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Directed Technical Change and Capital Deepening: A Reconsideration of Kaldor's Technical Progress Function*

Ekkehart Schlicht†

Abstract

This note proposes a growth model that is derived from the standard Solow growth model by replacing the neoclassical production function with Kaldor's technical progress function while maintaining a marginalist theory of factor prices in the spirit suggested by von Weizsäcker (1966, 1966b). The hybrid model so obtained explains balanced growth in a way that appears less arbitrary than possible explanations in the Solow model, especially because it directly accounts for Harrod neutral technical change, without any need for further assumptions. It complements the current neoclassical and AK models by offering a further perspective for interpreting economic growth.

Keywords: directed technical change, directed technological change, bias in innovation, technical progress function, neoclassical production function, Harrod neutrality, Hicks neutrality, Cambridge theory of distribution, marginal productivity theory, Kaldor, Kennedy, von Weizsäcker, Solow model, overdetermination

Journal of Economic Literature Classification: O30, O40, E12, E13, E25, B122, B31, B59

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1 Introduction

Most growth models - whether orthodox (neoclassical, AK) or heterodox (post-Keynesian, neo-Kaleckian, Classical, neo-Marxist) assume that technical change is purely labor augmenting, or "Harrod neutral".[1] For many questions analyzed with the aid of these models, such an assumption is perfectly legitimate, as it is sensible to focus on a certain topic and rule out other problems, such as issues relating to the direction of technical change.

Yet the assumption of Harrod neutrality is critical in the sense that the direction of technical change – whether capital augmenting or capital reducing ("Marx-biased") – will substantially affect most conclusions drawn, unless production technology is described by a Cobb-Douglas production function, where labor augmentation and capital augmentation are indistinguishable. Capital reducing technical change combined with a bounded savings rate could eventually induce stagnation, for instance, and may drive the shares of labor and capital to extremes, which seems hardly compatible with what we have observed in the past. Likewise, capital augmenting technical change may drive the shares of labor and capital to extreme values and may generate explosive rather than exponential growth.

Such problems are avoided by directly assuming from the outset either a Cobb-Douglas production function or Harrod-neutral technical change. Both assumptions refer, however, to highly singular, and, in this sense, utterly improbable cases. Regarding the Cobb-Douglas assumption, justifications like those by Houthakker (1955) or Jones (2005) require quite specific assumptions about the "input-output distribution" or the "distribution of ideas". Such distributional assumptions appear as singular as the direct Cobb-Douglas assumption itself. Further, the Cobb-Douglas assumption has been criticized on empirical grounds, as many studies suggest an elasticity of substitution different from unity. This would rule out the Cobb-Douglas case. (The development of the CES production function by Arrow et al. (1961) was motivated by such empirical findings. Antras (2004) provides up-to-date references and new estimates.)

Regarding Harrod-neutrality, von Weizsäcker (1962/2010, 1966a) and Kennedy (1964) have postulated a mechanism that may produce purely labor augmenting technical change and thereby reduce the arbitrariness of the assumption. They posit that factor prices govern the direction of technical change. The more abundant factor will become cheaper and technical progress will be directed towards increas-

[1] For orthodox growth models, see Aghion and Howitt (2009), for heterodox growth models, see Setterfield (2010).

ing the efficiency of the other, now increasingly scarce, factor. This mechanism has been added to the basic neoclassical model by von Weizsäcker (1962/2010), Samuelson (1965), and Drandakis and Phelps (1966) to rationalize Harrod neutrality. (Kennedy (1964) and von Weizsäcker (1966a) employ a Leontief production function.) The Kennedy-Weizsäcker mechanism has more recently gained renewed attention (Acemoglu 2003b, 2009, Ch. 15).

Another approach is feasible, however, that builds on von Weizsäcker's (1966; 1966b, Ch. iii) critique of Kaldor's (1957) growth theory and has been sketched in a different context in Schlicht (1974, Sect. 2.1). It is obtained by replacing the neoclassical production function in the standard Solow (1956) growth model by Kaldor's (1957) "technical progress function" and by employing a theory of distribution that flows from the assumption that firms select a cost minimizing rate of capital deepening. This theory of distribution, although marginalist in spirit, does not relate to marginal productivities, which do not exist in this construction, nor is it compatible with any Cambridge (post-Keynesian) theory of distribution.

The "hybrid" model that will be outlined in the following combines Kaldor's (1957) technical progress function with a Leontief production function and delivers Harrod-neutral technical change. It criss-crosses neoclassical and Post-Keynesian strands of though and turns von Weizsäcker's (1966; 1966b, Ch. iii) criticism of Kaldor's (1957) growth theory into a positive theory.

As von Weizsäcker's criticism of Kaldor's (1957) growth theory is, it appears to me, unduly disregarded, all this may be of some historical interest. Further, the hybrid model offers several advantages over other more recent approaches:

- The central assumption – that capital deepening induces an increase in labor productivity – appears to me intuitively quite convincing and less arbitrary, or singular, as compared to the alternative assumptions encountered in the literature: Cobb-Douglas, Harrod neutrality, or an invariant innovation possibility function.

- I assume that firms select the rate of capital deepening such as to maximize the decrease in unit costs. In this I follow von Weizsäcker (1962/2010), Kennedy (1964), and other earlier theories. I call this "gradient cost minimization". It permits an analytically very transparent and simple analysis but has been criticized recently as a heuristic theoretical shortcut and, essentially, an arbitrary optimization procedure that does not meet modern analytical standards and was enforced on the economists of the 1960s by their lack of appropriate theoretical tools (Acemoglu 2003a, 464, Jones 2005, 527). Such an assessment needs to be revised, though, as it can be shown that gradient cost minimization is, in equilibrium, equivalent to present value cost minimization, and both minimization procedures entail analogical results outside equilibrium. (This has been suspected by Samuelson (1965, 350) and is proved in Appendix 4.)

- The joint use of a production function and an innovation possibility function, as in the neoclassical versions of the induced technical change literature going back to von Weizsäcker (1962/2010) and Samuelson (1965) involves the problem of empirically separating substitution between labor and capital along the production function from substitution that occurs through a bias in technical change. This problem is absent in the theory to be presented, as no distinction is made between investment in machinery and investment in technology.

- Several authors have suggested that the possibility of substitution between capital and labor may be limited in the short run but could be ample in the long run (Johansen 1959, Foley and Michl 1999, Sect. 7.3, Jones 2003, Jones 2005, Michl 2009, Ch. 10, León-Ledesma and Satchi 2011). The hybrid model features this idea by combining fixed proportions in the short run with the possibility of substitution between capital and labor in the long run, through technical progress.

- Productivity growth, while positively associated with accumulation as in AK models, can also occur without accumulation, as in neoclassical models. The hybrid model offers, therefore, a middle ground between these extremes.

- It is sometimes emphasized that marginal productivity theory implies, together with a neoclassical production function, factor exhaustion: the product is distributed in its entirety to labor and capital and nothing is left to reward investment in technological progress: As a consequence, the neoclassical growth model is sometimes considered unsuited to explain endogenous growth (Nordhaus 1973, 210 Aghion and Howitt 2009, 47). In the hybrid model this problem does not occur because investment in capital and knowledge occur jointly, as in some AK models and also in the earlier neoclassical models of endogenous growth by Conlisk (1967, 1969) and Vogt (1968).

- In addition, the model offers a rather transparent way to highlight the fundamental incompatibility of a cost minimizing assumption with an independent investment function. In neoclassical models the problem is typically sidestepped by assuming that savings determine investment, and this modeling strategy is followed here in the basic model. In post-Keynesian models, the problem is avoided by disregarding that the market mechanism pushes for cost minimization. Yet both assumptions – cost minimization, and an independent investment function – appear mandatory components of any sensible growth theory, which poses a fundamental problem that any growth theory has to face. This "overdetermination" problem will be outlined with regard to the hybrid model and its neoclassical twin in Section 3.4.

As a caveat, let me add that this paper takes just one element of just one of Kaldor's approaches to economic growth and transplants it, as it were, into an alien patch. It is expressly not intended to do justice to Kaldor's broader view, nor to any other approach to economic growth that is mentioned. (For references on Kaldor's contributions to economic growth, see King (2010).)

