Supplementary information to The Kepler problem from a differential geometry point of view

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Introduction

This document consists of detailed calculations for the proofs in the main paper.

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(V.5.) Proof.

$\widetilde{\boldsymbol{p}} = (\mu+1)d\widetilde{\boldsymbol{q}} = (\mu+1)\left[\frac{\mu}{\mu+1}d\boldsymbol{q} + \frac{1}{\mu+1}d\boldsymbol{q}'\right] = \mu d\boldsymbol{q} + d\boldsymbol{q}' = \boldsymbol{p} + \boldsymbol{p}'.$

$\widetilde{\boldsymbol{p}}' = \frac{\mu}{\mu+1}d\widetilde{\boldsymbol{q}}' = \frac{\mu}{\mu+1}(d\boldsymbol{q} - d\boldsymbol{q}') = \frac{1}{\mu+1}(\boldsymbol{p} - \mu\boldsymbol{p}').$

$\widetilde{\boldsymbol{q}} = \frac{\mu\boldsymbol{q}+\boldsymbol{q}'}{\mu+1}$ (by definition)

$\Rightarrow (\mu+1)\widetilde{\boldsymbol{q}} = \mu\boldsymbol{q} + \boldsymbol{q}' = \mu(\widetilde{\boldsymbol{q}}' + \boldsymbol{q}') + \boldsymbol{q}' = \mu\widetilde{\boldsymbol{q}}' + (\mu+1)\boldsymbol{q}',$

$\Rightarrow \boldsymbol{q}' = \widetilde{\boldsymbol{q}} - \frac{\mu}{\mu+1}\widetilde{\boldsymbol{q}}'.$

$\boldsymbol{q} = \widetilde{\boldsymbol{q}}' + \boldsymbol{q}' = \widetilde{\boldsymbol{q}}' + \widetilde{\boldsymbol{q}} - \frac{\mu}{\mu+1}\widetilde{\boldsymbol{q}}' = \frac{(\mu+1)\widetilde{\boldsymbol{q}}' - \mu\widetilde{\boldsymbol{q}}'}{\mu+1} + \widetilde{\boldsymbol{q}} = \widetilde{\boldsymbol{q}} + \frac{1}{\mu+1}\widetilde{\boldsymbol{q}}'.$

$\widetilde{\boldsymbol{p}}' = \frac{1}{\mu+1}\boldsymbol{p} - \frac{\mu}{\mu+1}\boldsymbol{p}' = \frac{1}{\mu+1}(\widetilde{\boldsymbol{p}} - \boldsymbol{p}') - \frac{\mu}{\mu+1}\boldsymbol{p}' = \frac{1}{\mu+1}\widetilde{\boldsymbol{p}} - \boldsymbol{p}'$

$\Rightarrow \boldsymbol{p}' = \frac{1}{\mu+1}\widetilde{\boldsymbol{p}} - \widetilde{\boldsymbol{p}}'.$

$\boldsymbol{p} = \widetilde{\boldsymbol{p}} - \boldsymbol{p}' = \widetilde{\boldsymbol{p}} - \frac{1}{\mu+1}\widetilde{\boldsymbol{p}} + \widetilde{\boldsymbol{p}}' = \frac{\mu}{\mu+1}\widetilde{\boldsymbol{p}} + \widetilde{\boldsymbol{p}}'.$

$H = \frac{\|\boldsymbol{p}\|^2}{2\mu} + \frac{\|\boldsymbol{p}'\|^2}{2} - \frac{1}{\|\boldsymbol{q}-\boldsymbol{q}'\|}$

$H = \frac{1}{2\mu}\left(\frac{\mu}{\mu+1}\widetilde{\boldsymbol{p}} + \widetilde{\boldsymbol{p}}'\right)^2 + \frac{1}{2}\left(\frac{1}{\mu+1}\widetilde{\boldsymbol{p}} - \widetilde{\boldsymbol{p}}'\right)^2 - \frac{1}{\|\widetilde{\boldsymbol{q}}'\|}$

$H = \frac{1}{2\mu}\left[\left(\frac{\mu}{\mu+1}\right)^2\|\widetilde{\boldsymbol{p}}\|^2 + \frac{2\mu}{\mu+1}\|\widetilde{\boldsymbol{p}}\|\|\widetilde{\boldsymbol{p}}'\| + \|\widetilde{\boldsymbol{p}}'\|^2\right] + \frac{1}{2}\left[\left(\frac{\mu}{\mu+1}\right)^2\|\widetilde{\boldsymbol{p}}\|^2 - \frac{2}{\mu+1}\|\widetilde{\boldsymbol{p}}\|\|\widetilde{\boldsymbol{p}}'\| + \|\widetilde{\boldsymbol{p}}'\|^2\right] - \frac{1}{\|\widetilde{\boldsymbol{q}}'\|}$

$H = \left[\frac{\mu}{2(\mu+1)^2} + \frac{1}{2(\mu+1)^2}\right]\|\widetilde{\boldsymbol{p}}\|^2 + \left[\frac{1}{\mu+1} - \frac{1}{\mu+1}\right]\|\widetilde{\boldsymbol{p}}\|\|\widetilde{\boldsymbol{p}}'\| + \left[\frac{1}{2\mu} + \frac{1}{2}\right]\|\widetilde{\boldsymbol{p}}'\|^2 - \frac{1}{\|\widetilde{\boldsymbol{q}}'\|}$

$H = \frac{1}{2(\mu+1)}\|\widetilde{\boldsymbol{p}}\|^2 + \frac{(\mu+1)}{2\mu}\|\widetilde{\boldsymbol{p}}'\|^2 - \frac{1}{\|\widetilde{\boldsymbol{q}}'\|}.$ ■

VI. Stepwise Reduction

(VI.4.) Proof. (i) According to (II.4.) we have e.g.

$\{L_1, L_2\} = \sum_{i=1}^{n} \left(\frac{\partial L_1}{\partial q_i} \frac{\partial L_2}{\partial p_i} - \frac{\partial L_1}{\partial p_i} \frac{\partial L_2}{\partial q_i} \right)$

$= (0)(q_3) - (0)(-p_3) + (p_3)(0) - (0)(0) + (-p_2)(-q_1) - (q_2)(p_1)$

$= q_1 p_2 - q_2 p_1 = L_3.$

(ii) Because of the Jacobi identity (II.11.), we have $\{\{L_1, L_2\}, H\} + \{\{L_2, H\}, L_1\} + \{\{H, L_1\}, L_2\} = 0$. Since L_1 and L_2 are constants of the motion, $\{L_2, H\} = \{H, L_1\} = 0$, and so

$\{L_3, H\} = \{\{L_1, L_2\}, H\} = 0$. ■

(VI.8.) Proof.

First, remember, for the 2-dimensional Model III:

$H(\boldsymbol{q}, \boldsymbol{p}) = \frac{\|\boldsymbol{p}\|^2}{2\mu} - \frac{1}{\|\boldsymbol{q}\|},$

$L = q_1 p_2 - q_2 p_1.$

Then, a few intermediate steps:

$\frac{\partial L}{\partial q_1} = p_2,$

$\frac{\partial L}{\partial p_1} = -q_2,$

$\frac{\partial L}{\partial q_2} = -p_1,$

$\frac{\partial L}{\partial p_2} = q_1,$

$\frac{\partial H}{\partial q_i} = \frac{\partial}{\partial q_i} \left(\frac{-1}{\|\boldsymbol{q}\|} \right) = -\left(\frac{-1}{2} \right) \frac{1}{\|\boldsymbol{q}\|^3} (2q_i) = \frac{q_i}{\|\boldsymbol{q}\|^3}.$

$\frac{\partial H}{\partial p_i} = \frac{\partial}{\partial p_i} \left(\frac{\|\boldsymbol{p}\|^2}{2\mu} \right) = \frac{2p_i}{2\mu} = \frac{p_i}{\mu},$

The main result:

$\{L, H\} = \frac{\partial L}{\partial q_1} \frac{\partial H}{\partial p_1} - \frac{\partial L}{\partial p_1} \frac{\partial H}{\partial q_1} + \frac{\partial L}{\partial q_2} \frac{\partial H}{\partial p_2} - \frac{\partial L}{\partial p_2} \frac{\partial H}{\partial q_2}$

$= (p_2) \left(\frac{p_1}{\mu} \right) - (-q_2) \left(\frac{q_1}{\|\boldsymbol{q}\|^3} \right) + (-p_1) \left(\frac{p_2}{\mu} \right) - (q_1) \left(\frac{q_2}{\|\boldsymbol{q}\|^3} \right) = 0.$

Now a few more intermediate results:

$\frac{\partial A_i}{\partial q_i} = \frac{1}{\|\boldsymbol{q}\|} + q_i \left(\frac{-1}{2} \right) \frac{1}{\|\boldsymbol{q}\|^3} (2q_i) + \frac{1}{\mu} ({p_i}^2 - \|\boldsymbol{p}\|^2) = \frac{\|\boldsymbol{q}\|^2 - {q_i}^2}{\|\boldsymbol{q}\|^3} + \frac{1}{\mu} (-{p_j}^2) = \frac{{q_j}^2}{\|\boldsymbol{q}\|^3} + \frac{-{p_j}^2}{\mu}.$

$\frac{\partial A_i}{\partial q_j} = q_i \left(\frac{-1}{2} \right) \frac{1}{\|\boldsymbol{q}\|^3} (2q_j) + \frac{1}{\mu} (p_i p_j) = \frac{-q_i q_j}{\|\boldsymbol{q}\|^3} + \frac{p_i p_j}{\mu}.$

$\frac{\partial A_i}{\partial p_i} = \frac{1}{\mu} [(\boldsymbol{q} \cdot \boldsymbol{p}) + q_i p_i - q_i (2p_i)] = \frac{1}{\mu} [q_i p_i + q_j p_j + q_i p_i - q_i (2p_i)] = \frac{q_j p_j}{\mu}.$

$\frac{\partial A_i}{\partial p_j} = \frac{1}{\mu} (q_j p_i - 2q_i p_j).$

The main result:

$$\{A_1,H\} = \frac{\partial A_1}{\partial q_1}\frac{\partial H}{\partial p_1} - \frac{\partial A_1}{\partial p_1}\frac{\partial H}{\partial q_1} + \frac{\partial A_1}{\partial q_2}\frac{\partial H}{\partial p_2} - \frac{\partial A_1}{\partial p_2}\frac{\partial H}{\partial q_2}$$
$$= \left(\frac{q_2^2}{\|\boldsymbol{q}\|^3} + \frac{-p_2^2}{\mu}\right)\left(\frac{p_1}{\mu}\right) - \left(\frac{q_2p_2}{\mu}\right)\left(\frac{q_1}{\|\boldsymbol{q}\|^3}\right) + \left(\frac{-q_1q_2}{\|\boldsymbol{q}\|^3} + \frac{p_1p_2}{\mu}\right)\left(\frac{p_2}{\mu}\right) - \left(\frac{q_2p_1-2q_1p_2}{\mu}\right)\left(\frac{q_2}{\|\boldsymbol{q}\|^3}\right)$$
$$= \frac{1}{\mu\|\boldsymbol{q}\|^3}(q_2^2p_1 - q_1q_2p_2 - q_1q_2p_2 - q_2^2p_1 + 2q_1q_2p_2) + \frac{1}{\mu^2}(-p_1p_2^2 + p_1p_2^2) = 0.$$
$$\{A_1,H\} = 0.$$
$$\{A_2,H\} = \frac{\partial A_2}{\partial q_1}\frac{\partial H}{\partial p_1} - \frac{\partial A_2}{\partial p_1}\frac{\partial H}{\partial q_1} + \frac{\partial A_2}{\partial q_2}\frac{\partial H}{\partial p_2} - \frac{\partial A_2}{\partial p_2}\frac{\partial H}{\partial q_2}$$
$$= \left(\frac{-q_1q_2}{\|\boldsymbol{q}\|^3} + \frac{p_1p_2}{\mu}\right)\left(\frac{p_1}{\mu}\right) - \left(\frac{q_1p_2-2q_2p_1}{\mu}\right)\left(\frac{q_1}{\|\boldsymbol{q}\|^3}\right) + \left(\frac{q_1^2}{\|\boldsymbol{q}\|^3} - \frac{p_1^2}{\mu}\right)\left(\frac{p_2}{\mu}\right) - \left(\frac{q_1p_1}{\mu}\right)\left(\frac{q_2}{\|\boldsymbol{q}\|^3}\right)$$
$$= \frac{1}{\mu\|\boldsymbol{q}\|^3}(-q_1q_2p_1 - q_1^2p_2 + 2q_1q_2p_1 + q_1^2p_2 - q_1q_2p_1) + \frac{1}{\mu^2}(p_1^2p_2 - p_1^2p_2) = 0.$$
$$\{A_2,H\} = 0.$$
$$dA_1 = \frac{\partial A_1}{\partial q_1}dq_1 + \frac{\partial A_1}{\partial q_2}dq_2 + \frac{\partial A_1}{\partial p_1}dp_1 + \frac{\partial A_1}{\partial p_2}dp_2$$
$$= \left(\frac{q_2^2}{\|\boldsymbol{q}\|^3} + \frac{-p_2^2}{\mu}\right)dq_1 + \left(\frac{-q_1q_2}{\|\boldsymbol{q}\|^3} + \frac{p_1p_2}{\mu}\right)dq_2 + \left(\frac{q_2p_2}{\mu}\right)dp_1 + \left(\frac{q_2p_1-2q_1p_2}{\mu}\right)dp_2.$$
$$dA_2 = \frac{\partial A_2}{\partial q_1}dq_1 + \frac{\partial A_2}{\partial q_2}dq_2 + \frac{\partial A_2}{\partial p_1}dp_1 + \frac{\partial A_2}{\partial p_2}dp_2$$
$$= \left(\frac{-q_1q_2}{\|\boldsymbol{q}\|^3} + \frac{p_1p_2}{\mu}\right)dq_1 + \left(\frac{q_1^2}{\|\boldsymbol{q}\|^3} + \frac{-p_1^2}{\mu}\right)dq_2 + \left(\frac{q_1p_2-2q_2p_1}{\mu}\right)dp_1 + \left(\frac{q_1p_1}{\mu}\right)dp_2. \blacksquare$$

(VI.10.) Proof.

First, we'll solve one equation for x_3 and the other for x_4 and then substitute the expression for x_3 into the expression for x_4, resulting in x_4 in terms of x_1 and x_2. One more substitution then gives us x_3 in terms of x_1 and x_2.

$$dA_1(X) = \left(\frac{q_2^2}{\|\boldsymbol{q}\|^3} + \frac{-p_2^2}{\mu}\right)x_1 + \left(\frac{-q_1q_2}{\|\boldsymbol{q}\|^3} + \frac{p_1p_2}{\mu}\right)x_2 + \left(\frac{q_2p_2}{\mu}\right)x_3 + \left(\frac{q_2p_1-2q_1p_2}{\mu}\right)x_4,$$
$$dA_1(X) = 0 \Rightarrow$$
$$x_3 = \frac{\mu}{q_2p_2}\left[\left(\frac{-q_2^2}{\|\boldsymbol{q}\|^3} + \frac{p_2^2}{\mu}\right)x_1 + \left(\frac{q_1q_2}{\|\boldsymbol{q}\|^3} - \frac{p_1p_2}{\mu}\right)x_2 + \left(\frac{-q_2p_1+2q_1p_2}{\mu}\right)x_4\right].$$
$$dA_2(X) = 0 \Rightarrow$$
$$x_4 = \frac{\mu}{q_1p_1}\left[\left(\frac{q_1q_2}{\|\boldsymbol{q}\|^3} - \frac{p_1p_2}{\mu}\right)x_1 + \left(\frac{-q_1^2}{\|\boldsymbol{q}\|^3} + \frac{p_1^2}{\mu}\right)x_2 + \left(\frac{-q_1p_2+2q_2p_1}{\mu}\right)x_3\right].$$

Now we substitute the expression for x_3 into the expression for x_4 and use the common denominator of $q_1q_2p_1p_2$.

$$x_4 = \frac{\mu}{q_1q_2p_1p_2}\left(\frac{q_1q_2^2p_2}{\|\boldsymbol{q}\|^3} - \frac{q_2p_1p_2^2}{\mu}\right)x_1 + \frac{\mu}{q_1q_2p_1p_2}\left(\frac{-q_1^2q_2p_2}{\|\boldsymbol{q}\|^3} + \frac{q_2p_1^2p_2}{\mu}\right)x_2$$
$$+\frac{\mu}{q_1q_2p_1p_2}(-q_1p_2 + 2q_2p_1)\left(\frac{-q_2^2}{\|\boldsymbol{q}\|^3} + \frac{p_2^2}{\mu}\right)x_1 + \frac{\mu}{q_1q_2p_1p_2}(-q_1p_2 + 2q_2p_1)\left(\frac{q_1q_2}{\|\boldsymbol{q}\|^3} - \frac{p_1p_2}{\mu}\right)x_2$$
$$+\frac{\mu}{q_1q_2p_1p_2}(-q_1p_2 + 2q_2p_1)\left(\frac{-q_2p_1+2q_1p_2}{\mu}\right)x_4,$$
$$x_4 = \frac{\mu}{q_1q_2p_1p_2}\left(\frac{q_1q_2^2p_2}{\|\boldsymbol{q}\|^3} - \frac{q_2p_1p_2^2}{\mu} + \frac{q_1q_2^2p_2}{\|\boldsymbol{q}\|^3} - \frac{2q_2^3p_1}{\|\boldsymbol{q}\|^3} - \frac{q_1p_2^3}{\mu} + \frac{2q_2p_1p_2^2}{\mu}\right)x_1$$
$$+\frac{\mu}{q_1q_2p_1p_2}\left(\frac{-q_1^2q_2p_2}{\|\boldsymbol{q}\|^3} + \frac{q_2p_1^2p_2}{\mu} - \frac{q_1^2q_2p_2}{\|\boldsymbol{q}\|^3} + \frac{q_1p_1p_2^2}{\mu} + \frac{2q_1q_2^2p_1}{\|\boldsymbol{q}\|^3} - \frac{2q_2p_1^2p_2}{\mu}\right)x_2$$

$$+\frac{\mu}{q_1q_2p_1p_2}\left(\frac{q_1q_2p_1p_2-2q_1{}^2p_2{}^2-2q_2{}^2p_1{}^2+4q_1q_2p_1p_2}{\mu}\right)x_4,$$

Now we can multiply both sides by $\frac{q_1q_2p_1p_2}{\mu}$ and collect terms.

$$0=\left(\frac{2q_1q_2{}^2p_2-2q_2{}^3p_1}{\|\boldsymbol{q}\|^3}+\frac{-q_1p_2{}^3+q_2p_1p_2{}^2}{\mu}\right)x_1$$
$$+\left(\frac{-2q_1{}^2q_2p_2+2q_1q_2{}^2p_1}{\|\boldsymbol{q}\|^3}+\frac{q_1p_1p_2{}^2-q_2p_1{}^2p_2}{\mu}\right)x_2$$
$$+\left(\frac{-2(q_1p_2-q_2p_1)^2}{\mu}\right)x_4,$$

Now we divide both sides by $2L=2(q_1p_2-q_2p_1)$,

$$0=\left(\frac{q_2{}^2}{\|\boldsymbol{q}\|^3}+\frac{-p_2{}^2}{2\mu}\right)x_1+\left(\frac{-q_1q_2}{\|\boldsymbol{q}\|^3}+\frac{p_1p_2}{2\mu}\right)x_2+\left(\frac{-(q_1p_2-q_2p_1)}{\mu}\right)x_4,$$

$$x_4=\frac{\mu}{L}\left[\left(\frac{q_2{}^2}{\|\boldsymbol{q}\|^3}-\frac{p_2{}^2}{2\mu}\right)x_1+\left(\frac{-q_1q_2}{\|\boldsymbol{q}\|^3}+\frac{p_1p_2}{2\mu}\right)x_2\right],$$

$$x_4=\gamma_{41}x_1+\gamma_{42}x_2.$$

Now we can substitute this expression for x_4 into the original expression for x_3.

