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Regression models with functional responses and covariates constitute a powerful and increasingly important model class. However, regression with functional data poses well known and challenging problems of non-identifiability. This non-identifiability can manifest itself in arbitrarily large errors for coefficient surface estimates despite accurate predictions of the responses, thus invalidating substantial interpretations of the fitted models. We offer an accessible rephrasing of these identifiability issues in realistic applications of penalized linear function-on-function-regression and delimit the set of circumstances under which they are likely to occur in practice. Specifically, non-identifiability that persists under smoothness assumptions on the coefficient surface can occur if the functional covariate’s empirical covariance has a kernel which overlaps that of the roughness penalty of the spline estimator. Extensive simulation studies validate the theoretical insights, explore the extent of the problem and allow us to evaluate their practical consequences under varying assumptions about the data generating processes. Based on theoretical considerations and our empirical evaluation, we provide immediately applicable diagnostics for lack of identifiability and give recommendations for avoiding estimation artifacts in practice.

1 Introduction

The last two decades have seen rapid progress in regression methodology for high-dimensional data, largely driven by applications to genomic data in the “small n , large p ” paradigm. In regression models with functional predictors, similar problems arise from the fact that covariate information comes in the shape of high-dimensional, strongly auto-correlated vectors of function evaluations. Whenever the number of regression parameters to estimate exceeds the number of observations, estimates are not unique and the resulting model is not identifiable. To overcome this lack of identifiability, it then becomes necessary to use heuristics or prior knowledge to impose additional structural constraints like sparsity or smoothness on the estimators. Results then inherently depend – at least to some degree – on the assumptions underlying the chosen regularization method. In the following, we present a detailed analysis of the way in which smoothness assumptions combined with properties of the data generating process affect estimation results for function-on-function regression. As software capable of fitting increasingly complex models with functional data becomes available (e.g. `fda`, Ramsay et al. (2014); `refund`, Crainiceanu et al. (2014); `PACE`, Yang et al. (2012), `WFMM` Herrick (2013)), investigating the practical relevance of identifiability issues arising in these models is both timely and important in this rapidly developing field. The present work aims to perform such an investigation for the model class described in Scheipl et al. (2014) and implemented in `refund`’s `pffr` function, while results carry over to other penalized function-on-function regression approaches such as those implemented in the `fda` package.

A popular approach in regression for functional data restricts the functional coefficients to lie in the span of the first $K < n$ estimated eigenfunctions of a functional covariate’s covariance operator with the largest eigenvalues, (see Cardot et al., 1999, 2003; Yao et al., 2005; Reiss and Ogden, 2007; Yao and Müller, 2010; Wu et al., 2010, for example). This functional principal component regression (FPCR) approach solves the problem of over-parameterization (i.e., non-identifiability of the functional effect) by a – usually drastic – dimension reduction. The main challenges in this approach then become 1) achieving good estimates of the covariance’s eigenfunctions (“functional principal components” or FPCs), eigenvalues, and FPC scores from observed functional data and 2) choosing the regularization parameter K . In practice, the effect of the functional covariate is estimated by using the first K estimated FPC scores as synthetic covariates. However, the critical assumption that the true coefficient lies in the span of the first few empirical eigenfunctions of a suitable (cross-)covariance operator estimate is impossible to verify empirically. Due to the often wiggly and unsmooth nature of eigenfunctions of real data this assumption can also lead to estimates that are difficult to interpret or implausible to practitioners.

An alternative approach is to make assumptions on the functional coefficients informed by insights into the problem at hand, e.g., sparsity or smoothness of functional coefficients, and to estimate these functional coefficients subject to an appropriate penalty (e.g., LASSO or smoothness penalties). In this work, we will focus on smooth spline-based penalized regression models for functional responses with functional covariates as described in Ivanescu et al. (2015) and Scheipl et al. (2014), which constitute a powerful and flexible model class able to deal with the wealth of functional data increasingly collected in many fields of science. Nevertheless, our considerations carry over to other approaches to estimate smooth coefficient functions, such as approaches using derivative-based penalties as advocated by Ramsay and Silverman (2005).

Identifiability issues in functional regression have previously been discussed in Cardot et al. (2003) in the context of functional regression models for scalar responses and also, briefly, in the context of models with both functional responses and functional predictors by He et al. (2000), Chiou et al. (2004) and Prchal and Sarda (2007). While results therein provide conditions for the theoretical existence and unicity of solutions based on functional analysis arguments, they do not yield empirically verifiable criteria to determine whether the conditions for unicity are violated for a given data set. They also always assume that the true coefficient surface lies in the space spanned by the eigenfunctions of a (cross-)covariance operator. As far as we are aware, case studies in the previous literature have implicitly assumed that this assumption and the necessary conditions based on it will be satisfied for observed data. This is problematic since 1) the theoretical conditions found in the previous literature are impossible to satisfy, or at least verify, on finite samples of functional data in finite resolution, and 2) our experience with applications of functional regression models as well as results from simulation studies indicate that non-identifiability leading to spurious coefficient estimates may occur quite frequently. This is obviously a concern for applied statisticians desiring *interpretable* regression models associating functional covariates and (functional) responses.

Instead of relying on the functional analysis arguments suitable for investigations of asymptotic properties of the theoretical model, we use simple linear algebra to derive a condition for unicity of coefficient surface estimates in realistic, finite sample data settings in which functional covariates are observed with finite resolution. This allows us to give a necessary and sufficient condition for non-identifiability in penalized function-on-function regression that is empirically verifiable and thus applicable in realistic problems. The criterion is based on the amount of overlap between the kernels of the model’s penalty matrix and the model’s design matrix. Simulation studies indicate that, in practice, severe errors due to non-identifiability are strongly associated with our criterion; thus, this criterion is the first one that can be used in a wide variety of applications to assess identifiability

or lack thereof. Our analyses also indicate that many widely used preprocessing techniques for functional data which replace observed curves with spline-based or FPC-based low-rank approximations (i.e., pre-smoothing) will considerably increase the likelihood of identifiability issues in many settings.

Section 2 defines the model and data structure under discussion. We present an accessible rephrasing of the fundamental issue (Section 3) and follow up with an analysis of the scope of the problem based on simulated data in Section 4. Section 3.2 also presents necessary and sufficient conditions for settings in which $\beta(s, t)$ is or is not identifiable. The main conclusions we draw from our analysis are that the complexity of observed functional covariates puts hard limits on the identifiability of coefficient surfaces in a number of ways and that pre-processing of functional covariates may exacerbate identifiability issues.

2 Model and Data Structure

In the following, bold symbols denote vectors and matrices, and calligraphic letters denote function domains, function spaces, or sets of functions or vectors.

Define a simple function-on-function regression model as

$$Y_i(t) = \int_{\mathcal{S}} X_i(s)\beta(s, t)ds + \varepsilon_{it}, \quad \varepsilon_{it} \stackrel{\text{i.i.d.}}{\sim} N(0, \sigma^2), \quad (1)$$

where $Y_i(t)$ and $X_i(s)$, $i = 1, \dots, N$, are functional responses and covariates on closed intervals \mathcal{T} and \mathcal{S} in \mathbb{R} , respectively, and assume that they are realizations of zero-mean square integrable stochastic processes $Y(t) \in L^2[\mathcal{T}]$ and $X(s) \in L^2[\mathcal{S}]$ with continuous covariance functions, respectively. To simplify notation and exposition but without loss of generality, we assume that $E(Y_i(t)) \equiv 0$ and $E(X_i(s)) \equiv 0$ and that errors ε_{it} are independent and identically distributed Gaussian white noise and uncorrelated with $X_i(s)$.

Due to the assumptions on the functional covariates, they can be represented by a Karhunen-Loève expansion

$$X_i(s) = \sum_{m=1}^{\infty} \xi_{im}\phi_m(s), \quad (2)$$

with orthonormal $\phi_m(s)$, $\int_{\mathcal{S}} \phi_m(s)\phi_{m'}(s)ds = \delta_{mm'}$, and uncorrelated zero-mean FPC scores ξ_{im} with variances $\nu_1 \geq \nu_2 \geq \dots \geq 0$, $m \in \mathbb{N}$. The ν_m and $\phi_m(s)$, $m = 1, \dots, M$, are the ordered eigenvalues and eigenfunctions of the covariance operator K^X of $X(s)$, respectively, with the covariance function given by Mercer's theorem as

$$k^X(s, s') = E(X(s)X(s')) = \sum_{m=1}^{\infty} \nu_m\phi_m(s)\phi_m(s'). \quad (3)$$

Since estimating $\beta(s, t)$ is an inverse problem, some kind of regularization is required. Functional principal component based approaches like, for example, Yao et al. (2005) restrict $\beta(s, t)$ to lie in the span of the first K estimated eigenfunctions $\hat{\phi}_m(s)$, $m = 1, \dots, K$ for all t . The number of eigenfunctions K that is used serves as the (discrete) regularization parameter. In contrast, we will discuss and analyze a penalized approach. The underlying assumption is that $\beta(s, t)$ is a smooth function that can be well represented as a linear combination of suitable basis functions defined on $\mathcal{S} \times \mathcal{T}$.

In practice, functional responses $Y_i(t)$ and functional covariates $X_i(s)$ are observed on gridpoints $\mathbf{s}_i = (s_{i1}, \dots, s_{iS_i})$ and $\mathbf{t}_i = (t_{i1}, \dots, t_{iT_i})$. For simplicity, we assume those to be identical vectors \mathbf{s}, \mathbf{t} with lengths S and T , respectively, for each observation i . In the following, expressions $a(\mathbf{s})$ or $a(\mathbf{t})$ with a bold argument denote the vector of evaluations of $a(\cdot)$ on the respective grid, e.g., $a(\mathbf{s}) = (a(s_1), \dots, a(s_S))^\top$.

functional covariates are usually observed on a finite number of grid points S and the empirical covariance for N observations thus can have at most $\min(N, S)$ non-zero eigenvalues. As is exploited in functional principal component analysis (e.g. Ramsay and Silverman, 2005), functional observations are often simple enough to be represented accurately by a relatively small number of eigenfunctions, with eigenvalues of higher order small compared to noise or measurement error. It is also wide-spread practice to use pre-smoothed versions of observed functional covariates as inputs for models like (1) (e.g. James, 2002; Ramsay and Silverman, 2005), and these will have a non-empty kernel since they are represented as linear combinations of a limited number of basis functions. Basis function representations of $X(s)$ are also used when sparsely observed functional covariates have to be imputed on a grid of s -values to be used as inputs for model (1) (e.g. Goldsmith et al., 2011).