The paper is organized as follows. The next section outlines the hybrid model. Section 3 discusses some modeling question: how the hybrid model accounts for balanced growth in a more natural way than its neoclassical twin (Sections 3.1 - 3.2); that the concept of capital appropriate for the hybrid model differs from the usual one (Section 3.3); and that the overdetermination problem remains unresolved (Section 3.4). Section 4 provides a conclusion.

2 A Hybrid Model[2]

2.1 The Technical Progress Function

Consider a closed economy with two factors of production, labor N and capital K. Denote output by Y and labor productivity by $y = \frac{Y}{N}$. The development of labor productivity over time depends on the amount of capital employed per worker, denoted by $k = \frac{K}{N}$. The more the capital-labor ratio increases, the more will labor productivity increase, but even without any such capital deepening, labor productivity will increase somewhat. As Kaldor (1957, 596) put it, "*some* increases in productivity would take place even if capital per man remained constant over time, since there are always some innovations – improvements in factory lay-out and organization, for example – which enable production to be increased without additional investment".

[2] I take the term "hybrid model" from Marglin (1987) who used it for a number of different models. The present model may be added to his list.

The "technical progress function" formalizes these ideas. It gives the growth rate of labor productivity as an increasing function of capital deepening. Denoting time derivatives by a dot and growth rates by a hat, the growth rate of labor productivity is $\hat{y} = \frac{\dot{y}}{y} = \frac{1}{y}\frac{dy}{dt}$ and the rate of capital deepening is $\hat{k} = \left(\frac{\hat{K}}{N}\right) = \hat{K} - \hat{N}$. The technical progress function gives $\hat{y}$ as a function of $\hat{k}$:

$$\hat{y} = \varphi\left(\hat{k}\right). \tag{1}$$

For $\hat{k} = 0$ (a constant capital-labor ratio), the increase in labor productivity is positive, and it is increasing in capital deepening, but these increases are subject to diminishing returns. As Kaldor (1957, 596) explains, "there is likely to be some maximum beyond which the rate of growth in productivity could not be raised, however fast capital is being accumulated." Hence the technical progress function "is likely to be convex upwards and flatten out altogether beyond a certain point." These assumptions are formalized for the present purposes as follows:

$$\varphi(0) > 0,\ \varphi' > 0,\ \varphi'' < 0,\ \varphi'(\infty) = 0.$$

The technical progress function is depicted in Figure 1. It embodies the idea that capital accumulation and technical progress occur jointly. The idea has been taken up (and acknowledged) by Arrow (1962). It re-surfaced in some more recent AK theories, often in truncated form, namely that "aggregate productivity depends upon the aggregate capital stock".[3] In contrast, Kaldor assumes that even without capital accumulation, productivity increases over time. This is is known as the "Horndal effect" and appears to be an empirical regularity.[4] In this sense, Kaldor takes an intermediate position between the extremes of purely endogenous and purely exogenous technical progress, as encountered in the modern literature (Aghion and Howitt, 2009, Chs. 1 and 2). Further, the technical progress function is assumed to be convex ($\varphi'' < 0$). If it were linear, it could be integrated and into a Cobb-Douglas production function (Hahn and Matthews, 1964, 849). But the Cobb-Douglas production technology seems to be ruled out by empirical findings (Antras, 2004). So convexity appears to be an economically sensible assumption that has apparently obtained some empirical support (Bairam, 1995). Note, however,

[3] Aghion and Howitt (2009, 53). Basically the AK approach can be characterized by a production function with constant returns to scale of the form $Y = F(AN, K)$ with the efficiency of labor A being a function of K. This rules out Lundberg's (1961) Horndal effect.

[4] See Lundberg (1961), Ohlin (1962), David (1973), Lazonick and Brush (1985), Hendel and Spiegel (2013). This is also to be found in Verdoorn's Law where where labor productivity is held to increase somewhat even if the growth rate of production is zero; see *e.g.* **?**, 167.

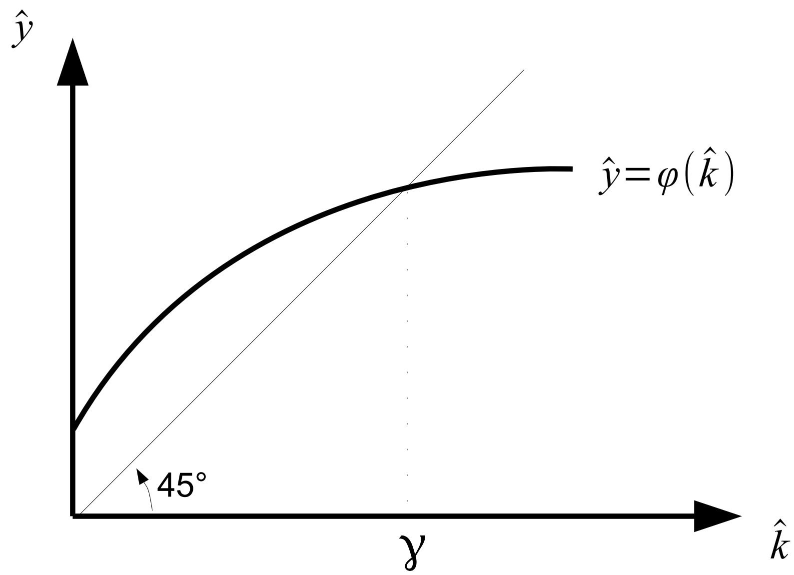

Figure 1: The technical progress function φ gives the increase in labor productivity $\hat{y}$ as a function of capital deepening $\hat{k}$.

that a convex technical progress function cannot be integrated into a neoclassical production function (Hahn and Matthews, 1964, 849). So it should be possible, in principle, to empirically check which view fits the facts better. Unfortunately, and to the best of my knowledge, this has never been tried.

Given labor productivity $y > 0$ and capital productivity $x > 0$, production can now be described by a Leontief production function

$$Y = \min\{yN, xK\} \tag{2}$$

Both productivities, x and y, will vary over time, and the technical progress function can be employed to describe these changes within an otherwise standard growth framework.[5]

Assume that labor grows with a rate $\nu \geq 0$, the savings rate s is constant and positive ($s > 0$), and the rate of depreciation δ is constant and positive as well ($\delta > 0$). Full employment of labor and capital implies $yN = xK = Y$. We start from such a

[5] So the coefficients in the Leontief production function (2) are assumed fixed in the short run, but can vary in the long run. This embodies Johansen's (1959) idea of *ex post* fixed and *ex ante* variable proportions in a non-vintage model.

situation. With a savings rate s, savings are $S = sY$ and the change in the capital stock is savings S minus depreciation δK.

$$\dot{K} = sY - \delta K. \tag{3}$$

Dividing this by K and noting $Y = xK$, yields

$$\hat{K} = sx - \delta.$$

From this we obtain the rate of capital deepening $\hat{k}$ as

$$\hat{k} = sx - \delta - \nu \tag{4}$$

which is the Solow equation, or accumulation equation, encountered in the standard growth model (Solow, 1956, eq. 6). It gives the rate of capital deepening as a function of the output-capital ratio.

By definition, capital productivity x (the output-capital ratio) is $x = \frac{Y}{K} = \frac{y}{k}$ and its growth rate is

$$\hat{x} = \hat{y} - \hat{k}. \tag{5}$$

The technical progress function (1) gives the increase in labor productivity as a function of the rate of capital deepening. Hence the growth of capital productivity can be written as a function of the rate of capital deepening as well:

$$\hat{x} = \varphi\left(\hat{k}\right) - \hat{k}. \tag{6}$$

Since the accumulation equation (4) gives the rate of capital deepening as a function of the output-capital ratio, we obtain finally

$$\hat{x} = \varphi\left(sx - \delta - \nu\right) - \left(sx - \delta - \nu\right). \tag{7}$$

This is a first-order autonomous differential equation that describes the development of capital productivity x over time. It can be analyzed easily.