$$x_3=\frac{\mu}{q_2p_2}\left(\frac{-q_2{}^2}{\|\boldsymbol{q}\|^3}+\frac{p_2{}^2}{\mu}\right)x_1+\frac{\mu}{q_2p_2}\left(\frac{q_1q_2}{\|\boldsymbol{q}\|^3}-\frac{p_1p_2}{\mu}\right)x_2$$
$$+\frac{\mu}{q_2p_2}\left(\frac{-q_2p_1+2q_1p_2}{\mu}\right)\frac{\mu}{L}\left(\frac{q_2{}^2}{\|\boldsymbol{q}\|^3}+\frac{-p_2{}^2}{2\mu}\right)x_1+\frac{\mu}{q_2p_2}\left(\frac{-q_2p_1+2q_1p_2}{\mu}\right)\frac{\mu}{L}\left(\frac{-q_1q_2}{\|\boldsymbol{q}\|^3}+\frac{p_1p_2}{2\mu}\right)x_2,$$

$$x_3=\frac{\mu}{q_2p_2}\left[\frac{-q_2{}^2}{\|\boldsymbol{q}\|^3}+\frac{p_2{}^2}{\mu}+\left(\frac{-q_2p_1+2q_1p_2}{\mu}\right)\frac{\mu}{L}\left(\frac{q_2{}^2}{\|\boldsymbol{q}\|^3}+\frac{-p_2{}^2}{2\mu}\right)\right]x_1$$
$$+\frac{\mu}{q_2p_2}\left[\frac{q_1q_2}{\|\boldsymbol{q}\|^3}-\frac{p_1p_2}{\mu}+\left(\frac{-q_2p_1+2q_1p_2}{\mu}\right)\frac{\mu}{L}\left(\frac{-q_1q_2}{\|\boldsymbol{q}\|^3}+\frac{p_1p_2}{2\mu}\right)\right]x_2,$$

$$x_3=\frac{\mu}{q_2p_2}\left[\frac{-q_2{}^2}{\|\boldsymbol{q}\|^3}+\frac{p_2{}^2}{\mu}+\frac{-q_2{}^3p_1+2q_1q_2{}^2p_2}{L\|\boldsymbol{q}\|^3}+\frac{q_2p_1p_2{}^2-2q_1p_2{}^3}{2\mu L}\right]x_1$$
$$+\frac{\mu}{q_2p_2}\left[\frac{q_1q_2}{\|\boldsymbol{q}\|^3}-\frac{p_1p_2}{\mu}+\frac{q_1q_2{}^2p_1-2q_1{}^2q_2p_2}{L\|\boldsymbol{q}\|^3}+\frac{-q_2p_1{}^2p_2+2q_1p_1p_2{}^2}{2\mu L}\right]x_2,$$

Now we need to multiply a few terms by $L=(q_1p_2-q_2p_1)$, in order to get the common denominators of $L\|\boldsymbol{q}\|^3$ and $2\mu L$.

$$x_3=\frac{\mu}{q_2p_2}\left[\frac{-q_1q_2{}^2p_2+q_2{}^3p_1-q_2{}^3p_1+2q_1q_2{}^2p_2}{L\|\boldsymbol{q}\|^3}+\frac{2q_1p_2{}^3-2q_2p_1p_2{}^2+q_2p_1p_2{}^2-2q_1p_2{}^3}{2\mu L}\right]x_1$$
$$+\frac{\mu}{q_2p_2}\left[\frac{q_1{}^2q_2p_2-q_1q_2{}^2p_1+q_1q_2{}^2p_1-2q_1{}^2q_2p_2}{L\|\boldsymbol{q}\|^3}+\frac{-2q_1p_1p_2{}^2+2q_2p_1{}^2p_2-q_2p_1{}^2p_2+2q_1p_1p_2{}^2}{2\mu L}\right]x_2,$$

$$x_3=\frac{\mu}{q_2p_2}\left[\frac{q_1q_2{}^2p_2}{L\|\boldsymbol{q}\|^3}+\frac{-q_2p_1p_2{}^2}{2\mu L}\right]x_1+\frac{\mu}{q_2p_2}\left[\frac{-q_1{}^2q_2p_2}{L\|\boldsymbol{q}\|^3}+\frac{+q_2p_1{}^2p_2}{2\mu L}\right]x_2,$$

$$x_3=\frac{\mu}{L}\left[\frac{q_1q_2}{\|\boldsymbol{q}\|^3}-\frac{p_1p_2}{2\mu}\right]x_1+\frac{\mu}{L}\left[\frac{-q_1{}^2}{\|\boldsymbol{q}\|^3}+\frac{p_1{}^2}{2\mu}\right]x_2,$$

$$x_3=\gamma_{31}x_1+\gamma_{32}x_2.\ \blacksquare$$

(VI.11.) Proof. Let X,Y be vector fields on $\boldsymbol{A}^{-1}(\boldsymbol{A_0})$. Then we have

$$\omega(X,Y)=x_1y_3+x_2y_4-x_3y_1-x_4y_2$$
$$\omega(X,Y)=x_1(\gamma_{31}y_1+\gamma_{32}y_2)+\ x_2(\gamma_{41}y_1+\gamma_{42}y_2)-y_1(\gamma_{31}x_1+\gamma_{32}x_2)-\ y_2(\gamma_{41}x_1+\gamma_{42}x_2)$$
$$\omega(X,Y)=(\gamma_{31}-\gamma_{31})x_1y_1+(\gamma_{32}-\gamma_{41})x_1y_2+(\gamma_{41}-\gamma_{32})x_2y_1+(\gamma_{42}-\gamma_{42})x_2y_2$$
$$\omega(X,Y)=(\gamma_{41}-\gamma_{32})(x_2y_1-x_1y_2)$$

$\omega(X,Y) = \frac{\mu}{L}\left(\frac{\|\boldsymbol{q}\|^2}{\|\boldsymbol{q}\|^3} - \frac{\|\boldsymbol{p}\|^2}{2\mu}\right)(x_2 y_1 - x_1 y_2)$

$\omega(X,Y) = \left(\frac{\mu H}{L}\right)(x_1 y_2 - x_2 y_1) \Rightarrow$

$\omega|\boldsymbol{A}^{-1}(\boldsymbol{A_0}) = \left(\frac{\mu H}{L}\right) dq_1 \wedge dq_2.$ ■

(VI.13.) Proof.

First, some intermediate steps:

$\|\boldsymbol{L}\|^2 = (\boldsymbol{q}\times\boldsymbol{p})\cdot(\boldsymbol{q}\times\boldsymbol{p}) = \|\boldsymbol{q}\|^2\|\boldsymbol{p}\|^2 - (\boldsymbol{q}\cdot\boldsymbol{p})^2,$

First, we show how the calculation would be done in 3 dimensions.

$\boldsymbol{A} = \frac{\boldsymbol{q}}{\|\boldsymbol{q}\|} - \frac{\boldsymbol{p}\times\boldsymbol{L}}{\mu},$

$\boldsymbol{q}\cdot(\boldsymbol{p}\times\boldsymbol{L}) = \boldsymbol{L}\cdot(\boldsymbol{q}\times\boldsymbol{p}) = \|\boldsymbol{L}\|^2,$

$\boldsymbol{q}\cdot\boldsymbol{A} = \frac{\|\boldsymbol{q}\|^2}{\|\boldsymbol{q}\|} - \frac{\boldsymbol{q}\cdot(\boldsymbol{p}\times\boldsymbol{L})}{\mu} = \|\boldsymbol{q}\| - \frac{\|\boldsymbol{L}\|^2}{\mu},$

$\|\boldsymbol{L}\|^2 = -\mu(\boldsymbol{q}\cdot\boldsymbol{A} - \|\boldsymbol{q}\|)$

The main result:

$$\|\boldsymbol{A}\|^2 = \frac{\|\boldsymbol{q}\|^2}{\|\boldsymbol{q}\|^2} - \frac{2\boldsymbol{q}\cdot(\boldsymbol{p}\times\boldsymbol{L})}{\mu\|\boldsymbol{q}\|} + \frac{(\boldsymbol{p}\times\boldsymbol{L})^2}{\mu^2}$$

$\|\boldsymbol{A}\|^2 = 1 - \frac{2\|\boldsymbol{L}\|^2}{\mu\|\boldsymbol{q}\|} + \frac{\|\boldsymbol{p}\|^2\|\boldsymbol{L}\|^2 - (\boldsymbol{p}\cdot\boldsymbol{L})^2}{\mu^2},$

$\|\boldsymbol{A}\|^2 = 1 + \left(\frac{\|\boldsymbol{p}\|^2}{\mu^2} - \frac{2}{\mu\|\boldsymbol{q}\|}\right)\|\boldsymbol{L}\|^2,$ (since $\boldsymbol{p}\cdot\boldsymbol{L} = \boldsymbol{0}$)

$\|\boldsymbol{A}\|^2 - 1 = \frac{2H\|\boldsymbol{L}\|^2}{\mu},$

Now, we repeat the calculations for 2 dimensions.

$\|\boldsymbol{L}\|^2 = (q_1 p_2 - q_2 p_1)^2,$

$\|\boldsymbol{L}\|^2 = {q_1}^2{p_2}^2 - 2q_1q_2p_1p_2 + {q_2}^2{p_1}^2,$

$\|\boldsymbol{L}\|^2 = {q_1}^2{p_1}^2 + {q_1}^2{p_2}^2 + {q_2}^2{p_1}^2 + {q_2}^2{p_2}^2 - {q_1}^2{p_1}^2 - {q_2}^2{p_2}^2 - 2q_1q_2p_1p_2,$

$\|\boldsymbol{L}\|^2 = ({q_1}^2 + {q_2}^2)({p_1}^2 + {p_2}^2) - (q_1p_1 + q_2p_2)^2,$

$\|\boldsymbol{L}\|^2 = \|\boldsymbol{q}\|^2\|\boldsymbol{p}\|^2 - (\boldsymbol{q}\cdot\boldsymbol{p})^2.$

$\boldsymbol{q}\cdot\boldsymbol{A} = \frac{\|\boldsymbol{q}\|^2}{\|\boldsymbol{q}\|} + \frac{1}{\mu}\left[(\boldsymbol{q}\cdot\boldsymbol{p})^2 - \|\boldsymbol{q}\|^2\|\boldsymbol{p}\|^2\right],$

$\boldsymbol{q}\cdot\boldsymbol{A} = \|\boldsymbol{q}\| - \frac{\|\boldsymbol{L}\|^2}{\mu},$

$\|\boldsymbol{L}\|^2 = -\mu(\boldsymbol{q}\cdot\boldsymbol{A} - \|\boldsymbol{q}\|).$

$\|\boldsymbol{A}\|^2 = \frac{\|\boldsymbol{q}\|^2}{\|\boldsymbol{q}\|^2} + \frac{2}{\mu\|\boldsymbol{q}\|}\left[(\boldsymbol{q}\cdot\boldsymbol{p})^2 - \|\boldsymbol{q}\|^2\|\boldsymbol{p}\|^2\right] + \frac{1}{\mu^2}\left[(\boldsymbol{q}\cdot\boldsymbol{p})^2\|\boldsymbol{p}\|^2 - 2(\boldsymbol{q}\cdot\boldsymbol{p})^2\|\boldsymbol{p}\|^2 + \|\boldsymbol{q}\|^2\|\boldsymbol{p}\|^4\right],$

$\|\boldsymbol{A}\|^2 = 1 + \frac{2}{\mu}\left[\frac{(\boldsymbol{q}\cdot\boldsymbol{p})^2}{\|\boldsymbol{q}\|} - \|\boldsymbol{q}\|\|\boldsymbol{p}\|^2 - \frac{(\boldsymbol{q}\cdot\boldsymbol{p})^2\|\boldsymbol{p}\|^2}{2\mu} + \frac{\|\boldsymbol{q}\|^2\|\boldsymbol{p}\|^4}{2\mu}\right],$

$\|\boldsymbol{A}\|^2 = 1 + \frac{2}{\mu}\left[\frac{-\|\boldsymbol{L}\|^2}{\|\boldsymbol{q}\|} + \frac{\|\boldsymbol{L}\|^2\|\boldsymbol{p}\|^2}{2\mu}\right],$

$\|\boldsymbol{A}\|^2 - 1 = \frac{2H\|\boldsymbol{L}\|^2}{\mu}$,

$H = \frac{\mu(\|\boldsymbol{A}\|^2-\mathbf{1})}{2\|\boldsymbol{L}\|^2} = \frac{-(\|\boldsymbol{A}\|^2-\mathbf{1})}{2(\boldsymbol{q}\cdot\boldsymbol{A}-\|\boldsymbol{q}\|)}$. ■

(VI.15.) Proof.

$\frac{\partial H}{\partial q_2} = \frac{+(\|\boldsymbol{A}\|^2-1)(A_2-q_2/\|\boldsymbol{q}\|)}{2(q_1A_1+q_2A_2-\|\boldsymbol{q}\|)^2} = \frac{-(A_2-q_2/\|\boldsymbol{q}\|)H}{(q_1A_1+q_2A_2-\|\boldsymbol{q}\|)}$,

$\dot{q}_1 = \{q_1, H\} = \left(\frac{\|\boldsymbol{L}\|}{\mu H}\right)\frac{\partial H}{\partial q_2}$, because of (VI.11.),

$\dot{q}_1 = \left(\frac{-\|\boldsymbol{L}\|}{\mu}\right)\frac{(A_2-q_2/\|\boldsymbol{q}\|)}{(q_1A_1+q_2A_2-\|\boldsymbol{q}\|)}$,

$\dot{q}_2 = \{q_2, H\} = \left(\frac{\|\boldsymbol{L}\|}{\mu H}\right)\frac{\partial H}{\partial q_1}$, because of (VI.11.),

$\dot{q}_2 = \left(\frac{+\|\boldsymbol{L}\|}{\mu}\right)\frac{(A_1-q_1/\|\boldsymbol{q}\|)}{(q_1A_1+q_2A_2-\|\boldsymbol{q}\|}$,

$\|\dot{\boldsymbol{q}}\| = \frac{q_1\dot{q}_1+q_2\dot{q}_2}{\|\boldsymbol{q}\|} = \left(\frac{\|\boldsymbol{L}\|}{\mu\|\boldsymbol{q}\|}\right)\left(\frac{-\|\boldsymbol{A}\|\sin\varphi}{1-\|\boldsymbol{A}\|\cos\varphi}\right)$,

$(\dot{\sin}\,\varphi) = \cos\varphi\,\dot{\varphi}$,

$(\dot{\sin}\,\varphi) = \frac{\dot{q}_1A_2-\dot{q}_2A_1}{\|\boldsymbol{q}\|\|\boldsymbol{A}\|} - \frac{(q_1A_2+q_2A_1)\dot{q}}{\|\boldsymbol{q}\|^2\|\boldsymbol{A}\|} = \left(\frac{\|\boldsymbol{L}\|}{\mu\|\boldsymbol{q}\|^2}\right)\cos\varphi \Rightarrow$

$\dot{\varphi} = \frac{\|\boldsymbol{L}\|}{\mu\|\boldsymbol{q}\|^2}$.

$\frac{d\|\boldsymbol{q}\|}{d\varphi} = \frac{\|\dot{\boldsymbol{q}}\|}{\dot{\varphi}} = \frac{-\|\boldsymbol{q}\|\|\boldsymbol{A}\|\sin\varphi}{1-\|\boldsymbol{A}\|\cos\varphi} \Rightarrow$

$\frac{d\|\boldsymbol{q}\|}{\|\boldsymbol{q}\|} = \frac{+\|\boldsymbol{A}\|d(\cos\varphi)}{1-\|\boldsymbol{A}\|\cos\varphi}$,

$\ln\|\boldsymbol{q}\| = -\ln(1-\|\boldsymbol{A}\|\cos\varphi) + const. \Rightarrow$

$\|\boldsymbol{q}\| = \frac{const.}{1-\|\boldsymbol{A}\|\cos\varphi}$. ■

VII. Calculation of the Trajectories of *H* via a Single Reduction

(VII.2.) Proof.

$\boldsymbol{A} = \frac{\boldsymbol{q}}{\|\boldsymbol{q}\|} - \frac{\boldsymbol{p}\times\boldsymbol{L}}{\mu} = \frac{\boldsymbol{q}}{\|\boldsymbol{q}\|} - \frac{\boldsymbol{p}\times(\boldsymbol{q}\times\boldsymbol{p})}{\mu} = \frac{\boldsymbol{q}}{\|\boldsymbol{q}\|} - \frac{\boldsymbol{q}(\boldsymbol{p}\cdot\boldsymbol{p})-\boldsymbol{p}(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}$,

$\boldsymbol{A} = \frac{\boldsymbol{q}}{\|\boldsymbol{q}\|} + \frac{\boldsymbol{p}(\boldsymbol{q}\cdot\boldsymbol{p})-\boldsymbol{q}\|\boldsymbol{p}\|^2}{\mu}$,

$A_i = \frac{q_i}{\|\boldsymbol{q}\|} + \frac{p_i(\boldsymbol{q}\cdot\boldsymbol{p})-q_i\|\boldsymbol{p}\|^2}{\mu}$.

$\|\boldsymbol{A}\|^2 = \boldsymbol{A}\cdot\boldsymbol{A} = \left(\frac{\boldsymbol{q}}{\|\boldsymbol{q}\|} - \frac{\boldsymbol{p}\times\boldsymbol{L}}{\mu}\right)\cdot\left(\frac{\boldsymbol{q}}{\|\boldsymbol{q}\|} - \frac{\boldsymbol{p}\times\boldsymbol{L}}{\mu}\right)$,

$\|\boldsymbol{A}\|^2 = \frac{\|\boldsymbol{q}\|^2}{\|\boldsymbol{q}\|^2} - \frac{2\boldsymbol{q}\cdot(\boldsymbol{p}\times\boldsymbol{L})}{\mu\|\boldsymbol{q}\|} + \frac{(\boldsymbol{p}\times\boldsymbol{L})\cdot(\boldsymbol{p}\times\boldsymbol{L})}{\mu^2}$,

$\|\boldsymbol{A}\|^2 = 1 - \frac{2\boldsymbol{L}\cdot(\boldsymbol{q}\times\boldsymbol{p})}{\mu\|\boldsymbol{q}\|} + \frac{\|\boldsymbol{p}\|^2\|\boldsymbol{L}\|^2-(\boldsymbol{p}\cdot\boldsymbol{L})^2}{\mu^2}$,

$\|\boldsymbol{A}\|^2 = 1 - \frac{2\|\boldsymbol{L}\|^2}{\mu\|\boldsymbol{q}\|} + \frac{\|\boldsymbol{p}\|^2\|\boldsymbol{L}\|^2-0}{\mu^2} = 1 + \frac{2\|\boldsymbol{L}\|^2}{\mu}\left(\frac{\|\boldsymbol{p}\|^2}{2\mu} - \frac{1}{\|\boldsymbol{q}\|}\right)$,

$\|\boldsymbol{A}\|^2 = 1 + \frac{2H\|\boldsymbol{L}\|^2}{\mu}$.

$\boldsymbol{L} \cdot \boldsymbol{A} = \boldsymbol{L} \cdot \left(\frac{\boldsymbol{q}}{\|\boldsymbol{q}\|} - \frac{\boldsymbol{p} \times \boldsymbol{L}}{\mu}\right)$, but $\boldsymbol{L} \cdot \boldsymbol{q} = \boldsymbol{q} \cdot \boldsymbol{L} = 0$, and $\boldsymbol{L} \cdot (\boldsymbol{p} \times \boldsymbol{L}) = \boldsymbol{p} \cdot (\boldsymbol{L} \times \boldsymbol{L}) = 0$,

$\boldsymbol{L} \cdot \boldsymbol{A} = 0.$

$\boldsymbol{L} \cdot \boldsymbol{M} = \sqrt{\frac{\mu}{2|H|}} \boldsymbol{L} \cdot \boldsymbol{A} = \boldsymbol{0}.$

For $H < 0$:

$\|\boldsymbol{M}\|^2 + \|\boldsymbol{L}\|^2 = \left(\sqrt{\frac{\mu}{2|H|}} \boldsymbol{A}\right)^2 + \|\boldsymbol{L}\|^2 = \left(\sqrt{\frac{-\mu}{2H}} \boldsymbol{A}\right)^2 + \|\boldsymbol{L}\|^2 = \frac{-\mu}{2H} \|\boldsymbol{A}\|^2 + \|\boldsymbol{L}\|^2,$

$\|\boldsymbol{M}\|^2 + \|\boldsymbol{L}\|^2 = \frac{-\mu}{2H}\left(1 + \frac{2H\|\boldsymbol{L}\|^2}{\mu}\right) + \|\boldsymbol{L}\|^2 = \frac{-\mu}{2H},$

$\|\boldsymbol{M}\|^2 + \|\boldsymbol{L}\|^2 = \frac{-\mu}{2H}.$

For $H > 0$:

$\|\boldsymbol{M}\|^2 - \|\boldsymbol{L}\|^2 = \left(\sqrt{\frac{\mu}{2|H|}} \boldsymbol{A}\right)^2 - \|\boldsymbol{L}\|^2 = \left(\sqrt{\frac{\mu}{2H}} \boldsymbol{A}\right)^2 - \|\boldsymbol{L}\|^2 = \frac{\mu}{2H} \|\boldsymbol{A}\|^2 - \|\boldsymbol{L}\|^2,$

$\|\boldsymbol{M}\|^2 - \|\boldsymbol{L}\|^2 = \frac{\mu}{2H}\left(1 + \frac{2H\|\boldsymbol{L}\|^2}{\mu}\right) - \|\boldsymbol{L}\|^2 = \frac{\mu}{2H},$

$\|\boldsymbol{M}\|^2 - \|\boldsymbol{L}\|^2 = \frac{+\mu}{2H}.$

$\{H, L_i\}$ is already known from (VI.4.), but here is a calculation:

$\{H, L_i\} = \{H, q_j p_k - q_k p_j\},$

$\{H, L_i\} = q_j\{H, p_k\} + p_k\{H, q_j\} - q_k\{H, p_j\} - p_j\{H, q_k\},$

$\{H, L_i\} = q_j\left(\frac{q_k}{\|\boldsymbol{q}\|^3}\right) + p_k\left(\frac{-p_j}{\mu}\right) - q_k\left(\frac{q_j}{\|\boldsymbol{q}\|^3}\right) - p_j\left(\frac{-p_k}{\mu}\right),$

(from the proof of (VI.8.))