In the following section, we thus investigate identifiability problems for finite-sample finite-resolution functional data and the interplay between the rank of the observed covariance, the rank of the basis used to represent $\beta(s, t)$ in s -direction and the penalty used in the penalized estimation approach for $\beta(s, t)$ introduced in Section 2.

3.2 Identifiability in practice

Rank-deficient design matrix

In practice, $\beta(s, t)$ is represented as a linear combination of a finite number $K_s K_t$ of basis functions, see (5). For the following, we will assume that the corresponding approximation error is negligible. Combining (4) and (5), we can write the model as

$$\mathbf{Y} = \mathbf{X} \mathbf{W} \mathbf{B}_s \Theta \mathbf{B}_t^T + \boldsymbol{\varepsilon}, \quad (6)$$

$\begin{matrix} N \times T & N \times S & S \times S & S \times K_s & K_s \times K_t & K_t \times T & N \times T \end{matrix}$

where $\mathbf{Y} = [Y_i(t_j)]_{\substack{i=1, \dots, N \\ j=1, \dots, T}}$, $\mathbf{X} = [X_i(s_l)]_{\substack{i=1, \dots, N \\ l=1, \dots, S}}$, $\mathbf{W} = \text{diag}(\mathbf{w})$ and $\boldsymbol{\varepsilon} = [\varepsilon_{it_j}]_{\substack{i=1, \dots, N \\ j=1, \dots, T}}$. Using $\text{vec}(\mathbf{ACB}) = (\mathbf{B}^T \otimes \mathbf{A}) \text{vec}(\mathbf{C})$ yields

$$\text{vec}(\mathbf{Y}) = [\mathbf{B}_t \otimes (\mathbf{X} \mathbf{W} \mathbf{B}_s)] \text{vec}(\Theta) + \text{vec}(\boldsymbol{\varepsilon}). \quad (7)$$

$\begin{matrix} NT \times 1 & T \times K_t & N \times S & S \times S & S \times K_s & K_t K_s \times 1 & NT \times 1 \end{matrix}$

In the linear regression model (7) for $\mathbf{y} = \text{vec}(\mathbf{Y})$, the parameter vector $\boldsymbol{\theta} = \text{vec}(\Theta)$ is identifiable if and only if the design matrix $\mathbf{D} = \mathbf{B}_t \otimes (\mathbf{X} \mathbf{W} \mathbf{B}_s)$ is of full column-rank.

The rank of \mathbf{D} is equal to $\text{rank}(\mathbf{D}) = \text{rank}(\mathbf{B}_t) \text{rank}(\mathbf{X} \mathbf{W} \mathbf{B}_s)$. \mathbf{B}_t will typically be of full rank K_t as long as $K_t \leq T$, as the K_t spline functions form a basis and the columns of \mathbf{B}_t are thus linearly independent for non-pathological cases. For \mathbf{X} , let $\mathbf{X} = \Xi \Phi$ be the empirical version of the Karhunen-Loève expansion (2), where $\mathbf{X}^T \mathbf{X} = \Phi^T \Lambda \Phi$ with Φ^T an $S \times M$ orthonormal matrix of eigenvectors, $M = \text{rank}(\mathbf{X})$, $\Lambda = \text{diag}(\hat{\lambda}_1, \dots, \hat{\lambda}_M)$ a diagonal matrix of ordered positive eigenvalues and Ξ containing M columns of estimated scores with empirical variances $\hat{\lambda}_1, \dots, \hat{\lambda}_M$. Then, by construction, the matrix $\mathbf{X} \mathbf{W} \mathbf{B}_s$ is at most of rank $\min(N, M, K_s, S) = \min(M, K_s)$, since $M \leq \min(N, S)$.

We then have the following proposition:

Proposition 3.2. *Assume that \mathbf{B}_t is of full rank K_t . Then, the design matrix $\mathbf{D} = \mathbf{B}_t \otimes (\mathbf{X} \mathbf{W} \mathbf{B}_s)$ in model (7) is rank-deficient if and only if*

- a) $M < K_s$ or
- b) if $M \geq K_s$, but $\text{rank}(\Phi \mathbf{W} \mathbf{B}_s) < K_s$.

Proof. As $\text{rank}(\mathbf{X}) = \text{rank}(\Xi \Phi) = \text{rank}(\Phi) = M$,

$$\begin{aligned} \text{rank}(\Phi \mathbf{W} \mathbf{B}_s) &\geq \text{rank}(\Xi \Phi \mathbf{W} \mathbf{B}_s) = \text{rank}(\mathbf{X} \mathbf{W} \mathbf{B}_s) \\ &\geq \text{rank}(\Xi \Phi) + \text{rank}(\Phi \mathbf{W} \mathbf{B}_s) - \text{rank}(\Phi) = \text{rank}(\Phi \mathbf{W} \mathbf{B}_s) \end{aligned}$$

using Harville (1997, Th. 17.5.1). Thus, $\text{rank}(\mathbf{X} \mathbf{W} \mathbf{B}_s) = \text{rank}(\Phi \mathbf{W} \mathbf{B}_s)$. The rank will be less than full if $\text{rank}(\Phi \mathbf{W} \mathbf{B}_s) < K_s$, including if $M < K_s$, as $\text{rank}(\Phi \mathbf{W} \mathbf{B}_s) \leq M$ by construction. \square

Proposition 3.2 yields a direct criterion to check for rank-deficient design matrices. Case a) corresponds to a low-rank covariance for the X -process. In this case, the functional predictor does not carry enough information, as measured by the number of eigenfunctions $\phi_m(s)$ with non-zero eigenvalues, compared to the number of parameters to estimate. Case b) means that even if $M \geq K_s$, non-identifiability can occur if the span of the basis used for $\beta(s, t)$ in s -direction contains functions in $\ker(K^X)$, as measured by numerical integration using the integration weights \mathbf{w} . More intuitively, this means that the basis for the s -direction of $\beta(s, t)$ accomodates modes of variation orthogonal to those of the $X(s)$ -process.

Note that identifiability is determined by the interplay between the complexity of the $X(s)$ and the coefficient basis for $\beta(s, t)$. Thus, more data will not necessarily resolve identifiability issues: A more dense grid \mathbf{s} will only eliminate problems with identifiability present with a coarser grid if there is sufficient small-scale structure in $X(s)$ also present in the basis used for $\beta(s, t)$ in s -direction. Increasing the sample size N will likewise eliminate problems with identifiability only if the low-rank of the covariate's covariance is due to small sample size.

The effect of the penalty

In cases of non-identifiability, the best we can hope for is partial identifiability of the parameters in a parameter subspace, i.e., identifiability under additional assumptions on the parameters. In this vein, functional principal component regression (e.g. Yao et al., 2005) restricts $\beta(s, t)$ to lie in the span of the first K eigenfunctions $\phi_m(s)$, $m = 1, \dots, K$, for all t . Remaining problems then include the fact that the $\hat{\phi}_m(s)$ are estimated quantities in practice, with corresponding measurement error, and the choice of K , which can strongly affect the shape of the resulting function estimate (Crainiceanu et al., 2009). Also, this approach couples assumptions on the shape of $\beta(s, t)$ to properties of the space spanned by the retained $\phi_m(s)$. In particular, the smoothness of $\beta(s, t)$ is determined by their smoothness, and inclusion of higher-order eigenfunctions can lead to wiggly function estimates that are hard to interpret and unstable under replication.

We here focus on a penalized approach and investigate the effects of the penalty on identifiability. While the use of a penalized approach is well-known to avoid identifiability problems due to high correlation between observations at neighboring grid points (e.g. Ramsay and Silverman, 2005, Ch. 15.2), the full interplay between penalty and identifiability is, we believe, not fully understood and underappreciated.

Consider again the design matrix $\mathbf{D} = \mathbf{B}_t \otimes (\mathbf{X}\mathbf{W}\mathbf{B}_s)$ of rank d with the singular value decomposition $\mathbf{D} = \mathbf{V}\boldsymbol{\Sigma}\mathbf{U}^\top = (\mathbf{V}_t \otimes \mathbf{V}_s)(\boldsymbol{\Sigma}_t \otimes \boldsymbol{\Sigma}_s)(\mathbf{U}_t^\top \otimes \mathbf{U}_s^\top)$, with $\mathbf{V}_t \boldsymbol{\Sigma}_t \mathbf{U}_t^\top$ and $\mathbf{V}_s \boldsymbol{\Sigma}_s \mathbf{U}_s^\top$ the singular value decompositions of \mathbf{B}_t and $\mathbf{D}_s := \mathbf{X}\mathbf{W}\mathbf{B}_s$, respectively. Let indices $_+$ and $_0$ denote the corresponding submatrices obtained by removing columns and/or rows corresponding to zero and non-zero singular values, respectively. We assume in the following that \mathbf{B}_t is of full rank K_t . Then, $\mathbf{D} = \mathbf{V}_+ \boldsymbol{\Sigma}_+ \mathbf{U}_+^\top = (\mathbf{V}_t \otimes \mathbf{V}_{s+})(\boldsymbol{\Sigma}_t \otimes \boldsymbol{\Sigma}_{s+})(\mathbf{U}_t^\top \otimes \mathbf{U}_{s+}^\top)$.

Thus, for any given $\boldsymbol{\theta}_*$ with $\mathbf{D}\boldsymbol{\theta}_* =: \mathbf{f}$, there exists a linear subspace $\mathcal{H}_f \subset \mathbb{R}^{K_s K_t}$ of dimension $(K_s K_t - d)$ given by $\mathcal{H}_f = \{\boldsymbol{\theta} \in \mathbb{R}^{K_s K_t} : \mathbf{D}\boldsymbol{\theta} = \mathbf{f}\} = \{\boldsymbol{\theta}_* + \boldsymbol{\theta}_0 : \boldsymbol{\theta}_0 \in \text{im}(\mathbf{U}_0)\}$ that yields identical fits \mathbf{f} .

If we assume our parameter function to come from a space of smooth functions, we can select the smoothest solution on a given hyperplane \mathcal{H}_f by minimizing $\boldsymbol{\theta}^\top \mathbf{P}\boldsymbol{\theta}$ for a suitable penalty matrix \mathbf{P} that penalizes roughness of the function parametrized by $\boldsymbol{\theta}$. We have the following proposition regarding uniqueness of the corresponding minimum.