Without capital deepening capital productivity is $x = \frac{1}{s}(\delta + \nu)$. Hence capital productivity grows at the rate $\hat{x} = \varphi(0)$ which is positive. On the other hand, for a sufficiently high rate of capital deepening, the technical progress function flattens out ($\lim_{\hat{k}\to\infty} \varphi'\left(\hat{k}\right) = 0$). The difference $\varphi\left(\hat{k}\right) - \hat{k}$ is dominated by the second term and becomes negative ($\lim_{\hat{k}\to\infty} \left\{\varphi\left(\hat{k}\right) - \hat{k}\right\} < 0$). In the context of equation (7) this translates into $\lim_{x\to\infty} \{\hat{x}\} < 0$. For continuity reasons there must exist a rate of capital deepening γ, implicitly defined by

$$\varphi\left(\gamma\right) = \gamma, \tag{8}$$

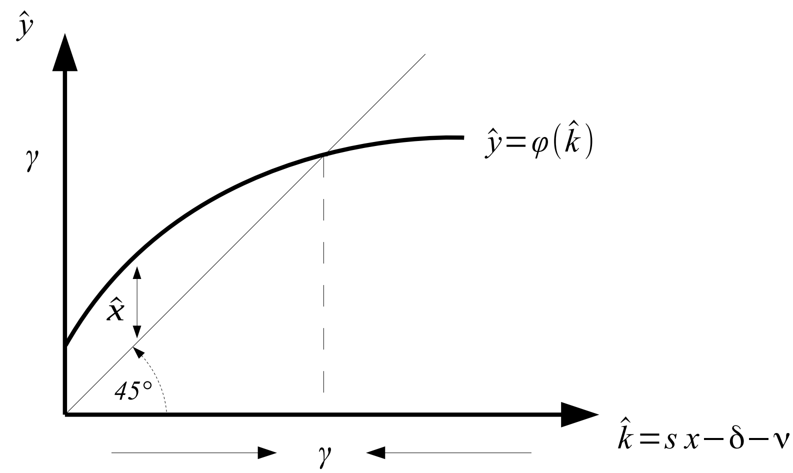

Figure 2: Capital deepening changes over time according to the difference between the change in labor productivity $\hat{y} = \varphi(\hat{k})$ and the rate of capital deepening $\hat{k}$. For values of $\hat{k}$ below γ, $\hat{k}$ increases and for values of $\hat{k}$ above γ, $\hat{k}$ decreases. The equilibrium at $\hat{k} = \gamma$ is stable.

that generates a constant output-capital ratio. As the second derivative $\frac{d^2}{d\gamma^2}\left(\varphi(\gamma) - \gamma\right) = \varphi''$ is negative, the expression $\left(\varphi(\hat{k}) - \hat{k}\right)(k - \gamma)$ is negative definite, and the root is unique.

With a rate of capital deepening of γ, equation (4) implies an output-capital ratio

$$\bar{x} = \frac{1}{s}\left(\gamma + \delta + \nu\right). \tag{9}$$

At this output-capital ratio we have $\hat{x} = 0$; so $\bar{x}$ is an equilibrium (critical point) of our differential equation (7). If the rate of capital deepening is γ, the output-capital ratio is such that the rate of capital deepening is equal to γ; further the output-capital ratio will remain constant at $x = \bar{x}$ over time.

Because $\left(\varphi(sx - \delta - \nu) - (sx - \delta - \nu)\right)(x - \bar{x})$ is negative definite, the equilibrium $\bar{x}$ is globally stable (in the sense of being asymptotically stable). Given any initial value of x, capital productivity will approach this equilibrium value over time. In equilibrium, capital productivity x will remain at $x = \bar{x}$ and labor productivity will increase by $\hat{y} = \gamma$. This is illustrated in Figure 2

2.2 The Direction of Technical Change

It is interesting to discuss the previous analysis within a standard framework, even if this does not do full justice to Kaldor's ideas.

Looking at the production function (2), $\hat{x}$ can be interpreted as the rate of capital augmenting technical change and $\hat{y}$ can be interpreted as the rate of labor augmenting technical change. The difference $\hat{y} - \hat{x}$ is the Hicksian bias in technical progress and $\hat{x}$ is the Harrod bias – it gives the deviation from Harrod neutral technical progress ($\hat{x} = 0$), either capital augmenting ($\hat{x} > 0$) or capital reducing ($\hat{x} < 0$). From (5) it can be seen that the Hicksian bias equals the rate of capital deepening and the Harrod bias is a function of capital deepening. In particular, for $\hat{k} < \gamma$, technical progress is capital augmenting and for $\hat{k} > \gamma$ it is capital reducing.[6] In this sense, the rate of capital deepening determines the direction of technical change.

If we follow Kaldor and assume that the rate of capital deepening is determined by the supply of savings in relation to population growth, the outcome will always tend to Harrod neutral technical change. In this sense, the technical progress function, embedded in a neoclassical framework, offers an alternative mechanism for generating Harrod-neutral technical change.

2.3 Factor Prices and the Choice of Technique

While the equilibrium discussed in Section 1 has been derived without reference to factor prices (the wage rate and the rate of interest), this does not imply that factor prices are irrelevant for equilibrium. Rather, any equilibrium must be compatible with cost minimization, and this implies specific factor prices. A simple way to discuss this in the hybrid model is obtained by importing von Weizsäcker's (1962/2010) and Kennedy's (1964) reasoning about cost minimization (or growth maximization) and assume that a firm that faces a choice between capital widening and capital deepening will try to settle for a combination of both that maximizes the decline in unit costs.[7]

[6] Foley and Michl (1999, Ch. 7) refer to $\hat{x} < 0$ that occurs at $\hat{k} < \gamma$ as "Marx biased" technical change.

[7] This kind of cost minimization may be termed "gradient cost minimization", as opposed to "present value cost minimization," *i.e.* the minimization of the present value of total costs. It has been proposed by Kennedy (1964) and von Weizsäcker (1962/2010) and is employed here mainly because of its simplicity and transparency. It carries some intuitive appeal because competition may be envisaged as a gradient process. Appendix 1 shows that present value cost minimization is equivalent to gradient cost minimization in equilibrium. Outside equilibrium, both forms of cost

The technical progress function implies that the firms have a choice between capital widening and capital deepening, and this will affect their costs. A certain amount of money can be invested in order to increase the number of workplaces while keeping the amount of capital invested in each workplace constant. This would be the case of pure capital widening. The capital-labor ratio would be left unchanged, and technical change would be Hicks-neutral. The other possibility is to invest into the existing workplaces in order to make them more productive. This would amount to capital deepening. Depending on the rate of capital deepening, the direction of technical change may turn out as capital augmenting ($\hat{k} < \gamma$), Harrod-neutral ($\hat{k} = \gamma$), or capital reducing ($\hat{k} > \gamma$). The individual firm faces, thus, a trade-off between the rates of labor and capital augmentation.[8]

Unit costs z are the sum of labor cost and capital user costs per unit. Denote the real wage rate by w, the real rate of interest by r and the rate of capital depreciation by δ. These are taken by the firm as exogenously given. Hence labor costs per unit are $\frac{w}{y}$ and capital user costs per unit are $\frac{r+\delta}{x}$. Unit costs are the sum of these:

$$z = \frac{w}{y} + \frac{r+\delta}{x}. \tag{10}$$

For a constant rate of depreciation, the change of unit costs over time is

$$\dot{z} = \frac{w}{y}\hat{y} + \frac{r+\delta}{x}\hat{x} + \frac{\dot{w}}{y} + \frac{\dot{r}}{x}.$$

In view of equations (1) and (6), the change in unit costs over time is then determined by the rate of capital deepening:

$$\dot{z} = z\varphi\left(\hat{k}\right) + \frac{r+\delta}{x}\hat{k} + \frac{\dot{w}}{y} + \frac{\dot{r}}{x}.$$

The firms take the factor prices, as well as their changes over time, as exogenous and aim to maximize the decline of unit costs over time. This amounts to minimizing

minimization lead to quite similar results in the sense that they induce biases in the same direction, but the bias is more pronounced, or "faster", with gradient cost minimization than with with present value cost minimization.

[8] This trade-off has been formalized in von Weizsäcker's (1962/2010) "new technical progress function" and Kennedy's (1964) "innovation possibility function". Its inverse is used in Appendix 1. Kennedy himself has noted the connection of the innovation possibility function and Kaldor's technical progress function: "Surprisingly enough … our innovation possibility function is really a disguised form of Kaldor's famous technical progress function. … if the technical progress function is known, the innovation possibility function can be derived from it." (Kennedy, 1964, 547n).

the expression $z\varphi\left(\hat{k}\right) + \frac{r+\delta}{x}\hat{k}$ by selecting an appropriate rate of capital deepening $\hat{k}$ and leads to the first-order condition for a minimum

$$\varphi'\left(\hat{k}\right) = \frac{r+\delta}{zx}.$$

The second order condition $z\varphi''\left(\hat{k}\right) < 0$ is satisfied.