$\{H, L_i\} = 0.$

Now for some intermediate results:

$\{H, q_i\} = -\frac{\partial H}{\partial p_i} = -\frac{p_i}{\mu}.$

$\{H, p_i\} = \frac{\partial H}{\partial q_i} = \frac{\partial}{\partial q_i}\left[(\|\boldsymbol{q}\|^2)^{-1/2}\right] = \frac{1}{2}(\|\boldsymbol{q}\|^2)^{-3/2} 2q_i,$

$\{H, p_i\} = \frac{q_i}{\|\boldsymbol{q}\|^3}.$

$\{H, \boldsymbol{q} \cdot \boldsymbol{p}\} = \{H, \boldsymbol{q} \cdot \boldsymbol{p}\} = \boldsymbol{q} \cdot \{H, \boldsymbol{p}\} + \boldsymbol{p} \cdot \{H, \boldsymbol{q}\},$

$\{H, \boldsymbol{q} \cdot \boldsymbol{p}\} = \boldsymbol{q} \cdot \left(\frac{\boldsymbol{q}}{\|\boldsymbol{q}\|^3}\right) + \boldsymbol{p} \cdot \left(\frac{-\boldsymbol{p}}{\mu}\right),$

$\{H, \boldsymbol{q} \cdot \boldsymbol{p}\} = \frac{\|\boldsymbol{q}\|^2}{\|\boldsymbol{q}\|^3} - \frac{\|\boldsymbol{p}\|^2}{\mu},$

$\{H, \boldsymbol{q} \cdot \boldsymbol{p}\} = \frac{1}{\|\boldsymbol{q}\|} - \frac{\|\boldsymbol{p}\|^2}{\mu}.$

$\left\{H, \frac{1}{\|\boldsymbol{q}\|}\right\} = \{H, (\|\boldsymbol{q}\|^2)^{-1/2}\},$

$\left\{H, \frac{1}{\|\boldsymbol{q}\|}\right\} = -\frac{1}{2}(\|\boldsymbol{q}\|^2)^{-3/2} 2\boldsymbol{q} \cdot \{H, \boldsymbol{q}\},$

$\left\{H, \frac{1}{\|\boldsymbol{q}\|}\right\} = -\frac{1}{2}\left(\frac{1}{\|\boldsymbol{q}\|^3}\right) 2\boldsymbol{q} \cdot \left(-\frac{\boldsymbol{p}}{\mu}\right)$,

$\left\{H, \frac{1}{\|\boldsymbol{q}\|}\right\} = \frac{\boldsymbol{q}\cdot\boldsymbol{p}}{\mu\|\boldsymbol{q}\|^3}$.

$\{H, \boldsymbol{p}\cdot\boldsymbol{p}\} = 2\boldsymbol{p}\cdot\{H, \boldsymbol{p}\}$,

$\{H, \boldsymbol{p}\cdot\boldsymbol{p}\} = 2\boldsymbol{p}\cdot\frac{\boldsymbol{q}}{\|\boldsymbol{q}\|^3}$,

$\{H, \boldsymbol{p}\cdot\boldsymbol{p}\} = \frac{2\boldsymbol{q}\cdot\boldsymbol{p}}{\|\boldsymbol{q}\|^3}$.

Now for the main result:

$\{H, A_i\} = \left\{H, \frac{q_i}{\|\boldsymbol{q}\|} + \frac{p_i(\boldsymbol{q}\cdot\boldsymbol{p}) - q_i\|\boldsymbol{p}\|^2}{\mu}\right\}$,

$\{H, A_i\} = q_i\left\{H, \frac{1}{\|\boldsymbol{q}\|}\right\} + \frac{1}{\|\boldsymbol{q}\|}\{H, q_i\} + \frac{p_i}{\mu}\{H, \boldsymbol{q}\cdot\boldsymbol{p}\} + \frac{\boldsymbol{q}\cdot\boldsymbol{p}}{\mu}\{H, p_i\} - \frac{q_i}{\mu}\{H, \boldsymbol{p}\cdot\boldsymbol{p}\} - \frac{\boldsymbol{p}\cdot\boldsymbol{p}}{\mu}\{H, q_i\}$,

$\{H, A_i\} = q_i\left(\frac{\boldsymbol{q}\cdot\boldsymbol{p}}{\mu\|\boldsymbol{q}\|^3}\right) + \frac{1}{\|\boldsymbol{q}\|}\left(\frac{-p_i}{\mu}\right) + \frac{p_i}{\mu}\left(\frac{1}{\|\boldsymbol{q}\|} - \frac{\|\boldsymbol{p}\|^2}{\mu}\right) + \frac{\boldsymbol{q}\cdot\boldsymbol{p}}{\mu}\left(\frac{q_i}{\|\boldsymbol{q}\|^3}\right) - \frac{q_i}{\mu}\left(\frac{2\boldsymbol{q}\cdot\boldsymbol{p}}{\|\boldsymbol{q}\|^3}\right) - \frac{\boldsymbol{p}\cdot\boldsymbol{p}}{\mu}\left(\frac{-p_i}{\mu}\right)$,

$\{H, A_i\} = q_i\left(\frac{\boldsymbol{q}\cdot\boldsymbol{p}}{\mu\|\boldsymbol{q}\|^3}\right)(1 + 1 - 2) + \frac{p_i}{\mu\|\boldsymbol{q}\|}(-1 + 1) + \frac{p_i\|\boldsymbol{p}\|^2}{\mu^2}(+1 - 1)$,

$\{H, A_i\} = 0$.

$\{H, M_i\} = 0$. ■

(VII.4.) Proof.

We begin by showing

$q_\parallel = \|\boldsymbol{q}\| \cos\theta$,

by definition (VII.3.),

$q_\perp = \|\boldsymbol{q}\| \sin\theta = \frac{\|\boldsymbol{q}\times\boldsymbol{A}\|}{\|\boldsymbol{A}\|}$.

$q_\perp = \frac{\boldsymbol{q}\cdot(\boldsymbol{L}\times\boldsymbol{A})}{\|\boldsymbol{L}\|\|\boldsymbol{A}\|}$,

by definition (VII.3.),

$q_\perp = -\frac{1}{\|\boldsymbol{L}\|\|\boldsymbol{A}\|}\,\boldsymbol{L}\cdot(\boldsymbol{q}\times\boldsymbol{A})$,

$q_\perp = -\frac{1}{\|\boldsymbol{L}\|\|\boldsymbol{A}\|}\,\boldsymbol{L}\cdot\left[\frac{\boldsymbol{q}\times\boldsymbol{q}}{\|\boldsymbol{q}\|} - \frac{\boldsymbol{q}\times(\boldsymbol{p}\times\boldsymbol{L})}{\mu}\right]$,

$q_\perp = \frac{1}{\mu\|\boldsymbol{L}\|\|\boldsymbol{A}\|}\,\boldsymbol{L}\cdot[-\boldsymbol{p}(\boldsymbol{q}\cdot\boldsymbol{L}) + \boldsymbol{L}(\boldsymbol{q}\cdot\boldsymbol{p})]$,

$q_\perp = \frac{\|\boldsymbol{L}\|(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu\|\boldsymbol{A}\|}$,

$\|\boldsymbol{q}\|^2 \sin^2\theta = \frac{\|\boldsymbol{q}\times\boldsymbol{A}\|^2}{\|\boldsymbol{A}\|^2}$,

$\|\boldsymbol{q}\|^2 \sin^2\theta = \frac{1}{\|\boldsymbol{A}\|^2}\left[\frac{\boldsymbol{q}\times(\boldsymbol{p}\times\boldsymbol{L})}{\mu}\right]^2$,

$\|\boldsymbol{q}\|^2 \sin^2\theta = \frac{1}{\mu^2\|\boldsymbol{A}\|^2}[-\boldsymbol{p}(\boldsymbol{q}\cdot\boldsymbol{L}) + \boldsymbol{L}(\boldsymbol{q}\cdot\boldsymbol{p})]^2$,

$\|\boldsymbol{q}\|^2 \sin^2\theta = \frac{\|\boldsymbol{L}\|^2}{\mu^2\|\boldsymbol{A}\|^2}(\boldsymbol{q}\cdot\boldsymbol{p})^2$,

$\|\boldsymbol{q}\| \sin\theta = \pm \frac{\|\boldsymbol{L}\|(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu\|\boldsymbol{A}\|}$,

$q_{\perp} = \pm\|\boldsymbol{q}\| \sin\theta$.

Main results:

First, we'll calculate $\|\boldsymbol{q}\|$.

$\boldsymbol{q}\cdot\boldsymbol{A} = \|\boldsymbol{q}\| - \frac{1}{\mu}\boldsymbol{q}\cdot(\boldsymbol{p}\times\boldsymbol{L})$,
$\boldsymbol{q}\cdot\boldsymbol{A} = \|\boldsymbol{q}\| - \frac{1}{\mu}\|\boldsymbol{L}\|\cdot(\boldsymbol{q}\times\boldsymbol{p})$,
$\boldsymbol{q}\cdot\boldsymbol{A} = \|\boldsymbol{q}\| - \frac{\|\boldsymbol{L}\|^2}{\mu}$,
$\cos\theta = \frac{\boldsymbol{q}\cdot\boldsymbol{A}}{\|\boldsymbol{q}\|\|\boldsymbol{A}\|}$,

by definition (VII.3.),

$\cos\theta = \frac{1}{\|\boldsymbol{A}\|} - \frac{\|\boldsymbol{L}\|^2}{\mu\|\boldsymbol{A}\|\|\boldsymbol{q}\|} \Rightarrow$
$1 - \|\boldsymbol{A}\|\cos\theta = \frac{\|\boldsymbol{L}\|^2}{\mu\|\boldsymbol{q}\|} \Rightarrow$
$\|\boldsymbol{q}\| = \frac{\|\boldsymbol{L}\|^2}{\mu(1-\|\boldsymbol{A}\|\cos\theta)}$.

$\sin\theta = \frac{q_{\perp}}{\|\boldsymbol{q}\|} = \frac{\boldsymbol{q}\cdot(\boldsymbol{L}\times\boldsymbol{A})}{\|\boldsymbol{q}\|\|\boldsymbol{L}\|\|\boldsymbol{A}\|}$,

by definition (VII.3.),

$\sin\theta = \frac{1}{\|\boldsymbol{q}\|\|\boldsymbol{L}\|\|\boldsymbol{A}\|}[-\boldsymbol{L}\cdot(\boldsymbol{q}\times\boldsymbol{A})]$,

$\sin\theta = -\frac{1}{\|\boldsymbol{q}\|\|\boldsymbol{L}\|\|\boldsymbol{A}\|}\boldsymbol{L}\cdot\left[\frac{(\boldsymbol{q}\times\boldsymbol{q})}{\|\boldsymbol{q}\|} - \frac{\boldsymbol{q}\times(\boldsymbol{p}\times\boldsymbol{L})}{\mu}\right]$,

$\sin\theta = -\frac{1}{\|\boldsymbol{q}\|\|\boldsymbol{L}\|\|\boldsymbol{A}\|}\boldsymbol{L}\cdot\left[\frac{-\boldsymbol{p}(\boldsymbol{q}\cdot\boldsymbol{L})+\boldsymbol{L}(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right]$,

$\sin\theta = -\frac{\|\boldsymbol{L}\|^2(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu\|\boldsymbol{q}\|\|\boldsymbol{L}\|\|\boldsymbol{A}\|}$,

$\sin\theta = -\frac{\|\boldsymbol{L}\|(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu\|\boldsymbol{A}\|}\left(\frac{\mu(1-\|\boldsymbol{A}\|\cos\theta)}{\|\boldsymbol{L}\|^2}\right)$,

$(\boldsymbol{q}\cdot\boldsymbol{p}) = -\frac{\|\boldsymbol{L}\|\|\boldsymbol{A}\|\sin\theta}{(1-\|\boldsymbol{A}\|\cos\theta)}$.

$q_{\parallel} = \|\boldsymbol{q}\|\cos\theta$,

by definition (VII.3.),

$q_{\parallel} = \frac{\|\boldsymbol{L}\|^2\cos\theta}{\mu(1-\|\boldsymbol{A}\|\cos\theta)}$.

$q_{\perp} = \|\boldsymbol{q}\|\sin\theta$,
$q_{\perp} = \frac{\|\boldsymbol{L}\|^2\sin\theta}{\mu(1-\|\boldsymbol{A}\|\cos\theta)}$.

$p_{\parallel} = \frac{\boldsymbol{p}\cdot\boldsymbol{A}}{\|\boldsymbol{A}\|}$,

by definition (VII.3.),

$p_{\parallel} = \frac{1}{\|\boldsymbol{A}\|}\left[\frac{\boldsymbol{p}\cdot\boldsymbol{q}}{\|\boldsymbol{q}\|} - \frac{\boldsymbol{p}\cdot(\boldsymbol{p}\times\boldsymbol{L})}{\mu}\right]$,

$$p_{\parallel} = \frac{1}{\|\boldsymbol{A}\|}\left[\frac{\boldsymbol{q}\cdot\boldsymbol{p}}{\|\boldsymbol{q}\|} - \frac{\boldsymbol{L}\cdot(\mathbf{p}\times\boldsymbol{p})}{\mu}\right],$$
$$\mathrm{p}_{\parallel} = \frac{\boldsymbol{q}\cdot\boldsymbol{p}}{\|\mathbf{A}\|\|\boldsymbol{q}\|},$$
$$\mathrm{p}_{\parallel} = \frac{1}{\|\mathbf{A}\|}\left(-\frac{\|\boldsymbol{L}\|\|\boldsymbol{A}\|\sin\theta}{(1-\|\boldsymbol{A}\|\cos\theta)}\right)\left(\frac{\mu(1-\|\boldsymbol{A}\|\cos\theta)}{\|\boldsymbol{L}\|^2}\right),$$
$$\mathrm{p}_{\parallel} = -\frac{\sin\theta}{\|\boldsymbol{L}\|}.$$

$$\|\boldsymbol{p}\|^2 = \frac{1}{\|\boldsymbol{q}\|^2}\left(\|\boldsymbol{L}\|^2 + (\boldsymbol{q}\cdot\boldsymbol{p})^{\mathbf{2}}\right),$$

cf. remark after (VII.3.),

$$\|\boldsymbol{p}\|^2 = \frac{\mu^2(1-\|\boldsymbol{A}\|\cos\theta)^2}{\|\boldsymbol{L}\|^4}\left(\|\boldsymbol{L}\|^2 + \frac{\|\boldsymbol{L}\|^2\|\boldsymbol{A}\|^2\sin^2\theta}{(1-\|\boldsymbol{A}\|\cos\theta)^2}\right),$$
$$\|\boldsymbol{p}\|^2 = \frac{\mu^2}{\|\boldsymbol{L}\|^2}[(1-\|\boldsymbol{A}\|\cos\theta)^2 + \|\boldsymbol{A}\|^2\sin^2\theta],$$
$$\|\boldsymbol{p}\|^2 = \frac{\mu^2}{\|\boldsymbol{L}\|^2}\left[1 - 2\|\boldsymbol{A}\|\cos\theta + \|\boldsymbol{A}\|^{\mathbf{2}}\cos^2\theta + \|\boldsymbol{A}\|^2\sin^2\theta\right],$$
$$\|\boldsymbol{p}\|^2 = \frac{\mu^2}{\|\boldsymbol{L}\|^2}\left[1 - 2\|\boldsymbol{A}\|\cos\theta + \|\boldsymbol{A}\|^{\mathbf{2}}\right].$$

$$p_{\perp} = \frac{\mathbf{p}\cdot(\boldsymbol{L}\times\boldsymbol{A})}{\|\boldsymbol{L}\|\|\boldsymbol{A}\|},$$

by definition (VII.3.),

$$p_{\perp} = \frac{1}{\|\boldsymbol{L}\|\|\boldsymbol{A}\|}\left[\frac{\mathbf{p}\cdot(\boldsymbol{L}\times\boldsymbol{q})}{\|\boldsymbol{q}\|} - \frac{\mathbf{p}\cdot\left(\boldsymbol{L}\times(\boldsymbol{p}\times\boldsymbol{L})\right)}{\mu}\right],$$
$$p_{\perp} = \frac{1}{\|\boldsymbol{L}\|\|\boldsymbol{A}\|}\left[\frac{\boldsymbol{L}\cdot(\boldsymbol{q}\times\boldsymbol{p})}{\|\boldsymbol{q}\|} - \frac{\mathbf{p}\cdot\left(\boldsymbol{p}(\boldsymbol{L}\cdot\boldsymbol{L})-\boldsymbol{L}(\boldsymbol{L}\cdot\boldsymbol{p})\right)}{\mu}\right],$$
$$p_{\perp} = \frac{1}{\|\boldsymbol{L}\|\|\boldsymbol{A}\|}\left[\frac{\|\boldsymbol{L}\|^2}{\|\boldsymbol{q}\|} - \frac{\|\boldsymbol{p}\|^2\|\boldsymbol{L}\|^2}{\mu}\right],$$
$$p_{\perp} = \frac{\|\boldsymbol{L}\|}{\|\boldsymbol{A}\|}\left[\frac{1}{\|\boldsymbol{q}\|} - \frac{\|\boldsymbol{p}\|^2}{\mu}\right],$$
$$p_{\perp} = \frac{\|\boldsymbol{L}\|}{\|\boldsymbol{A}\|}\left[\frac{\mu(1-\|\boldsymbol{A}\|\cos\theta)}{\|\boldsymbol{L}\|^2} - \frac{\mu}{\|\boldsymbol{L}\|^2}\left(1 - 2\|\boldsymbol{A}\|\cos\theta + \|\boldsymbol{A}\|^{\mathbf{2}}\right)\right],$$
$$p_{\perp} = \frac{\mu}{\|\boldsymbol{L}\|\|\boldsymbol{A}\|}\left[\frac{\mu(1-\|\boldsymbol{A}\|\cos\theta)}{\|\boldsymbol{L}\|^2} - \frac{\mu}{\|\boldsymbol{L}\|^2}\left(1 - 2\|\boldsymbol{A}\|\cos\theta + \|\boldsymbol{A}\|^{\mathbf{2}}\right)\right],$$
$$p_{\perp} = \frac{\mu}{\|\boldsymbol{L}\|\|\boldsymbol{A}\|}\left[\|\boldsymbol{A}\|\cos\theta - \|\boldsymbol{A}\|^{\mathbf{2}}\right],$$
$$p_{\perp} = -\frac{\mu}{\|\boldsymbol{L}\|}[\|\boldsymbol{A}\| - \cos\theta]. \blacksquare$$

(VII.5.) Proof.

For $q_{\parallel}$, we have the Hamilton equation:

$$\frac{dq_{\parallel}}{dt} = \{q_{\parallel}, H\} = \frac{\partial H}{\partial p_{\parallel}} = \frac{p_{\parallel}}{\mu} \Rightarrow$$
$$\frac{dq_{\parallel}}{d\theta}\frac{d\theta}{dx} = \frac{p_{\parallel}}{\mu}\left(\theta(t)\right) \Rightarrow$$
$$\left(\frac{\|\boldsymbol{L}\|^2}{\mu}\right)\frac{d}{d\theta}\left(\frac{\cos\theta}{1-\|\boldsymbol{A}\|\cos\theta}\right)\left(\frac{d\theta}{dt}\right) = \left(\frac{-1}{\|\boldsymbol{L}\|}\right)\sin\theta.$$
$$\frac{d}{d\theta}\left(\frac{\cos\theta}{1-\|\boldsymbol{A}\|\cos\theta}\right) = \frac{-\sin\theta}{(1-\|\boldsymbol{A}\|\cos\theta)^2} \Rightarrow$$
$$\left(\frac{\|\boldsymbol{L}\|^2}{\mu}\right)\left(\frac{-\sin\theta + \|\boldsymbol{A}\|\sin\theta\cos\theta - \|\boldsymbol{A}\|\sin\theta\cos\theta}{(1-\|\boldsymbol{A}\|\cos\theta)^2}\right)\left(\frac{d\theta}{dt}\right) = \left(\frac{-1}{\|\boldsymbol{L}\|}\right)\sin\theta \Rightarrow$$
$$\left(\frac{\|\boldsymbol{L}\|^2}{\mu}\right)\left(\frac{-\sin\theta}{(1-\|\boldsymbol{A}\|\cos\theta)^2}\right)\left(\frac{d\theta}{dt}\right) = \left(\frac{-1}{\|\boldsymbol{L}\|}\right)\sin\theta \Rightarrow$$

$\frac{d\theta}{dt} = \frac{\mu}{\|\boldsymbol{L}\|^3}(1 - \|\boldsymbol{A}\| \cos\theta)^2.$

The other cases are analogous, and produce the same result. We will calculate one of them.