Proposition 3.3. *Let $\mathbf{P} = \lambda_s(\mathbf{I}_{K_t} \otimes \mathbf{P}_s) + \lambda_t(\mathbf{P}_t \otimes \mathbf{I}_{K_s})$, with \mathbf{P}_s and \mathbf{P}_t positive semi-definite matrices. Assume that \mathbf{B}_t is of full rank K_t , that $\text{rank}(\mathbf{P}_t) < K_t$ and that $\lambda_s > 0, \lambda_t \geq 0$. Then, for any $\mathbf{f} \in \text{im}(\mathbf{D})$ there is a unique minimum $\min_{\boldsymbol{\theta} \in \mathcal{H}_f} \{\boldsymbol{\theta}^\top \mathbf{P}\boldsymbol{\theta}\}$ if and only if $\ker(\mathbf{D}_s^\top \mathbf{D}_s) \cap \ker(\mathbf{P}_s) = \{\mathbf{0}\}$.*

Proof. We have

$$\begin{aligned}
& \min\{\boldsymbol{\theta}^\top \mathbf{P}\boldsymbol{\theta}\} && \text{s.t. } \boldsymbol{\theta} \in \mathcal{H}_f \\
& = \min\{\boldsymbol{\theta}^\top \mathbf{U}\mathbf{U}^\top \mathbf{P}\mathbf{U}\mathbf{U}^\top \boldsymbol{\theta}\} && \text{s.t. } \mathbf{D}\boldsymbol{\theta} = \mathbf{f} \\
& = \min\{(\boldsymbol{\theta}^\top \mathbf{U}_+ | \boldsymbol{\theta}^\top \mathbf{U}_0) \begin{pmatrix} \mathbf{U}_+^\top \mathbf{P}\mathbf{U}_+ & \mathbf{U}_+^\top \mathbf{P}\mathbf{U}_0 \\ \mathbf{U}_0^\top \mathbf{P}\mathbf{U}_+ & \mathbf{U}_0^\top \mathbf{P}\mathbf{U}_0 \end{pmatrix} \begin{pmatrix} \mathbf{U}_+^\top \boldsymbol{\theta} \\ \mathbf{U}_0^\top \boldsymbol{\theta} \end{pmatrix}\} && \text{s.t. } \mathbf{U}_+^\top \boldsymbol{\theta} = \boldsymbol{\Sigma}_+^{-1} \mathbf{V}_+^\top \mathbf{f}.
\end{aligned}$$

Denote $\mathbf{v}_+ = \mathbf{U}_+^\top \boldsymbol{\theta}$ and $\mathbf{v}_0 = \mathbf{U}_0^\top \boldsymbol{\theta}$, with $(\mathbf{v}_+^\top, \mathbf{v}_0^\top)^\top = \mathbf{U}^\top \boldsymbol{\theta}$ a bijective re-parametrization of $\boldsymbol{\theta}$. Note that \mathbf{v}_+ is fixed while \mathbf{v}_0 is free to vary within the hyperplane. Setting the derivative with respect to \mathbf{v}_0 equal to zero yields

$$\mathbf{U}_0^\top \mathbf{P}\mathbf{U}_0 \mathbf{v}_0 = -\mathbf{U}_0^\top \mathbf{P}\mathbf{U}_+ \mathbf{v}_+. \quad (8)$$

Now, if $\ker(\mathbf{D}_s^\top \mathbf{D}_s) \cap \ker(\mathbf{P}_s) \neq \{\mathbf{0}\}$, choose $\mathbf{u}_s \neq \mathbf{0}$ with $\mathbf{U}_{s0} \mathbf{u}_s \in \ker(\mathbf{D}_s^\top \mathbf{D}_s) \cap \ker(\mathbf{P}_s)$ and $\mathbf{u}_t \neq \mathbf{0}$ with $\mathbf{U}_t \mathbf{u}_t \in \ker(\mathbf{P}_t)$. As $\mathbf{U}_0 = \mathbf{U}_t \otimes \mathbf{U}_{s0}$, we have

$$(\mathbf{u}_t^\top \otimes \mathbf{u}_s^\top) \mathbf{U}_0^\top \mathbf{P}\mathbf{U}_0 (\mathbf{u}_t \otimes \mathbf{u}_s) = \lambda_s (\mathbf{u}_t^\top \mathbf{u}_t) \mathbf{u}_s^\top \mathbf{U}_{s0}^\top \mathbf{P}_s \mathbf{U}_{s0} \mathbf{u}_s + \lambda_t \mathbf{u}_t^\top \mathbf{U}_t^\top \mathbf{P}_t \mathbf{U}_t \mathbf{u}_t (\mathbf{u}_s^\top \mathbf{u}_s) = 0.$$

Thus, $\mathbf{U}_0^\top \mathbf{P}\mathbf{U}_0$ is not of full rank, no unique solution \mathbf{v}_0 of (8) exists, so there is no unique minimum $\min_{\boldsymbol{\theta} \in \mathcal{H}_f} \{\boldsymbol{\theta}^\top \mathbf{P}\boldsymbol{\theta}\}$.

On the other hand, if $\ker(\mathbf{D}_s^\top \mathbf{D}_s) \cap \ker(\mathbf{P}_s) = \{\mathbf{0}\}$, for any \mathbf{x} with $\mathbf{x}^\top \mathbf{U}_{s0}^\top \mathbf{P}_s \mathbf{U}_{s0} \mathbf{x} = 0$ we have $\mathbf{U}_{s0} \mathbf{x} \in \ker(\mathbf{D}_s^\top \mathbf{D}_s) \cap \ker(\mathbf{P}_s) = \{\mathbf{0}\}$ and thus $\mathbf{x} = \mathbf{U}_{s0}^\top \mathbf{U}_{s0} \mathbf{x} = \mathbf{0}$. As this means that $\mathbf{U}_{s0}^\top \mathbf{P}_s \mathbf{U}_{s0}$ is positive definite, we also have that $\mathbf{U}_0^\top \mathbf{P}\mathbf{U}_0 = \lambda_s \mathbf{I}_{K_t} \otimes (\mathbf{U}_{s0}^\top \mathbf{P}_s \mathbf{U}_{s0}) + \lambda_t (\mathbf{U}_t^\top \mathbf{P}_t \mathbf{U}_t) \otimes \mathbf{I}_{(K_s - d/K_t)}$ is positive definite and thus invertible. Therefore, there is a unique minimum $\mathbf{v}_0 = -(\mathbf{U}_0^\top \mathbf{P}\mathbf{U}_0)^{-1} \mathbf{U}_0^\top \mathbf{P}\mathbf{U}_+ \mathbf{v}_+$ and a unique smoothest point

$$\boldsymbol{\theta}_f = \mathbf{U}(\mathbf{v}_+^\top, -[(\mathbf{U}_0^\top \mathbf{P}\mathbf{U}_0)^{-1} \mathbf{U}_0^\top \mathbf{P}\mathbf{U}_+ \mathbf{v}_+]^\top)^\top = \mathbf{H}\mathbf{U}_+ \mathbf{v}_+ = \mathbf{H}\boldsymbol{\theta}, \quad (9)$$

with $\mathbf{H} = (\mathbf{I}_{K_s K_t} - \mathbf{U}_0 (\mathbf{U}_0^\top \mathbf{P}\mathbf{U}_0)^{-1} \mathbf{U}_0^\top \mathbf{P})$ and $\boldsymbol{\theta}_f^\top \mathbf{P}\boldsymbol{\theta}_f = \min_{\boldsymbol{\theta} \in \mathcal{H}_f} \boldsymbol{\theta}^\top \mathbf{P}\boldsymbol{\theta}$. \square

The assumption that \mathbf{P}_t is of less-than-full rank is natural in the context of derivative-based penalties and excludes cases like the ridge penalty $\mathbf{P}_t = \mathbf{I}_{K_t}$, which would have the same effect as a full-rank penalty $\mathbf{P}_s = \mathbf{I}_{K_s}$, even in cases of a kernel overlap between $\mathbf{D}_s^\top \mathbf{D}_s$ and \mathbf{P}_s . A potentially full-rank \mathbf{P}_t would change the 'if and only if' in Propositions 3.3 and 3.4 below to 'if'.

Proposition 3.3 shows that in the case of a kernel overlap $\ker(\mathbf{D}_s^\top \mathbf{D}_s) \cap \ker(\mathbf{P}_s) \neq \{\mathbf{0}\}$, the additional side condition $\boldsymbol{\theta}^\top \mathbf{P}\boldsymbol{\theta} \rightarrow \min$ does not yield a unique smoothest point on the hyperplane and the model remains unidentifiable. On the other hand, if there is no kernel overlap, there is a unique smoothest point $\boldsymbol{\theta}_f \in \mathcal{H}_f$ and this unique point has the form of a projection of $\boldsymbol{\theta}$ along the hyperplane. Note that for the ridge penalty $\mathbf{P} = \lambda \mathbf{I}_{K_s K_t}$, one obtains the projection onto the image $\text{im}(\mathbf{D})$, which sets the part in the kernel of $\mathbf{D}^\top \mathbf{D}$ to zero. More generally, a smoothness penalty \mathbf{P} generates a projection that may have a non-zero component in the kernel of $\mathbf{D}^\top \mathbf{D}$ if this yields a smaller overall penalty value. In this case of no kernel overlap, we thus have a weak form of identifiability, which guarantees that there is a unique smoothest representative on any hyperplane of parameters giving the same conditional distribution for \mathbf{Y} .

This characterization, which only requires checking of design matrix and penalty in s -direction and not for the full model, carries over to the penalized maximum likelihood or least squares estimation problem

$$\min_{\boldsymbol{\theta}} \{\|\mathbf{y} - \mathbf{D}\boldsymbol{\theta}\|^2 + \lambda_s \boldsymbol{\theta}^\top (\mathbf{I}_{K_t} \otimes \mathbf{P}_s) \boldsymbol{\theta} + \lambda_t \boldsymbol{\theta}^\top (\mathbf{P}_t \otimes \mathbf{I}_{K_s}) \boldsymbol{\theta}\} \quad (10)$$

for some $\lambda_s > 0, \lambda_t \geq 0$.