With free entry, competition will eliminate pure profits, and unit costs will be equalized to unit price, which is one. Hence we obtain $z = 1$ and the condition

$$\varphi'\left(\hat{k}\right) = \frac{r+\delta}{x} \tag{11}$$

results.[9] The expression $\frac{r+\delta}{x}$ gives the share of capital costs in total costs. Equation (11) determines the optimal rate of capital deepening by the condition that the slope of the technical progress function equals the profit share $\pi = \frac{r+\delta}{x}$. This can be written as

$$\pi = \varphi'\left(\hat{k}\right). \tag{12}$$

This is the condition given by von Weizsäcker (1962/2010, 251) and Kennedy (1964, 544) for an optimal choice of the direction of technical change in a different guise.[10]

We may think that such choices will be made by different firms. As the technical progress function is assumed to be convex ($\varphi'' < 0$), equation (11) tells us that an increase in capital's share will reduce the rate of capital deepening, and an increase in labor's share – the complement to capital's share – will increase the rate of capital deepening selected by each firm. This carries over to the aggregate. In equilibrium, capital's share π is given by the slope of the technical progress function at the equilibrium growth rate γ (Figure 3).

The hybrid growth model can be described by the two equations (7) and (12) which give the system

$$\hat{x} = \varphi(sx - \delta - \nu) - (sx - \delta - \nu) \tag{13}$$

$$\pi = \varphi'(sx - \delta - \nu). \tag{14}$$

[9] This applies also to monopolistic competition with free entry – Chamberlin's (1939) tangency solution. With a positive markup m, unit costs would be $z = \frac{1}{1+m}$ and equation (11) would read $\varphi'\left(\hat{k}\right) = (1+m)\frac{r+\delta}{x}$.

[10] As the trade off between capital augmentation $\hat{x}$ and labor augmentation $\hat{y}$ is $\frac{d\hat{x}}{d\hat{y}} = \frac{\varphi'-1}{\varphi'}$, the optimality condition (11) implies that this trade-off is equal to the ratio of labor's share to capital's share.

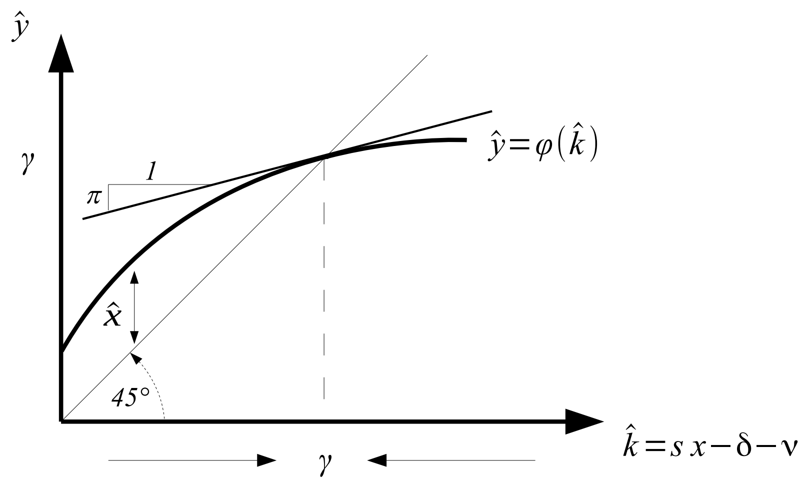

Figure 3: At the stable rate of capital deepening $\hat{k}=\gamma$ the equilibrium profit share π equals the slope of the technical progress function $\varphi'(\gamma)$.

The equation (14) may be further rationalized by considering the following adjustment process. Denote the inverse function of the first derivative of the technical progress function by $\kappa(\cdot)$. This amounts to

$$\varphi'(\kappa(\pi)) = \pi.$$

Hence $\kappa(\pi)$ gives the rate of capital deepening desired by the firms if the profit share is π. As $\varphi''\pi' = 1$ and $\varphi'' < 0$, we have $\kappa' < 0$. The desired rate of capital deepening is a decreasing function of the profit share. If we postulate that a supply of capital deepening $\hat{k}$ in excess of the desired rate of capital deepening $\kappa(\pi)$ entails an excess supply of capital relative to labor, capital costs will decline and the profit share will be reduced, and we arrive at the adjustment equation

$$\dot{\pi} = \mu\left(\hat{k} - \kappa(\pi)\right) \tag{15}$$

for some speed of adjustment $\mu > 0$. As $\frac{\partial\dot{\pi}}{\partial\pi} = -\mu\kappa' < 0$, a sufficiently high speed of adjustment μ guarantees that this adjustment to any time path of $\hat{k}$ is stable.[11]

[11] The function $\phi(x,\pi) = \left(sx-\gamma-\delta-\nu\right)^2$ is a Ljapunov function for (13) and the function $\varphi(x,\pi) = \left(sx-\gamma-\delta-\nu-\kappa(\pi)\right)^2$ is a partial Ljapunov function for (15). Together they satisfy the requirements for the moving equilibrium theorem given in Schlicht (1985, 40). Hence the system (13), (15) is globally asymptotically stable.

3 Discussion

3.1 Kaldor's Stylized Facts

Kaldor's (1957) has listed a number of "stylized facts" about economic growth. These facts provide the starting point for neoclassical growth theory. It has been suggested that any theory of growth should, as a first approximation, account for these "facts" – it should be able account for balanced growth (Durlauf and Johnson 2008, Barro and i Martin 2003, 12-16). Regardless of whether this is considered a sensible modelling requirement or not, it is interesting to note that the hybrid model (13), (14) accounts for these "facts" easily, without the need for additional assumptions:

1. The capital/output ratio remains roughly constant. (Capital productivity x converges to $\bar{x} = \frac{1}{s}\left(\gamma + \delta + \nu\right)$, see (9) and Figure 2.)

2. The profit share remains roughly constant. (As x converges to $\bar{x}$, the profit share converges to $\varphi'\left(\gamma\right)$, see equation (14). This implies also that labor's share $1 - \pi$ remains constant.)

3. The growth of labor productivity remains roughly constant. (It tends to γ, see Figure 2.)

4. The capital-labor ratio grows at a roughly constant rate. (It grows with $s\bar{x} - \delta - \nu = \gamma$, see Figure 2.)

5. The rate of return on investment remains roughly constant over time. (Equations (11) and (9) imply an equilibrium rate of interest $r = \frac{1}{s}\varphi'\left(\gamma\right)\left(\gamma + \delta + \nu\right) - \delta$.)

6. The real wage grows over time. (As labor's share $\frac{w}{y}$ remains constant, the real wage w will grow with the same rate as labor productivity y; both grow with γ.)

Thus the hybrid model presented here actually implies Kaldor's "facts." A further "fact" may be added to Kaldor's list and is implied by the hybrid model:

7. The share of profits is less than 50 per cent. (The technical progress function must cut the 45-degree line from above. Its slope at the intersection gives the profit share π and must be less than .5, see Figure (3).) This proposition is empirically supported (Giovannoni, 2010).

The requirement that a growth model should be able to account for the above stylized facts does not imply, of course, that growth is always balanced. The increase in the profit share observed over the last decades suggest otherwise and requires suitable modifications (Rodriguez and Jayadev 2010, Schneider 2011). In this regard, the hybrid model does not differ from other models that deliver, in their elementary form, balanced growth. It has been suggested, for instance, that a proper distinction between productive capital and financial wealth may help to explain such developments (Vollrath, 2014), but a discussion of such matters outranges the present compass.

3.2 The Neoclassical Twin

Much insight can be gained by abandoning model monism and interpreting actual growth processes from several perspectives, such as the neoclassical or AK. This is nicely done in Aghion and Howitt (2009), for example. The hybrid model offers a third perspective that may complement the others for such purposes.

The differences between the three approaches relate mainly to the modeling of production and technological change, because all three approaches don't differ much with regard to consumer behavior: consumers who want to maximize lifetime utility (or something else) are, in a steady state, basically faced with the same data: an exponential growth of the real wage and a fixed rate of interest. Hence their intertemporal decisions can always be modeled in the same manner. Regarding issues like convergence between different economies, spillovers, and the long-run determinants of growth, these model differ somewhat, but a detailed discussion of these matters goes beyond the scope of the present paper and must be left to future research.