$\frac{dq_\perp}{dt} = \{q_\perp, H\} = \frac{\partial H}{\partial p_\perp} = \frac{p_\perp}{\mu} \Rightarrow$

$\frac{dq_\perp}{d\theta}\frac{d\theta}{dt} = \frac{p_\perp}{\mu}\big(\theta(t)\big) \Rightarrow$

$\left(\frac{\|\boldsymbol{L}\|^2}{\mu}\right)\frac{d}{d\theta}\left(\frac{\sin\theta}{1-\|\boldsymbol{A}\|\cos\theta}\right)\left(\frac{d\theta}{dt}\right) = \left(\frac{-1}{\|\boldsymbol{L}\|}\right)(\|\boldsymbol{A}\| - \cos\theta).$

$\frac{\|\boldsymbol{L}\|^2}{\mu}\left[\frac{\cos\theta(1-\|\boldsymbol{A}\|\cos\theta)-\sin\theta(\|\boldsymbol{A}\|\sin\theta)}{(1-\|\boldsymbol{A}\|\cos\theta)^2}\right]\frac{d\theta}{dt} = \frac{-1}{\|\boldsymbol{L}\|}(\|\boldsymbol{A}\| - \cos\theta).$

$\frac{\|\boldsymbol{L}\|^2}{\mu}\left[\frac{\cos\theta-\|\boldsymbol{A}\|\cos^2\theta-\|\boldsymbol{A}\|\sin^2\theta}{(1-\|\boldsymbol{A}\|\cos\theta)^2}\right]\frac{d\theta}{dt} = \frac{-1}{\|\boldsymbol{L}\|}(\|\boldsymbol{A}\| - \cos\theta).$

$\frac{\|\boldsymbol{L}\|^2}{\mu}\left[\frac{\cos\theta-\|\boldsymbol{A}\|}{(1-\|\boldsymbol{A}\|\cos\theta)^2}\right]\frac{d\theta}{dt} = \frac{-1}{\|\boldsymbol{L}\|}(\|\boldsymbol{A}\| - \cos\theta) \Rightarrow$

$\frac{d\theta}{dt} = \frac{\mu}{\|\boldsymbol{L}\|^3}(1 - \|\boldsymbol{A}\| \cos\theta)^2.$ ■

(VII.7.) Proof.

$\cos u = \frac{-\|\boldsymbol{A}\| + \cos\theta}{1-\|\boldsymbol{A}\|\cos\theta},$

by definition (VII.6.),

$\cos u - \|\boldsymbol{A}\| \cos u \cos\theta = \cos\theta - \|\boldsymbol{A}\|,$

$\cos\theta\,(1 + \|\boldsymbol{A}\| \cos u) = \cos u + \|\boldsymbol{A}\|,$

$\cos\theta = \frac{\|\boldsymbol{A}\|+\cos u}{1+\|\boldsymbol{A}\|\cos u}.$

$\sin u = \frac{\sqrt{1-\|\boldsymbol{A}\|^2}\sin\theta}{1-\|\boldsymbol{A}\|\cos\theta},$

by definition (VII.6.),

$\sin u - \|\boldsymbol{A}\| \sin u \cos\theta = \sqrt{1-\|\boldsymbol{A}\|^2}\sin\theta,$

$\sin\theta = \frac{\sin u(1-\|\boldsymbol{A}\|\cos\theta)}{\sqrt{1-\|\boldsymbol{A}\|^2}},$

$\sin\theta = \frac{\sin u}{\sqrt{1-\|\boldsymbol{A}\|^2}}\left(1 - \|\boldsymbol{A}\|\frac{\|\boldsymbol{A}\|+\cos u}{1+\|\boldsymbol{A}\|\cos u}\right),$

$\sin\theta = \frac{\sin u}{\sqrt{1-\|\boldsymbol{A}\|^2}}\left(\frac{1+\|\boldsymbol{A}\|\cos u-\|\boldsymbol{A}\|^2-\|\boldsymbol{A}\|\cos u}{1+\|\boldsymbol{A}\|\cos u}\right),$

$\sin\theta = \frac{\sqrt{1-\|\boldsymbol{A}\|^2}\sin u}{1+\|\boldsymbol{A}\|\cos u}.$

$(1 - \|\boldsymbol{A}\| \cos u)(1 + \|\boldsymbol{A}\| \cos u) = 1 - \|\boldsymbol{A}\|^2,$

$1 - \|\boldsymbol{A}\| \cos u = \frac{1-\|\boldsymbol{A}\|^2}{1+\|\boldsymbol{A}\|\cos u}.$ ■

(VII.8.) Proof.

$\cos\theta = \frac{\|\mathbf{A}\|+\cos u}{1+\|\mathbf{A}\|\cos u},$

from (VII.7.),

$(*)\ \frac{d(\cos\theta)}{dt} = \frac{-(1-\|\mathbf{A}\|^2)\sin u\dot{u}}{(1+\|\mathbf{A}\|\cos u)^2}.$

$\frac{d(\cos\theta)}{dt} = -\sin\theta\frac{d\theta}{dt},$

$\frac{d(\cos\theta)}{dt} = -\frac{\mu}{\|\boldsymbol{L}\|^3}\sin\theta\,(1-\|\mathbf{A}\|\cos\theta)^2,$

from (VII.5.),

$\frac{d(\cos\theta)}{dt} = -\frac{\mu}{\|\boldsymbol{L}\|^3}\left(\frac{\sqrt{1-\|\mathbf{A}\|^2}\sin u}{1+\|\mathbf{A}\|\cos u}\right)\left(\frac{(1-\|\mathbf{A}\|^2)}{1+\|\mathbf{A}\|\cos u}\right)^2,$

from (VII.7.),

$(**)\frac{d(\cos\theta)}{dt} = -\frac{\mu}{\|\boldsymbol{L}\|^3}\frac{(1-\|\mathbf{A}\|^2)^{5/2}\sin u}{(1+\|\mathbf{A}\|\cos u)^3}.$

$(*) = (**) \Rightarrow \dot{u} = \frac{du}{dt} = -\frac{\mu}{\|\boldsymbol{L}\|^3}\frac{(1-\|\mathbf{A}\|^2)^{3/2}}{1+\|\mathbf{A}\|\cos u},$

$dt = \frac{\|\boldsymbol{L}\|^3}{\mu(1-\|\mathbf{A}\|^2)^{3/2}}(1+\|\mathbf{A}\|\cos u)du = \frac{\|\boldsymbol{L}\|^3}{\mu(1-\|\mathbf{A}\|^2)^{3/2}}d(u+\|\mathbf{A}\|\sin u),$

$\mathrm{t} = \frac{\|\boldsymbol{L}\|^3}{\mu(1-\|\mathbf{A}\|^2)^{3/2}}(u+\|\mathbf{A}\|\sin u).$ ■

VIII. Discussion of the Symmetry and Trajectories of $\boldsymbol{M}$

(VIII.1.) Proof.

For convenience, we will repeat a few things here:

$\boldsymbol{A} = \frac{\boldsymbol{q}}{\|\boldsymbol{q}\|} + \frac{\boldsymbol{p}(\boldsymbol{q}\cdot\boldsymbol{p})-\boldsymbol{q}\|\boldsymbol{p}\|^2}{\mu},$

$A_1 = \frac{q_1}{\|\boldsymbol{q}\|} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})-q_1\|\boldsymbol{p}\|^2}{\mu},$

$\boldsymbol{M} = \sqrt{\frac{\mu}{2|H|}}\boldsymbol{A},$

$H = \frac{\|\boldsymbol{p}\|^2}{2\mu} - \frac{1}{\|\boldsymbol{q}\|},$

Then, a few intermediate steps:

$\{p_i, H\} = -\frac{\partial H}{\partial q_i} = -\frac{\partial}{\partial q_i}\left(\frac{-1}{\|\boldsymbol{q}\|}\right) = -\left(\frac{-1}{2}\right)\frac{-1}{\|\boldsymbol{q}\|^3}(2q_i) = \frac{-q_i}{\|\boldsymbol{q}\|^3},$

$\{p_i, H\} = -\frac{\partial H}{\partial q_i} = \frac{-q_i}{\|\boldsymbol{q}\|^3}.$

$\{q_i, H\} = \frac{\partial H}{\partial p_i} = \frac{\partial}{\partial p_i}\left(\frac{\|\boldsymbol{p}\|^2}{2\mu}\right) = \frac{2p_i}{2\mu} = \frac{p_i}{\mu}.$

Now for the main results.

$\{q_1, A_1\} = \sum_{i=1}^{3}\left(\frac{\partial q_1}{\partial q_i}\frac{\partial A_1}{\partial p_i} - \frac{\partial q_1}{\partial p_i}\frac{\partial A_1}{\partial q_i}\right),$

$\{q_1, A_1\} = \frac{\partial A_1}{\partial p_1} = \left(\frac{1}{\mu}\right)(q_2p_2 + q_3p_3).$

$\{q_2, A_1\} = \frac{\partial A_1}{\partial p_2} = \left(\frac{1}{\mu}\right)(q_2 p_1 - 2q_1 p_2).$

$\{q_3, A_1\} = \frac{\partial A_1}{\partial p_3} = \left(\frac{1}{\mu}\right)(q_3 p_1 - 2q_1 p_3).$

$\{p_1, A_1\} = \sum_{i=1}^{3}\left(\frac{\partial p_1}{\partial q_i}\frac{\partial A_1}{\partial p_i} - \frac{\partial p_1}{\partial p_i}\frac{\partial A_1}{\partial q_i}\right) = -\frac{\partial A_1}{\partial q_1},$

$\{p_1, A_1\} = -\frac{1}{\|\boldsymbol{q}\|} - q_1\left(\frac{-1}{2}\right)\frac{2q_1}{\|\boldsymbol{q}\|^3} + \frac{1}{\mu}({p_2}^2 + {p_3}^2),$

$\{p_1, A_1\} = \frac{1}{\|\boldsymbol{q}\|^3}(-{q_1}^2 - {q_2}^2 - {q_3}^2 + {q_1}^2) + \frac{1}{\mu}({p_2}^2 + {p_3}^2),$

$\{p_1, A_1\} = \frac{1}{\|\boldsymbol{q}\|^3}(-{q_2}^2 - {q_3}^2) + \frac{1}{\mu}({p_2}^2 + {p_3}^2).$

$\{p_2, A_1\} = -\frac{\partial A_1}{\partial q_2} = -q_1\left(\frac{-1}{2}\right)\frac{2q_2}{\|\boldsymbol{q}\|^3} - \frac{1}{\mu}(p_1 p_2),$

$\{p_2, A_1\} = \frac{1}{\|\boldsymbol{q}\|^3} q_1 q_2 + \frac{1}{\mu}(-p_1 p_2).$

$\{p_3, A_1\} = -\frac{\partial A_1}{\partial q_3} = -q_1\left(\frac{-1}{2}\right)\frac{2q_3}{\|\boldsymbol{q}\|^3} - \frac{1}{\mu}(p_1 p_3),$

$\{p_3, A_1\} = \frac{1}{\|\boldsymbol{q}\|^3} q_1 q_3 + \frac{1}{\mu}(-p_1 p_3).$

$\{q_1, M_1\} = \left\{q_1, \sqrt{\frac{\mu}{2|H|}} A_1\right\},$

$\{q_1, M_1\} = \sqrt{\frac{\mu}{2}}\left(\frac{-1}{2}\right)|H|^{-3/2}\{q_1, \lfloor H \rceil\} A_1 + \sqrt{\frac{\mu}{2|H|}}\{q_1, A_1\},$

$\{q_1, M_1\} = -\sqrt{\frac{\mu}{2}}|H|^{-3/2}\left[\{q_1, \lfloor H \rceil\}\frac{A_1}{2} - |H|\{q_1, A_1\}\right],$

$\{q_1, M_1\} = -\sqrt{\frac{\mu}{2}}|H|^{-3/2}\left[\pm\{q_1, H\}\frac{A_1}{2} - \pm H\{q_1, A_1\}\right]$, with $+$ sign for $H > 0$,

$\{q_1, M_1\} = \pm\sqrt{\frac{\mu}{2}}|H|^{-3/2}\left[\frac{p_1}{\mu}\frac{A_1}{2} - \frac{H}{\mu}(q_2 p_2 + q_3 p_3)\right]$, with $+$ sign for $H < 0$.

$\{q_2, M_1\}$ and $\{\mathrm{q}_3, \mathrm{M}_1\}$ are analogous, so we can present a shorter calculation.

$\{q_2, M_1\} = \left\{q_2, \sqrt{\frac{\mu}{2|H|}} A_1\right\},$

$\{q_2, M_1\} = -\sqrt{\frac{\mu}{2}}|H|^{-3/2}\left[\pm\{q_2, H\}\frac{A_1}{2} - \pm H\{q_2, A_1\}\right]$, with $+$ sign for $H > 0$,

$\{q_2, M_1\} = \pm\sqrt{\frac{\mu}{2}}|H|^{-3/2}\left[\frac{p_2}{\mu}\frac{A_1}{2} - H\left(\frac{1}{\mu}\right)(q_2 p_1 - 2q_1 p_2)\right]$, with $+$ sign for $H < 0$.

$\{q_3, M_1\} = \left\{q_3, \sqrt{\frac{\mu}{2|H|}} A_1\right\},$

$\{q_3, M_1\} = -\sqrt{\frac{\mu}{2}}|H|^{-3/2}\left[\pm\{q_3, H\}\frac{A_1}{2} - \pm H\{q_3, A_1\}\right]$, with $+$ sign for $H > 0$,

$\{q_3, M_1\} = \pm\sqrt{\frac{\mu}{2}}|H|^{-3/2}\left[\frac{p_3}{\mu}\frac{A_1}{2} - H\left(\frac{1}{\mu}\right)(q_3 p_1 - 2q_1 p_3)\right]$, with $+$ sign for $H < 0$.

$\{p_1, M_1\} = \left\{p_1, \sqrt{\frac{\mu}{2|H|}} A_1\right\},$

$\{p_1, M_1\} = \sqrt{\frac{\mu}{2}}\left(\frac{-1}{2}\right)|H|^{-3/2}\{p_1, \lfloor H \rceil\}A_1 + \sqrt{\frac{\mu}{2|H|}}\{p_1, A_1\}$,

$\{p_1, M_1\} = -\sqrt{\frac{\mu}{2}}|H|^{-3/2}\left[\{p_1, \lfloor H \rceil\}\frac{A_1}{2} - |H|\{p_1, A_1\}\right]$,

$\{p_1, M_1\} = -\sqrt{\frac{\mu}{2}}|H|^{-3/2}\left[\pm\{p_1, H\}\frac{A_1}{2} - \pm H\{p_1, A_1\}\right]$, with + sign for $H > 0$,

$\{p_1, M_1\} = \pm\sqrt{\frac{\mu}{2}}|H|^{-3/2}\left[\{p_1, H\}\frac{A_1}{2} - H\{p_1, A_1\}\right]$, with + sign for $H < 0$,

$\{p_1, M_1\} = \pm\sqrt{\frac{\mu}{2}}|H|^{-3/2}\left[\left(\frac{-q_1}{\|\boldsymbol{q}\|^3}\right)\frac{A_1}{2} - H\left(\frac{1}{\|\boldsymbol{q}\|^3}(-q_2{}^2 - q_3{}^2) + \frac{1}{\mu}(p_2{}^2 + p_3{}^2)\right)\right]$,

with + sign for $H < 0$.

$\{p_2, M_1\}$ and $\{p_3, M_1\}$ are analogous, so we can present a shorter calculation.

$\{p_2, M_1\} = \left\{p_2, \sqrt{\frac{\mu}{2|H|}}A_1\right\}$,

$\{p_2, M_1\} = \pm\sqrt{\frac{\mu}{2}}|H|^{-3/2}\left[\{p_2, H\}\frac{A_1}{2} - H\{p_2, A_1\}\right]$, with + sign for $H < 0$,

$\{p_2, M_1\} = \pm\sqrt{\frac{\mu}{2}}|H|^{-3/2}\left[\left(\frac{-q_2}{\|\boldsymbol{q}\|^3}\right)\frac{A_1}{2} - H\left(\frac{1}{\|\boldsymbol{q}\|^3}q_1q_2 + \frac{1}{\mu}(-p_1p_2)\right)\right]$, with + sign for $H < 0$.

$\{p_3, M_1\} = \pm\sqrt{\frac{\mu}{2}}|H|^{-3/2}\left[\{p_3, H\}\frac{A_1}{2} - H\{p_3, A_1\}\right]$, with + sign for $H < 0$,

$\{p_3, M_1\} = \pm\sqrt{\frac{\mu}{2}}|H|^{-3/2}\left[\left(\frac{-q_3}{\|\boldsymbol{q}\|^3}\right)\frac{A_1}{2} - H\left(\frac{1}{\|\boldsymbol{q}\|^3}q_1q_3 + \frac{1}{\mu}(-p_1p_3)\right)\right]$, with + sign for $H < 0$. ■

(VIII.2.) Proof. We can introduce six new coordinates: $q_1, p_1, \left(\frac{1}{\|\mathbf{q}\|}\right), L_1, H,$ and A_1. These six functions are independent, and three of them are constant, namely $L_1, H,$ and A_1. As a result, we need to find the differential equations for $q_1, p_1,$ and $\left(\frac{1}{\|\mathbf{q}\|}\right)$, as functions of these six new coordinates.