Proposition 3.4. *Assume that \mathbf{B}_t is of full rank K_t , that $\text{rank}(\mathbf{P}_t) < K_t$ and that $\lambda_s > 0, \lambda_t \geq 0$. Then, there is a unique penalized least squares solution for (10) if and only if $\ker(\mathbf{D}_s^\top \mathbf{D}_s) \cap \ker(\mathbf{P}_s) = \{\mathbf{0}\}$.*

Proof. Problem (10) has a unique solution iff $(\mathbf{D}^\top \mathbf{D} + \lambda_s(\mathbf{I}_{K_t} \otimes \mathbf{P}_s) + \lambda_t(\mathbf{P}_t \otimes \mathbf{I}_{K_s})) \geq 0$ is invertible, i.e., positive definite. Now, suppose that $\ker(\mathbf{D}_s^\top \mathbf{D}_s) \cap \ker(\mathbf{P}_s) = \{\mathbf{0}\}$. For any $\mathbf{x} \in \mathbb{R}^{K_t K_s}$ with

$$\mathbf{x}^\top (\mathbf{D}^\top \mathbf{D} + \lambda_s(\mathbf{I}_{K_t} \otimes \mathbf{P}_s) + \lambda_t(\mathbf{P}_t \otimes \mathbf{I}_{K_s})) \mathbf{x} = 0,$$

we have, with $\mathbf{b} = \mathbf{U}^\top \mathbf{x} = (b_{kj})_{kj \in \{11, 12, \dots, K_t K_s\}}$,

$$\begin{aligned} \mathbf{x}^\top \mathbf{D}^\top \mathbf{D} \mathbf{x} &= \mathbf{b}^\top \left(\boldsymbol{\Sigma}_t^2 \otimes \begin{pmatrix} \boldsymbol{\Sigma}_{s+}^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \right) \mathbf{b} = 0 \\ \Rightarrow b_{kj} &= 0 \forall 1 \leq j \leq d/K_t; 1 \leq k \leq K_t, \end{aligned} \quad (11)$$

and also

$$\mathbf{b}^\top [\mathbf{I}_{K_t} \otimes ((\mathbf{U}_{s+} | \mathbf{U}_{s0})^\top \mathbf{P}_s (\mathbf{U}_{s+} | \mathbf{U}_{s0}))] \mathbf{b} \stackrel{(11)}{=} \tilde{\mathbf{b}}^\top [\mathbf{I}_{K_t} \otimes (\mathbf{U}_{s0}^\top \mathbf{P}_s \mathbf{U}_{s0})] \tilde{\mathbf{b}} = 0,$$

where $\tilde{\mathbf{b}}$ is obtained by removing the zero entries given by (11) which correspond to \mathbf{U}_{s+} from \mathbf{b} . Thus, for all $1 \leq k \leq K_t$, and letting $\tilde{\mathbf{b}}_k = (b_{k(d/K_t+1)}, \dots, b_{kK_s})$, we have $\mathbf{U}_{s0} \tilde{\mathbf{b}}_k \in \ker(\mathbf{D}_s^\top \mathbf{D}_s) \cap \ker(\mathbf{P}_s) = \{\mathbf{0}\}$. Thus, $\tilde{\mathbf{b}} = \mathbf{0}, \mathbf{b} = \mathbf{0}, \mathbf{x} = \mathbf{0}$ and $(\mathbf{D}^\top \mathbf{D} + \lambda_s(\mathbf{I}_{K_t} \otimes \mathbf{P}_s) + \lambda_t(\mathbf{P}_t \otimes \mathbf{I}_{K_s}))$ is of full rank.

On the other hand, if $\ker(\mathbf{D}_s^\top \mathbf{D}_s) \cap \ker(\mathbf{P}_s) \neq \{\mathbf{0}\}$, there is a $\mathbf{u}_s \neq \mathbf{0}$ with $\mathbf{D}_s \mathbf{u}_s = \mathbf{0}$ and $\mathbf{P}_s \mathbf{u}_s = \mathbf{0}$. Choose $\mathbf{0} \neq \mathbf{u}_t \in \ker \mathbf{P}_t$. Then,

$$\begin{aligned} (\mathbf{u}_t^\top \otimes \mathbf{u}_s^\top) (\mathbf{D}^\top \mathbf{D} + \lambda_s(\mathbf{I}_{K_t} \otimes \mathbf{P}_s) + \lambda_t(\mathbf{P}_t \otimes \mathbf{I}_{K_s})) (\mathbf{u}_t \otimes \mathbf{u}_s) \\ = \mathbf{u}_t^\top \mathbf{B}_t^\top \mathbf{B}_t \mathbf{u}_t \cdot 0 + \lambda_s \mathbf{u}_t^\top \mathbf{u}_t \cdot 0 + \lambda_t \cdot 0 \cdot \mathbf{u}_s^\top \mathbf{u}_s = 0. \end{aligned}$$

Thus, $(\mathbf{D}^\top \mathbf{D} + \lambda_s(\mathbf{I}_{K_t} \otimes \mathbf{P}_s) + \lambda_t(\mathbf{P}_t \otimes \mathbf{I}_{K_s}))$ is singular and not invertible. \square

Proposition 3.4 gives a criterion for the uniqueness of the penalized least squares solution. We can show how the penalty achieves this uniqueness by writing (10) as a nested minimization problem. Here, the outer minimization finds the $\mathbf{f} = \mathbf{D}\boldsymbol{\theta}$ with optimal fit to the data, and the inner minimization minimizes the penalty term over \mathcal{H}_f to obtain the smoothest solution for a given level of residual variation.

$$\begin{aligned} & \min_{\boldsymbol{\theta}} \{ \|\mathbf{y} - \mathbf{D}\boldsymbol{\theta}\|^2 + \lambda_s \boldsymbol{\theta}^\top (\mathbf{I}_{K_t} \otimes \mathbf{P}_s) \boldsymbol{\theta} + \lambda_t \boldsymbol{\theta}^\top (\mathbf{P}_t \otimes \mathbf{I}_{K_s}) \boldsymbol{\theta} \} \\ &= \min_{\mathbf{f} \in \text{im}(\mathbf{D})} \min_{\boldsymbol{\theta}} \{ \|\mathbf{y} - \mathbf{f}\|^2 + \boldsymbol{\theta}^\top \mathbf{P} \boldsymbol{\theta} \quad \text{s.t.} \quad \mathbf{D}\boldsymbol{\theta} = \mathbf{f} \} \\ &= \min_{\mathbf{f} \in \text{im}(\mathbf{D})} \{ \|\mathbf{y} - \mathbf{D}\boldsymbol{\theta}_f\|^2 + \boldsymbol{\theta}_f^\top \mathbf{P} \boldsymbol{\theta}_f \} \\ &= \min_{\mathbf{v}_+ \in \mathbb{R}^d} \{ \|\mathbf{y} - \mathbf{V}_+ \boldsymbol{\Sigma}_+ \mathbf{v}_+\|^2 + \mathbf{v}_+^\top \mathbf{U}_+^\top \mathbf{H}^\top \mathbf{P} \mathbf{H} \mathbf{U}_+ \mathbf{v}_+ \} \end{aligned}$$

where $\boldsymbol{\theta}_f = \mathbf{H} \mathbf{U}_+ \mathbf{v}_+$ for a given \mathbf{f} is uniquely defined as in (9), with $\mathbf{v}_+ = \boldsymbol{\Sigma}_+^{-1} \mathbf{V}_+^\top \mathbf{f}$ and $\mathbf{H} = (\mathbf{I}_{K_s K_t} - \mathbf{U}_0 (\mathbf{U}_0^\top \mathbf{P} \mathbf{U}_0)^{-1} \mathbf{U}_0^\top \mathbf{P})$, if $\ker(\mathbf{D}_s^\top \mathbf{D}_s) \cap \ker(\mathbf{P}_s) = \{\mathbf{0}\}$.

As $\mathbf{V}_+ \boldsymbol{\Sigma}_+$ is a matrix of full column rank d , this minimization problem has a unique solution that, for given λ_s, λ_t , balances the fit to the data and smoothness.

To summarize, in the case of no kernel overlap, i.e., $\ker(\mathbf{D}_s^\top \mathbf{D}_s) \cap \ker(\mathbf{P}_s) = \{\mathbf{0}\}$, we obtain a weaker form of identifiability even when the design matrix \mathbf{D} is not of full rank, which guarantees that there is a unique smoothest representative on any hyperplane of parameters giving the same conditional distribution for \mathbf{y} . Then, there will also be a unique

solution to the penalized estimation problem, which is the smoothest representative on the hyperplane of possible solutions with equally good fit.

In practice, λ_s and λ_t are estimated from the data. We here do not investigate the more complex case when λ_s, λ_t are not fixed. It should also be noted that $(\mathbf{D}^\top \mathbf{D} + \lambda_s(\mathbf{I}_{K_t} \otimes \mathbf{P}_s) + \lambda_t(\mathbf{P}_t \otimes \mathbf{I}_{K_s}))$ may still be close to singular even in cases of no kernel overlap if smoothing parameters are very small, with corresponding reduced stability in estimation.

3.3 Diagnostics, practical recommendations and countermeasures

In order to safeguard against misleading coefficient estimates in practical applications of functional regression, it is necessary 1) to develop empirical criteria for diagnosing problematic data settings in which the coefficient function is not identifiable based on the available data, or where only the penalty ensures unique estimates, 2) to avoid pre-processing protocols and tuning parameters that increase the likelihood of identifiability issues and 3) to develop improved algorithms for estimating function-on-function effects that are less prone to severely misleading estimates in problematic settings.

Diagnostics We are interested in identifying settings in which only the penalty term guarantees the existence of a unique solution. Following Proposition 3.2, the most direct approach to do so is to compute the condition number of $\mathbf{D}_s^\top \mathbf{D}_s = (\mathbf{X} \mathbf{W} \mathbf{B}_s)^\top \mathbf{X} \mathbf{W} \mathbf{B}_s$ and choose a suitable cut-off (10^6 , in the following) for numeric rank deficiency.

In addition, propositions 3.3 and 3.4 indicate that a measure of the degree of overlap between the spans of $\ker(\mathbf{D}_s^\top \mathbf{D}_s)$ and $\ker(\mathbf{P}_s)$ can be used to detect non-identifiability. In our empirical evaluation of such measures, we found that a measure for the distance between the spans of two matrices introduced in Larsson and Villani (2001), when modified for our setting, showed the most promise as the resulting measure is free of tuning parameters and can be computed quickly from the data.

