpha_p^2 + \delta_{3,r_0}\alpha_p^3 + \delta_{4,r_0}\alpha_p^4 \\
&\quad + \left(2\log\left(\frac{r_5}{r_0}\right) + 2\right)\left(\alpha_p^2 + 2\delta_{2,r_0}\alpha_p^3 + \left(\delta_{2,r_0}^2 + 2\delta_{3,r_0}\right)\alpha_p^4\right) \\
&\quad + \left(4\log^2\left(\frac{r_5}{r_0}\right) + 14\log\left(\frac{r_5}{r_0}\right) + 13\right)\left(\alpha_p^3 + 3\delta_{2,r_0}\alpha_p^4\right) \\
&\quad + \left(8\log^3\left(\frac{r_5}{r_0}\right) + 54\log^2\left(\frac{r_5}{r_0}\right) + 144\log\left(\frac{r_5}{r_0}\right) + 132\right)\alpha_p^4 \\
&\quad + O(\alpha_p^5)
\end{aligned} \tag{41}$$

This implies, neglecting order α_p^5 terms:

$$\delta_{4,r_0} = -8\log^3\left(\frac{r_5}{r_0}\right) + 6\log^2\left(\frac{r_5}{r_0}\right) + 6\log\left(\frac{r_5}{r_0}\right) - 42 \tag{42}$$

In terms of the physical coupling α_p the regulated effective coupling α_{r_0} takes the form:

$$
\begin{aligned}
\alpha_{r_0}(r) = \alpha_0 & \\
&+ \left(2\log\left(\frac{r}{r_0}\right) + 2\right)\alpha_0^2 \\
&+ \left(4\log^2\left(\frac{r}{r_0}\right) + 14\log\left(\frac{r}{r_0}\right) + 13\right)\alpha_0^3 \\
&+ \cdots \\
= {}& \alpha_p + \delta_{2,r_0}\alpha_p^2 + \delta_{3,r_0}\alpha_p^3 + \cdots \\
&+ \left(2\log\left(\frac{r}{r_0}\right) + 2\right)\left(\alpha_p + \delta_{2,r_0}\alpha_p^2 + \delta_{3,r_0}\alpha_p^3 + \cdots\right)^2 \\
&+ \left(4\log^2\left(\frac{r}{r_0}\right) + 14\log\left(\frac{r}{r_0}\right) + 13\right)\left(\alpha_p + \delta_{2,r_0}\alpha_p^2 + \delta_{3,r_0}\alpha_p^3 + \cdots\right)^3 \\
&+ \cdots \\
= {}& \alpha_p \\
&+ 2\log\left(\frac{r}{r_5}\right)\alpha_p^2 \\
&+ \left(4\log^2\left(\frac{r}{r_5}\right) + 6\log\left(\frac{r}{r_5}\right)\right)\alpha_p^3 \\
&+ \left(8\log^3\left(\frac{r}{r_5}\right) + 30\log^2\left(\frac{r}{r_5}\right) + 48\log\left(\frac{r}{r_5}\right)\right)\alpha_p^4 \\
&+ O(\alpha_p^5)
\end{aligned}
\tag{43}
$$

The effective coupling α expressed as a power series in the physical coupling α_p is independent of the regulator r_0. Removing the regulator by sending it to zero, $r_0 \to 0$, gives the same, finite expression, called the renormalized effective coupling α.

4 Renormalization using dimensional regularization

In this chapter the integral equation is solved iteratively for f in $d = 3 + \epsilon$ spatial dimensions, yielding the regulated effective coupling α_ϵ as a power series in the bare coupling α_0,

$$\alpha_\epsilon(r) = \alpha_0 f^2(r). \tag{44}$$

Then the renormalization procedure is done like in the case of cutoff regularization. The resulting power series in the physical coupling α_p of the regulated effective coupling α_ϵ depends on the regulator ϵ. The renormalized effective coupling α is obtained by taking the limit $\epsilon \to 0$,

$$\alpha(r) = \lim_{\epsilon \to 0} \alpha_\epsilon(r). \tag{45}$$

4.1 Regularization

In $d = 3 + \epsilon$ spatial dimensions the integral equation is

$$f(r) = 1 + \alpha_0 r^{1-\epsilon} \int_r^\infty \frac{1}{r_2^{2-\epsilon}} \int_0^{r_2} \frac{f^3(r_1)}{r_1^{1-2\epsilon}}\, dr_1\, dr_2. \tag{46}$$

The regulator is removed in the limit of three spatial dimensions, $\epsilon \to 0$.

4.2 Iteration

The integral equation can be solved iteratively,

$$f_n(r) = 1 + \alpha_0 r^{1-\epsilon} \int_r^\infty \frac{1}{r_2^{2-\epsilon}} \int_0^{r_2} \frac{f_{n-1}^3(r_1)}{r_1^{1-2\epsilon}}\, dr_1\, dr_2, \tag{47}$$

starting with

$$f_0(r) = 1. \tag{48}$$

First iteration:

$$\begin{aligned} f_1(r) &= 1 + \alpha_0 r^{1-\epsilon} \int_r^\infty \frac{1}{r_2^{2-\epsilon}} \int_0^{r_2} \frac{1}{r_1^{1-2\epsilon}}\, dr_1\, dr_2 \\ &= 1 + \alpha_0 r^{1-\epsilon} \frac{1}{2\epsilon} \int_r^\infty \frac{1}{r_2^{2-3\epsilon}}\, dr_2 \\ &= 1 + \frac{1}{1-3\epsilon} \frac{r^{2\epsilon}}{2\epsilon} \alpha_0 \end{aligned} \tag{49}$$

Second iteration, keeping only terms of order α_0^2:

$$\begin{aligned} f_2(r) &= 1 + \alpha_0 r^{1-\epsilon} \int_r^\infty \frac{1}{r_2^{2-\epsilon}} \int_0^{r_2} \frac{1}{r_1^{1-2\epsilon}} \left(1 + \frac{1}{1-3\epsilon} \frac{r_1^{2\epsilon}}{2\epsilon} \alpha_0\right)^3 dr_1\, dr_2 \\ &= 1 + \alpha_0 r^{1-\epsilon} \int_r^\infty \frac{1}{r_2^{2-\epsilon}} \int_0^{r_2} \frac{1}{r_1^{1-2\epsilon}} \left(1 + 3\frac{1}{1-3\epsilon} \frac{r_1^{2\epsilon}}{2\epsilon} \alpha_0\right) dr_1\, dr_2 + O(\alpha_0^3) \\ &= 1 + \frac{1}{1-3\epsilon} \frac{r^{2\epsilon}}{2\epsilon} \alpha_0 + 3\frac{1}{1-3\epsilon} \frac{1}{1-5\epsilon} \frac{r^{4\epsilon}}{8\epsilon^2} \alpha_0^2 + O(\alpha_0^3) \end{aligned} \tag{50}$$

Third iteration, keeping only terms of order α_0^3:

$$\begin{aligned} f_3(r) = 1 &+ \frac{1}{1-3\epsilon}\frac{r^{2\epsilon}}{2\epsilon}\alpha_0 + 3\frac{1}{1-3\epsilon}\frac{1}{1-5\epsilon}\frac{r^{4\epsilon}}{8\epsilon^2}\alpha_0^2 \\ &+ 3\left(\frac{1}{(1-3\epsilon)^2} + \frac{3}{2(1-3\epsilon)(1-5\epsilon)}\right)\frac{1}{1-7\epsilon}\frac{r^{6\epsilon}}{24\epsilon^3}\alpha_0^3 + O(\alpha_0^4) \end{aligned} \tag{51}$$

It can be observed that like in the case of cutoff regularization each iteration determines one higher order coefficient.