$A_1 = \frac{q_1}{\|\boldsymbol{q}\|} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p}) - q_1\|\boldsymbol{p}\|^2}{\mu} = \frac{q_1}{\|\boldsymbol{q}\|} + \frac{p_1(q_1p_1 + q_2p_2 + q_3p_3) - q_1(p_1{}^2 + p_2{}^2 + p_3{}^2)}{\mu}$,

$A_1 = \frac{q_1}{\|\boldsymbol{q}\|} + \frac{p_1(q_2p_2 + q_3p_3) - q_1(p_2{}^2 + p_3{}^2)}{\mu}$,

$H = \frac{\|\boldsymbol{p}\|^2}{2\mu} - \frac{1}{\|\boldsymbol{q}\|} = \frac{p_1{}^2 + p_2{}^2 + p_3{}^2}{2\mu} - \frac{1}{\|\boldsymbol{q}\|}$,

$p_2{}^2 + p_3{}^2 = 2\mu\left(H + \frac{1}{\|\boldsymbol{q}\|}\right) - p_1{}^2$,

$A_1 = \frac{q_1}{\|\boldsymbol{q}\|} + \frac{1}{\mu}\left[p_1(q_2p_2 + q_3p_3) - 2\mu q_1\left(H + \frac{1}{\|\boldsymbol{q}\|}\right) + q_1p_1{}^2\right]$,

$p_1(q_2p_2 + q_3p_3) = 2\mu q_1\left(H + \frac{1}{\|\boldsymbol{q}\|}\right) - q_1p_1{}^2 + \mu A_1 - \frac{\mu q_1}{\|\boldsymbol{q}\|}$,

$p_1(q_2p_2 + q_3p_3) = \mu A_1 + 2\mu q_1 H + \frac{\mu q_1}{\|\boldsymbol{q}\|} - q_1p_1{}^2$,

$q_2p_2 + q_3p_3 = \frac{\mathbf{1}}{p_1}\left[\mu A_1 + 2\mu q_1 H + \frac{\mu q_1}{\|\boldsymbol{q}\|} - q_1p_1{}^2\right]$.

$$q_1' = \{q_1, M_1\} = \pm\sqrt{\frac{\mu}{2}}|H|^{-3/2}\left[\frac{p_1 A_1}{2\mu} - \frac{H}{\mu}(q_2 p_2 + q_3 p_3)\right],$$

$$q_1' = \pm\sqrt{\frac{\mu}{2}}|H|^{-3/2}\left[\frac{p_1 A_1}{2\mu} - \frac{H}{\mu p_1}\left(\mu A_1 + 2\mu q_1 H + \frac{\mu q_1}{\|\boldsymbol{q}\|} - q_1 {p_1}^2\right)\right],$$

$$q_1' = \pm\sqrt{\frac{\mu}{2}}|H|^{-3/2}\left[\frac{p_1 A_1}{2\mu} - \frac{H}{p_1}\left(A_1 + 2q_1 H + \frac{q_1}{\|\boldsymbol{q}\|} - \frac{q_1 {p_1}^2}{\mu}\right)\right],$$

$$q_1' = \pm\sqrt{\frac{\mu}{2}}|H|^{-3/2}\left[\frac{p_1}{\mu}\left(\frac{A_1}{2} + Hq_1\right) - \frac{H}{p_1}\left(\frac{q_1}{\|\boldsymbol{q}\|} + A_1 + 2Hq_1\right)\right].$$

$$p_1' = \{p_1, M_1\} = \pm\sqrt{\frac{\mu}{2}}|H|^{-3/2}\left[\left(\frac{-q_1}{\|\boldsymbol{q}\|^3}\right)\frac{A_1}{2} - H\left(\frac{1}{\|\boldsymbol{q}\|^3}(-{q_2}^2 - {q_3}^2) + \frac{1}{\mu}({p_2}^2 + {p_3}^2)\right)\right],$$

$$p_1' = \pm\sqrt{\frac{\mu}{2}}|H|^{-3/2}\left[\left(\frac{-q_1}{\|\boldsymbol{q}\|^3}\right)\frac{A_1}{2} - H\left(\frac{{q_1}^2 - \|\boldsymbol{q}\|^2}{\|\boldsymbol{q}\|^3} + \frac{1}{\mu}(\|\boldsymbol{p}\|^2 - {p_1}^2)\right)\right],$$

$$p_1' = \pm\sqrt{\frac{\mu}{2}}|H|^{-3/2}\left[\left(\frac{-q_1}{\|\boldsymbol{q}\|^3}\right)\frac{A_1}{2} - H\left(\frac{{q_1}^2 - \|\boldsymbol{q}\|^2}{\|\boldsymbol{q}\|^3} + \frac{1}{\mu}\left(2\mu\left(H + \frac{1}{\|\boldsymbol{q}\|}\right) - {p_1}^2\right)\right)\right],$$

$$p_1' = \pm\sqrt{\frac{\mu}{2}}|H|^{-3/2}\left[\left(\frac{1}{\|\boldsymbol{q}\|^3}\right)\left(\frac{-q_1 A_1}{2} - H{q_1}^2\right) - H\left(\frac{-1}{\|\boldsymbol{q}\|} + \frac{2}{\|\boldsymbol{q}\|} + 2H - \frac{{p_1}^2}{\mu}\right)\right],$$

$$p_1' = \pm\sqrt{\frac{\mu}{2}}|H|^{-3/2}\left[\left(\frac{1}{\|\boldsymbol{q}\|}\right)^3\left(\frac{-A_1 q_1}{2} - H{q_1}^2\right) - H\left(\frac{1}{\|\boldsymbol{q}\|} + 2H - \left(\frac{{p_1}^2}{\mu}\right)\right)\right].$$

$$\left(\frac{1}{\|\boldsymbol{q}\|}\right)' = \left(\frac{-1}{2}\right)\left(\frac{1}{\|\boldsymbol{q}\|^3}\right)(2q_1 q_1' + 2q_2 q_2' + 2q_3 q_3'),$$

$$\left(\frac{1}{\|\boldsymbol{q}\|}\right)' = \left(\frac{-1}{\|\boldsymbol{q}\|^3}\right)(q_1\{q_1, M_1\} + q_2\{q_2, M_1\} + q_3\{q_3, M_1\}),$$

$$\left(\frac{1}{\|\boldsymbol{q}\|}\right)' = \mp\sqrt{\frac{\mu}{2}}|H|^{-\frac{3}{2}}\left(\frac{1}{\|\boldsymbol{q}\|^3}\right)\left[\begin{matrix}\frac{A_1(q_1 p_1 + q_2 p_2 + q_3 p_3)}{2\mu} \\ -\frac{H}{\mu}(q_1 q_2 p_2 + q_1 q_3 p_3 + {q_2}^2 p_1 - 2q_1 q_2 p_2 + {q_3}^2 p_1 - 2q_1 q_3 p_3)\end{matrix}\right],$$

$$\left(\frac{1}{\|\boldsymbol{q}\|}\right)' = \mp\sqrt{\frac{\mu}{2}}|H|^{-\frac{3}{2}}\left(\frac{1}{\|\boldsymbol{q}\|^3}\right)\left[\begin{matrix}\frac{A_1 q_1 p_1}{2\mu} + \frac{A_1}{2\mu}(q_2 p_2 + q_3 p_3) \\ -\frac{H}{\mu}(q_1(q_2 p_2 + q_3 p_3) + ({q_2}^2 + {q_3}^2)p_1 - 2q_1(q_2 p_2 + q_3 p_3))\end{matrix}\right],$$

$$\left(\frac{1}{\|\boldsymbol{q}\|}\right)' = \mp\sqrt{\frac{\mu}{2}}|H|^{-\frac{3}{2}}\left(\frac{1}{\|\boldsymbol{q}\|^3}\right)\left[\begin{matrix}\frac{A_1 q_1 p_1}{2\mu} - \frac{H}{\mu}\|\boldsymbol{q}\|^2 p_1 + \frac{H}{\mu}{q_1}^2 p_1 \\ +\left(\frac{A_1}{2\mu} + \frac{H}{\mu}q_1\right)\left(\frac{1}{p_1}\right)\left(\mu A_1 + 2\mu H q_1 + \frac{\mu q_1}{\|\boldsymbol{q}\|} - q_1 {p_1}^2\right)\end{matrix}\right],$$

$$\left(\frac{1}{\|\boldsymbol{q}\|}\right)' = \mp\sqrt{\frac{\mu}{2}}|H|^{-\frac{3}{2}}\left(\frac{1}{\|\boldsymbol{q}\|^3}\right)\left[\begin{matrix}\left(\frac{A_1}{2} + Hq_1\right)\left(\frac{q_1 p_1}{\mu}\right) - \frac{H}{\mu}\|\boldsymbol{q}\|^2 p_1 \\ +\left(\frac{A_1}{2} + Hq_1\right)\left(\frac{1}{p_1}\right)\left(A_1 + 2Hq_1 + \frac{q_1}{\|\boldsymbol{q}\|} - \frac{q_1 {p_1}^2}{\mu}\right)\end{matrix}\right],$$

$$\left(\frac{1}{\|\boldsymbol{q}\|}\right)' = \mp\sqrt{\frac{\mu}{2}}|H|^{-3/2}\left(\frac{1}{\|\boldsymbol{q}\|}\right)^3\left[\left(\frac{A_1}{2} + Hq_1\right)\left(\frac{1}{p_1}\right)\left(\frac{q_1}{\|\boldsymbol{q}\|} + A_1 + 2Hq_1\right) + \left(\frac{-Hp_1\|\boldsymbol{q}\|^2}{\mu}\right)\right]. \blacksquare$$

(VIII.3.) Proof.

$\{H, L_i\} = 0$, from (VI.2.),

$\{L_i, L_j\} = L_k$, from (VI.4.),

$\{H, A_i\} = 0$, from (VI.8.).

$\{L_i, A_j\} = \{(\boldsymbol{q}\times\boldsymbol{p})_i, A_j\} = \{q_j p_k - q_k p_j, A_j\}$,

$\{L_i, A_j\} = q_j\{p_k, A_j\}+p_k\{q_j, A_j\} - q_k\{p_j, A_j\} - p_j\{q_k, A_j\}$,

$\{L_i, A_j\} = q_j \frac{1}{\|\boldsymbol{q}\|^3} q_j q_k + \frac{1}{\mu} q_j(-p_j p_k) + p_k\left(\frac{1}{\mu}\right)(q_i p_i + q_k p_k)$

$\quad -q_k\left(\frac{1}{\|\boldsymbol{q}\|^3}\right)(-q_i{}^2 - q_k{}^2) - q_k(1/\mu)(p_i{}^2 + p_k{}^2) - p_j(1/\mu)(q_k p_j - 2q_j p_k)$,

$\{L_i, A_j\} = \frac{1}{\|\boldsymbol{q}\|^3}(q_j{}^2 q_k + q_i{}^2 q_k + q_k{}^3)$

$\quad +\frac{1}{\mu}(-q_j p_j p_k + q_i p_i p_k + q_k p_k{}^2 - q_k p_i{}^2 - q_k p_k{}^2 - q_k p_j{}^2 - 2q_j p_j p_k)$,

$\{L_i, A_j\} = \frac{1}{\|\boldsymbol{q}\|^3}(q_k\|\boldsymbol{q}\|^2) + \frac{1}{\mu}(q_i p_i p_k + q_j p_j p_k - q_k p_i{}^2 - q_k p_j{}^2)$,

$\{L_i, A_j\} = \frac{q_k}{\|\boldsymbol{q}\|} + \frac{1}{\mu}(p_k(\boldsymbol{q}\cdot\boldsymbol{p}) - q_k\|\boldsymbol{p}\|^2)$,

$\{L_i, A_j\} = A_k$.

$\{L_i, A_i\} = \{(\boldsymbol{q}\times\boldsymbol{p})_i, A_i\} = \{q_j p_k - q_k p_j, A_i\}$,

$\{L_i, A_i\} = q_j\{p_k, A_i\}+p_k\{q_j, A_i\} - q_k\{p_j, A_i\} - p_j\{q_k, A_i\}$,

$\{L_i, A_i\} = q_j \frac{1}{\|\boldsymbol{q}\|^3} q_i q_k + \frac{1}{\mu} q_j(-p_i p_k) + p_k\left(\frac{1}{\mu}\right)(q_j p_i - 2q_i p_j)$

$\quad -q_k\left(\frac{1}{\|\boldsymbol{q}\|^3}\right)(-q_i q_j) + q_k(1/\mu)(-p_i p_j) - p_j(1/\mu)(q_k p_i - 2q_i p_k)$,

$\{L_i, A_i\} = \frac{1}{\|\boldsymbol{q}\|^3}(q_i q_j q_k - q_i q_j q_k)$

$\quad +\frac{1}{\mu}(-q_j p_i p_k + q_j p_i p_k - 2q_i p_{j p_k} + q_k p_i p_j - q_k p_i \mathrm{p}_j + 2q_i p_j p_k)$,

$\{L_i, A_i\} = 0$.

Now we need some more intermediate results.

$\left\{\frac{1}{\|\boldsymbol{q}\|}, A_i\right\} = \sum_{m=1}^{3}\left[\frac{\partial}{\partial q_m}\left(\frac{1}{\|\boldsymbol{q}\|}\right)\frac{\partial}{\partial p_m}(A_i) - \frac{\partial}{\partial p_m}\left(\frac{1}{\|\boldsymbol{q}\|}\right)\frac{\partial}{\partial q_m}(A_i)\right]$,

$\left\{\frac{1}{\|\boldsymbol{q}\|}, A_i\right\} = \sum_{m=1}^{3}\left[\left(\frac{-1}{2}\right)\left(\frac{1}{\|\boldsymbol{q}\|^3}\right)(2q_m)\frac{\partial}{\partial p_m}(A_i) - 0\right]$,

$\left\{\frac{1}{\|\boldsymbol{q}\|}, A_i\right\} = \sum_{m=1}^{3}\left[\left(\frac{-q_m}{\|\boldsymbol{q}\|^3}\right)\frac{\partial}{\partial p_m}(A_i)\right]$,

$\left\{\frac{1}{\|\boldsymbol{q}\|}, A_i\right\}=\left(\frac{-1}{\|\boldsymbol{q}\|^3}\right)\left(\frac{1}{\mu}\right)(q_i\boldsymbol{q}\cdot\boldsymbol{p} + q_i{}^2 p_i - 2q_i{}^2 p_i + q_j{}^2 p_i - 2q_i q_j p_j + q_k{}^2 p_i - 2q_i q_k p_k)$,

$\left\{\frac{1}{\|\boldsymbol{q}\|}, A_i\right\}=\left(\frac{-1}{\|\boldsymbol{q}\|^3}\right)\left(\frac{1}{\mu}\right)(q_i\boldsymbol{q}\cdot\boldsymbol{p} - q_i{}^2 p_i + q_j{}^2 p_i + q_k{}^2 p_i - 2q_i\boldsymbol{q}\cdot\boldsymbol{p} + 2q_i{}^2 p_i)$,

$\left\{\frac{1}{\|\boldsymbol{q}\|}, A_i\right\}=\left(\frac{-1}{\|\boldsymbol{q}\|^3}\right)\left(\frac{1}{\mu}\right)(-q_i\boldsymbol{q}\cdot\boldsymbol{p} + \|q\|^2 p_i)$,

$\left\{\frac{1}{\|\boldsymbol{q}\|}, A_i\right\}=\left(\frac{1}{\mu}\right)\left(\frac{+q_i\boldsymbol{q}\cdot\boldsymbol{p}}{\|\boldsymbol{q}\|^3} - \frac{p_i}{\|\boldsymbol{q}\|}\right)$.

$\{\boldsymbol{q}\cdot\boldsymbol{p}, A_i\} = \sum_{m=1}^{3}[q_m\{p_m, A_i\} + p_m\{q_m, A_i\}]$,

$\{\boldsymbol{q}\cdot\boldsymbol{p}, A_i\} = \frac{q_i}{\|\boldsymbol{q}\|^3}(-q_j{}^2 - q_k{}^2) + \frac{q_i}{\mu}(p_j{}^2 + p_k{}^2)$

$\quad +\frac{q_j}{\|\boldsymbol{q}\|^3}(q_i q_j) + \frac{q_j}{\mu}(-p_i p_j)$

$\quad +\frac{q_k}{\|\boldsymbol{q}\|^3}(q_i q_k) + \frac{q_k}{\mu}(-p_i p_k)$

$$+\tfrac{p_i}{\mu}\left(q_i p_j + q_k p_k\right) + \tfrac{p_j}{\mu}\left(q_j p_i - 2q_i p_j\right) + \tfrac{p_k}{\mu}\left(q_k p_i - 2q_i p_k\right),$$

$$\{\boldsymbol{q}\cdot\boldsymbol{p}, A_i\} = \tfrac{1}{\|\boldsymbol{q}\|^3}\left(-q_i {q_j}^2 - q_i {q_k}^2 + q_i {q_j}^2 + q_i {q_k}^2\right)$$

$$+\tfrac{1}{\mu}\begin{pmatrix} q_i {p_j}^2 + q_i {p_k}^2 - q_j p_i p_j - q_k p_i p_k + q_i p_i p_j \\ +q_k p_i p_k + q_j p_i p_j - 2q_i {p_j}^2 + q_k p_i p_k - 2q_i {p_k}^2 \end{pmatrix},$$

$$\{\boldsymbol{q}\cdot\boldsymbol{p}, A_i\} = \tfrac{1}{\mu}\left(-q_i {p_j}^2 + q_i {p_k}^2 + q_j p_i p_j + q_k p_i p_k\right),$$

$$\{\boldsymbol{q}\cdot\boldsymbol{p}, A_i\} = \tfrac{1}{\mu}(-q_i\|p\|^2 + p_i\boldsymbol{q}\cdot\boldsymbol{p}).$$

$$\{\|\boldsymbol{p}\|^2, A_i\} = 2p_i\{p_i, A_i\} + 2p_j\{p_j, A_i\} + 2p_k\{p_k, A_i\},$$

$$\{\|\boldsymbol{p}\|^2, A_i\} = \frac{2}{\|\boldsymbol{q}\|^3}\left[p_i\left(-{q_j}^2 - {q_k}^2\right)\right] + \frac{2}{\mu}\left[p_i\left({p_j}^2 + {p_k}^2\right)\right]$$

$$+\tfrac{2}{\|\boldsymbol{q}\|^3}\left[p_j\left(q_i q_j\right)\right] + \tfrac{2}{\mu}\left[p_j\left(-p_i p_j\right)\right]$$

$$+\tfrac{2}{\|\boldsymbol{q}\|^3}\left[p_k(q_i q_k)\right] + \tfrac{2}{\mu}\left[p_k(-p_i p_k)\right],$$

$$\{\|\boldsymbol{p}\|^2, A_i\} = \tfrac{2}{\|\boldsymbol{q}\|^3}\left(-{q_j}^2 p_i - {q_k}^2 p_i + q_i q_j p_j + q_i q_k p_k\right) + \tfrac{2}{\mu}\left(p_i {p_j}^2 + p_i {p_k}^2 - p_i {p_j}^2 - p_i {p_k}^2\right),$$

$$\{\|\boldsymbol{p}\|^2, A_i\} = \tfrac{2}{\|\boldsymbol{q}\|^3}\left(-{q_i}^2 p_i - {q_j}^2 p_i - {q_k}^2 p_i + {q_i}^2 p_i + q_i q_j p_j + q_i q_k p_k\right) + 0,$$

$$\{\|\boldsymbol{p}\|^2, A_i\} = \tfrac{2}{\|\boldsymbol{q}\|^3}\left(q_i(\boldsymbol{q}\cdot\boldsymbol{p}) - p_i\|\boldsymbol{q}\|^2\right),$$

$$\{\|\boldsymbol{p}\|^2, A_i\} = \tfrac{2q_i(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|^3} - \tfrac{2p_i}{\|\boldsymbol{q}\|}.$$

Now back to the main results.

$$\{A_i, A_j\} = \left\{\tfrac{q_i}{\|\boldsymbol{q}\|} + \tfrac{p_i(\boldsymbol{q}\cdot\boldsymbol{p}) - q_i\|\boldsymbol{p}\|^2}{\mu}, A_j\right\},$$

$$\{A_i, A_j\} = \tfrac{1}{\|\boldsymbol{q}\|}\{q_i, A_j\} + q_i\left\{\tfrac{1}{\|\boldsymbol{q}\|}, A_j\right\} + \tfrac{p_i}{\mu}\{(\boldsymbol{q}\cdot\boldsymbol{p}), A_j\} + \tfrac{\boldsymbol{q}\cdot\boldsymbol{p}}{\mu}\{p_i, A_j\} - \tfrac{q_i}{\mu}\{\|\boldsymbol{p}\|^2, A_j\} - \tfrac{\|\boldsymbol{p}\|^2}{\mu}\{q_i, A_j\},$$

$$\{A_i, A_j\} = \tfrac{1}{\|\boldsymbol{q}\|}\left(\tfrac{1}{\mu}\right)\left[q_i p_j - 2q_j p_i\right] + q_i\left(\tfrac{1}{\mu}\right)\left[\tfrac{+q_j\boldsymbol{q}\cdot\boldsymbol{p}}{\|\boldsymbol{q}\|^3} - \tfrac{p_j}{\|\boldsymbol{q}\|}\right] + \tfrac{p_i}{\mu}\left(\tfrac{1}{\mu}\right)\left[-q_j\|p\|^2 + p_j\boldsymbol{q}\cdot\boldsymbol{p}\right]$$

$$+\tfrac{\boldsymbol{q}\cdot\boldsymbol{p}}{\mu}\left[\tfrac{1}{\|\boldsymbol{q}\|^3}q_i q_j + \tfrac{1}{\mu}\left(-p_i p_j\right)\right] - \tfrac{q_i}{\mu}\left[\tfrac{2q_j(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|^3} - \tfrac{2p_j}{\|\boldsymbol{q}\|}\right] - \tfrac{\|\boldsymbol{p}\|^2}{\mu}\left(\tfrac{1}{\mu}\right)\left[\left(q_i p_j - 2q_j p_i\right)\right],$$

$$= \tfrac{1}{\mu\|\boldsymbol{q}\|^3}\left(q_i p_j\|\boldsymbol{q}\|^2 - 2q_j p_i\|\boldsymbol{q}\|^2 + q_i q_j\boldsymbol{q}\cdot\boldsymbol{p} - q_i p_j\|\boldsymbol{q}\|^2 + q_i q_j\boldsymbol{q}\cdot\boldsymbol{p} - 2q_i q_j\boldsymbol{q}\cdot\boldsymbol{p} + 2q_i p_j\|\boldsymbol{q}\|^2\right)$$

$$+\tfrac{1}{\mu^2}\left(-q_j p_i\|\boldsymbol{p}\|^2 + p_i p_j\boldsymbol{q}\cdot\boldsymbol{p} - p_i p_j\boldsymbol{q}\cdot\boldsymbol{p} - q_i p_j\|\boldsymbol{p}\|^2 + 2q_j p_i\|\boldsymbol{p}\|^2\right),$$

$$\{A_i, A_j\} = \tfrac{1}{\mu\|\boldsymbol{q}\|^3}\left(2\left(q_i p_j - q_j p_i\right)\|\boldsymbol{q}\|^2\right) + \tfrac{1}{\mu^2}\left(-\left(q_i p_j - q_j p_i\right)\|\boldsymbol{p}\|^2\right),$$

$$\{A_i, A_j\} = \tfrac{2}{\mu\|\boldsymbol{q}\|}(L_k) - \tfrac{1}{\mu^2}(L_k\|\boldsymbol{p}\|^2),$$

$$\{A_i, A_j\} = \tfrac{2}{\mu}\left(\tfrac{1}{\|\boldsymbol{q}\|} - \tfrac{\|\boldsymbol{p}\|^2}{2\mu}\right)L_k,$$

$$\{A_i, A_j\} = \tfrac{-2H}{\mu}L_k.$$

For $H < 0$:

$$\{M_i, M_j\} = \left\{\sqrt{\tfrac{-\mu}{2H}}A_i, \sqrt{\tfrac{-\mu}{2H}}A_j\right\} = \tfrac{-\mu}{2H}\{A_i, A_j\} = L_k.$$

Now we'll prepare some more intermediate results.