In particular, we modify the original definition of Larsson-Villani in order to accommodate two matrices of unequal column numbers. We then define the amount of overlap \bigcap_{LV} between the span of two matrices $\mathbf{A} \in \mathbb{R}^{n \times p_A}$, $\mathbf{B} \in \mathbb{R}^{n \times p_B}$, $n > p_A, p_B$, by

$$\bigcap_{LV}(\mathbf{A}, \mathbf{B}) = \text{trace}(\mathbf{V}_B^\top \mathbf{V}_A \mathbf{V}_A^\top \mathbf{V}_B).$$

Here, \mathbf{V}_Z is a matrix containing the left singular vectors of the matrix \mathbf{Z} and is thus an orthogonal matrix spanning the same column space as \mathbf{Z} , $\mathbf{Z} \in \{\mathbf{A}, \mathbf{B}\}$. It is easy to see that this measure is symmetric, $\bigcap_{LV}(\mathbf{A}, \mathbf{B}) = \bigcap_{LV}(\mathbf{B}, \mathbf{A})$. Similarly to Theorem 2 in Larsson and Villani (2001), one can also show that $\bigcap_{LV}(\mathbf{A}, \mathbf{B}) \in [0, \min(p_A, p_B)]$, with the overlap assuming its maximum of $\min(p_A, p_B)$ iff $\mathbf{A} \in \text{im}(\mathbf{B})$ or $\mathbf{B} \in \text{im}(\mathbf{A})$ and its minimum of 0 iff $\mathbf{A} \in \text{im}(\mathbf{B}_\perp)$ or $\mathbf{B}_\perp \in \text{im}(\mathbf{A})$, where the $n \times (n - p_Z)$ matrix \mathbf{Z}_\perp is the orthonormal complement of \mathbf{Z} , $\mathbf{Z} \in \{\mathbf{A}, \mathbf{B}\}$.

To measure the degree of overlap between the kernels of $\mathbf{D}_s^\top \mathbf{D}_s$ and of \mathbf{P}_s , we could use $\bigcap_{LV}((\mathbf{D}_s^\top \mathbf{D}_s)_\perp, \mathbf{P}_{s\perp})$, as the span of $(\mathbf{D}_s^\top \mathbf{D}_s)_\perp$ corresponds to the kernel of $\mathbf{D}_s^\top \mathbf{D}_s$ and the span of $\mathbf{P}_{s\perp}$ corresponds to the kernel of \mathbf{P}_s . In the following, we will however use

$$\bigcap_{X_\perp \mathcal{P}_\perp} = \bigcap_{LV}((\mathbf{X}^\top \mathbf{X})_\perp, \mathbf{W} \mathbf{B}_s \mathbf{P}_{s\perp}) = \bigcap_{LV}((\mathbf{X}^\top)_\perp, \mathbf{W} \mathbf{B}_s \mathbf{P}_{s\perp}), \quad (12)$$

as this choice obtained slightly better sensitivity and specificity for the detection of problematic settings in our simulations in Section 4. Moreover, one can easily show that

$$\ker(\mathbf{D}_s^\top \mathbf{D}_s) \cap \ker(\mathbf{P}_s) = \{\mathbf{0}\} \quad \Leftrightarrow \quad \ker(\mathbf{X}^\top \mathbf{X}) \cap \{\mathbf{W} \mathbf{B}_s \mathbf{x} \mid \mathbf{x} \in \ker(\mathbf{P}_s)\} = \{\mathbf{0}\}$$

such that the two formulations address the same question. For $\mathbf{W} = \mathbf{I}_S$, this measure has the interpretation of the overlap between the empirical nullspace of the observed $X(s)$ process or $\ker(K^X)$ and $\mathcal{P}_{s\perp}$, the space of functions not penalized by the penalty defined

by \mathbf{P}_s , evaluated on the grid given by \mathbf{s} . It can be determined quickly and accurately before the model is fit. Problematic cases are indicated by overlap measures ≥ 1 , as this is indicative of an at least one-dimensional sub-space of functions contained in the kernel overlap.

Practical recommendations The theoretical results suggest several recommendations for pre-processing of functional covariates and choice of the penalty in practice.

1. Pre-smoothing of functional covariates is commonly done to remove measurement error and/or obtain functions on a common grid (e.g. James, 2002; Ramsay and Silverman, 2005; Goldsmith et al., 2011). If the resulting (effective) rank of the smoothed covariate process drops below K_s , this will lead to models that are only identifiable through the penalty term. We thus recommend to use a sufficiently large number of FPCs and/or spline basis functions if such pre-processing is required.

As a peculiar consequence of this point it may be preferable in some cases to accept a small amount of measurement error-induced attenuation in $\hat{\beta}(s, t)$ based on noisy, unprocessed $X(s)$ in order to avoid a potentially much larger non-identifiability-induced error in $\hat{\beta}(s, t)$ based on low-rank, pre-processed $X(s)$.

2. Curve-wise centering of functional covariates such that $\sum_{l=1}^S X_i(s_l) = 0$ for all i is sometimes used e.g. in the context of spectroscopy data to remove the optical offset (c.f. Fuchs et al., 2015). Then, constant functions lie in the kernel of K^X , $\ker(K^X)$. This is not recommended if a penalty is used that does not penalize constant functions (as most difference or derivative-based penalties do) to avoid non-identifiability.
3. Penalties with larger nullspaces increase the likelihood of a kernel overlap and resulting non-identifiability problems. For difference or derivative-based penalties, for example, a penalty penalizing deviations from constant functions (first order differences or derivatives) would thus be preferable in this sense to higher-order differences/derivatives. Constant coefficient functions, which then span the penalty nullspace, correspond to models with the mean over the functions as covariate - which are often used by practitioners - and thus also lend themselves to intuitive interpretations.

In particular, for penalties where constant coefficient functions span the penalty nullspace, it is straightforward to see that only X-processes that are centered curve-wise will result in a kernel overlap (unless smoothing parameters are estimated to be very small). Unless curve-wise centering is performed as discussed in 2., such processes will typically occur rarely and using first-order difference/derivative penalties should thus guard against many if not most serious identifiability issues in practice.

Countermeasures The third point above suggests modifications of the penalty nullspace to avoid non-identifiability caused by potential overlap between the penalty nullspace $\mathcal{P}_{s\perp}$ and $\ker(K^X)$. We describe three approaches using penalties with empty null-spaces; a systematic comparison of their performance is given in Section 4.2.

1. The simplest approach is the use of a simple ridge penalty $\mathbf{P}_s = \mathbf{I}_{K_s}$. However, the resulting estimates will typically not have good smoothness properties as the ridge penalty is not a roughness penalty in the conventional sense and its bias towards small absolute values of $\beta(s, t)$ tends to increase estimation error, cf. Figures 3 and 4.
2. A second approach uses a modified marginal penalty matrix without nullspace along the lines of the so-called “shrinkage approach” described in Marra and Wood (2011) and originally developed for the purpose of variable selection in generalized additive models. Marra and Wood (2011) replace the marginal penalty $\mathbf{P}_s = \mathbf{\Gamma} \text{diag}(\rho_1, \dots, \rho_{K_s}) \mathbf{\Gamma}^T$, with eigenvectors contained in $\mathbf{\Gamma}$, eigenvalues $\rho_1, \dots, \rho_{K_s}$ and rank $\tilde{K}_s < K_s$, by a full

rank marginal penalty

$$\tilde{\mathbf{P}}_s = \mathbf{\Gamma} \text{diag}(\rho_1, \dots, \rho_{\tilde{K}_s}, \epsilon \rho_{\tilde{K}_s}, \dots, \epsilon \rho_{\tilde{K}_s}) \mathbf{\Gamma}^T.$$

This substitutes the zero eigenvalues $\rho_{\tilde{K}_s+1}, \dots, \rho_{K_s}$ with $\epsilon \rho_{\tilde{K}_s}$ for all $k = \tilde{K}_s + 1, \dots, K_s$ using $0 < \epsilon \ll 1$, and thereby adds a small amount of penalization to parameter vectors in the null space of the original penalty. In the following, we will refer to such a modified penalty as a full-rank penalty. By imposing a small degree of regularization on functions in $\mathcal{P}_{s\perp}$ we can preserve the attractive smoothing properties of the original penalty while still avoiding large artefacts due to non-identifiability resulting from a kernel overlap. We use $\epsilon = 0.1$ as suggested in Marra and Wood (2011).

3. In a similar effort to avoid spurious estimates in scalar-on-function regression, James and Silverman (2005, their eq. (16)) suggested using the empirical FPCs of $X(s)$ scaled by their inverse eigenvalues as a penalty. This penalizes coefficient functions with large variability in directions in which $X(s)$ varies very little or not at all (i.e., in $\ker(K^X)$). We adapted this approach to our function-on-function setting by replacing the conventional difference operator based B-spline penalty matrix with

$$\mathbf{P}_s = \mathbf{B}_s^T \sum_{m=1}^{\min(N,S)} \hat{\nu}_m^{-1} \text{diag}(\mathbf{w} \cdot \hat{\phi}_m(s)^2) \mathbf{B}_s$$

with estimated FPCs and eigenvalues $\hat{\phi}_m(s)$ and $\hat{\nu}_m$, $m = 1, \dots, \min(N, S)$. For a given t_0 , this penalty matrix approximates the marginal penalty term $\sum_{m=1}^{\min(N,S)} \int (\hat{\nu}_m^{-1/2} \hat{\phi}_m(s) \beta(s, t_0))^2 ds$ suggested by James and Silverman (2005).

The empirical $\hat{\phi}_m(s)$ and $\hat{\nu}_m$ have to be estimated from a singular value decomposition of \mathbf{X} . It is unclear, however, how to compute the inverse eigenvalues if \mathbf{X} is of low rank, i.e., if some of the $\hat{\nu}_m$ are (numerically) zero, which is of course precisely the setting in which this penalty might yield more stable estimates. In our experiments, we tried replacement of the zero eigenvalues with the smallest non-zero eigenvalue and a variety of other replacement schemes, but none seemed to offer improvements compared to the full-rank penalty approach. Results from this approach seem to be fairly sensitive to both the chosen replacement scheme for zero eigenvalues and to the estimated FPCs themselves.

4 Simulation study

This section presents results on the practical consequences of the theoretical development in Section 3. Specifically, we investigate the performance of the tensor product spline-based approach given in Section 2 on artificial data of varying complexity and noise levels in terms of estimation accuracy and use the simulation results to validate the diagnostics for problematic settings we have developed. Subsequently, we present results for the modified full-rank penalties of Marra and Wood (2011) and the FPC-based penalty of James and Silverman (2005) described in Section 3.3. All models were fit with the `pffr()` function available in the `refund` package, which estimates the smoothing parameters using restricted maximum likelihood (REML).

4.1 Simulation setup

We simulate data from data generating process (1), with $n = 50$ subjects, $\mathcal{T} = \mathcal{S} = [0, 1]$ and $S = 100$ gridpoints for $X_i(s) = \sum_{m=1}^M \xi_{im} \phi_m(s)$. The effect surface $\beta(s, t)$ is estimated using tensor product cubic B-splines. We set $K_t = 10$ and use a marginal first order

difference penalty for the t -direction. Test runs showed that results are insensitive to the number of gridpoints for the response; we used $T = 50$ gridpoints for $Y(t)$. Twenty replications were simulated for all sensible combinations of the following parameters (144000 replicates in total). For the fitting algorithm, we vary

- the **marginal roughness penalties** for the spline coefficients: either second order difference penalties (“ Δ^2 ”) or first order difference penalties (“ Δ^1 ”). For the second order differences, coefficient vectors in the penalty’s nullspace $\ker(\mathbf{P}(\lambda_s, \lambda_t))$ parameterize surfaces that are constant or linear in both directions. For the first order differences, coefficient vectors in $\ker(\mathbf{P}(\lambda_s, \lambda_t))$ parameterize constant surfaces. Note that rank deficiencies can increase if associated smoothing parameters become sufficiently small. In that case, the penalty nullspace effectively becomes the entire span of the associated basis functions.
- and the **number of basis functions** over \mathcal{S} : $K_s \in \{5, 8, 12\}$.