Terms of the form $1-3\epsilon$, $1-5\epsilon$, ... appear in all of the coefficients. They are close to one for small ϵ but they can't be set to one because these small deviations from one will make a difference after the renormalization when the limit $\epsilon \to 0$ is taken. This is because these small deviations appear in products containing terms like $\frac{1}{\epsilon}$, $\frac{1}{\epsilon^2}$, ... that become large for small ϵ.

The regulated effective coupling α_ϵ as a power series in the bare coupling α_0 is:

$$\begin{aligned} \alpha_\epsilon(r) &= \alpha_0 f^2(r) \\ &= \alpha_0 \\ &\quad + \frac{1}{1-3\epsilon}\frac{r^{2\epsilon}}{\epsilon}\alpha_0^2 \\ &\quad + \frac{2-7\epsilon}{2(1-3\epsilon)^2(1-5\epsilon)}\frac{r^{4\epsilon}}{\epsilon^2}\alpha_0^3 \\ &\quad + \frac{1}{(1-3\epsilon)^2(1-7\epsilon)}\frac{r^{6\epsilon}}{\epsilon^3}\alpha_0^4 \\ &\quad + O(\alpha_0^5) \end{aligned} \tag{52}$$

4.3 Renormalization

Using the renormalization condition

$$\alpha_\epsilon(r_5) = \alpha_p \tag{53}$$

the coefficients δ of the power series of the bare coupling α_0,

$$\alpha_0 = \alpha_p + \delta_{2,\epsilon}\alpha_p^2 + \delta_{3,\epsilon}\alpha_p^3 + \cdots , \tag{54}$$

can be determined. Imposing the renormalization condition at order α_p^2,

$$\begin{aligned} \alpha_p &= \alpha_\epsilon(r_5) \\ &= \alpha_0 + \frac{1}{1-3\epsilon}\frac{r_5^{2\epsilon}}{\epsilon}\alpha_0^2 + O(\alpha_0^3) \\ &= \alpha_p + \delta_{2,\epsilon}\alpha_p^2 + \frac{1}{1-3\epsilon}\frac{r_5^{2\epsilon}}{\epsilon}\alpha_p^2 + O(\alpha_p^3), \end{aligned} \tag{55}$$

implies, neglecting order α_p^3 terms:

$$\delta_{2,\epsilon} = -\frac{1}{1-3\epsilon}\frac{r_5^{2\epsilon}}{\epsilon} \tag{56}$$

Impose the renormalization condition at order α_p^3 to determine $\delta_{3,\epsilon}$:

$$\begin{aligned}
\alpha_p &= \alpha_\epsilon(r_5) \\
&= \alpha_0 + \frac{1}{1-3\epsilon}\frac{r_5^{2\epsilon}}{\epsilon}\alpha_0^2 + \frac{2-7\epsilon}{2(1-3\epsilon)^2(1-5\epsilon)}\frac{r_5^{4\epsilon}}{\epsilon^2}\alpha_0^3 + O(\alpha_0^4) \\
&= \alpha_p + \delta_{2,\epsilon}\alpha_p^2 + \delta_{3,\epsilon}\alpha_p^3 + \frac{1}{1-3\epsilon}\frac{r_5^{2\epsilon}}{\epsilon}\left(\alpha_p^2 + 2\delta_{2,\epsilon}\alpha_p^3\right) \\
&\quad + \frac{2-7\epsilon}{2(1-3\epsilon)^2(1-5\epsilon)}\frac{r_5^{4\epsilon}}{\epsilon^2}\alpha_p^3 + O(\alpha_p^4)
\end{aligned} \tag{57}$$

This implies, neglecting order α_p^4 terms:

$$\delta_{3,\epsilon} = \frac{2-13\epsilon}{2(1-3\epsilon)^2(1-5\epsilon)}\frac{r_5^{4\epsilon}}{\epsilon^2} \tag{58}$$

Imposing the renormalization condition at order α_p^4 one obtains:

$$\delta_{4,\epsilon} = -\frac{2-31\epsilon+135\epsilon^2}{2(1-3\epsilon)^3(1-5\epsilon)(1-7\epsilon)}\frac{r_5^{6\epsilon}}{\epsilon^3} \tag{59}$$

In terms of the physical coupling α_p the regulated effective coupling α_ϵ takes the form:

$$
\begin{aligned}
\alpha_\epsilon(r) &= \alpha_0 \\
&\quad + \frac{1}{1-3\epsilon}\frac{r^{2\epsilon}}{\epsilon}\alpha_0^2 \\
&\quad + \frac{2-7\epsilon}{2(1-3\epsilon)^2(1-5\epsilon)}\frac{r^{4\epsilon}}{\epsilon^2}\alpha_0^3 \\
&\quad + \cdots \\
&= \alpha_p + \delta_{2,\epsilon}\alpha_p^2 + \delta_{3,\epsilon}\alpha_p^3 + \cdots \\
&\quad + \frac{1}{1-3\epsilon}\frac{r^{2\epsilon}}{\epsilon}\left(\alpha_p + \delta_{2,\epsilon}\alpha_p^2 + \delta_{3,\epsilon}\alpha_p^3 + \cdots\right)^2 \\
&\quad + \frac{2-7\epsilon}{2(1-3\epsilon)^2(1-5\epsilon)}\frac{r^{4\epsilon}}{\epsilon^2}\left(\alpha_p + \delta_{2,\epsilon}\alpha_p^2 + \delta_{3,\epsilon}\alpha_p^3 + \cdots\right)^3 \\
&\quad + \cdots \\
&= \alpha_p \\
&\quad + \frac{1}{1-3\epsilon}\frac{r^{2\epsilon} - r_5^{2\epsilon}}{\epsilon}\alpha_p^2 \\
&\quad + \frac{\left(r^{2\epsilon} - r_5^{2\epsilon}\right)\left((2-7\epsilon)r^{2\epsilon} - (2-13\epsilon)r_5^{2\epsilon}\right)}{2(1-3\epsilon)^2(1-5\epsilon)\epsilon^2}\alpha_p^3 \\
&\quad + \Big(2\left(1-8\epsilon+15\epsilon^2\right)r^{4\epsilon} - \left(4-47\epsilon+117\epsilon^2\right)r^{2\epsilon}r_5^{2\epsilon} \\
&\qquad + \left(2-31\epsilon+135\epsilon^2\right)r_5^{4\epsilon}\Big)\frac{\left(r^{2\epsilon} - r_5^{2\epsilon}\right)}{2(1-3\epsilon)^3(1-5\epsilon)(1-7\epsilon)\epsilon^3}\alpha_p^4 \\
&\quad + O(\alpha_p^5)
\end{aligned} \tag{60}
$$