$\{q_i, q_j\} = \sum_{m=1}^{3}\left(\frac{\partial q_i}{\partial q_m}\frac{\partial q_j}{\partial p_m} - \frac{\partial q_i}{\partial p_m}\frac{\partial q_j}{\partial q_m}\right) = 0,$

$\{q_i, q_j\} = 0,$

$\{p_i, p_j\} = 0,$

$\{q_i, p_i\} = 1,$

$\{q_i, p_j\} = 0,$

$\{q_i, L_i\} = \{q_i, q_j p_k - q_k p_j\} = q_j\{q_i, p_k\} + p_k\{q_i, q_j\} - q_k\{q_i, p_j\} - p_j\{q_i, q_k\}$

$= q_j(0) + p_k(0) - q_k(0) - p_j(0) = 0,$

$\{q_i, L_i\} = 0.$

$\{q_i, L_j\} = \{q_i, q_k p_i - q_i p_k\} = q_k\{q_i, p_i\} + p_i\{q_i, q_k\} - q_i\{q_i, p_k\} - p_k\{q_i, q_i\}$

$\{q_i, L_j\} = q_k(1) + p_i(0) - q_i(0) - p_k(0) = q_k,$

$\{q_i, L_j\} = q_k.$

$\{p_i, L_i\} = 0.$

$\{p_i, L_j\} = \{p_i, q_k p_i - q_i p_k\} = q_k\{p_i, p_i\} + p_i\{p_i, q_k\} - q_i\{p_i, p_k\} - p_k\{p_i, q_i\}$

$\{p_i, L_j\} = q_k(0) + p_i(0) - q_i(0) - p_k(-1) = p_k,$

$\{q_i, L_j\} = p_k.$

$\{L_i, L_j\} = \{q_j p_k - q_k p_j, L_j\} = q_j\{p_k, L_j\} + p_k\{q_j, L_j\} - q_k\{p_j, L_j\} - p_j\{q_k, L_j\}$

$\{L_i, L_j\} = q_j(-p_i) + p_k(0) - q_k(0) - p_j(-q_i) = -q_j p_i + q_i p_j = L_k,$

$\{L_i, L_j\} = L_k.$

Now back to the main results.

For $H < 0$:

$\left\{\left(\frac{L\pm M}{2}\right)_i, \left(\frac{L\pm M}{2}\right)_j\right\} = \left\{\frac{L_i}{2} \pm \frac{M_i}{2}, \frac{L_j}{2} \pm \frac{M_j}{2}\right\} = \frac{1}{4}\{L_i, L_j\} \pm \frac{1}{4}\{L_i, M_j\} \pm \frac{1}{4}\{M_i, L_j\} + \frac{1}{4}\{M_i, M_j\}$

$\left\{\left(\frac{L\pm M}{2}\right)_i, \left(\frac{L\pm M}{2}\right)_j\right\} = \frac{1}{4}L_k \pm \frac{1}{4}\sqrt{\frac{-\mu}{2H}}\{L_i, A_j\} \pm \frac{1}{4}\sqrt{\frac{-\mu}{2H}}\{A_i, L_j\} + \frac{1}{4}\left(\frac{-\mu}{2H}\right)\{A_i, A_j\}$

$\left\{\left(\frac{L\pm M}{2}\right)_i, \left(\frac{L\pm M}{2}\right)_j\right\} = \frac{1}{4}L_k \pm \frac{1}{4}\sqrt{\frac{-\mu}{2H}}A_k \pm \frac{1}{4}\sqrt{\frac{-\mu}{2H}}A_k + \frac{1}{4}\left(\frac{-\mu}{2H}\right)\left(\frac{-2H}{\mu}\right)L_k$

$\left\{\left(\frac{L\pm M}{2}\right)_i, \left(\frac{L\pm M}{2}\right)_j\right\} = \frac{1}{2}L_k \pm \frac{1}{2}M_k.$

$\left\{\left(\frac{L\pm M}{2}\right)_i, \left(\frac{L\pm M}{2}\right)_j\right\} = \left(\frac{L\pm M}{2}\right)_k.$

$\left\{\left(\frac{L+M}{2}\right)_i, \left(\frac{L-M}{2}\right)_j\right\} = \left\{\frac{L_i}{2} + \frac{M_i}{2}, \frac{L_j}{2} - \frac{M_j}{2}\right\} = \frac{1}{4}\{L_i, L_j\} - \frac{1}{4}\{L_i, M_j\} + \frac{1}{4}\{M_i, L_j\} - \frac{1}{4}\{M_i, M_j\}$

$\left\{\left(\frac{L+M}{2}\right)_i, \left(\frac{L-M}{2}\right)_j\right\} = \frac{1}{4}L_k - \frac{1}{4}\sqrt{\frac{-\mu}{2H}}\{L_i, A_j\} + \frac{1}{4}\sqrt{\frac{-\mu}{2H}}\{A_i, L_j\} - \frac{1}{4}\left(\frac{-\mu}{2H}\right)\{A_i, A_j\}$

$\left\{\left(\frac{L+M}{2}\right)_i, \left(\frac{L-M}{2}\right)_j\right\} = \frac{1}{4}L_k - \frac{1}{4}\sqrt{\frac{-\mu}{2H}}A_k + \frac{1}{4}\sqrt{\frac{-\mu}{2H}}A_k - \frac{1}{4}\left(\frac{-\mu}{2H}\right)\left(\frac{-2H}{\mu}\right)L_k$

$\left\{\left(\frac{L+M}{2}\right)_i, \left(\frac{L-M}{2}\right)_j\right\} = 0.$

For $H > 0$:

$\{M_i, M_j\} = \left(\frac{\mu}{2H}\right)\{A_i, A_j\} = \left(\frac{\mu}{2H}\right)\frac{-2H}{\mu}L_k = -L_k,$

$\{M_i, M_j\} = -L_k.$

For $H = 0$:

$\{A_i, A_j\} = \frac{-2H}{\mu}L_k{=}0.$ ■

(VIII.4.) Proof. <<Citation>>

(VIII.5.) Proof. Because of (I.27.) and (II.8.), we need to calculate the Poisson brackets.

for $H \neq 0$:

$\{L_i, M_j\} = \sqrt{\frac{\mu}{2|H|}}\{L_i, A_j\} = \sqrt{\frac{\mu}{2|H|}}A_k = M_k,$ (VIII.3.)

$\{L_i, M_j\} = M_k.$

$\{L_1, M_1\} = \sqrt{\frac{\mu}{2|H|}}\{L_1, A_1\} = 0,$ (VIII.3.),

$\{A_1, M_1\} = \sqrt{\frac{\mu}{2|H|}}\{A_1, A_1\} = 0,$

$\{H, M_1\} = 0,$ (VII.2.)

$\{L_2M_2 + L_3M_3, M_1\} = L_2\{M_2, M_1\} + M_2\{L_2, M_1\} + L_3\{M_3, M_1\} + M_3\{L_3, M_1\}$

$\{L_2M_2 + L_3M_3, M_1\} = L_2\frac{\mu}{2|H|}\{A_2, A_1\} + M_2(-M_3) + L_3\frac{\mu}{2|H|}\{A_3, A_1\} + M_3(M_2),$

$\{L_2M_2 + L_3M_3, M_1\} = L_2\frac{\mu}{2|H|}\frac{-2H}{\mu}(-L_3) - M_2M_3 + L_3\frac{\mu}{2|H|}\frac{-2H}{\mu}(L_2) + M_2M_3 = 0.$

$\{L_2M_2 + L_3M_3, M_1\} = 0.$

For $H = 0$:

$\{L_2A_2 + L_3A_3, A_1\} = L_2\{A_2, A_1\} + A_2\{L_2, A_1\} + L_3\{A_3, A_1\} + A_3\{L_3, A_1\}$

$\{L_2A_2 + L_3A_3, A_1\} = L_2(0) + A_2(-A_3) + L_3(0) + A_3(A_2) = 0,$

$\{L_2A_2 + L_3A_3, A_1\} = 0.$

For $H < 0$:

$\{{L_2}^2 + {M_3}^2, M_1\} = 2L_2\{L_2, M_1\} + 2M_3\{M_3, M_1\}$

$\{{L_2}^2 + {M_3}^2, M_1\} = 2L_2(-M_3) + 2M_3(L_2) = 0.$

$\{{L_2}^2 + {M_3}^2, M_1\} = 0.$

$\{{L_3}^2 + {M_2}^2, M_1\} = 2L_3\{L_3, M_1\} + 2M_2\{M_2, M_1\}$

$\{{L_3}^2 + {M_2}^2, M_1\} = 2L_3(M_2) + 2M_2(-L_3) = 0.$

$\{{L_3}^2 + {M_2}^2, M_1\} = 0.$

$\{M_2M_3 - L_2L_3, M_1\} = M_2\{M_3, M_1\} + M_3\{M_2, M_1\} - L_2\{L_3, M_1\} - L_3\{L_2, M_1\} =$

$\{M_2M_3 - L_2L_3, M_1\} = M_2(L_2) + M_3(-L_3) - L_2(M_2) - L_3(-M_3) = 0,$

$\{M_2M_3 - L_2L_3, M_1\} = 0.$

For $H > 0$:

$\{{L_2}^2 - {M_3}^2, M_1\} = 2L_2\{L_2, M_1\} - 2M_3\{M_3, M_1\}$

$\{{L_2}^2 - {M_3}^2, M_1\} = 2L_2(-M_3) - 2M_3(-L_2) = 0.$

$\{{L_2}^2 - {M_3}^2, M_1\} = 0.$

$\{{L_3}^2 - {M_2}^2, M_1\} = 2L_3\{L_3, M_1\} - 2M_2\{M_2, M_1\}$

$\{{L_3}^2 - {M_2}^2, M_1\} = 2L_3(M_2) - 2M_2(L_3) = 0.$

$\{{L_3}^2 - {M_2}^2, M_1\} = 0.$

$\{M_2M_3 + L_2L_3, M_1\} = M_2\{M_3, M_1\} + M_3\{M_2, M_1\} + L_2\{L_3, M_1\} + L_3\{L_2, M_1\} =$

$\{M_2M_3 + L_2L_3, M_1\} = M_2(-L_2) + M_3(L_3) + L_2(M_2) + L_3(-M_3) = 0,$

$\{M_2M_3 + L_2L_3, M_1\} = 0.$

For $H = 0$:

$\{A_1, A_1\} = 0.$

$\{A_2, A_1\} = \frac{-2H}{\mu}(-L_3) = 0$, (VIII.3.).

$\{L_1, A_1\} = \{q_2p_3 - q_3p_2, A_1\} = q_2\{p_3, A_1\} + p_3\{q_2, A_1\} - q_3\{p_2, A_1\} - p_2\{q_3, A_1\},$

$\{L_1, A_1\} = q_2\left[\frac{1}{\|\boldsymbol{q}\|^3}q_1q_3 + \frac{1}{\mu}(-p_1p_3)\right] + p_3\left[\left(\frac{1}{\mu}\right)(q_2p_1 - 2q_1p_2)\right]$

$\quad -q_3\left[\frac{1}{\|\boldsymbol{q}\|^3}q_1q_2 + \frac{1}{\mu}(-p_1p_2)\right] - p_2\left[\left(\frac{1}{\mu}\right)(q_3p_1 - 2q_1p_3)\right],$

$\{L_1, A_1\} = \left(\frac{1}{\|\boldsymbol{q}\|^3}\right)[q_1q_2q_3 - q_1q_2q_3]$

$\quad + \left(\frac{1}{\mu}\right)[-q_2p_1p_3 + q_2p_1p_3 - 2q_1p_2p_3 + q_3p_1p_2 - q_3p_1p_2 + 2q_1p_2p_3] = 0,$

$\{L_1, A_1\} = 0.$

$\{H, A_1\} = 0$, (VI.2.). ■

(VIII.6.) Proof.

(i) For $H < 0$: Because of (II.8.), we have $\frac{dL_2}{ds} = \mathbf{L}_{X_{M_1}}L_2 = \{L_2, M_1\} = -M_3$, and so

$L_2' = \{L_2, M_1\} = -M_3,$

$M_3' = \{M_3, M_1\} = +L_2$

$(L_2 + iM_3)' = -M_3 + iL_2 = i(L_2 + iM_3)$

$\Rightarrow L_2 + iM_3 = C_1e^{i(s-\alpha_1)}$

$L_2 = C_1\, cos(s - \alpha_1),$

$M_3 = C_1\, sin(s - \alpha_1).$

$L_3' = \{L_3, M_1\} = M_2,$

$M_2' = \{M_2, M_1\} = -L_3$

$(L_3 - iM_2)' = M_2 + iL_3 = i(L_3 - iM_2)$

$\Rightarrow L_3 - iM_2 = C_2 e^{i(s-\alpha_2)}$

$L_3 = C_2 \, cos(s - \alpha_2)$,

$M_2 = -C_2 \, sin(s - \alpha_2)$.

$(L \times M)_1 = L_2 M_3 - L_3 M_2$,

$(L \times M)_1 = C_1 \, cos(s - \alpha_1) C_1 \, sin(s - \alpha_1) - C_2 \, cos(s - \alpha_2)\,(-1) C_2 \, sin(s - \alpha_2)$,

$(L \times M)_1 = \left(C_1{}^2/2\right) sin\, 2(s - \alpha_1) + \left(C_2{}^2/2\right) sin\, 2(s - \alpha_2)$, $+$ sign disagrees with original paper,

(ii) For $H > 0$:

$L_2' = \{L_2, M_1\} = -M_3$,

$M_3' = \{M_3, M_1\} = -L_2$,

$(L_2 + M_3)' = -M_3 - L_2 = -(L_2 + M_3)$,

$L_2 + M_3 = C_3 e^{-(s-\alpha_3)}$,

$(L_2 - M_3)' = -M_3 + L_2 = (L_2 - M_3)$,

$L_2 - M_3 = C_3 e^{+(s-\alpha_3)}$,

$L_2 = C_3 \left(\frac{1}{2}\right) \left[e^{-(s-\alpha_3)} + e^{+(s-\alpha_3)}\right] = C_3 cosh(s - \alpha_3)$,

$M_3 = C_3 \left(\frac{1}{2}\right) \left[e^{-(s-\alpha_3)} - e^{+(s-\alpha_3)}\right] = -C_3 sinh(s - \alpha_3)$,

$L_3' = \{L_3, M_1\} = M_2$,

$M_2' = \{M_2, M_1\} = L_3$,

$(L_3 + M_2)' = M_2 + L_3 = (L_3 + M_2)$,

$L_3 + M_2 = C_4 e^{(s-\alpha_4)}$,

$(L_3 - M_2)' = M_2 - L_3 = -(L_3 - M_2)$,

$L_3 - M_2 = C_4 e^{-(s-\alpha_4)}$,

$L_3 = C_4 \left(\frac{1}{2}\right) \left[e^{+(s-\alpha_4)} + e^{-(s-\alpha_4)}\right] = C_4 cosh(s - \alpha_4)$,

$M_2 = C_4 \left(\frac{1}{2}\right) \left[e^{(s-\alpha_4)} - e^{-(s-\alpha_4)}\right] = C_4 sinh(s - \alpha_4)$,

$(L \times M)_1 = L_2 M_3 - L_3 M_2$,

$(L \times M)_1 = C_3 cosh(s - \alpha_3)(-1) C_3 sinh(s - \alpha_3) - C_4 cosh(s - \alpha_4) C_4 sinh(s - \alpha_4)$,

$(L \times M)_1 = -\left(C_3{}^2/2\right) sinh2(s - \alpha_3) - \left(C_4{}^2/2\right) sinh2(s - \alpha_4)$, - sign disagrees with original paper,

(iii) Since $A_3' = \{A_3, A_1\} = 0$, A_3 is constant.

$L_2' = \{L_2, A_1\} = -A_3$,

$L_2 = -A_3 s + C_5$.

$L_3' = \{L_3, A_1\} = A_2$,

$L_3 = A_2 s + C_6$.

$(L \times A)_1 = L_2 A_3 - L_3 A_2$,

$(L\times A)_1 = (-A_3 s + C_5)A_3 - (A_2 s + C_6)A_2,$

$(L\times A)_1 = -\left({A_2}^2 + {A_3}^2\right)s - A_2 C_6 + A_3 C_5.$ ■

(VIII.7.) Proof.

For $H < 0$:

$\frac{d\|\mathbf{M}\|}{ds} = \frac{1}{2\|\mathbf{M}\|}\frac{d\|\mathbf{M}\|^2}{ds} = \frac{1}{2\|\mathbf{M}\|}\{\|\mathbf{M}\|^2, M_1\},$

$\frac{d\|\mathbf{M}\|}{ds} = \left(\frac{1}{2\|\mathbf{M}\|}\right)[\{M_1^2, M_1\} + \{M_2^2, M_1\} + \{M_3^2, M_1\}],$

$\frac{d\|\mathbf{M}\|}{ds} = \left(\frac{1}{\|\mathbf{M}\|}\right)[M_1\{M_1, M_1\} + M_2\{M_2, M_1\} + M_3\{M_3, M_1\}],$

$\frac{d\|\mathbf{M}\|}{ds} = \left(\frac{1}{\|\mathbf{M}\|}\right)[M_2(-L_3) + M_3(L_2)],$

$\frac{d\|\mathbf{M}\|}{ds} = \frac{+(\boldsymbol{L}\times\boldsymbol{M})_1}{\|\boldsymbol{M}\|}.$

For $H > 0$:

$\frac{d\|\mathbf{M}\|}{ds} = \frac{1}{2\|\mathbf{M}\|}\frac{d\|\mathbf{M}\|^2}{ds} = \frac{1}{2\|\mathbf{M}\|}\{\|\mathbf{M}\|^2, M_1\},$

$\frac{d\|\mathbf{M}\|}{ds} = \left(\frac{1}{2\|\mathbf{M}\|}\right)[\{M_1^2, M_1\} + \{M_2^2, M_1\} + \{M_3^2, M_1\}],$

$\frac{d\|\mathbf{M}\|}{ds} = \left(\frac{1}{\|\mathbf{M}\|}\right)[M_1\{M_1, M_1\} + M_2\{M_2, M_1\} + M_3\{M_3, M_1\}],$

$\frac{d\|\mathbf{M}\|}{ds} = \left(\frac{1}{\|\mathbf{M}\|}\right)[M_2(L_3) + M_3(-L_2)],$

$\frac{d\|\mathbf{M}\|}{ds} = \frac{-(\boldsymbol{L}\times\boldsymbol{M})_1}{\|\boldsymbol{M}\|}.$

For all H, along A_1:

$\frac{d\|\boldsymbol{L}\|}{ds} = \frac{1}{2\|\boldsymbol{L}\|}\frac{d\|\boldsymbol{L}\|^2}{ds} = \frac{1}{2\|\boldsymbol{L}\|}\{\|\boldsymbol{L}\|^2, A_1\},$

$\frac{d\|\boldsymbol{L}\|}{ds} = \left(\frac{1}{2\|\boldsymbol{L}\|}\right)[\{L_1^2, A_1\} + \{L_2^2, A_1\} + \{L_3^2, A_1\}],$

$\frac{d\|\boldsymbol{L}\|}{ds} = \left(\frac{1}{\|\boldsymbol{L}\|}\right)[L_1\{L_1, A_1\} + L_2\{L_2, A_1\} + L_3\{L_3, A_1\}],$

$\frac{d\|\boldsymbol{L}\|}{ds} = \left(\frac{1}{\|\boldsymbol{L}\|}\right)[L_2(-A_3) + L_3(A_2)],$