For the data generating process, we vary the following parameters:

- **number of eigenfunctions for the $X(s)$ -trajectories** with non-zero eigenvalues: $M \in \{3, 5, 8, 12, 20\}$. This means we have settings with $M \leq K_s$ and $M > K_s$ for most M . Note that the effective numerical rank of the simulated values can be (much) lower than M depending on the speed with which the eigenvalues decrease.
- **signal-to-noise ratio**: $\text{SNR}_\varepsilon = \frac{\text{sd}(\int_{\mathcal{S}} X_i(s)\beta(s,t)ds)}{\text{sd}(\varepsilon_{it})} \in \{2, 10, 1000\}$, where $\text{sd}(x)$ is the empirical standard deviation of x . This corresponds to high and intermediate noise levels for realistic scenarios as well as settings with almost no noise to check the theoretical properties.
- **FPC systems** for $X_i(s) = \sum_{m=1}^M \phi_m(s)\xi_{im}$ with $\xi_{im} \sim N(0, \nu_m)$, with different patterns of decrease in the eigenvalues ν_m of the covariance operators: either a linear decrease ($\nu_m = \frac{M+1-m}{M}$) or an exponential decrease ($\nu_m = \exp(-\frac{m-1}{2})$). Some of these processes are constructed so that their covariance operator kernels $\ker(K^X)$ include functions in the penalty nullspace $\mathcal{P}_{s\perp}$.
 - *Poly*: eigenfunctions are orthogonal polynomials of degree 0 to $M-1$ with linear or exponentially decreasing eigenvalues (*Poly, Lin* and *Poly, Exp*, respectively). For *Poly*, $\ker(K^X)$ is disjunct from $\mathcal{P}_{s\perp}$, since the first and second eigenfunctions are constant and linear polynomials.
 - *Fourier*: eigenfunctions are those of a standard Fourier basis. Although a complete Fourier basis is a basis for all square-integrable functions, in practice the kernel $\ker(K^X)$ of a truncated Fourier basis contains functions that are very close to the constant since no linear combination of a finite set of Fourier basis functions yields an exactly constant function, so $\ker(K^X)$ is not disjunct from $\mathcal{P}_{s\perp}$. We used this basis with constant ($\nu_m \equiv 1$ for *Fourier, Const*) or exponentially decreasing (for *Fourier, Exp*) eigenvalues ν_m .
 - *Wiener*: eigenfunctions and eigenvalues are those of the standard Wiener process on $[0, 1]$: $\phi_m(s) = \sqrt{2} \sin(\pi(m-0.5)s)$; $\nu_m = (\frac{\pi}{2}(2m+1))^{-2}$. The $\ker(K^X)$ is close to $\mathcal{P}_{s\perp}$ in this case as no linear combination of a finite set of these basis functions yields an exactly constant or linear function.
 - *BrownBridge*: eigenfunctions and eigenvalues are those of the standard Brownian bridge on $[0, 1]$: $\phi_m(s) = \sqrt{2} \sin(\pi ms)$; $\nu_m = \frac{1}{\pi m}$. The $\ker(K^X)$ is close to $\mathcal{P}_{s\perp}$ in this case as no linear combination of a finite set of these basis functions yields an exactly constant or linear function.
 - *Poly(1+)*, *Poly(2+)*, *Poly(-2)*: eigenfunctions are orthogonal polynomials of degree 1 [2] to M [$M+1$] for *Poly(1+)* [*Poly(2+)*], so that $\ker(K^X)$ includes the (complete) nullspace of the rank-deficient penalties, i.e., the constant [and linear] functions. *Poly(-2)* has polynomial eigenfunctions of degree $\{0, 2, 3, 4, \dots, M+1\}$

so that $\ker(K^X)$ overlaps the nullspace of the second differences penalty but not the first differences penalty. All three processes are associated with linearly decreasing ν_m .

From top to bottom, these processes become increasingly more “antagonistic” in the sense that 1) the kernels of these eigenfunction systems move increasingly closer to the kernels of the penalties we consider and 2) more quickly decreasing eigenvalues result in lower effective rank of the observed $X(s)$.

- **coefficient functions** $\beta(s, t) = \mathbf{B}_s \Theta \mathbf{B}_t^T$ are drawn randomly for each setting. The associated coefficients are drawn as $\text{vec}(\Theta) \sim N(\mathbf{0}, (0.1\mathbf{I} + \mathbf{P}(\lambda_s, \lambda_t))^{-1})$, where $\mathbf{P}(\lambda_s, \lambda_t)$ is a first order difference tensor penalty matrix.
 - The marginal B-spline bases B_s and B_t have either 4 or 8 basis functions for each direction.
 - $\lambda_s = \lambda_t$ are either 0.1 or 1.

This generates coefficient surfaces of varying complexity and roughness. We do not fit models where the basis used to generate $\beta(s, t)$ is larger than that used to estimate $\beta(s, t)$ since that could introduce a distracting approximation error not relevant to the issues at hand.

In order to make results comparable across the different settings, we use the relative integrated mean squared errors $\text{rIMSE}_\beta = \frac{\int (\beta(s, t) - \hat{\beta}(s, t))^2 ds dt}{\int \beta(s, t)^2 ds dt}$ and $\text{rIMSE}_Y = \frac{1}{N} \sum_{i=1}^N \frac{\int (\hat{Y}_i(t) - E(Y_i(t)))^2 dt}{\int (Y_i(t) - \overline{Y_i(t)})^2 dt}$, where $\overline{Y_i(t)}$ is the mean of $Y_i(t)$ over t .

4.2 Results

Identifiability The estimation accuracy for $\hat{Y}(t)$ (not shown) is excellent across the board even for the very noisy settings, with a maximal relative integrated mean square error of 0.41 and a median of 0.00036. Estimation accuracy for $\hat{\beta}(s, t)$, however, varies wildly over a range of 18 magnitudes between 7.2×10^{-9} and 1.8×10^{10} . Further analysis shows that the simulation study design succeeds in creating the identifiability issues described by the results in Section 3.2. To quantify the severity of identifiability issues, we compute rank correlations between rIMSE_β and rIMSE_Y over the 20 replicates of each simulation setting. As expected, we observe low or even negative correlations mostly for settings in which \mathbf{D}_s is rank-deficient. This effect increases both for lower signal-to-noise ratios and for more complex true shapes of $\beta(s, t)$. For intuition, consider that the “best” solution for (10) for any given error will be the smoothest surface (i.e., the one with the smallest penalty term) in the set of surfaces that can be generated by adding functions from $\ker(K^X)$ to any initial $\beta(s, t)$ with the given error. This may be quite close or quite far from the true $\beta(s, t)$, depending on the specific setting, with more noisy data and more complex true shapes more likely to result in fits that are quite far from the truth and still producing good model fit.

Estimation performance for $\beta(s, t)$ Figure 1 shows the estimation errors for coefficient surfaces generated with $\text{SNR}_\epsilon = 10$. The right column shows results for numerically rank deficient \mathbf{D}_s (i.e., condition number $\kappa(\mathbf{D}_s^\top \mathbf{D}_s) \geq 10^6$), the left column for designs with $\kappa(\mathbf{D}_s^\top \mathbf{D}_s) < 10^6$. The top row shows results for first differences penalty, bottom row for second differences penalty. Boxplots are grouped by the amount of overlap between $\ker(K^X)$ and $\mathcal{P}_{s\perp}$ as computed by $\bigcap_{X\perp} \mathcal{P}_{s\perp}$ (see (12)), color-coded for the different processes the $X(s)$ -trajectories are sampled from. Results for $\text{SNR}_\epsilon = 2$ and 10^3 were qualitatively very similar – errors obviously become larger for noisier data but the pattern shown in Figure 1 remains the same. Note that relative estimation errors below ≈ 0.01 correspond to estimates that are visually indistinguishable from the true surfaces, and that errors below ≈ 0.1 (thick black horizontal line) usually preserve most essential features of the true $\beta(s, t)$

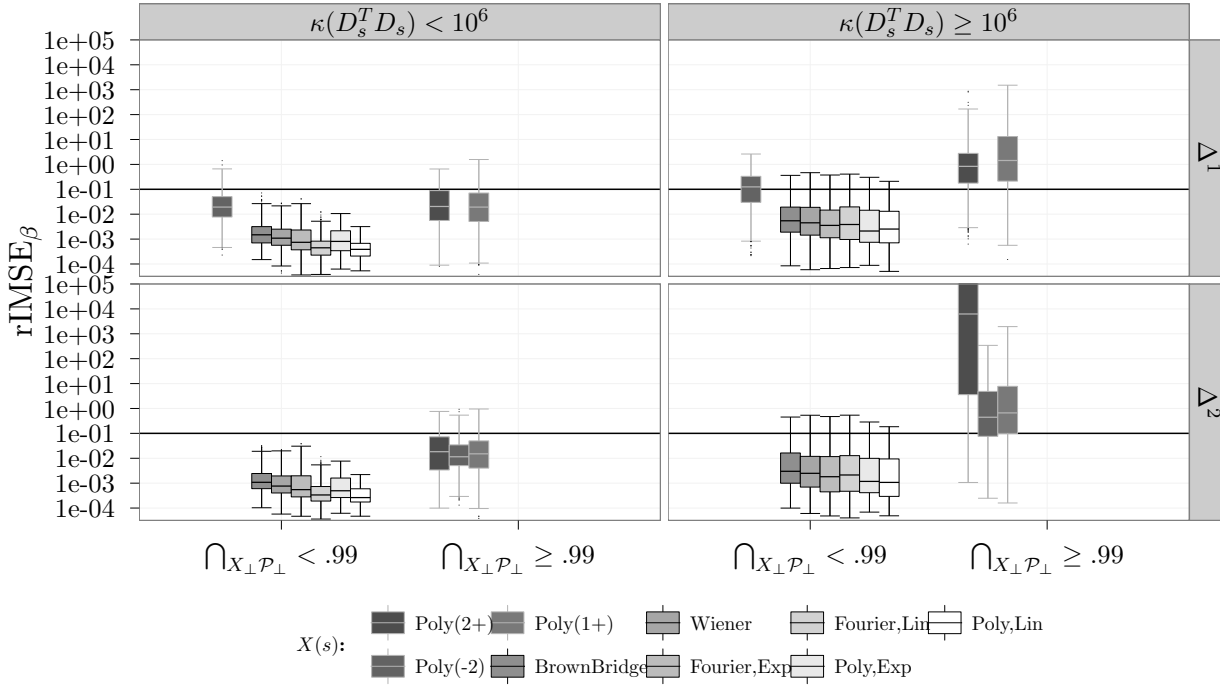


Figure 1: Boxplots for relative integrated mean square error rIMSE_β for all 18000 results for $\text{SNR}_\epsilon = 10$. Columns show results for full rank \mathbf{D}_s versus numerically rank deficient \mathbf{D}_s . Rows show results for the first and second order difference penalties. Boxplots grouped by overlap between $\ker(K^X)$ and $\mathcal{P}_{s\perp}$ as computed by $\bigcap_{X\perp\mathcal{P}_{s\perp}}$, color-coded for the different processes the $X(s)$ are sampled from. Vertical axis on \log_{10} -scale, extreme errors $> 10^5$ for $\text{Poly}(2+)$ are cut off.