The central theoretical difference between the hybrid model and both the neoclassical models and the AK models concerns the direction of technical change. The problems pose themselves in similar ways in the AK models and in the neoclassical models, but the discussion is better developed for the neoclassical case. For this reason, it is perhaps apposite to illustrate this aspect by juxtaposing the hybrid model and an analogous neoclassical model, its "neoclassical twin". This will be done in the following.

The neoclassical twin of the hybrid model is obtained by replacing the Leontief production function (2) by a neoclassical production function. This production function gives output Y as a smoothly differentiable function of labor input N and capital input K. In order to account for growth, it must be time-dependent: $Y = F(N, K, t)$. Further, $F(\cdot)$ is assumed to be linear homogeneous in N and

K. This permits to define the associated per-capita production function $f(\cdot)$ as $f(k,t) := F(1,k,t)$ which gives per-capita production y as a function of capital intensity k: $y = f(k,t)$. As the output-capital ratio is $x = \frac{y}{k}$, we obtain from (4) the Solow model in its standard form.

$$\dot{k} = sf(k,t) - (\nu + \delta)k. \tag{16}$$

For any given initial capital-labor ratio k_0, equation (16) determines the time paths of the capital-labor ratio k and labor productivity y. Although it appears that factor prices do not enter the model (16), this is not quite correct. In any equilibrium, factor prices must be compatible with cost minimization. Given factor prices w and r, the firms will determine a cost minimizing technique by selecting a capital intensity that minimizes unit costs $\frac{w+(r+\delta)k}{f(k,t)}$. This implies the marginal productivity theory according to which the profit share equals the production elasticity of capital

$$\pi = \frac{f'(k,t)\,k}{f(k,t)}. \tag{17}$$

This corresponds to condition (12) in the hybrid model. Equations (16), (17) define the neoclassical twin of the hybrid model (13), (14). Whereas the hybrid model accounts for Kaldor's stylized facts without ado, this is not true for the neoclassical twin. Indeed, the key dilemma of the neoclassical twin is that it does not imply anything. By postulating a suitable shifting of the production function over time, the model can be made compatible with practically all conceivable developments, including developments that conform to Kaldor's stylized facts. In order to obtain time-paths that conform to those "facts," however, it is necessary to assume a very specific shifting of the production function over time: we need to assume Harrod neutral technical change in the relevant range (Uzawa 1961, Schlicht 2006, Jones and Scrimgeour 2008). The sole justification for this assumption is that it generates time-paths that accommodate Kaldor's facts. By this assumption the model is tweaked to deliver the desired result. The model itself contributes nothing in this regard. Harrod neutrality "is just a special case" (Hahn and Matthews, 1964, 831). As Aghion and Howitt (2009, 28n) put it:

> There is no good reason to think that technological change takes [the Harrod neutral] form; it just leads to tractable steady-state results.

More specifically, the production function must be specified as $F(N,K,t) = \Psi\left(e^{\gamma t}N,K\right)$ which translates for the per-capita production function to $f(k,t) = e^{\gamma t}\psi\left(e^{-\gamma t}k\right)$. The thus adjusted twin model now reads:

$$
\begin{aligned}
\dot{k}_t &= s e^{\gamma t}\psi\left(e^{-\gamma t}k_t\right) - (\nu+\delta)\,k_t && (18)\\
\pi_t &= \frac{\psi'\left(e^{-\gamma t}k_t\right)k_t}{\psi\left(e^{-\gamma t}k_t\right)}. && (19)
\end{aligned}
$$

This adjusted model (18), (19) is the only formal solution that generates results fitting Kaldor's "facts." With any production function $f(\cdot)$ that cannot be written as $e^{\gamma t}\psi\left(e^{-\gamma t}k\right)$, the model is incompatible with these "facts."[12]

It is easy to check that the time path

$$\bar{k}_t = e^{\gamma t}\bar{k}_0$$

with $\bar{k}_0$ as the root of $\psi\left(\bar{k}_0\right) = \frac{1}{s}\left(\nu+\delta+\gamma\right)$ is a solution to (18). This is the balanced growth path. Under the usual assumptions, $\bar{k}_0$ is unique and all solutions k_t of (18) converge to k_t in the sense that the ratio $\frac{k_t}{\bar{k}_t}$ approaches one for $t \to \infty$. [13]

Yet the assumption that technical progress takes the very special form $e^{\gamma t}\psi\left(e^{-\gamma t}k\right)$ appears arbitrary.[14] One way out is to assume right away that the production function is Cobb-Douglas, but this conflicts with empirical evidence (Antras, 2004). Another way out has been proposed by Irmen (2013) who shows that capital-augmenting technical progress can be accommodated with Kaldor's "facts" if adjustment costs of capital grow by a rate that happens to just compensate the bias. However, this assumption appears as special as the straightforward assumption of Harrod neutrality. A third, and perhaps more preferable, way to reduce this arbitrariness has been proposed by von Weizsäcker (1962/2010,1966a) and Kennedy (1964). They assume that that factor prices govern the direction of technical change. The more abundant factor will become cheaper and technical progress will be directed towards increasing the efficiency of the scarce factor. This mechanism has been added to the basic neoclassical model by von Weizsäcker (1962/2010), Samuelson (1965), and Drandakis and Phelps (1966) to rationalize

[12] More precisely: with any other production function, equation (18) violates Kaldor's "facts." The underlying theorem is Uzawa's (1961) steady state theorem. It has originally been proved under the assumption that the marginal productivity theory (19) holds true. Schlicht (2006) has shown that the theorem can be generalized and holds true regardless of the theory of distribution employed. In other words, the necessity of Harrod neutrality persists even if equation (19) that embodies the marginal productivity theory is replaced by something else.

[13] See Appendix A 2.

[14] In the canonical AK model mentioned in footnote 3, the per-capita production would be required to have the form $y = ak$ which appears very arbitrary, too.

Harrod neutrality. (Kennedy (1964) and von Weizsäcker (1966a) employ a Leontief production function.)

The argument is that capital augmenting technical change would make capital increasingly abundant and labor increasingly scarce. Technical change will therefore tend to eventually become Harrod neutral. This argument appears problematic because the assumption of Harrod neutrality is now replaced the "innovation possibility function" that describes the trade off between labor augmenting and capital augmenting technical change. As this trade off at the Harrod-neutral position determines the shares of capital and labor, the trade off is critical but there is again no good reason to assume that this trade-off is roughly stable. Such an assumption would presuppose a knowledge about trade-offs among yet unknown future technologies.

The results "depend on the invariance over time of the innovation possibility functions, an invariance that is ... difficult to swallow" (Kennedy, 1973, 56). It may even be argued that the "induced innovation ... model has let a very restrictive assumption slip in the back door" and that the direct macro assumption of Harrod neutrality is preferable over the trade-off argument because both would appear equally arbitrary, yet the former is more transparent (Nordhaus 1973, 213, Schlicht 2006, n. 1).

Further, the assumption of a neoclassical production function is open to the criticism raised in the capital controversy of the 'sixties. This is a severe shortcoming that has induced some leading proponents of the neoclassical growth model to turn to Austrian capital theory (von Weizsäcker 1971, Hicks 1973b, 1973a), and others to leave the field (Samuelson 1966). The hybrid theory sidesteps this problem. Kennedy (1973, 53) saw this as an advantage of his theory of technical progress (in its multi-sector version):

> ... the theory neatly sidesteps all the difficulties that arise when relative prices alter as a result of changes in the rate of interest, difficulties exemplified by the recent concern about re-switching. Since in real life changes in relative prices are brought about much more significantly by technical progress than by changes in the rate of interest, it is reassuring to have a theory in which the rise in the relative price of a factor leads unequivocally to an economy in its use!

This carries over to the hybrid model.

3.3 The Concept of Capital

One reason for Kaldor to develop the concept of the technical progress function relates to the concept of capital. He argues that it is not useful to separate investment in physical capital from investment in new technologies, because both usually go together:

> … the present model … eschews any distinction between changes in techniques (and in productivity) which are induced by changes in the supply of capital relative to labor and those induced by technical invention or innovation – i.e., the introduction of new knowledge.