$\frac{d\|\boldsymbol{L}\|}{ds} = \frac{-(\boldsymbol{L}\times\boldsymbol{A})_1}{\|\boldsymbol{L}\|}.$

For $H \neq 0$, along M_1:

$\frac{d\|\boldsymbol{L}\|}{ds} = \frac{1}{2\|\boldsymbol{L}\|}\frac{d\|\boldsymbol{L}\|^2}{ds} = \frac{1}{2\|\boldsymbol{L}\|}\{\|\boldsymbol{L}\|^2, M_1\},$

$\frac{d\|\boldsymbol{L}\|}{ds} = \left(\frac{1}{2\|\boldsymbol{L}\|}\right)\sqrt{\frac{\mu}{2|H|}}[\{L_1^2, A_1\} + \{\mathrm{L}_2^2, A_1\} + \{L_3^2, A_1\}],$

$\frac{d\|\boldsymbol{L}\|}{ds} = \sqrt{\frac{\mu}{2|H|}}\left(\frac{-(\boldsymbol{L}\times\boldsymbol{A})_1}{\|\boldsymbol{L}\|}\right).$

$\frac{d\|\boldsymbol{L}\|}{ds} = \frac{-(\boldsymbol{L}\times\boldsymbol{M})_1}{\|\boldsymbol{L}\|}.$ ■

(VIII.8.) Proof. Because of (VII.4.), we can express $\vec{q}$ and $\vec{p}$ as functions of $\boldsymbol{M}$, $\boldsymbol{L}$, and θ. Because of (VIII.6.), $\boldsymbol{M}$ and $\boldsymbol{L}$ can be expressed as functions of s. Then the differential equation for $\theta(s)$ follows from (VIII.1.). The proof is analogous to (VII.5.). Here, we use the equation for $p_{\|}$, because it is the simplest. We sketch the proof for $H \neq 0$:

$\frac{dp_{\|}}{ds} = \frac{d}{ds}\left(\frac{-\mu}{\|\boldsymbol{L}\|}sin\theta\right) = \frac{+\mu}{\|\boldsymbol{L}\|^2}\frac{d\|\boldsymbol{L}\|}{ds}sin\theta - \frac{\mu}{\|\boldsymbol{L}\|}cos\theta\frac{d\theta}{ds}.$

With (VIII.7.), we then have:

$(*)\,\frac{dp_{\|}}{ds} = \left(-\frac{\mu}{L^3}\right)(\boldsymbol{L}\times\boldsymbol{M})_1 sin\theta - \frac{\mu}{\|\boldsymbol{L}\|}cos\theta\frac{d\theta}{ds}.$

On the other hand, we have

$\frac{dp_{\|}}{ds} = \frac{d}{ds}\left(\frac{\boldsymbol{p}\cdot\boldsymbol{M}}{\|\boldsymbol{M}\|}\right) = \frac{d\boldsymbol{p}}{ds}\cdot\frac{\boldsymbol{M}}{\|\boldsymbol{M}\|} + \frac{\boldsymbol{p}}{\|\boldsymbol{M}\|}\cdot\frac{d\boldsymbol{M}}{ds} - \frac{\boldsymbol{p}\cdot\boldsymbol{M}}{\|\boldsymbol{M}\|^2}\frac{d\boldsymbol{M}}{ds}$, and

$\frac{\boldsymbol{p}}{\|\boldsymbol{M}\|}\cdot\frac{d\boldsymbol{M}}{ds} = \frac{\boldsymbol{p}}{\|\boldsymbol{M}\|}\cdot\{\boldsymbol{M}, M_1\} = \left(\frac{1}{\|\boldsymbol{M}\|}\right)(\mp(\boldsymbol{p}\times\boldsymbol{L})_1) = \pm\left(A_1 - \frac{q_1}{\|\boldsymbol{q}\|}\right)\left(\frac{\mu}{\|\boldsymbol{M}\|}\right)$, with the upper sign for $H < 0$ and the lower sign for $H > 0$. This implies

$\frac{dp_{\|}}{ds} = \frac{d\vec{p}}{ds}\cdot\frac{\vec{M}}{\|\boldsymbol{M}\|} \pm \left(A_1 - \frac{q_1}{\|\boldsymbol{q}\|}\right)\left(\frac{\mu}{\|\boldsymbol{M}\|}\right) \mp \frac{p_{\|}}{\|\boldsymbol{M}\|^2}(\boldsymbol{L}\times\boldsymbol{M})_1$. From (VIII.1.) we get:

$\frac{d\vec{p}}{ds}\cdot\frac{\vec{M}}{\|\boldsymbol{M}\|} = \frac{q_{\|}M_1}{2H\|\boldsymbol{q}\|^3} + \sqrt{\frac{\mu}{2|H|}}\left(\frac{-M_1}{\|\boldsymbol{M}\|\|\boldsymbol{q}\|} + \frac{q_1 q_{\|}}{\|\boldsymbol{q}\|^3} + \frac{\|\boldsymbol{p}\|^2 M_1}{\mu\|\boldsymbol{M}\|} - \frac{p_1 p_{\|}}{\mu}\right) \Rightarrow$

$\frac{dp_{\|}}{ds} = \mp\frac{p_{\|}}{\|\boldsymbol{M}\|^2}(\boldsymbol{L}\times\boldsymbol{M})_1 \mp \frac{\mu q_1}{\|\boldsymbol{M}\|\|\boldsymbol{q}\|} + \frac{q_{\|}M_1}{2H\|\boldsymbol{q}\|^3} + \sqrt{\frac{\mu}{2|H|}}\left(\frac{+M_1}{\|\boldsymbol{M}\|\|\boldsymbol{q}\|} + \frac{q_1 q_{\|}}{\|\boldsymbol{q}\|} - \frac{p_1 p_{\|}}{\mu}\right).$

Now, we use (VII.4.) and (VII.2.) and the expression

$q_1 = q_{\|}\left(\frac{1}{\|\boldsymbol{M}\|}\right)M_1 + q_{\perp}\left(\frac{1}{\|\boldsymbol{L}\|\|\boldsymbol{M}\|}\right)(\boldsymbol{L}\times\boldsymbol{M})_1$, and arrive at

$(**)\,\frac{dp_{\|}}{ds} = cos\theta\left[\mp\frac{\mu^3 M_1}{4H^2\|\boldsymbol{M}\|^2\|\boldsymbol{L}\|^4}\right] + cos^2\theta\left[-\sqrt{\frac{\mu}{2|H|}}\frac{\mu^2 M_1}{H\|\boldsymbol{M}\|\|\boldsymbol{L}\|^4}\right] + cos^3\theta\left[\frac{\mu^2 M_1}{2H\|\boldsymbol{L}\|^4}\right]$

$+sin\theta cos\theta\left[\sqrt{\frac{\mu}{2|H|}}\frac{2\mu(\boldsymbol{L}\times\boldsymbol{M})_1}{\|\boldsymbol{L}\|^3\|\boldsymbol{M}\|}\right] + sin\theta cos^2\theta\left[\frac{-\mu(\boldsymbol{L}\times\boldsymbol{M})_1}{\|\boldsymbol{L}\|^3}\right] + sin\theta\left[\frac{-\mu(\boldsymbol{L}\times\boldsymbol{M})_1}{\|\boldsymbol{L}\|^3}\right].$

A comparison of (*) and (**) yields the result. ■

(VIII.9.) Proof. (i) With the help of (VIII.6.), we can see that the functions in (VIII.8.) are continuous and bounded everywhere. Therefore, the existence theorem of Peano (cf. Kamke [8], page 126) holds.

(ii) In (VII.4.), $\boldsymbol{q}$ and $\boldsymbol{p}$ were expressed as functions of $\boldsymbol{M}$, $\boldsymbol{L}$, and θ. In (VIII.6.), $\boldsymbol{M}$ and $\boldsymbol{L}$ were expressed as functions of s. Because of (VIII.8.) and (VIII.9.) (i), the desired function $\theta(s)$ exists. With that, we have $\boldsymbol{q}$ and $\boldsymbol{p}$ as functions of the integral curve parameter s. ■

IX. Stereographic Projection and global SO(4) (resp. SO(3,1)) Symmetry

(IX.2.) Proof.

(i) For $H = H_0 < 0$:

$\xi_0 = \frac{\|\boldsymbol{p}\|^2+2H_0\mu}{\|\boldsymbol{p}\|^2-2H_0\mu}$,

$1-\xi_0 = \frac{\|\boldsymbol{p}\|^2-2H_0\mu-\|\boldsymbol{p}\|^2-2H_0\mu}{\|\boldsymbol{p}\|^2-2H_0\mu} = \frac{-4H_0\mu}{\|\boldsymbol{p}\|^2-2H_0\mu}$,

$\xi_k = \frac{-2\sqrt{-2H_0\mu}p_k}{\|\boldsymbol{p}\|^2-2H_0\mu}$,

$p_k = \frac{-\xi_k(\|\boldsymbol{p}\|^2-2H_0\mu)}{2\sqrt{-2H_0\mu}}$,

$p_k = \left(\frac{-\xi_k}{2\sqrt{-2H_0\mu}}\right)\left(\frac{-4H_0\mu}{1-\xi_0}\right)$,

$p_k = -\sqrt{-2H_0\mu}\frac{\xi_k}{1-\xi_0}$.

$\eta_k = \left(\frac{\|\boldsymbol{p}\|^2-2H_0\mu}{2\mu}\right)q_k - \frac{(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}p_k$,

$q_k = \left[\eta_k + \frac{(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}p_k\right]\left[\frac{2\mu}{\|\boldsymbol{p}\|^2-2H_0\mu}\right]$,

$q_k = \left[\eta_k + \left(\frac{\eta_0}{-\sqrt{-2H_0}}\right)\left(\frac{-\xi_k}{2\sqrt{-2H_0\mu}}\right)\left(\frac{-4H_0\mu}{1-\xi_0}\right)\right]\left(\frac{1-\xi_0}{-2H_0}\right)$,

$q_k = \left[\eta_k + \left(\frac{\eta_0\xi_k}{1-\xi_0}\right)\right]\left(\frac{1-\xi_0}{-2H_0}\right)$,

$q_k = \left[\left(\frac{\eta_k(1-\xi_0)+\eta_0\xi_k}{1-\xi_0}\right)\right]\left(\frac{1-\xi_0}{-2H_0}\right)$,

$q_k = \left(\frac{1}{-2H_0}\right)(\eta_k(1-\xi_0)+\eta_0\xi_k)$.

(ii) For $H = H_0 > 0$:

$\xi_k = \frac{-2\sqrt{2H_0\mu}p_k}{\|\boldsymbol{p}\|^2-2H_0\mu}$,

$p_k = \frac{-\xi_k(\|\boldsymbol{p}\|^2-2H_0\mu)}{4H_0\mu}\sqrt{2H_0\mu}$,

$p_k = \sqrt{2H_0\mu}\frac{\xi_k}{1-\xi_0}$.

$\eta_k = -\left(\frac{\|\boldsymbol{p}\|^2-2H_0\mu}{2\mu}\right)q_k + \frac{(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}p_k$,

$q_k = \left[\eta_k - \frac{(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}p_k\right]\left[\frac{-2\mu}{\|\boldsymbol{p}\|^2-2H_0\mu}\right]$,

$q_k = \left[\eta_k - \left(\frac{\eta_0}{-\sqrt{2H_0\mu}}\right)\left(\sqrt{2H_0\mu}\right)\left(\frac{\xi_k}{1-\xi_0}\right)\right]\left(\frac{1-\xi_0}{2H_0}\right)$,

$q_k = \left[\eta_k + \left(\frac{\eta_0\xi_k}{1-\xi_0}\right)\right]\left(\frac{1-\xi_0}{2H_0}\right)$,

$q_k = \left[\left(\frac{\eta_k(1-\xi_0)+\eta_0\xi_k}{1-\xi_0}\right)\right]\left(\frac{1-\xi_0}{2H_0}\right)$,

$q_k = \left(\frac{1}{2H_0}\right)(\eta_k(1-\xi_0)+\eta_0\xi_k)$. ■

(IX.3.) Proof.

First, we'll get some intermediate results:

Preliminary results for $H = H_0 \neq 0$:

$\frac{1}{1-\xi_0} = \frac{\|\boldsymbol{p}\|^2-2H_0\mu}{-4H_0\mu}$,

$\|\boldsymbol{p}\|^2 = 2H_0\mu - \frac{4H_0\mu}{1-\xi_0}$,

$\|\boldsymbol{p}\|^2 = 2H_0\mu\left(\frac{1-\xi_0-2}{1-\xi_0}\right)$,

$\|\boldsymbol{p}\|^2 = -2H_0\mu\left(\frac{1+\xi_0}{1-\xi_0}\right)$.

$H_0 = \frac{\|\boldsymbol{p}\|^2}{2\mu} - \frac{1}{\|\boldsymbol{q}\|}$,

$\frac{1}{\|\boldsymbol{q}\|} = \frac{\|\boldsymbol{p}\|^2}{2\mu} - H_0$,

$\frac{1}{\|\boldsymbol{q}\|} = -H_0\left(\frac{1+\xi_0}{1-\xi_0}\right) - H_0\left(\frac{1-\xi_0}{1-\xi_0}\right)$,

$\frac{1}{\|\boldsymbol{q}\|} = \frac{-2H_0}{1-\xi_0}$.

$\|\boldsymbol{q}\|^2 = \left(\frac{1-\xi_0}{-2H_0}\right)^2$.

$\|\boldsymbol{p}\|^2 - 2H_0\mu = \frac{-2H_0\mu(1+\xi_0)-2H_0\mu(1-\xi_0)}{1-\xi_0}$,

$\|\boldsymbol{p}\|^2 - 2H_0\mu = -4H_0\mu\frac{1}{1-\xi_0}$.

$\|\boldsymbol{p}\|^2 + 2H_0\mu = \frac{-2H_0\mu(1+\xi_0)+2H_0\mu(1-\xi_0)}{1-\xi_0}$,

$\|\boldsymbol{p}\|^2 + 2H_0\mu = -4H_0\mu\frac{\xi_0}{1-\xi_0}$.

$\frac{\|\boldsymbol{p}\|^2+2H_0\mu}{\|\boldsymbol{p}\|^2-2H_0\mu} = (-4H_0\mu)\left(\frac{\xi_0}{1-\xi_0}\right)\left(\frac{1}{-4H_0\mu}\right)(1-\xi_0)$,

$\frac{\|\boldsymbol{p}\|^2+2H_0\mu}{\|\boldsymbol{p}\|^2-2H_0\mu} = \xi_0$.

(i) For $H = H_0 < 0$:

$\eta_0 = \frac{-\sqrt{-2H_0\mu}}{\mu}(\boldsymbol{q}\cdot\boldsymbol{p})$,

$(\boldsymbol{q}\cdot\boldsymbol{p}) = \left(\frac{\mu}{-\sqrt{-2H_0\mu}}\right)\eta_0$.

$A_1 = \frac{q_1}{\|\boldsymbol{q}\|} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})-q_1\|\boldsymbol{p}\|^2}{\mu}$, (VIII.1.),

$A_1 = q_1\left(\frac{1}{\|\boldsymbol{q}\|} - \frac{\|\boldsymbol{p}\|^2}{\mu}\right) + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}$,

$A_1 = \left(\frac{1}{-2H_0}\right)[\eta_1(1-\xi_0)+\eta_0\xi_1]\left[\frac{-2H_0}{1-\xi_0} + 2H_0\left(\frac{1+\xi_0}{1-\xi_0}\right)\right] - \left(\frac{\sqrt{-2H_0\mu}}{\mu}\right)\left(\frac{\xi_1}{1-\xi_0}\right)\left(\frac{\mu}{-\sqrt{-2H_0\mu}}\right)\eta_0$,

$$A_1 = [\eta_1(1-\xi_0) + \eta_0\xi_1]\frac{+1-(1+\xi_0)}{1-\xi_0} + \frac{\xi_1\eta_0}{1-\xi_0}$$

$$A_1 = [\eta_1(1-\xi_0) + \eta_0\xi_1]\left(\frac{-\xi_0}{1-\xi_0}\right) + \frac{\xi_1\eta_0}{1-\xi_0}$$

$$A_1 = [-\eta_1\xi_0(1-\xi_0) - \eta_0\xi_0\xi_1 + \xi_1\eta_0]\left(\frac{1}{1-\xi_0}\right)$$

$$A_1 = [-\eta_1\xi_0(1-\xi_0) + \xi_1\eta_0(1-\xi_0)]\left(\frac{1}{1-\xi_0}\right)$$

$$A_1 = \xi_1\eta_0 - \xi_0\eta_1.$$

$$\xi_0{}^2 + \sum_{k=1}^{3}\xi_k{}^2 = \left(\frac{1}{\|\boldsymbol{p}\|^2 - 2H_0\mu}\right)^2\left[(\|\boldsymbol{p}\|^2 + 2H_0\mu)^2 + \left(-2\sqrt{-2H_0\mu}\right)^2\|\boldsymbol{p}\|^2\right],$$

$$\xi_0{}^2 + \sum_{k=1}^{3}\xi_k{}^2 = \left(\frac{1-\xi_0}{-4H_0\mu}\right)^2\left[(-4H_0\mu)^2\left(\frac{\xi_0}{1-\xi_0}\right)^2 + (-8H_0\mu)(-2H_0\mu)\left(\frac{1+\xi_0}{1-\xi_0}\right)\right],$$

$$\xi_0{}^2 + \sum_{k=1}^{3}\xi_k{}^2 = \left[\xi_0{}^2 + (1+\xi_0)(1-\xi_0)\right],$$

$$\xi_0{}^2 + \sum_{k=1}^{3}\xi_k{}^2 = 1.$$

$$\begin{aligned}\xi_0\eta_0 + \sum_{k=1}^{3}\xi_k\eta_k &= \left(\frac{\|\boldsymbol{p}\|^2 + 2H_0\mu}{\|\boldsymbol{p}\|^2 - 2H_0\mu}\right)\left(\frac{-\sqrt{-2H_0\mu}}{\mu}(\boldsymbol{q}\cdot\boldsymbol{p})\right)\\ &+ \left(\frac{-2\sqrt{-2H_0\mu}}{\|\boldsymbol{p}\|^2 - 2H_0\mu}\right)\left(\frac{\|\boldsymbol{p}\|^2 - 2H_0\mu}{2\mu}\right)(\boldsymbol{q}\cdot\boldsymbol{p})\\ &+ \left(\frac{-2\sqrt{-2H_0\mu}}{\|\boldsymbol{p}\|^2 - 2H_0\mu}\right)\left(-\frac{(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right)\|\boldsymbol{p}\|^2,\end{aligned}$$

$$\begin{aligned}\xi_0\eta_0 + \sum_{k=1}^{3}\xi_k\eta_k &= \xi_0\left(\frac{-\sqrt{-2H_0\mu}}{\mu}\right)\left(\frac{\mu}{-\sqrt{-2H_0\mu}}\right)\eta_0\\ &+ \left(\frac{-\sqrt{-2H_0\mu}}{\mu}\right)\left(\frac{\mu}{-\sqrt{-2H_0\mu}}\right)\eta_0\\ &+ \left(\frac{-2\sqrt{-2H_0\mu}}{\mu}\right)\left(\frac{1}{-4H_0\mu}\right)(1-\xi_0)\left(\frac{\mu}{-\sqrt{-2H_0\mu}}\right)\eta_0(-2H_0\mu)\left(\frac{1+\xi_0}{1-\xi_0}\right),\end{aligned}$$

$$\xi_0\eta_0 + \sum_{k=1}^{3}\xi_k\eta_k = \eta_0(1+\xi_0) - \eta_0(1+\xi_0),$$

$$\xi_0\eta_0 + \sum_{k=1}^{3}\xi_k\eta_k = 0.$$

$$\begin{aligned}\eta_0{}^2 + \sum_{k=1}^{3}\eta_k{}^2 &= \left(\frac{-\sqrt{-2H_0\mu}}{\mu}\right)^2(\boldsymbol{q}\cdot\boldsymbol{p})^2 + \left(\frac{\|\boldsymbol{p}\|^2 - 2H_0\mu}{2\mu}\right)^2\|\boldsymbol{q}\|^2\\ &-2\left(\frac{\|\boldsymbol{p}\|^2 - 2H_0\mu}{2\mu}\right)\left(\frac{1}{\mu}\right)(\boldsymbol{q}\cdot\boldsymbol{p})^2 + \left(\frac{1}{\mu^2}\right)(\boldsymbol{q}\cdot\boldsymbol{p})^2\|\boldsymbol{p}\|^2,\end{aligned}$$

$$\begin{aligned}\eta_0{}^2 + \sum_{k=1}^{3}\eta_k{}^2 &= \left(\frac{-2H_0\mu}{\mu^2}\right)\left(\frac{\mu^2}{-2H_0\mu}\right)\eta_0{}^2 + (-2H_0)^2\left(\frac{1}{1-\xi_0}\right)^2\left(\frac{1-\xi_0}{-2H_0}\right)^2\\ &-2(-2H_0)\left(\frac{1}{1-\xi_0}\right)\left(\frac{1}{\mu}\right)\left(\frac{\mu^2}{-2H_0\mu}\right)\eta_0{}^2 + \left(\frac{1}{\mu^2}\right)\left(\frac{\mu^2}{-2H_0\mu}\right)\eta_0{}^2(-2H_0\mu)\left(\frac{1+\xi_0}{1-\xi_0}\right),\end{aligned}$$

$$\eta_0{}^2 + \sum_{k=1}^{3}\eta_k{}^2 = \eta_0{}^2 + 1 - 2\left(\frac{\eta_0{}^2}{1-\xi_0}\right) + \frac{\eta_0{}^2(1+\xi_0)}{1-\xi_0},$$

$$\eta_0{}^2 + \sum_{k=1}^{3}\eta_k{}^2 = \frac{1}{1-\xi_0}\left(\eta_0{}^2 - \xi_0\eta_0{}^2 + 1 - \xi_0 - 2\eta_0{}^2 + \eta_0{}^2 + \xi_0\eta_0{}^2\right)$$

$$\eta_0{}^2 + \sum_{k=1}^{3}\eta_k{}^2 = 1.$$

(ii) For $H = H_0 > 0$:

$\eta_0 = \frac{-\sqrt{2H_0\mu}}{\mu}(\boldsymbol{q}\cdot\boldsymbol{p})$,

$(\boldsymbol{q}\cdot\boldsymbol{p}) = \left(\frac{\mu}{-\sqrt{2H_0\mu}}\right)\eta_0$.