The renormalized effective coupling α is obtained by taking the limit $\epsilon \to 0$:

$$
\begin{aligned}
\alpha(r) &= \lim_{\epsilon\to 0}\alpha_\epsilon(r) \\
&= \alpha_p \\
&\quad + 2\log\left(\frac{r}{r_5}\right)\alpha_p^2 \\
&\quad + \left(4\log^2\left(\frac{r}{r_5}\right) + 6\log\left(\frac{r}{r_5}\right)\right)\alpha_p^3 \\
&\quad + \left(8\log^3\left(\frac{r}{r_5}\right) + 30\log^2\left(\frac{r}{r_5}\right) + 48\log\left(\frac{r}{r_5}\right)\right)\alpha_p^4 \\
&\quad + O(\alpha_p^5)
\end{aligned} \tag{61}
$$

This expression coincides with the one obtained using cutoff regularization. That is:

$$\alpha(r) = \lim_{r_0 \to 0} \alpha_{r_0}(r) = \lim_{\epsilon \to 0} \alpha_\epsilon(r) \tag{62}$$

5 Resummation

In this chapter the renormalization group is explained. The terms of the renormalized effective coupling α are resummed by collecting leading, subleading, ... logarithms. It is explained how from the lowest order expression of the effective coupling α containing only one leading logarithm more terms in the series of leading logarithms can be obtained by enforcing the renormalization group structure. Then the beta function β, which is the logarithmic derivative of the physical coupling α_p with respect to the scale r_5, is introduced. It is then demonstrated how to integrate it to obtain all the leading, subleading and subsubleading logarithms.

5.1 Renormalization group

Instead of a renormalization condition

$$\alpha(r_{5,1}) = \alpha_{p,1} \tag{63}$$

another one can be used:

$$\alpha(r_{5,2}) = \alpha_{p,2} \tag{64}$$

The set of tuples

$$\{(r_{5,1}, \alpha_{p,1}), (r_{5,2}, \alpha_{p,2}), \ldots\} \tag{65}$$

of scales r_5 and associated physical couplings α_p constitutes the representation space of a group: The renormalization group. The group operation f, a representation of the additive group of the reals [30], relates different physical couplings α_p corresponding to different scales r_5:

$$(r_{5,2}, \alpha_{p,2}) = f_{r_{5,2} - r_{5,1}}((r_{5,1}, \alpha_{p,1})) \tag{66}$$

The group relation

$$f_{\Delta r_{5,2}}\left(f_{\Delta r_{5,1}}\left((r_5, \alpha_p)\right)\right) = f_{\Delta r_{5,1} + \Delta r_{5,2}}\left((r_5, \alpha_p)\right) \tag{67}$$

means that going from one scale r_5 to a second scale $r_5 + \Delta r_{5,1}$ and from there to a third scale $r_5 + \Delta r_{5,1} + \Delta r_{5,2}$ is the same as going from the first to the third scale directly. By enforcing this group relation some terms in higher order coefficients of the perturbative expansion of the effective coupling α can be obtained.

5.2 Resummation

The expression for the renormalized effective coupling α,

$$\begin{aligned}\alpha(r) = {}& \alpha_p \\ & + 2\log\left(\frac{r}{r_5}\right)\alpha_p^2 \\ & + \left(4\log^2\left(\frac{r}{r_5}\right) + 6\log\left(\frac{r}{r_5}\right)\right)\alpha_p^3 \\ & + \left(8\log^3\left(\frac{r}{r_5}\right) + 30\log^2\left(\frac{r}{r_5}\right) + 48\log\left(\frac{r}{r_5}\right)\right)\alpha_p^4 \\ & + \left(16\log^4\left(\frac{r}{r_5}\right) + 104\log^3\left(\frac{r}{r_5}\right) + 342\log^2\left(\frac{r}{r_5}\right) + 570\log\left(\frac{r}{r_5}\right)\right)\alpha_p^5 \\ & + \cdots \end{aligned} \tag{68}$$

can be rearranged by grouping together leading logarithms, that is, from each coefficient the term with the highest power of a logarithm, subleading logarithms, subsubleading logarithms, and so on:

$$\begin{aligned}\alpha(r) = {}& \left(1 + 2\log\left(\frac{r}{r_5}\right)\alpha_p + 4\log^2\left(\frac{r}{r_5}\right)\alpha_p^2 \right. \\ & \qquad \left. + 8\log^3\left(\frac{r}{r_5}\right)\alpha_p^3 + 16\log^4\left(\frac{r}{r_5}\right)\alpha_p^4 + \cdots\right)\alpha_p \\ & + \left(6\log\left(\frac{r}{r_5}\right)\alpha_p + 30\log^2\left(\frac{r}{r_5}\right)\alpha_p^2 + 104\log^3\left(\frac{r}{r_5}\right)\alpha_p^3 + \cdots\right)\alpha_p^2 \\ & + \left(48\log\left(\frac{r}{r_5}\right)\alpha_p + 342\log^2\left(\frac{r}{r_5}\right)\alpha_p^2 + \cdots\right)\alpha_p^3 \\ & + \left(570\log\left(\frac{r}{r_5}\right)\alpha_p + \cdots\right)\alpha_p^4 \\ & + \cdots \end{aligned} \tag{69}$$

There are, apart from guessing, two ways to determine more terms of the series of leading logarithms

$$1 + 2\log\left(\frac{r}{r_5}\right)\alpha_p + 4\log^2\left(\frac{r}{r_5}\right)\alpha_p^2 + 8\log^3\left(\frac{r}{r_5}\right)\alpha_p^3 + 16\log^4\left(\frac{r}{r_5}\right)\alpha_p^4 + \cdots, \tag{70}$$

subleading logarithms,

$$6 \log \left(\frac{r}{r_5} \right) \alpha_p + 30 \log^2 \left(\frac{r}{r_5} \right) \alpha_p^2 + 104 \log^3 \left(\frac{r}{r_5} \right) \alpha_p^3 + \cdots , \tag{71}$$

and so on. One way consists of determining higher order terms perturbatively and collecting the leading, subleading, ... logarithms. Each additional perturbative order determines one additional term in the series of leading logarithms, one additional term in the series of subleading logarithms and so on.