well. Results with $\text{rIMSE}_\beta > 1$ bear little resemblance to the “true” function. Also recall that all of these fits, with rIMSE_β values varying by more than 14 magnitudes, resulted in a comparatively small range of rIMSE_Y between 10^{-5} and 10^{-3} . Closer inspection of results shows that the extremely large errors for $\text{Poly}(1+)$, $\text{Poly}(-2)$ and $\text{Poly}(2+)$ are caused by the expected behavior: estimates are shifted by functions from the overlap of $\ker(K^X)$ and $\mathcal{P}_{s\perp}$. The top row of Figure 3 shows an example of this behaviour: the estimate for first order difference penalty is shifted by a large constant, while the estimate for the second order difference penalty is shifted by both a constant and a huge linear trend in s -direction. The fitted values of all models shown in Figure 3 are practically identical.

These results mostly corroborate the results derived in Section 3 – we see that:

- Serious errors $\text{rIMSE}_\beta > 0.1$ are very rare for both full-rank and rank-deficient \mathbf{D}_s if the generating process for $X(s)$ is not antagonistic in the sense that $\ker(K^X) \cap \mathcal{P}_{s\perp} = \{0\}$, i.e. for the *Poly*, *Fourier*, *Wiener* and *BrownBridge* processes.
- As long as $\ker(K^X) \cap \mathcal{P}_{s\perp} = \{0\}$ (approximated numerically by the criterion that $\bigcap_{X\perp\mathcal{P}_{s\perp}} < 0.99$), regularization of the estimated coefficient surface allows us to achieve good estimates even if the unpenalized regression model *per se* would not be identifiable due to rank deficiency of \mathbf{D}_s .
- The larger $\mathcal{P}_{s\perp}$ (top to bottom), and the closer $\ker(K^X)$ is to $\mathcal{P}_{s\perp}$ (boxplots on the left in each panel), the larger the likelihood of very wrong estimates and the average rIMSE_β .

Diagnostics As in Figure 1, we use $\bigcap_{X\perp\mathcal{P}_{s\perp}}$ (see (12)) as a measure of the overlap between $\ker(K^X)$ and $\mathcal{P}_{s\perp}$. We consider a replicate to be “flagged” as problematic if both $\bigcap_{X\perp\mathcal{P}_{s\perp}} \geq 0.99$ and $\kappa(\mathbf{D}_s^T, \mathbf{D}_s) > 10^6$.

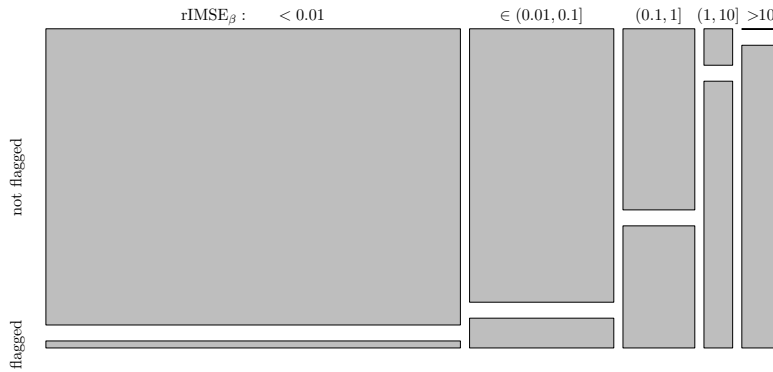


Figure 2: Mosaicplot for contingency table of "flagged" replicates and categorized $rIMSE_\beta$.

Figure 2 shows a mosaic plot of the contingency table of "flagged" replicates and categorized $rIMSE_\beta$. While the sensitivity for identifying replicates with $rIMSE_\beta > 1$ is 0.95, the specificity is only 0.45. The sensitivity for identifying replicates with $rIMSE_\beta > 0.1$ is 0.67, the specificity is 0.2. These fairly low specificities indicate that the penalized approach to function-on-function regression discussed here can outperform theoretical expectations and frequently finds good solutions even in very difficult settings. Total accuracy for identifying settings with $rIMSE_\beta > 0.1$ is 0.9 and 0.92 for $rIMSE_\beta > 1$. The positive predictive value (precision) of the criterion for $rIMSE_\beta > 1$ is 0.55, while it is 0.8 for $rIMSE_\beta > .1$. The negative predictive value of the criterion for $rIMSE_\beta > 1$ is 0.99, while it is 0.92 for $rIMSE_\beta > .1$. Also note that certainly not all errors in the $(0.1, 1]$ -range are due to identifiability issues, so not all of the (rare) non-detections are failures of the criterion.

Performance of modified penalties We broadened the scope of our simulation study by additionally comparing the performance of the non-standard penalties introduced in Section 3.3:

- a full-rank ridge penalty (" Δ^0 "),
- the modified full-rank roughness penalties as suggested by Marra and Wood (2011); in our case we used both full-rank first order differences penalties (" $\tilde{\Delta}^1$ ") and full-rank second order differences penalties (" $\tilde{\Delta}^2$ "),
- and the FPC-based penalty of James and Silverman (2005) (" $ke(K^X)$ (FAME)"). We replace $\hat{\nu}_m$ by $\max(\hat{\nu}_m, 10^{-10}\hat{\nu}_1)$ in order to remove any (numerically) zero or negative eigenvalues.

Figure 4 shows the $rIMSE_\beta$ for $SNR_\epsilon = 10$ for the different $X(s)$ -processes and penalties. Note that, in contrast to Figure 1, colours now represent the different penalties, not the different $X(s)$ -processes. Boxplots for Δ^1 and Δ^2 contain the same results as those shown in Figure 1.

The full-rank difference penalties $\tilde{\Delta}^1$ and $\tilde{\Delta}^2$ seem to drastically reduce the size and likelihood of severe estimation errors in the difficult settings, especially compared to Δ^2 . We also incur slightly worse estimates for the easier settings, but these differences are small and hardly relevant in practical terms. To see this, compare the four rightmost boxplots (lighter greys) in each group. Both the ridge (Δ^0) and FAME penalties are not competitive for numerically rank deficient D_s for most of the $X(s)$ -processes. To see this, compare the two leftmost boxplots (dark greys) in each group with the remainder.

Figure 3 shows illustrative exemplary fits for these modified penalties in the rightmost panel of the top row and the panels in the bottom row. While $rIMSE_Y$ is very similar for all 6 penalties, the shapes of the corresponding coefficient surface estimates differ from each other in the expected fashion: It is clear to see that the ridge penalty induces a strong bias towards small $|\beta(s, t)|$ and tends to undersmooth the estimated surface, with