As his reason he gives:

> The use of more capital per worker (whether measured in terms of the value of capital at constant prices, in terms of tons of weight of the equipment, mechanical power, etc.) inevitably entails the introduction of superior techniques which require "inventiveness" of some kind, though these need not necessarily represent the application of basically new principles or ideas. On the other hand, most, though not all, technical innovations which are capable of raising the productivity of labor require the use of more capital per man – more elaborate equipment and/or more mechanical power

and he continues:

> It follows that any sharp or clear-cut distinction between movements *along* a "production function" with a given state of knowledge, and a *shift* in the"production function" caused by a change in the state of knowledge is arbitrary and artificial. Hence instead of assuming that some given rate of increase in productivity is attributable to technical progress which is superimposed, so to speak, on the growth of productivity attributable to capital accumulation, we shall postulate a single relationship between the growth of capital and the growth of productivity which incorporates the influence of both factors (Kaldor, 1957, 393f).

As a consequence, the concept of capital must be seen as involving all outlays for investment. The idea is that investment spending is optimally allocated between development of new technology, and the installment of new production facilities.

(Such division has been modeled in the early neoclassical endogenous growth models by Conlisk (1967, 1969) and Vogt (1968).) This view seems to accord with current business practice, as the price paid for a new machine will cover both R&D expenditure and production costs for that product. So our statistical data lump these expenses together. From a practical point of view it appears, thus, reasonable to employ Kaldor's concept of capital instead of drawing a distinction between physical and intellectual capital.

On the other hand, not all forms of wealth accumulation are to be counted as outlays for investment. The issuing of government debt creates financial wealth even if productive investment remains unchanged (Schlicht, 2006, 497, 505). From this point of view, the issuing of government debt should not be counted as inducing capital deepening.

3.4 Criticism: The Missing Investment Function and the Overdetermination Problem

The hybrid model described in this paper has been devised and presented in a neoclassical spirit. It does employ neither a neoclassical production function nor marginal productivity theory, yet it shares the other central shortcoming of the orthodox (neoclassical and AK) models: there is no independent investment function. Rather it is assumed that the consumers' savings decision automatically translate into investment. In a decentralized economy, saving decisions are made by households, however, while investment decisions are made by firms. So these decisions are made independently of each other, and it is necessary to include in any model of a decentralized economy a mechanism that equates savings with investment.

Regarding the neoclassical model, Hahn and Matthews (1964, 790) put the criticism as follows:

> In its basic form the neo-classical model depends on the assumption that it is always possible and consistent with equilibrium that investment should be undertaken of an amount equal to full-employment savings. The mechanism that ensures this is as a rule not specified.

Such a a negligence leads to severe problems regarding logical consistency, both of the hybrid model and its neoclassical twin.[15] In the following I shall simply outline

[15] Heterodox models meet the reverse problem: They usually include an investment function, but neglect the desideratum of cost minimization.

this problem for both models. As the problem remains unsolved, I cannot offer any solution, but it may become apparent that taking the problem seriously might open interesting theoretical prospects.

The problem involved here is that, by adding another equation to a fully specified model, the model becomes "overdetermined" in the sense that it contains more equations than unknowns (Sen 1963, Schlicht 1983, Marglin 1987, 14). One solution is to introduce another variable that can assume a value such that the new equation can be made consistent with the initial model by a suitable adjustment of this variable. In this case, the investment function would be inessential and could simply be dropped. Hahn and Matthews (1964, 790) have described this approach:

> Most neo-classical writers have, however, had in mind some financial mechanism. In the ideal neo-classical world one may think of there being a certain level of the rate of interest (r) that will lead entrepreneurs, weighing interest cost against expected profits, to carry out investment equal to full-employment savings. In the absence of risk, etc., the equilibrium rate of interest would equal the rate of profit on investment; otherwise the rate of profit will be higher by the requisite risk premium.

While such an argument sounds convincing, it is feasible neither for the hybrid model nor its neoclassical twin.

In the hybrid model, the equilibrium rate of interest is determined by the slope condition $\frac{s(r+\delta)}{(\gamma+\delta+v)} = \varphi'\left(\gamma\right)$ and the equilibrium output capital ratio, see equations (11) and (9). This implies an equilibrium rate of interest

$$r = \frac{1}{s}\varphi'\left(\gamma\right)\left(\gamma+\delta+v\right)-\delta.$$

So there is no room for varying the rate of interest such that the volume of investment is adjusted to savings. To achieve this, two rates of interest would be needed: one to induce the correct choice of technique, the other to induce the correct volume of investment.

In the neoclassical twin the problem is similar. The equilibrium rate of interest that induces a cost minimizing choice of capital intensity in equilibrium is fixed as

$$r = \psi'\left(\bar{k}_0\right)$$

with $\bar{k}_0$ determined as the root of $\psi\left(\bar{k}_0\right) = \frac{1}{s}\left(v+\delta+\gamma\right)$. So there is no room for varying the rate of interest in order to adjust investment to savings here, just as in the hybrid model.

Including a risk premium would not change matters, because the capital costs relevant for the choice of technique will be the same as the capital costs relevant for determining the level of investment: they are simply capital costs, whether with or without a risk premium. From this point of view, solutions like those proposed by Beckmann (1965, eq. 12), von Weizsäcker (1966, eq. 9), or Fischer (1972, eq. 16) appear problematic.

Kaldor was aware of this problem. He thought that the technical progress function would permit getting rid of the over-determination problem by eliminating marginal productivity theory, and many heterodox writers argued in similar ways. Getting rid of marginal productivity theory would permit dropping the equations that determine factor prices (equation (14) in the hybrid model or equation (19) in the neoclassical twin) and thereby make room for the Cambridge theory of factor prices that builds on the equalization of saving and investment. But this position is not tenable, as von Weizsäcker (1966, 1966b) has shown: the coice of technique remains a problem in Kaldor's original model, and heterodox models have to take account of this problem as well.

The classical assumptions about saving and investment would avoid the over-determination problem: if the savings rate is equal to the profit share and all profits are re-invested, savings and investment are always equal, and the problem vanishes. Similarly, if the social planner decides about savings and investment simultaneously, the problem disappears. The problem emerges only with an independent investment function.

Yet an independent investment function seems to be required in order to make the argument that savings and investment are adjusted to each other. The assertion that this happens automatically is appropriate for the classical assumptions about savings and investment, or for the planning solution (the Ramsey-Malinvaud-Cass-Koopmans models), but inappropriate in a monetary economy where saving decisions and investment decisions are made independently of each other by different actors.[16] Many heterodox writers are aware of this issue and introduce, just

[16] The treatment by Blanchard and Fischer (1989, 51) is typical for orthodox texts: "Equations (29, (21), and (22) chracterize the behavior of the decentralized economy. Note that they are identical to equations (8), (2), and (7) which characterze the behavior of the economy as chosen by a central planner. Thus the dynamic behavior of the decentralzed economy will be the same as that of the centrally planned one. Our analysis of dynamics carries over to the decentralized economy." This assumes that the optimal savings decisions of households translate automatically into investment decisions by the firms. Yet this cannot be determined by profit maximization, as a neoclassical production function with constant returns does not permit a unique profit maximum. (Acemoglu, 2009, 32-33).

like Kaldor, a distributive mechanism that equates savings and investment over the business cycle, but, to the best of my knowledge, disregard the aspect of selecting a cost minimizing technology.

The introduction of an independent investment function may lead to interesting prospects, though. To illustrate, consider the case that the equalization of savings and investment requires a rate of interest r_1, and that the proper choice of technique requires a different interest rate $r_2 > r_1$. If monetary policy succeeds to establish the interest rate r_1, the desired rate of capital deepening would be too large. The newly created jobs would be endowed with too much capital, and not enough workplaces can be created with the given amount of investment; unemployment of labor through capital shortage would result. In the converse case $r_1 > r_2$, the rate of capital deepening would be too low, more jobs would be newly created than could be manned, and a labor shortage would result.

Despite these potentially interesting and promising aspects, no systematic theoretical work has taken up these problems as yet and these and related ideas (for instance, the possible role of the business cycle in solving the puzzle as in Schlicht 1983) remain speculation.

4 Conclusion

The present note has been written in order to draw attention to Kaldor's technical progress function and to acknowledge it as a pioneering contribution to endogenous growth that, although largely forgotten, provides an interesting and still relevant alternative to current modeling. The substitution of the neoclassical production function by Kaldor's technical progress function in a standard growth model leads to a hybrid model that accounts for balanced growth without any further assumptions, while the standard growth models need to be tweaked by assumptions in a way that amounts to assuming the result.