The other way consists of using the perturbative expression of the physical coupling α to relate different physical couplings $\alpha_{p,1}$ and $\alpha_{p,2}$ corresponding to different scales $r_{5,1}$ and $r_{5,2}$, respectively. The requirement that the operation f of changing the scale satisfies a group law allows determining more terms that have not been present in the perturbative expression of the effective coupling α. In fact, given only the first term in the series of leading logarithms all terms of this series can be determined by enforcing the group law. If the first term of the subleading logarithms is also given, the full series of subleading logarithms can be determined and so on.

5.3 Leading logarithms: Order by order

Assume the effective coupling α has been computed to second order in the physical coupling α_p,

$$\alpha(r) = \alpha_p + 2 \log \left(\frac{r}{r_5} \right) \alpha_p^2 + O(\alpha_p^3), \tag{72}$$

yielding the following relation between physical couplings α_p and $\alpha_{p,1}$, corresponding to scales r_5 and $r_{5,1}$, respectively:

$$\begin{aligned} \alpha_{p,1} &= \alpha(r_{5,1}) \\ &= \alpha_p + 2 \log \left(\frac{r_{5,1}}{r_5} \right) \alpha_p^2 + O(\alpha_p^3) \end{aligned} \tag{73}$$

To this order the group law is satisfied:

$$\begin{aligned}
& f_{\Delta r_{5,2}}\left(f_{\Delta r_{5,1}}\left((r_5,\alpha_p)\right)\right) \\
&= f_{\Delta r_{5,2}}\left(\left(r_5+\Delta r_{5,1}, \alpha_p + 2\log\left(\frac{r_5+\Delta r_{5,1}}{r_5}\right) + O(\alpha_p^3)\right)\right) \\
&= \left(r_5+\Delta r_{5,1}+\Delta r_{5,2}, \alpha_p + 2\log\left(\frac{r_5+\Delta r_{5,1}}{r_5}\right)\alpha_p^2\right. \\
&\qquad \left. +2\log\left(\frac{r_5+\Delta r_{5,1}+\Delta r_{5,2}}{r_5+\Delta r_{5,1}}\right)\left(\alpha_p + 2\log\left(\frac{r_5+\Delta r_{5,1}}{r_5}\right)\alpha_p^2\right)^2 + O(\alpha_p^3)\right) \\
&= \left(r_5+\Delta r_{5,1}+\Delta r_{5,2}, \alpha_p + 2\log\left(\frac{r_5+\Delta r_{5,1}+\Delta r_{5,2}}{r_5}\right) + O(\alpha_p^3)\right) \\
&= f_{\Delta r_{5,1}+\Delta r_{5,2}}\left((r_5,\alpha_p)\right)
\end{aligned} \tag{74}$$

The unknown higher order coefficient k_3 in the expression of the effective coupling α,

$$\alpha(r) = \alpha_p + 2\log\left(\frac{r}{r_5}\right)\alpha_p^2 + k_3\left(\frac{r}{r_5}\right)\alpha_p^3 + O(\alpha_p^4) \tag{75}$$

contains, apart from subleading logarithms, the next term in the series of leading logarithms. It can be determined by enforcing the group law at this order:

$$\begin{aligned}
& f_{\Delta r_{5,2}}\left(f_{\Delta r_{5,1}}\left((r_5,\alpha_p)\right)\right) \\
&= f_{\Delta r_{5,2}}\left(\left(r_5+\Delta r_{5,1}, \alpha_p + 2\log\left(\frac{r_5+\Delta r_{5,1}}{r_5}\right)\alpha_p^2\right.\right. \\
&\qquad \left.\left. +k_3\left(\frac{r_5+\Delta r_{5,1}}{r_5}\right) + O(\alpha_p^4)\right)\right) \\
&= \left(r_5+\Delta r_{5,1}+\Delta r_{5,2}, \alpha_p + 2\log\left(\frac{r_5+\Delta r_{5,1}+\Delta r_{5,2}}{r_5}\right)\alpha_p^2\right. \\
&\qquad + \left(k_3\left(\frac{r_5+\Delta r_{5,1}}{r_5}\right) + k_3\left(\frac{r_5+\Delta r_{5,1}+\Delta r_{5,2}}{r_5+\Delta r_{5,1}}\right)\right. \\
&\qquad \left.\left. +\ 8\log\left(\frac{r_5+\Delta r_{5,1}}{r_5}\right)\log\left(\frac{r_5+\Delta r_{5,1}+\Delta r_{5,2}}{r_5+\Delta r_{5,1}}\right)\right)\alpha_p^3 + O(\alpha_p^4)\right)
\end{aligned} \tag{76}$$

should equal

$$\begin{aligned} & f_{\Delta r_{5,1}+\Delta r_{5,2}}\left((r_5,\alpha_p)\right) \\ =& \left(r_5+\Delta r_{5,1}+\Delta r_{5,2}, \alpha_p + 2\log\left(\frac{r_5+\Delta r_{5,1}+\Delta r_{5,2}}{r_5}\right)\alpha_p^2\right. \\ & \left. + k_3\left(\frac{r_5+\Delta r_{5,1}+\Delta r_{5,2}}{r_5}\right)\alpha_p^3 + O(\alpha_p^4)\right) \end{aligned} \tag{77}$$

Neglecting order α_p^4 terms this implies the following functional equation for the coefficient k_3:

$$k_3\left(\frac{r_{5,2}}{r_5}\right) = k_3\left(\frac{r_{5,1}}{r_5}\right) + k_3\left(\frac{r_{5,2}}{r_{5,1}}\right) + 8\log\left(\frac{r_{5,1}}{r_5}\right)\log\left(\frac{r_{5,2}}{r_{5,1}}\right), \tag{78}$$

where $r_{5,1} = r_5 + \Delta r_{5,1}$ and $r_{5,2} = r_5 + \Delta r_{5,1} + \Delta r_{5,2}$. The solution is [4]

$$k_3(x) = 4\log^2 x + q\log x, \tag{79}$$

where q is an arbitrary constant. Thus the next term in the series of leading logarithms, $4\log^2 x$, has been obtained. More leading logarithms can be obtained by enforcing the group law at higher orders.