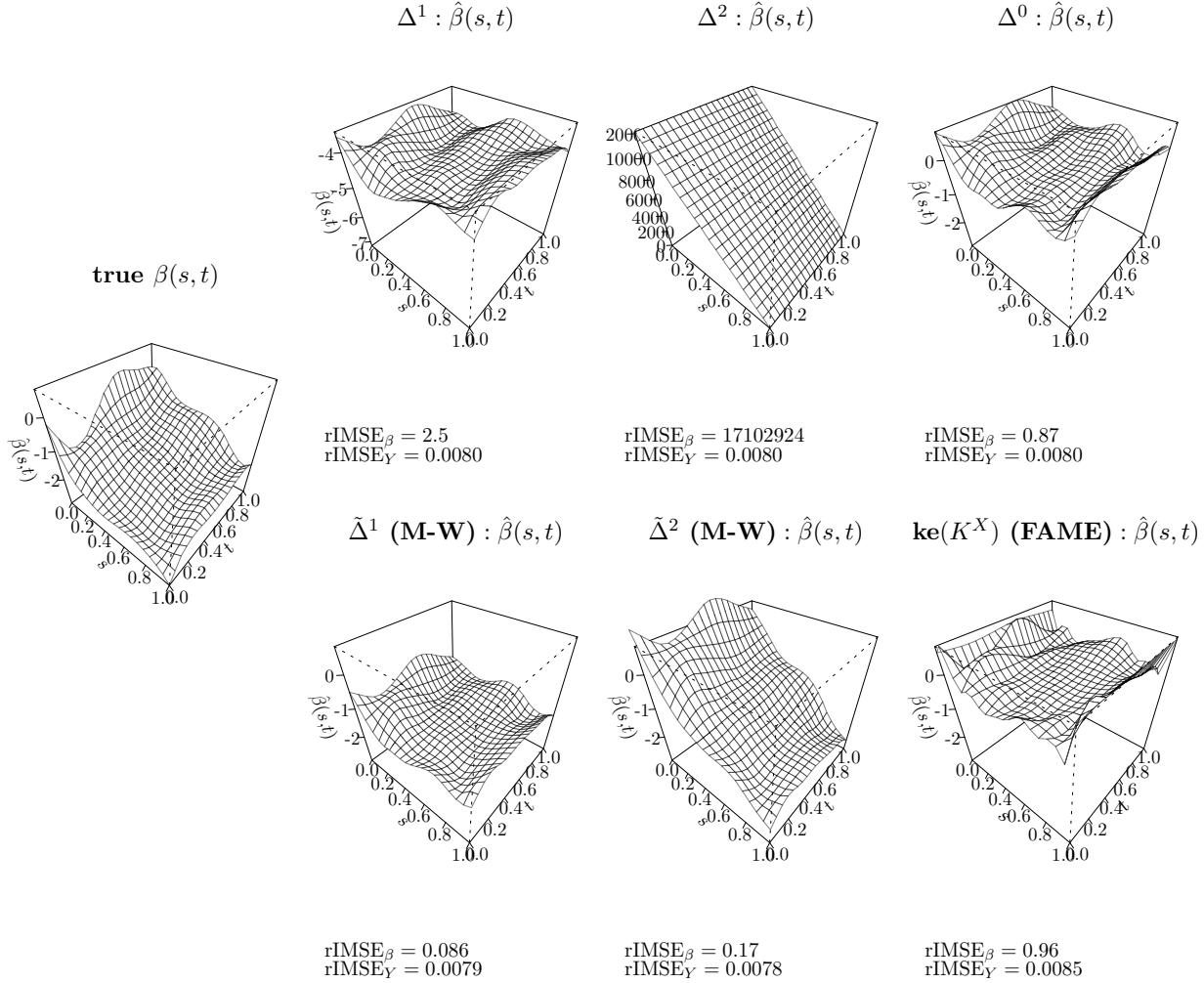


Figure 3: Example estimates for different penalties in a very difficult setting with $X(s)$ from $\text{Poly}(2+)$, $M = 5$, $K_s = 12$, $\text{SNR}_\varepsilon = 2$. Center left panel: true coefficient surface. Top row, left to right: first order difference penalty, second order difference penalty, ridge penalty. Bottom row, left to right: full-rank first order difference penalty, full-rank second order difference penalty, FAME penalty. Note the different z-axis scales in the left and middle panels of the top row. Subtitles give rIMSE_β and rIMSE_γ for each fit.

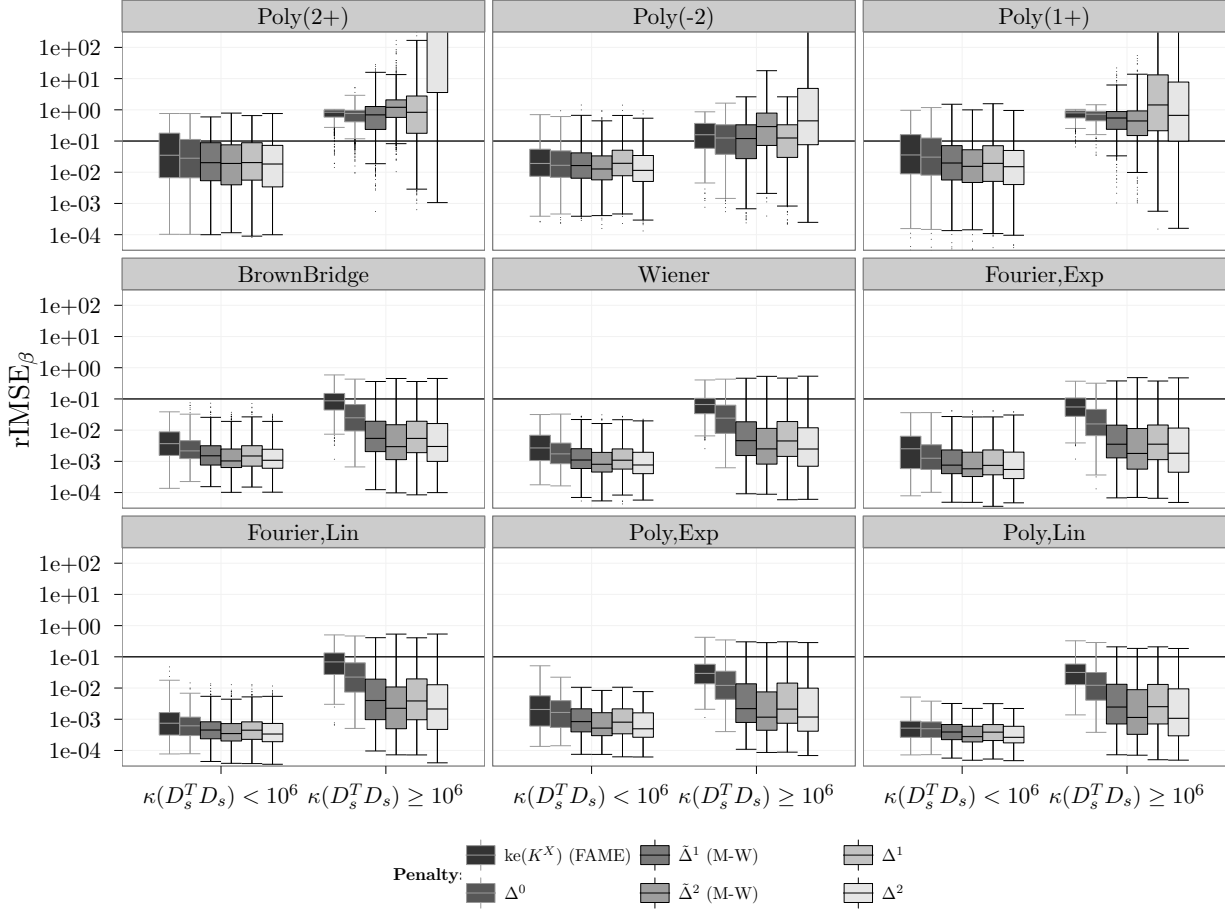


Figure 4: rIMSE_{β} for $\text{SNR}_{\varepsilon} = 10$ for the 8 $X(s)$ -processes (panels) and the 6 different penalties (color). Separate boxplots in each panel for settings with numerically rank deficient D_s ($\kappa(D_s^T D_s) \geq 10^6$) versus settings with full rank D_s .

a spurious wiggle for $s > .9$. We observe similar effects for the FAME-penalty in this example. Compared to the result for Δ^1 , the fit for $\tilde{\Delta}^1$ is much closer to the level of the true $\beta(s, t)$ and does not add a spurious constant to the fit. Similar remarks apply for the comparison between Δ^2 and $\tilde{\Delta}^2$: The huge spurious linear trend in s is almost completely removed. For all 6 penalties, $\kappa(\mathbf{D}_s^T, \mathbf{D}_s) > 10^6$ in this example. The overlap criterion $\bigcap_{X \perp \mathcal{P}_{s \perp}}$ is 1 for Δ_1 , 2 for Δ_2 and 0 for all others.

Summary of simulation results We can draw the following conclusions based on the entirety of simulation results:

- the potential for extreme estimation errors is large for $X(s)$ -processes with low effective rank whose $\ker(K^X)$ is not disjoint from $\mathcal{P}_{s \perp}$.
- there is no strong positive correlation between accuracy of fitted values (rIMSE_Y) and accuracy of the estimated coefficient surface (rIMSE_β) for rank deficient \mathbf{D}_s . Even extremely wrong estimates of $\beta(s, t)$ can yield good model fits.
- calculating $\bigcap_{X \perp \mathcal{P}_{s \perp}}$ and $\kappa(\mathbf{D}_s^T, \mathbf{D}_s)$ yields a suitable criterion that can diagnose problematic settings reliably, albeit with a substantial rate of false alarms.
- the full-rank roughness penalties stabilize estimates in non-identifiable settings without diminishing accuracy in other settings by a relevant amount.

5 Conclusion and Discussion

Coefficient surface estimates in spline-based function-on-function-regression (1) can suffer from identifiability problems if the span of the marginal basis for the coefficient surface over a functional covariate's domain overlaps the kernel of the covariance operator of the functional covariates. A rank deficient design matrix can occur in particular if the covariance operator is of effective rank smaller than the number of marginal basis functions - either because the number of eigenfunctions with non-zero eigenvalues is truly below the number of marginal basis functions, or if eigenvalues of the covariance operator decrease too rapidly compared to the noise level of the data.

In practice, spline based approaches are fitted with a regularization penalty corresponding to a smoothness assumption on the coefficient surface. We have shown that identifiability problems persist if, and only if, in addition to a rank deficiency of the design matrix, the kernel of the functional predictor's covariance $\ker(K^X)$ overlaps the function space $\mathcal{P}_{s \perp}$ spanned by parameter vectors in the nullspace of the spline's roughness penalty.

A lack of identifiability also implies a lack of correlation between accuracy of the coefficient estimates and goodness of fit for the responses. As this extends to prediction errors for out-of-sample data from the same process, it is usually not possible to detect identifiability issues for a given data set based on subsampling or cross-validation schemes. Instead, based on theoretical considerations and simulation results, we have identified two easily computable diagnostic criteria in order to detect non-identifiable model specifications before estimation. The criteria combine the condition number of a partial design matrix with a measure of the amount of overlap between the kernel of the functional predictor's covariance and the nullspace of the penalty. Non-identifiability may in particular be an issue if both criteria are indicative of a problematic setting, or if the partial design matrix is numerically rank deficient and the penalty smoothing parameter is estimated to be close to zero. If a non-identifiable model specification is discovered, we recommend that practitioners choose a modified full-rank roughness penalty to safeguard against spurious estimates. The `pffr`-function in the `refund` package uses a full-rank first order differences penalty by default and incorporates the diagnostic checks developed and evaluated in this work.

Another practical consequence of our results is that pre-processing methods for functional covariates should be avoided if they reduce their effective rank (such as pre-smoothing with low-dimensional bases) or if they increase the amount of overlap between $\ker(K^X)$ and $\mathcal{P}_{s\perp}$ (such as curve-wise centering).

Jointly, these provisions seem to be sufficient to diagnose and safeguard against most serious artifacts of non-identifiability in practice. Our results indicate that in many cases, penalization allows the reasonable estimation of coefficient surfaces that are not identifiable in the theoretical model under an additional smoothness assumption, avoiding instead the common assumption that the estimated coefficient surface lies in the span of the covariance operator of the covariate.

This work drives home the point that we cannot hope to reliably estimate arbitrarily complex effect shapes from functional covariates with low information content. In that sense, the potential oversmoothing of a truly complex surface shape by a full-rank penalty is simply following a principle of parsimony. At the same time, substantial interpretation of coefficient surface estimates derived from rank-deficient designs has the potential to be very misleading. This is a significant challenge for the emerging field of functional regression methods, at least in applications where these methods are used not only for prediction, but also for inferring and understanding the underlying processes. Similar results are expected to hold for the simpler case of penalized scalar-on-function regression models.

Functional principal component regression approaches do not suffer from the potential identifiability issues discussed here, but they do so at the price of restricting the estimated coefficient surface to the span of the estimated functional principal components. The authors have begun work on a systematic comparison of the strengths and weaknesses of penalized spline based and functional principal component based approaches for function-on-function regression in a variety of scenarios.

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