The proof of the pie is in the eating, however. The usefulness of the proposed model, as that of others, will be decided by using it for analyzing questions of interest: What determines the shape and position of the technical progress function? How to incorporate human capital formation?[17] How do technological spillovers work in the context of international trade? How does optimal growth look like? What about the knife-edge problem? What about the increase in capital's share over the recent decades? etc. The approach may yield answers that differ somewhat from

[17] On that, see Bergheim (2007).

those obtainable from other models, and this may help to understand the issues better. In any case, and at the current state of analysis, I think we ought refrain from model monism and not insist that one particular model is the correct one and the others are wrong. Rather we should appreciate various different approaches to growth processes in their own right and discuss empirical findings in the light of alternative interpretations. It is hoped that the hybrid model outlined here may broaden our theoretic menu in this regard.

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Appendix 1: Cost Minimization

The model (13), (14) has been derived, mainly for analytical convenience, under the assumption that the choice of capital deepening maximizes the decline in unit costs at each point in time ("gradient cost minimization"). In the following the solution for cost minimization will be provided. It will turn out that gradient cost minimization and full cost minimization are equivalent in a steady state, but differ outside a steady state somewhat. In this sense, the older literature that relies on gradient cost minimization (Kennedy 1964, Samuelson 1965, von Weizsäcker 1966b, Drandakis and Phelps 1966, Conlisk 1967, Vogt 1968) as well as the approach taken in this paper is vindicated.

We start with the problem of minimizing unit costs at some future point in time by selecting an appropriate time-path of capital deepening. The problem has been originally posed (but not solved) by Samuelson (1965, 350) in his version of the Kennedy-Weizsäcker theory. For the hybrid model it can be solved by a straightforward variational argument.

Define the function ϕ that describes the Kennedy-Weizsäcker trade-off between the growth rates of productivities for capital $\hat{x}$ and labor $\hat{y}$:

$$\hat{y} = \phi(\hat{x}). \tag{A 1}$$

This frontier is implied by the identity $\hat{x} = \hat{y} - \hat{k}$ and the technical progress function $\hat{y} = \varphi\left(\hat{k}\right)$ with $\gamma = \varphi\left(\gamma\right)$. The function is implicitly defined by

$$\phi(\hat{x}) = \varphi\left(\phi(\hat{x}) - \hat{x}\right) \tag{A 2}$$

and has the properties

$$\phi(0) = \gamma \tag{A 3}$$

$$\phi' = -\frac{\varphi'}{1-\varphi'} \in (-1, 0) \tag{A 4}$$

$$\phi'' = \frac{\varphi''}{\left(1-\varphi'\right)^3} < 0. \tag{A 5}$$

Consider the problem to minimize unit costs at a future date $T > 0$ when starting with labor productivity y_0 and capital productivity x_0 at time $t = 0$. Wages grow along the steady state path according to

$$w_t = w_0 e^{\gamma t}, \tag{A 6}$$

the rate of interest r remains constant over time, and initial unit costs are one:

$$z_0 = \frac{r+\delta}{x_0} + \frac{w_0}{y_0} = 1. \tag{A 7}$$

The firm wants to minimize unit costs at some point in time $T > 0$ by selecting suitable time-paths of the increases in productivity growth $\hat{x}$ and $\hat{y}$. As these time-paths are constrained by the trade-off (A 1), the problem reduces to selecting just a time path $\hat{x}_t$. This entails the time path $\hat{y}_t = \phi(\hat{x}_t)$ of labor productivity and the time path $\hat{k}_t = \hat{y}_t - \hat{x}_t$ of capital deepening. For ease of notation we denote the change in capital productivity by

$$u_t = \hat{x}_t \tag{A 8}$$

and take this as the control variable that is used to minimize costs at time T.

Lemma. *For any given $T > 0$, an optimal control u_t^* that minimizes unit costs at time T over the set of piecewise continuous controls is a constant control.*

Proof. With control u_t, the productivities at $t = T$ are given by

$$x_T = x_0 e^{\int_0^T u_t dt} \tag{A 9}$$

$$y_T = y_0 e^{\int_0^T \phi(u_t) dt} \tag{A 10}$$

and the implied unit costs at time T are

$$z_T = \frac{r+\delta}{x_0 e^{\int_0^T u_t dt}} + \frac{w_0 e^{\gamma T}}{y_0 e^{\int_0^T \phi(u_t) dt}}. \tag{A 11}$$

Assume that u_t^* is optimal and consider any other possible control. It differs from u_t^* by

$$\triangle_t = u_t - u_t^*. \tag{A 12}$$

We refer to $\triangle_t$ as a *variation*. Consider now the set of controls parametrized by ε:

$$\mathfrak{U} = \left\{ u_t^* + \varepsilon \triangle_t \middle| \varepsilon \in [-1, 1] \right\}. \tag{A 13}$$

This set contains all convex combinations of controls u_t^* and u_t. In particular it contains u_t^* (for $\varepsilon = 0$) and u_t (for $\varepsilon = 1$). Given some control u_t^* and any variation $\triangle_t$, the unit costs resulting from controls taken out of the set $\mathfrak{U}$ are a function of ε:

$$z_T(\varepsilon) = \frac{r+\delta}{x_0} e^{-\int_0^T (u_t^* + \varepsilon \triangle_t) dt} + \frac{w_0}{y_0} e^{-\int_0^T \phi(u_t^* + \varepsilon \triangle_t - \gamma) dt}. \tag{A 14}$$

The first derivative is

$$\begin{aligned} \frac{\partial z_T}{\partial \varepsilon} &= -\frac{r+\delta}{x_0} e^{-\int_0^T (u_t^* + \varepsilon \triangle_t) dt} \int_0^T \triangle_t dt - \frac{w_0}{y_0.} e^{-\int_0^T \phi(u_t^* + \varepsilon \triangle_t - \gamma) dt} \int_0^T \phi' \triangle_t dt \\ &= -\int_0^T \left(\frac{r+\delta}{x_T} + \frac{w_T}{y_T} \phi' \left(u_t^* + \varepsilon \triangle_t \right) \right) \triangle_t dt \end{aligned}$$

The second derivative is strictly positive:

$$\frac{\partial z_T^2}{\partial \varepsilon^2} = -\int_0^T \frac{w_T}{y_T} \phi'' \left(u_t^* + \varepsilon \triangle_t \right) \triangle_t^2 dt > 0$$

A necessary condition for a minimum is that the first derivative of z_t vanishes at $\varepsilon = 0$:

$$\left. \frac{\partial z_T}{\partial \varepsilon} \right|_{\varepsilon=0} = -\int_0^T \left(\frac{r+\delta}{x_T} + \phi' \left(u_t^* \right) \right) \triangle_t dt = 0 \tag{A 15}$$

Consider the possible variation

$$\triangle_t = \frac{r+\delta}{x_T} + \phi' \left(u_t^* \right). \tag{A 16}$$

With this variation, the necessary condition for a minimum (A 15) reads

$$\left. \frac{\partial z_T}{\partial \varepsilon} \right|_{\varepsilon=0} = -\int_0^T \left(\frac{r+\delta}{x_T} + \phi' \left(u_t^* \right) \right)^2 dt = 0.$$

This implies $\phi' \left(u_t^* \right) = -\frac{r+\delta}{x_T}$ for almost all $t \in [0, T]$ and hence that u_t^* is the same for almost all t. Write this as

$$u_t^* = \bar{u} \text{ for almost all } t \in [0, T]. \tag{A 17}$$

□

Proposition. *Denote the initial profit share by* $\pi_0 = \frac{r+\delta}{x_0}$.

If $\pi_0 = \varphi' \left(\gamma \right)$, *the optimal control is* $u_t = \bar{u} = 0$.

If $\pi_0 > \varphi' \left(\gamma \right)$, *the optimal control is* $u_t = \bar{u} > 0$.

If $\pi_0 < \varphi' \left(\gamma \right)$, *the optimal control is* $u_t = \bar{u} < 0$.

For T sufficiently large, the optimal control is arbitrarily close to $u_t = \bar{u} = 0$.