5.4 Leading logarithms: The beta function

Instead of enforcing the group law order by order to obtain more terms in the series of leading logarithms the differential relation between physical couplings α_p and $\alpha_{p,1}$ corresponding to nearby scales r_5 and $r_{5,1}$ can be integrated. The integration corresponds to the limit in which the scale r_5 is changed in infinitely many infinitely small steps. In this limit the group law is satisfied automatically to all orders and therefore the result of the integration must contain an expression for all the leading logarithms. The logarithmic derivative of the physical coupling α_p with respect to the scale r_5 is called the beta function β:

$$\beta(\alpha_p) = r_5 \frac{\partial \alpha_p}{\partial r_5} \tag{80}$$

It can be computed from the renormalized effective coupling α as

$$\beta(\alpha_p) = r_5\ \alpha'(r_{5,1})\big|_{r_{5,1}=r_5}. \tag{81}$$

To second order in the physical coupling α_p it is

$$\begin{aligned} \beta(\alpha_p) &= r_5 \frac{\partial}{\partial r_{5,1}}\left(\alpha_p + 2\log\left(\frac{r_{5,1}}{5_5}\right)\alpha_p^2 + O(\alpha_p^3)\right)\bigg|_{r_{5,1}=r_5} \\ &= 2\alpha_p^2 + O(\alpha_p^3). \end{aligned} \tag{82}$$

The differential equation

$$r_5 \frac{\partial \alpha_p}{\partial r_5} = 2\alpha_p^2 \tag{83}$$

can be integrated to relate physical couplings α_p corresponding to different scales r_5:

$$\alpha_{p,2} = \frac{\alpha_{p,1}}{1 - 2\log\left(\frac{r_{5,2}}{r_{5,1}}\right)\alpha_{p,1}} \tag{84}$$

A power series expansion in the physical coupling $\alpha_{p,1}$ shows that this expression contains all the leading logarithms:

$$\alpha_{p,2} = \left(1 + 2\log\left(\frac{r_{5,2}}{r_{5,1}}\right)\alpha_{p,1} + 4\log^2\left(\frac{r_{5,2}}{r_{5,1}}\right)\alpha_{p,1}^2 + \cdots\right)\alpha_{p,1} \tag{85}$$

It exactly satisfies the group law:

$$\begin{aligned}
&f_{\Delta r_{5,2}}\left(f_{\Delta r_{5,1}}\left((r_5, \alpha_p)\right)\right) \\
&= f_{\Delta r_{5,2}}\left(\left(r_5 + \Delta r_{5,1}, \frac{\alpha_p}{1 - 2\log\left(\frac{r_5+\Delta r_{5,1}}{r_5}\right)\alpha_p}\right)\right) \\
&= \left(r_5 + \Delta r_{5,1} + \Delta r_{5,2}, \frac{\frac{\alpha_p}{1-2\log\left(\frac{r_5+\Delta r_{5,1}}{r_5}\right)\alpha_p}}{1 - 2\log\left(\frac{r_5+\Delta r_{5,1}+\Delta r_{5,2}}{r_5+\Delta r_{5,1}}\right)\frac{\alpha_p}{1-2\log\left(\frac{r_5+\Delta r_{5,1}}{r_5}\right)\alpha_p}}\right) \\
&= \left(r_5 + \Delta r_{5,1} + \Delta r_{5,2}, \frac{\alpha_p}{1 - 2\log\left(\frac{r_5+\Delta r_{5,1}+\Delta r_{5,2}}{r_5}\right)\alpha_p}\right) \\
&= f_{\Delta r_{5,1}+\Delta r_{5,2}}\left((r_5, \alpha_p)\right)
\end{aligned} \tag{86}$$

5.5 Subleading logarithms

All the leading and subleading logarithms can be obtained by integrating [4, 11] the third order beta function,

$$\beta(\alpha_p) = 2\alpha_p^2 + 6\alpha_p^3, \tag{87}$$

that is, by solving the differential equation

$$r_5 \frac{\partial \alpha_p}{\partial r_5} = 2\alpha_p^2 + 6\alpha_p^3. \tag{88}$$

After rearranging terms,

$$\frac{\frac{\partial \alpha_p}{\partial r_5}}{2\alpha_p^2 + 6\alpha_p^3} = \frac{1}{r_5}, \tag{89}$$

both sides can be integrated:

$$\int_{r_{5,1}}^{r_{5,2}} \frac{\frac{\partial \alpha_p}{\partial r_5}}{2\alpha_p^2 + 6\alpha_p^3}\, dr_5 = \int_{r_{5,1}}^{5_{5,2}} \frac{1}{r_5}\, dr_5 \tag{90}$$

Changing integration variables,

$$\int_{\alpha_{p,1}}^{\alpha_{p,2}} \frac{1}{2\alpha_p^2 + 6\alpha_p^3}\, d\alpha_p = \log\left(\frac{r_{5,2}}{r_{5,1}}\right), \tag{91}$$

and integrating leads to the following transcendental equation:

$$\frac{1}{2}\left(\frac{1}{\alpha_{p,1}} - \frac{1}{\alpha_{p,2}} + 3\log\left(\frac{\alpha_{p,1}(1+3\alpha_{p,2})}{\alpha_{p,2}(1+3\alpha_{p,1})}\right)\right) = \log\left(\frac{r_{5,2}}{r_{5,1}}\right) \tag{92}$$

This equation can be solved approximately for the physical coupling $\alpha_{p,2}$ by replacing the instances of $\alpha_{p,2}$ appearing in the logarithm with the solution obtained from the second order beta function β:

$$\begin{aligned}
\alpha_{p,2} &= \frac{\alpha_{p,1}}{1 - 2\log\left(\frac{r_{5,2}}{r_{5,1}}\right)\alpha_{p,1} + 3\log\left(\frac{\alpha_{p,1}(1+3\alpha_{p,2})}{\alpha_{p,2}(1+3\alpha_{p,1})}\right)\alpha_{p,1}} \\
&\approx \frac{\alpha_{p,1}}{1 - 2\log\left(\frac{r_{5,2}}{r_{5,1}}\right)\alpha_{p,1} + 3\log\left(1 - \frac{2\log\left(\frac{r_{5,2}}{r_{5,1}}\right)\alpha_{p,1}}{1+3\alpha_{p,1}}\right)\alpha_{p,1}} \\
&\approx \frac{\alpha_{p,1}}{1 - 2\log\left(\frac{r_{5,2}}{r_{5,1}}\right)\alpha_{p,1}} \\
&\quad - 3\log\left(1 - 2\log\left(\frac{r_{5,2}}{r_{5,1}}\right)\alpha_{p,1}\right)\frac{\alpha_{p,1}^2}{\left(1 - 2\log\left(\frac{r_{5,2}}{r_{5,1}}\right)\alpha_{p,1}\right)^2}
\end{aligned} \tag{93}$$

In the last step terms containing subsubleading and lower order logarithms have been neglected. A power series expansion shows that this expression contains exactly the leading and subleading logarithms:

$$\begin{aligned}
\alpha_{p,2} = &\left(1 + 2\log\left(\frac{r_{5,2}}{r_{5,1}}\right)\alpha_{p,1} + 4\log^2\left(\frac{r_{5,2}}{r_{5,1}}\right)\alpha_{p,1}^2 \right. \\
&\qquad \left. + 8\log^2\left(\frac{r_{5,2}}{r_{5,1}}\right)\alpha_{p,1}^3 + \cdots\right)\alpha_{p,1} \\
&+ \left(6\log\left(\frac{r_{5,2}}{r_{5,1}}\right)\alpha_{p,1} + 30\log^2\left(\frac{r_{5,2}}{r_{5,1}}\right)\alpha_{p,1}^2 \right. \\
&\qquad \left. + 104\log^3\left(\frac{r_{5,2}}{r_{5,1}}\right)\alpha_{p,1}^3 + \cdots\right)\alpha_{p,1}^2
\end{aligned} \tag{94}$$