Proof. With

$$\begin{aligned} \frac{r+\delta}{x_0} &= \pi_0 \\ \frac{w_0}{y_0} &= 1 - \pi_0 \end{aligned}$$

and a constant $u_t = \bar{u}$ (as implied by the Lemma), we obtain from (A 11) unit costs at time T as

$$z_T = \pi_0 e^{-\bar{u}T} + (1 - \pi_0)\, e^{-(\phi(\bar{u}) - \gamma) T}. \tag{A 18}$$

This is to be minimized with respect to $\bar{u}$. The derivatives are

$$\frac{\partial z_T}{\partial \bar{u}} = -T\pi_0 e^{-\bar{u}T} - T(1-\pi_0) e^{-(\phi(\bar{u})-\gamma)T} \phi'(\bar{u}) \tag{A 19}$$

$$\begin{aligned}\frac{\partial^2 z_T}{\partial \bar{u}^2} &= T^2 \pi_0 e^{-\bar{u}T} + T^2 (1-\pi_0) e^{-(\phi(\bar{u})-\gamma)T} \left(\phi'(\bar{u})\right)^2 + \\ &\quad -T(1-\pi_0) e^{-(\phi(\bar{u})-\gamma)T} \phi''(\bar{u}).\end{aligned} \tag{A 20}$$

As all terms in (A 20) are strictly positive, any solution $\bar{u}$ to $\frac{\partial z_T}{\partial \bar{u}} = 0$ gives a unique minimum of z_T.

At $\bar{u} = 0$ we obtain

$$\left.\frac{\partial z_T}{\partial \bar{u}}\right|_{\bar{u}=0} = -T\pi_0 - T(1-\pi_0)\phi'(0).$$

This implies

$$\left.\frac{\partial z_T}{\partial \bar{u}}\right|_{\bar{u}=0} \gtreqless 0 \Leftrightarrow -\phi'(0) \gtreqless \frac{\pi_0}{1-\pi_0}$$

and implies for the cost-minimizing solution $\bar{u}^*$

$$\bar{u}^* \gtreqless 0 \Leftrightarrow -\phi'(0) \lesseqgtr \frac{\pi_0}{1-\pi_0}.$$

As $-\phi' = \frac{\varphi'}{1-\varphi'}$, this can be expressed in terms the technical progress function as

$$\bar{u}^* \gtreqless 0 \Leftrightarrow \varphi'(\gamma) \lesseqgtr \pi_0.$$

If we start with the equilibrium profit share $\pi_0 = \varphi'(\gamma)$, it is optimal to continue with the rate of capital deepening γ. This will keep capital productivity constant and labor productivity growing at the rate γ. If we start with a profit share π_0 that exceeds γ, it is optimal to select a rate of capital deepening less than γ that entails growing capital productivity and a growth in labor productivity less than γ. Conversely an initial profit share $\pi_0 < \varphi'(\gamma)$ would require a rate of capital deepening exceeding γ. All this is qualitatively similar to gradient cost minimization, but the reaction will be much less pronounced.

To see this, consider the first-order condition for the minimizing solution $\bar{u}^*$ more closely. It can be written as

$$\pi_0 e^{(\phi(\bar{u}^*)-\gamma-\bar{u}^*)T} + (1-\pi_0)\phi'(\bar{u}^*) = 0. \tag{A 21}$$

This gives $\bar{u}^*$ implicitly as a function of π_0 and T. The partial derivatives of this function are

$$\begin{aligned}
\frac{\partial \bar{u}^*}{\partial \pi_0} &= -\frac{e^{(\phi(\bar{u}^*)-\gamma-\bar{u}^*)T} - \phi'(\bar{u}^*)}{T\pi_0 e^{(\phi(\bar{u}^*)-\gamma-\bar{u}^*)T}\left(\phi'-1\right) + (1-\pi_0)\,\phi''(\bar{u}^*)} \\
\frac{\partial \bar{u}^*}{\partial T} &= -\frac{\left(\phi(\bar{u}^*)-\gamma-\bar{u}^*\right)e^{(\phi(\bar{u}^*)-\gamma-\bar{u}^*)T}}{T\pi_0 e^{(\phi(\bar{u}^*)-\gamma-\bar{u}^*)T}\left(\phi'-1\right) + (1-\pi_0)\,\phi''(\bar{u}^*)}
\end{aligned}$$

As the denominator in both expressions is strictly negative we have

$$\begin{aligned}
\frac{\partial \bar{u}^*}{\partial \pi_0} &> 0 \\
\frac{\partial \bar{u}^*}{\partial T} &\gtreqless 0 \Leftrightarrow \bar{u}^* \lesseqgtr 0.
\end{aligned}$$

Therefore a higher initial profit share leads to a higher increase in capital productivity. This goes along with smaller rate of capital deepening. Conversely a smaller share of profits leads to a higher rate of capital deepening. The larger the planning horizon T, the less pronounced will be this reaction.

From (A 21) we see further that for $T \to \infty$, the expression $\left(\phi(\bar{u}^*)-\gamma-\bar{u}^*\right)$ must go to zero, because $\left(\phi(\bar{u}^*)-\gamma-\bar{u}^*\right)T$ must remain bounded and we conclude that the optimal control $\bar{u}^*$ must go to zero:

$$\lim_{T\to\infty} \bar{u}^* = 0.$$

In other words: If the firm wants to minimize costs in the very distant future, it will select a rate of capital deepening very close to the equilibrium rate γ. □

Hence gradient cost minimization used in Section 2.3 is only optimal in the steady state. Outside the steady state it is optimal to react to differences between the profit share and the slope of the technical progress function $\varphi'(\gamma)$ in a less pronounced, but qualitatively similar way. This qualitative result carries over to the minimization of the present value of total costs, as this involves minimization of a weighted average of future costs.

Appendix 2: Convergence in the Neoclassical Twin

In the following, the relative convergence of different solutions to the differential equation (18) describing the neoclassical twin is shown. The function f is assumed to satisfy the Inada conditions, and ψ inherits them: $\psi(0) = 0$, $\psi' > 0$, $\psi'' < 0$, $\psi'(0) = \infty$, $\psi'(0) = \infty$, $\psi'(\infty) = 0$. This implies that for $\frac{1}{s}(\nu + \delta + \gamma) > 0$ the equation

$$\psi(\bar{k}_0) = \frac{1}{s}(\nu + \delta + \gamma)\bar{k}_0 \tag{A 22}$$

has a positive root $\bar{k}_0$ that is unique, and that $\frac{\psi(\xi)}{\xi}$ is a decreasing function of ξ. Hence the expression$(\log\xi - \log\bar{k}_0)\left(\frac{\psi(\xi)}{\xi} - \frac{\psi(\bar{k}_0)}{\bar{k}_0}\right)$ is negative definite:

$$(\log\xi - \log\bar{k}_0)\left(\frac{\psi(\xi)}{\xi} - \frac{\psi(\bar{k}_0)}{\bar{k}_0}\right) < 0 \text{ for all } \xi > 0 \text{ with } \xi \neq \bar{k}_0. \tag{A 23}$$

It is easy to check that the time-path

$$\bar{k}_t = e^{\gamma t}\bar{k}_0 \tag{A 24}$$

satisfies (18).

Define

$$\xi_t = e^{-\gamma t}k_t. \tag{A 25}$$

Equations (18) and (A 22) imply

$$\dot{\xi}_t = s\psi(\xi_t) - (\nu + \delta + \gamma)\xi_t. \tag{A 26}$$

Consider now the relative distance between any solution k_t of (18) and $\bar{k}_t$:

$$V_t = (\log k_t - \log\bar{k}_t)^2.$$

As $k_t = e^{\gamma t}\xi_t$ and $\bar{k}_t = e^{\gamma t}\bar{k}_0$, this is identical to

$$V_t = (\log\xi_t - \log\bar{k}_0)^2.$$

The time derivative of this distance is

$$\begin{aligned} \dot{V}_t &= 2(\log\xi_t - \log\bar{k}_0)\hat{\xi}_t \\ &= 2s(\log\xi_t - \log\bar{k}_0)\left(\frac{\psi(\xi_t)}{\xi_t} - \frac{(\nu + \delta + \gamma)}{s}\right). \end{aligned}$$

Equation (A 22) implies $\frac{(\nu+\delta+\gamma)}{s} = \psi(\bar{k}_0)$ and we can write

$$\dot{V}_t = 2s(\log\xi_t - \log\bar{k}_0)\left(\frac{\psi(\xi_t)}{\xi_t} - \frac{\psi(\bar{k}_0)}{\bar{k}_0}\right)$$

which is negative whenever ξ_t differs from $\bar{k}_0$, see (A 23). Hence all solutions of (18) converge in the sense that the ratio of two solutions k'_t and k''_t approach unity. (This does not imply that the distance between such solutions shrinks over time.)