5.6 Subsubleading logarithms

The fourth order beta function β is:

$$\beta(\alpha_p) = 2\alpha_p^2 + 6\alpha_p^3 + 48\alpha_p^4 + O(\alpha_p^5) \tag{95}$$

Integrating

$$\int_{\alpha_{p,1}}^{\alpha_{p,2}} \frac{1}{2\alpha_p^2 + 6\alpha_p^3 + 48\alpha_p^4} d\alpha_p = \log\left(\frac{r_{5,2}}{r_{5,1}}\right) \tag{96}$$

leads to the following transcendental equation:

$$\begin{aligned}
&\frac{1}{2}\left(\frac{1}{\alpha_{p,1}} - \frac{1}{\alpha_{p,2}} - 3\log\left(\frac{\alpha_{p,2}}{\alpha_{p,1}}\right) + \frac{3}{2}\log\left(\frac{1+3\alpha_{p,2}+24\alpha_{p,2}^2}{1+3\alpha_{p,1}+24\alpha_{p,1}^2}\right)\right.\\
&\left.- 13\sqrt{\frac{3}{29}}\left(\tan^{-1}\left(\sqrt{\frac{3}{29}}(1+16\alpha_{p,2})\right) - \tan^{-1}\left(\sqrt{\frac{3}{29}}(1+16\alpha_{p,1})\right)\right)\right)\\
&= \log\left(\frac{r_{5,2}}{5_{5,1}}\right)
\end{aligned} \tag{97}$$

It can be solved approximately for the physical coupling $\alpha_{p,2}$ by first expanding in powers of $\frac{\alpha_{p,2}}{1-2\log\left(\frac{r_{5,2}}{r_{5,1}}\right)}$ to third order, replacing $\alpha_{p,2}$ with the solution obtained from the third order beta function β whenever it appears in a logarithm or an inverse arc tangent and finally by partially expanding in $\alpha_{p,1}$ in such a way that products containing $\log\left(\frac{r_{5,2}}{r_{5,1}}\right)$ are preserved:

$$\begin{aligned}
\alpha_{p,2} \approx\ & \frac{\alpha_{p,1}}{1-2\log\left(\frac{r_{5,2}}{r_{5,1}}\right)\alpha_{p,1}}\\
& - 3\log\left(1-2\log\left(\frac{r_{5,2}}{r_{5,1}}\right)\alpha_{p,1}\right)\frac{\alpha_p^2}{\left(1-2\log\left(\frac{r_{5,2}}{r_{5,1}}\right)\alpha_{p,1}\right)^2}\\
& + \left(9\log^2\left(1-2\log\left(\frac{r_{5,2}}{r_{5,1}}\right)\alpha_{p,1}\right) - 9\log\left(1-2\log\left(\frac{r_{5,2}}{r_{5,1}}\right)\alpha_{p,1}\right)\right.\\
& \left. +30\log\left(\frac{r_{5,2}}{r_{5,1}}\right)\alpha_{p,1}\right)\frac{\alpha_p^3}{\left(1-2\log\left(\frac{r_{5,2}}{r_{5,1}}\right)\alpha_{p,1}\right)^3}
\end{aligned} \tag{98}$$

The series of subsubsubleading logarithms has also been obtained and is given in the appendix. For the lower and lower order logarithms it becomes increasingly difficult to find the right approximations in order to solve the transcendental equation. The first difficulty is to figure out which terms to expand in order not to loose too much accuracy. The second complication is to find compact expressions that exactly contain the series of logarithms and no additional terms.

6 Discussion

In this text renormalization has been treated as a reparametrization procedure that is used when infinite parameters are involved. The interpretation that renormalization is merely a reparametrization is also given in [5]. The existence of the renormalization group structure has been interpreted as a consequence of the fact that every theory given in terms of some set of parameters can be rewritten in terms of any other set of parameters that is related to the original set of parameters by an invertible function. The physical coupling measured at one scale has been treated as one parameter while the physical coupling measured at another scale has been treated as another parameter. Some comments:

In [7] the renormalizability has been checked for the classical massless scalar field theory with quartic self-interaction using cutoff regularization. This has been done by showing that the effective coupling can be made independent of the cutoff by letting the bare coupling become a function of the cutoff. In this text the explicit form of the bare coupling has been determined for both cutoff and dimensional regularization. Plugging their explicit forms into the respective expressions for the regulated effective couplings two different expressions were obtained. While in the case of cutoff regularization the regulator canceled the dimensional regulator did not cancel. After removing the regulator both expressions coincided. This can be seen as a consistency check: The renormalized expression is independent of the regularization scheme.

The authors of [7] have used the renormalization group to resum the cutoff-regulated effective coupling to obtain exact expressions of all the leading, subleading and subsubleading logarithms. These expressions are similar to the ones obtained in this text. The difference is that in this text the *renormalized* effective coupling has been resummed. The similarity of the expressions comes from the fact that in the case of cutoff regularization and with their particular choice of constants of integration the renormalization procedure simply replaces each instance of the cutoff with the scale. For another example of a theory in which renormalization consists of replacing the cutoff with the scale see [4].

The examples of renormalization that deal with the Casimir effect [30], electric fields [3] and the ionization energy of a classical electron [25] have in common that a quantity appearing in an intermediate step of calculation like the energy of vacuum fluctuations of the electromagnetic field, an electric potential or the energy of the electric field around a singularity are infinite. The renormalization procedure consists of regulating these infinite quantities. At the end of the calculations the regulators are removed and finite results are obtained. The difference to renormalization as done in this text is that no parameters have to be re-expressed through physical parameters obtained from experiment. The problems don't contain free parameters. All the parameters that appear are already given and finite.

In quantum field theories there are more renormalization conditions than there are independent parameters. For instance for the massive scalar field theory with cubic self-interaction there are four renormalization conditions [21]: Two for the mass and the coupling and two for the scale and the shift of the field. The former two are taken from experiment and relate the bare mass and the bare coupling to the physical mass and physical coupling, respectively. The latter two determine the scale and the shift of the field. The field is scaled and shifted such that two particular matrix elements of the field have certain values. This comes from the requirement that a creation operator in the interacting theory behaves like a creation operator in the free theory: It creates a one-particle state when it acts on the vacuum. It should not create a superposition containing the vacuum and multi-particle states. This field renormalization is necessary whenever an interaction is introduced into a quantum field theory and not only when there are infinities. This point is emphasized for instance in [26].

Instead of making some bare parameters infinite to fit experiment, they can be written as the sum of a finite and an infinite number. In quantum field theory this is called the method of counter terms. The infinite numbers correspond to counter terms. While the full quantum theory would require this infinite coupling, at tree level experiment dictates that these couplings are finite. When taking into account the infinite loop contributions counter terms are added to cancel these infinities. With each additional loop order new infinities appear and are absorbed by a change in the counter term. The method of counter terms is equivalent to renormalization as done in this text, which is called multiplicative renormalization [11]. In the context of counter term renormalization the scale, that is, the position or momentum at which measurements are done, is also called subtraction point.

When using dimensional regularization to renormalize a quantum field theory then sometimes a dimensionful parameter is introduced [23]. This is done to give the action, which is a spacetime integral over the Lagrangian density, the dimension of Planck's constant. In this text this dimensionful parameter has not been introduced because the renormalization procedure would cancel it. The result for the renormalized effective coupling would be the same.

In [7] dimensional transmutation has been demonstrated by constructing a dimensionful parameter from the cutoff regulated effective coupling and the cutoff. The dimensionful parameter turns out to be independent of the cutoff. In order to construct a dimensionful parameter without referring to any particular regularization scheme the physical coupling α_p and the corresponding scale r_5 can be used, giving the same result.

7 Appendix

7.1 Renormalization using cutoff regularization

Solution f of the cutoff regulated integral equation:

$$
\begin{aligned}
f(r) = 1 & \\
&+ \left(\log\left(\frac{r}{r_0}\right) + 1\right)\alpha_0 \\
&+ \left(\frac{3}{2}\log^2\left(\frac{r}{r_0}\right) + 6\log\left(\frac{r}{r_0}\right) + 6\right)\alpha_0^2 \\
&+ \left(\frac{5}{2}\log^3\left(\frac{r}{r_0}\right) + \frac{39}{2}\log^2\left(\frac{r}{r_0}\right) + 60\log\left(\frac{r}{r_0}\right) + 60\right)\alpha_0^3 \\
&+ \left(\frac{35}{8}\log^4\left(\frac{r}{r_0}\right) + 53\log^3\left(\frac{r}{r_0}\right)\right. \\
&\qquad \left. + \frac{573}{2}\log^2\left(\frac{r}{r_0}\right) + 790\log\left(\frac{r}{r_0}\right) + 790\right)\alpha_0^4 \\
&+ O\left(\alpha_0^5\right)
\end{aligned}
\tag{99}
$$

Regulated effective coupling α_{r_0} as a power series in the bare coupling α_0:

$$
\begin{aligned}
\alpha_{r_0}(r) = \alpha_0 & \\
&+ \left(2\log\left(\frac{r}{r_0}\right) + 2\right)\alpha_0^2 \\
&+ \left(4\log^2\left(\frac{r}{r_0}\right) + 14\log\left(\frac{r}{r_0}\right) + 13\right)\alpha_0^3 \\
&+ \left(8\log^3\left(\frac{r}{r_0}\right) + 54\log^2\left(\frac{r}{r_0}\right) + 144\log\left(\frac{r}{r_0}\right) + 132\right)\alpha_0^4 \\
&+ \left(16\log^4\left(\frac{r}{r_0}\right) + 168\log^3\left(\frac{r}{r_0}\right)\right. \\
&\qquad \left. + 786\log^2\left(\frac{r}{r_0}\right) + 1892\log\left(\frac{r}{r_0}\right) + 1736\right)\alpha_0^5 \\
&+ O\left(\alpha_0^6\right)
\end{aligned}
\tag{100}
$$

Bare coupling α_0 as a power series in the physical coupling α_p:

$$\alpha_0 = \alpha_p + \delta_{2,r_0}\alpha_p^2 + \delta_{3,r_0}\alpha_p^3 + \cdots \tag{101}$$

with

$$\delta_{2,r_0} = -2\log\left(\frac{r_5}{r_0}\right) - 2 \tag{102}$$

$$\delta_{3,r_0} = 4\log^2\left(\frac{r_5}{r_0}\right) + 2\log\left(\frac{r_5}{r_0}\right) - 5 \tag{103}$$

$$\delta_{4,r_0} = -8\log^3\left(\frac{r_5}{r_0}\right) + 6\log^2\left(\frac{r_5}{r_0}\right) + 6\log\left(\frac{r_5}{r_0}\right) - 42 \tag{104}$$

$$\delta_{5,r_0} = 16\log^4\left(\frac{r_5}{r_0}\right) - 40\log^3\left(\frac{r_5}{r_0}\right) + 54\log^2\left(\frac{r_5}{r_0}\right) + 48\log\left(\frac{r_5}{r_0}\right) - 513 \tag{105}$$

Regulated effective coupling α_{r_0} as a power series in the physical coupling α_p:

$$\begin{aligned}
\alpha_{r_0}(r) = \alpha_p \\
&+ 2\log\left(\frac{r}{r_5}\right)\alpha_p^2 \\
&+ \left(4\log^2\left(\frac{r}{r_5}\right) + 6\log\left(\frac{r}{r_5}\right)\right)\alpha_p^3 \\
&+ \left(8\log^3\left(\frac{r}{r_5}\right) + 30\log^2\left(\frac{r}{r_5}\right) + 48\log\left(\frac{r}{r_5}\right)\right)\alpha_p^4 \\
&+ \left(16\log^4\left(\frac{r}{r_5}\right) + 104\log^3\left(\frac{r}{r_5}\right)\right. \\
&\qquad \left. +342\log^2\left(\frac{r}{r_5}\right) + 570\log\left(\frac{r}{r_5}\right)\right)\alpha_p^5 \\
&+ O(\alpha_p^6)
\end{aligned} \tag{106}$$

7.2 Renormalization using dimensional regularization

Solution f of the dimensionally regulated integral equation:

$$\begin{aligned} f(r) = 1 & \\ & + \frac{1}{2(1-3\epsilon)} \frac{r^{2\epsilon}}{\epsilon} \alpha_0 \\ & + \frac{3}{8(1-3\epsilon)(1-5\epsilon)} \frac{r^{4\epsilon}}{\epsilon^2} \alpha_0^2 \\ & + \frac{5-19\epsilon}{16(1-3\epsilon)^2(1-5\epsilon)(1-7\epsilon)} \frac{r^{6\epsilon}}{\epsilon^3} \alpha_0^3 \\ & + \frac{35-306\epsilon+619\epsilon^2}{128(1-3\epsilon)^3(1-5\epsilon)(1-7\epsilon)(1-9\epsilon)} \frac{r^{8\epsilon}}{\epsilon^4} \alpha_0^4 \\ & + O(\alpha_0^5) \end{aligned} \tag{107}$$

Regulated effective coupling α_ϵ as a power series in the bare coupling α_0:

$$\begin{aligned} \alpha_\epsilon(r) = \alpha_0 & \\ & + \frac{1}{1-3\epsilon} \frac{r^{2\epsilon}}{\epsilon} \alpha_0^2 \\ & + \frac{2-7\epsilon}{2(1-3\epsilon)^2(1-5\epsilon)} \frac{r^{4\epsilon}}{\epsilon^2} \alpha_0^3 \\ & + \frac{1}{(1-3\epsilon)^2(1-7\epsilon)} \frac{r^{6\epsilon}}{\epsilon^3} \alpha_0^4 \\ & + \frac{8-126\epsilon+639\epsilon^2-1027\epsilon^3}{8(1-5\epsilon)^2(1-3\epsilon)^3(1-7\epsilon)(1-9\epsilon)} \frac{r^{8\epsilon}}{\epsilon^4} \alpha_0^5 \\ & + O(\alpha_0^6) \end{aligned} \tag{108}$$

Bare coupling α_0 as a power series in the physical coupling α_p:

$$\alpha_0 = \alpha_p + \delta_{2,\epsilon}\alpha_p^2 + \delta_{3,\epsilon}\alpha_p^3 + \cdots \tag{109}$$

with

$$\delta_{2,\epsilon} = -\frac{1}{1-3\epsilon} \frac{r_5^{2\epsilon}}{\epsilon} \tag{110}$$

$$\delta_{3,\epsilon} = \frac{2-13\epsilon}{2(1-3\epsilon)^2(1-5\epsilon)} \frac{r_5^{4\epsilon}}{\epsilon^2} \tag{111}$$

$$\delta_{4,\epsilon} = -\frac{2-31\epsilon+135\epsilon^2}{2(1-3\epsilon)^3(1-5\epsilon)(1-7\epsilon)} \frac{r_5^{6\epsilon}}{\epsilon^3} \tag{112}$$

$$\delta_{5,\epsilon} = \frac{8-254\epsilon+3137\epsilon^2-18060\epsilon^3+39021\epsilon^4}{8(1-5\epsilon)^2(1-3\epsilon)^4(1-7\epsilon)(1-9\epsilon)} \frac{r_5^{8\epsilon}}{\epsilon^4} \tag{113}$$

Regulated effective coupling α_ϵ as a power series in the physical coupling α_p:

$$
\begin{aligned}
\alpha_\epsilon(r) = \alpha_p& \\
&+ \frac{1}{1-3\epsilon}\frac{r^{2\epsilon}-r_5^{2\epsilon}}{\epsilon}\alpha_p^2 \\
&+ \frac{\left(r^{2\epsilon}-r_5^{2\epsilon}\right)\left((2-7\epsilon)r^{2\epsilon}-(2-13\epsilon)r_5^{2\epsilon}\right)}{2(1-3\epsilon)^2(1-5\epsilon)\epsilon^2}\alpha_p^3 \\
&+ \Big(\left(r^{2\epsilon}-r_5^{2\epsilon}\right)\left(2\left(1-8\epsilon+15\epsilon^2\right)r^{4\epsilon}\right. \\
&\quad \left. - \left(4-47\epsilon+117\epsilon^2\right)r^{2\epsilon}r_5^{2\epsilon} + \left(2-31\epsilon+135\epsilon^2\right)r_5^{4\epsilon}\right)\Big) \\
&\quad \times 1/\left(2(1-3\epsilon)^3(1-5\epsilon)(1-7\epsilon)\epsilon^3\right)\alpha_p^4 \\
&+ \Big(\left(8-150\epsilon+1017\epsilon^2-2944\epsilon^3+3081\epsilon^4\right)r^{8\epsilon} \\
&\quad - 32(1-5\epsilon)^2\left(1-12\epsilon+27\epsilon^2\right)r^{6\epsilon}r_5^{2\epsilon} \\
&\quad + 6\left(8-202\epsilon+1849\epsilon^2-7238\epsilon^3+10143\epsilon^4\right)r^{4\epsilon}r_5^{4\epsilon} \\
&\quad - 16\left(2-57\epsilon+609\epsilon^2-2887\epsilon^3+5085\epsilon^4\right)r^{2\epsilon}r_5^{6\epsilon} \\
&\quad + \left(8-254\epsilon+3137\epsilon^2-18060\epsilon^3+39021\epsilon^4\right)r_5^{8\epsilon}\Big) \\
&\quad \times 1/\left(8(1-5\epsilon)^2(1-3\epsilon)^4(1-7\epsilon)(1-9\epsilon)\epsilon^4\right)\alpha_p^5 \\
&+ O(\alpha_p^6)
\end{aligned}
\tag{114}
$$

7.3 Renormalized effective coupling

Renormalized effective coupling α:

$$
\begin{aligned}
\alpha(r) = \alpha_p& \\
&+ 2\log\left(\frac{r}{r_5}\right)\alpha_p^2 \\
&+ \left(4\log^2\left(\frac{r}{r_5}\right) + 6\log\left(\frac{r}{r_5}\right)\right)\alpha_p^3 \\
&+ \left(8\log^3\left(\frac{r}{r_5}\right) + 30\log^2\left(\frac{r}{r_5}\right) + 48\log\left(\frac{r}{r_5}\right)\right)\alpha_p^4 \\
&+ \left(16\log^4\left(\frac{r}{r_5}\right) + 104\log^3\left(\frac{r}{r_5}\right) + 342\log^2\left(\frac{r}{r_5}\right) + 570\log\left(\frac{r}{r_5}\right)\right)\alpha_p^5 \\
&+ O(\alpha_p^6)
\end{aligned}
\tag{115}
$$

7.4 Beta function

Beta function β:

$$\begin{aligned}\beta(\alpha_p) = {} & 2\alpha_p^2 + 6\alpha_p^3 + 48\alpha_p^4 + 570\alpha_p^5 + 8568\alpha_p^6 + 151956\alpha_p^7 \\ & + 3061440\alpha_p^8 + 68474970\alpha_p^9 + 1674962280\alpha_p^{10} \\ & + 44341481052\alpha_p^{11} + 1260798169824\alpha_p^{12} + O(\alpha_p^{13})\end{aligned} \tag{116}$$

7.5 Resummed effective coupling

Resummed renormalized effective coupling α:

$$\begin{aligned}\alpha(r) = {} & \frac{\alpha_p}{1-2\alpha_p x} \\ & - 3\log(1-2\alpha_p x)\frac{\alpha_p^2}{(1-2\alpha_p x)^2} \\ & + \left(9\log^2(1-2\alpha_p x) - 9\log(1-2\alpha_p x) + 30\alpha_p x\right)\frac{\alpha_p^3}{(1-2\alpha_p x)^3} \\ & - \Bigg(27\log^3(1-2\alpha_p x) - \frac{135}{2}\log^2(1-2\alpha_p x) + 336\alpha_p^2 x^2 \\ & \quad + 180\alpha_p x\log(1-2\alpha_p x) + 72\log(1-2\alpha_p x) - 426\alpha_p x\Bigg)\frac{\alpha_p^4}{(1-2\alpha_p x)^4} \\ & + \cdots\end{aligned} \tag{117}$$

with

$$x = \log\left(\frac{r}{r_5}\right) \tag{118}$$

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