$A_1 = \frac{q_1}{\|\boldsymbol{q}\|} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p}) - q_1\|\boldsymbol{p}\|^2}{\mu}$,

from (VIII.1.),

$A_1 = q_1\left(\frac{1}{\|\boldsymbol{q}\|} - \frac{\|\boldsymbol{p}\|^2}{\mu}\right) + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}$,

$A_1 = \left(\frac{1}{2H_0}\right)[\eta_1(1-\xi_0) + \eta_0\xi_1]\left[\frac{-2H_0}{1-\xi_0} + 2H_0\left(\frac{1+\xi_0}{1-\xi_0}\right)\right] + \left(\frac{\sqrt{2H_0\mu}}{\mu}\right)\left(\frac{\xi_1}{1-\xi_0}\right)\left(\frac{\mu}{-\sqrt{2H_0\mu}}\right)\eta_0$,

$$A_1 = [\eta_1(1-\xi_0) + \eta_0\xi_1]\frac{-1 + (1+\xi_0)}{1-\xi_0} - \frac{\xi_1\eta_0}{1-\xi_0}$$

$$A_1 = [\eta_1(1-\xi_0) + \eta_0\xi_1]\left(\frac{\xi_0}{1-\xi_0}\right) - \frac{\xi_1\eta_0}{1-\xi_0}$$

$$A_1 = [\eta_1\xi_0(1-\xi_0) + \eta_0\xi_0\xi_1 - \xi_1\eta_0]\left(\frac{1}{1-\xi_0}\right)$$

$$A_1 = [\eta_1\xi_0(1-\xi_0) - \xi_1\eta_0(1-\xi_0)]\left(\frac{1}{1-\xi_0}\right)$$

$A_1 = \xi_0\eta_1 - \xi_1\eta_0$.

${\xi_0}^2 - \sum_{k=1}^{3}{\xi_k}^2 = \left(\frac{1}{\|\boldsymbol{p}\|^2 - 2H_0\mu}\right)^2\left[(\|\boldsymbol{p}\|^2 + 2H_0\mu)^2 + \left(-2\sqrt{2H_0\mu}\right)^2\|\boldsymbol{p}\|^2\right]$,

${\xi_0}^2 - \sum_{k=1}^{3}{\xi_k}^2 = \left(\frac{1-\xi_0}{-4H_0\mu}\right)^2\left[(-4H_0\mu)^2\left(\frac{\xi_0}{1-\xi_0}\right)^2 + (-8\mathrm{H}_0\mu)(-2H_0\mu)\left(\frac{1+\xi_0}{1-\xi_0}\right)\right]$,

${\xi_0}^2 - \sum_{k=1}^{3}{\xi_k}^2 = \left[{\xi_0}^2 + (1+\xi_0)(1-\xi_0)\right]$,

${\xi_0}^2 - \sum_{k=1}^{3}{\xi_k}^2 = 1$.

$$\xi_0\eta_0 - \sum_{k=1}^{3}\xi_k\eta_k = \left(\frac{\|\boldsymbol{p}\|^2 + 2H_0\mu}{\|\boldsymbol{p}\|^2 - 2H_0\mu}\right)\left(\frac{-\sqrt{2H_0\mu}}{\mu}(\boldsymbol{q}\cdot\boldsymbol{p})\right)$$
$$+ \left(\frac{-2\sqrt{2H_0\mu}}{\|\boldsymbol{p}\|^2 - 2H_0\mu}\right)\left(\frac{\|\boldsymbol{p}\|^2 - 2H_0\mu}{2\mu}\right)(\boldsymbol{q}\cdot\boldsymbol{p})$$
$$+ \left(\frac{-2\sqrt{2H_0\mu}}{\|\boldsymbol{p}\|^2 - 2H_0\mu}\right)\left(\frac{(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right)\|\boldsymbol{p}\|^2,$$

$$\xi_0\eta_0 - \sum_{k=1}^{3}\xi_k\eta_k = \xi_0\left(\frac{-\sqrt{2H_0\mu}}{\mu}\right)\left(\frac{\mu}{-\sqrt{2H_0\mu}}\right)\eta_0 + \left(\frac{-\sqrt{2H_0\mu}}{\mu}\right)\left(\frac{\mu}{-\sqrt{2H_0\mu}}\right)\eta_0$$
$$+ \left(\frac{-2\sqrt{2H_0\mu}}{\mu}\right)\left(\frac{1}{-4H_0\mu}\right)(1-\xi_0)\left(\frac{-\mu}{-\sqrt{2H_0\mu}}\right)\eta_0(-2H_0\mu)\left(\frac{1+\xi_0}{1-\xi_0}\right),$$

$\xi_0\eta_0 - \sum_{k=1}^{3}\xi_k\eta_k = \eta_0(1+\xi_0) - \eta_0(1+\xi_0)$,

$\xi_0\eta_0 - \sum_{k=1}^{3}\xi_k\eta_k = 0$.

$$\eta_0^{\ 2} - \sum_{k=1}^{3}{\eta_k}^2 = \left(\frac{-\sqrt{2H_0\mu}}{\mu}\right)^2(\boldsymbol{q}\cdot\boldsymbol{p})^2$$
$$- \left(\frac{\|\boldsymbol{p}\|^2 - 2H_0\mu}{2\mu}\right)^2\|\boldsymbol{q}\|^2$$

$$+2\left(\frac{\|\boldsymbol{p}\|^2-2H_0\mu}{2\mu}\right)\left(\frac{1}{\mu}\right)(\boldsymbol{q}\cdot\boldsymbol{p})^2$$
$$-\left(\frac{1}{\mu^2}\right)(\boldsymbol{q}\cdot\boldsymbol{p})^2\|\boldsymbol{p}\|^2,$$
$$\eta_0{}^2-\textstyle\sum_{k=1}^3\eta_k{}^2=\left(\frac{2H_0\mu}{\mu^2}\right)\left(\frac{\mu^2}{2H_0\mu}\right)\eta_0{}^2-(-2H_0)^2\left(\frac{1}{1-\xi_0}\right)^2\left(\frac{1-\xi_0}{-2H_0}\right)^2$$
$$+2(-2H_0)\left(\frac{1}{1-\xi_0}\right)\left(\frac{1}{\mu}\right)\left(\frac{\mu^2}{2H_0\mu}\right)\eta_0{}^2-\left(\frac{1}{\mu^2}\right)\left(\frac{\mu^2}{2H_0\mu}\right)\eta_0{}^2(-2H_0\mu)\left(\frac{1+\xi_0}{1-\xi_0}\right),$$
$$\eta_0{}^2-\textstyle\sum_{k=1}^3\eta_k{}^2=\eta_0{}^2-1-2\left(\frac{\eta_0{}^2}{1-\xi_0}\right)+\frac{\eta_0{}^2(1+\xi_0)}{1-\xi_0},$$
$$\eta_0{}^2-\textstyle\sum_{k=1}^3\eta_k{}^2=\frac{1}{1-\xi_0}\left(\eta_0{}^2-\xi_0\eta_0{}^2-1+\xi_0-2\eta_0{}^2+\eta_0{}^2+\xi_0\eta_0{}^2\right),$$
$$\eta_0{}^2-\textstyle\sum_{k=1}^3\eta_k{}^2=-1.\ \blacksquare$$

(IX.4.) Proof. In each case, the proof consists of a direct application of the definitions and a lengthy calculation. To reduce the volume, we introduce a parameter σ=sign(H), i.e. σ=+1 if $H>0$ and σ=+1 if $H<0$.

Intermediate results for $H\neq 0$:

$$q_1'=-\sigma\sqrt{\frac{\mu}{2}}(\sigma H_0)^{-3/2}\left[\left(\frac{1}{2\mu}\right)(A_1)(p_1)+(-H_0)\left(\frac{1}{\mu}\right)(q_2p_2+q_3p_3)\right],$$

from (VIII.1.),

$$q_1'=-\sigma\sqrt{\frac{\mu}{2}}(\sigma H_0)^{-3/2}\begin{bmatrix}\left(\frac{1}{2\mu}\right)\sigma(\xi_0\eta_1-\xi_1\eta_0)\left(\sigma\sqrt{\sigma 2H_0\mu}\frac{\xi_1}{1-\xi_0}\right)\\+(-H_0)\left(\frac{1}{\mu}\right)\left(\frac{1}{\sigma 2H_0}\right)(\eta_2-\xi_0\eta_2+\xi_2\eta_0)\left(\sigma\sqrt{\sigma 2H_0\mu}\frac{\xi_2}{1-\xi_0}\right)\\+(-H_0)\left(\frac{1}{\mu}\right)\left(\frac{1}{\sigma 2H_0}\right)(\eta_3-\xi_0\eta_3+\xi_3\eta_0)\left(\sigma\sqrt{\sigma 2H_0\mu}\frac{\xi_3}{1-\xi_0}\right)\end{bmatrix},$$

because of (IX.2.) and (IX.3.),

$$q_1'=\left(\frac{1}{-2H_0}\right)\left(\frac{[(\xi_0\eta_1-\xi_1\eta_0)(-\xi_1)+(\eta_2-\xi_0\eta_2+\xi_2\eta_0)(-\xi_2)+(\eta_3-\xi_0\eta_3+\xi_3\eta_0)(-\xi_3)]}{1-\xi_0}\right),$$
$$q_1'=\left(\frac{1}{-2H_0}\right)\left(\frac{-\xi_1{}^2\eta_0+\xi_0\xi_1\eta_1-\xi_2\eta_2+\xi_0\xi_2\eta_2-\xi_2{}^2\eta_0-\xi_3\eta_3+\xi_0\xi_3\eta_3-\xi_3{}^2\eta_0}{1-\xi_0}\right),$$
$$q_1'=\left(\frac{1}{-2H_0}\right)\left(\frac{-(\xi_1{}^2+\xi_2{}^2+\xi_3{}^2)\eta_0+\xi_0(\xi_1\eta_1+\xi_2\eta_2+\xi_3\eta_3)-(\xi_2\eta_2+\xi_3\eta_3)}{1-\xi_0}\right),$$
$$q_1'=\left(\frac{1}{-2H_0}\right)\left(\frac{-(-\sigma(1-\xi_0{}^2))\eta_0+\xi_0(\sigma\xi_0\eta_0)-(\sigma\xi_0\eta_0-\xi_1\eta_1)}{1-\xi_0}\right),$$

because of (IX.3.),

$$q_1'=\left(\frac{1}{-2H_0}\right)\left(\frac{\sigma\eta_0-\sigma\xi_0{}^2\eta_0+\sigma\xi_0{}^2\eta_0-\sigma\xi_0\eta_0+\xi_1\eta_1}{1-\xi_0}\right),$$
$$q_1'=\left(\frac{1}{-2H_0}\right)\left(\frac{\sigma\eta_0(1-\xi_0)+\xi_1\eta_1}{1-\xi_0}\right),$$
$$q_1'=\left(\frac{1}{-2H_0}\right)\left(\sigma\eta_0+\frac{\xi_1\eta_1}{1-\xi_0}\right),$$

$$q_2'=-\sigma\sqrt{\frac{\mu}{2}}(\sigma H_0)^{-3/2}\left[\left(\frac{1}{2\mu}\right)(A_1)(p_2)+(-H_0)\left(\frac{1}{\mu}\right)(q_2p_1-2q_1p_2)\right],$$

from (VIII.1.),

$$q_2' = -\sigma\sqrt{\frac{\mu}{2}}(\sigma H_0)^{-3/2}\begin{bmatrix} \left(\frac{1}{2\mu}\right)\sigma(\xi_0\eta_1-\xi_1\eta_0)\left(\sigma\sqrt{2\sigma H_0\mu}\frac{\xi_2}{1-\xi_0}\right) \\ +(-H_0)\left(\frac{1}{\mu}\right)\left(\frac{1}{2\sigma H_0}\right)(\eta_2-\xi_0\eta_2+\xi_2\eta_0)\left(\sigma\sqrt{2\sigma H_0\mu}\frac{\xi_1}{1-\xi_0}\right) \\ +(-H_0)\left(\frac{1}{\mu}\right)(-2)\left(\frac{1}{2\sigma H_0}\right)(\eta_1-\xi_0\eta_1+\xi_1\eta_0)\left(\sigma\sqrt{2\sigma H_0\mu}\frac{\xi_2}{1-\xi_0}\right)\end{bmatrix},$$

because of (IX.2.) and (IX.3.),

$$q_2' = \left(\frac{1}{-2H_0}\right)\left(\frac{[(\xi_0\eta_1-\xi_1\eta_0)(\xi_2)+(\eta_2-\xi_0\eta_2+\xi_2\eta_0)(-\xi_1)+(-2)(\eta_1-\xi_0\eta_1+\xi_1\eta_0)(-\xi_2)]}{1-\xi_0}\right),$$

$$q_2' = \left(\frac{1}{-2H_0}\right)\left(\frac{+\xi_0\xi_2\eta_1-\xi_1\xi_2\eta_0-\xi_1\eta_2+\xi_0\xi_1\eta_2-\xi_1\xi_2\eta_0+2\xi_2\eta_1-2\xi_0\xi_2\eta_1+2\xi_1\xi_2\eta_0}{1-\xi_0}\right),$$

$$q_2' = \left(\frac{1}{-2H_0}\right)\left(\frac{-\xi_0\xi_2\eta_1-\xi_1\eta_2+\xi_0\xi_1\eta_2+2\xi_2\eta_1}{1-\xi_0}\right),$$

$$q_3' = -\sigma\sqrt{\frac{\mu}{2}}(\sigma H_0)^{-3/2}\left[\left(\frac{1}{2\mu}\right)(A_1)(p_3)+(-H_0)\left(\frac{1}{\mu}\right)(q_3p_1-2q_1p_3)\right],$$

from (VIII.1.),

$$q_3' = -\sigma\sqrt{\frac{\mu}{2}}(\sigma H_0)^{-3/2}\begin{bmatrix} \left(\frac{1}{2\mu}\right)\sigma(\xi_0\eta_1-\xi_1\eta_0)\left(\sigma\sqrt{2\sigma H_0\mu}\frac{\xi_3}{1-\xi_0}\right) \\ +(-H_0)\left(\frac{1}{\mu}\right)\left(\frac{1}{2\sigma H_0}\right)(\eta_3-\xi_0\eta_3+\xi_3\eta_0)\left(\sigma\sqrt{2\sigma H_0\mu}\frac{\xi_1}{1-\xi_0}\right) \\ +(-H_0)\left(\frac{1}{\mu}\right)(-2)\left(\frac{1}{2\sigma H_0}\right)(\eta_1-\xi_0\eta_1+\xi_1\eta_0)\left(\sigma\sqrt{2\sigma H_0\mu}\frac{\xi_3}{1-\xi_0}\right)\end{bmatrix},$$

because of (IX.2.) and (IX.3.),

$$q_3' = \left(\frac{1}{-2H_0}\right)\left(\frac{[(\xi_0\eta_1-\xi_1\eta_0)(\xi_3)+(\eta_3-\xi_0\eta_3+\xi_3\eta_0)(-\xi_1)+(-2)(\eta_1-\xi_0\eta_1+\xi_1\eta_0)(-\xi_3)]}{1-\xi_0}\right),$$

$$q_3' = \left(\frac{1}{-2H_0}\right)\left(\frac{+\xi_0\xi_3\eta_1-\xi_1\xi_3\eta_0-\xi_1\eta_3+\xi_0\xi_1\eta_3-\xi_1\xi_3\eta_0+2\xi_3\eta_1-2\xi_0\xi_3\eta_1+2\xi_1\xi_3\eta_0}{1-\xi_0}\right),$$

$$q_3' = \left(\frac{1}{-2H_0}\right)\left(\frac{-\xi_0\xi_3\eta_1-\xi_1\eta_3+\xi_0\xi_1\eta_3+2\xi_3\eta_1}{1-\xi_0}\right),$$

$$p_1' = -\sigma\sqrt{\frac{\mu}{2}}(\sigma H_0)^{-3/2}\begin{bmatrix} \left(\frac{1}{2}\right)(A_1)(-q_1)\left(\frac{1}{\|\boldsymbol{q}\|^3}\right) \\ +(-H_0)(-\|\boldsymbol{q}\|^2+{q_1}^2)\left(\frac{1}{\|\boldsymbol{q}\|^3}\right) \\ +(-H_0)\left(\frac{1}{\mu}\right)(\|\boldsymbol{p}\|^2-{p_1}^2)\end{bmatrix},$$

from (VIII.1.),

$$p_1' = -\sigma\sqrt{\frac{\mu}{2}}(\sigma H_0)^{-3/2}\begin{bmatrix} \left(\frac{1}{2}\right)\sigma(\xi_0\eta_1-\xi_1\eta_0)(-1)\left(\frac{1}{\sigma 2H_0}\right)(\eta_1-\xi_0\eta_1+\xi_1\eta_0)\left(\frac{-2H_0}{1-\xi_0}\right)^3 \\ +(-H_0)(-1)\left(\frac{-2H_0}{1-\xi_0}\right)+(-H_0)\left(\frac{1}{\sigma 2H_0}\right)^2(\eta_1-\xi_0\eta_1+\xi_1\eta_0)^2\left(\frac{-2H_0}{1-\xi_0}\right)^3 \\ +(-H_0)\left(\frac{1}{\mu}\right)(-2H_0\mu)\left(\frac{1+\xi_0}{1-\xi_0}\right)+(-H_0)\left(\frac{1}{\mu}\right)(-1)\left(\sigma\sqrt{\sigma 2H_0\mu}\frac{\xi_1}{1-\xi_0}\right)^2\end{bmatrix},$$

because of (IX.2.) and (IX.3.),

$$p_1' = -\left(\frac{\sigma\sqrt{2\sigma H_0\mu}}{(1-\xi_0)^3}\right)\begin{bmatrix}(\xi_0\eta_1-\xi_1\eta_0)(\eta_1-\xi_0\eta_1+\xi_1\eta_0) \\ +(-1)(1-\xi_0)^2+(\eta_1-\xi_0\eta_1+\xi_1\eta_0)^2 \\ +(1+\xi_0)(1-\xi_0)^2-\sigma{\xi_1}^2(1-\xi_0)\end{bmatrix},$$

$$p_1' = -\left(\frac{\sigma\sqrt{2\sigma H_0\mu}}{(1-\xi_0)^3}\right)\begin{bmatrix} -\xi_1\eta_0\eta_1 + \xi_0\xi_1\eta_0\eta_1 - \xi_1{}^2\eta_0{}^2 + \xi_0\eta_1{}^2 - \xi_0{}^2\eta_1{}^2 + \xi_0\xi_1\eta_0\eta_1 - 1 \\ +2\xi_0 - \xi_0{}^2 + \eta_1{}^2 + \xi_0{}^2\eta_1{}^2 + \xi_1{}^2\eta_0{}^2 \\ -2\xi_0\eta_1{}^2 + 2\xi_1\eta_0\eta_1 - 2\xi_0\xi_1\eta_0\eta_1 + 1 - 2\xi_0 + \xi_0{}^2 \\ +\xi_0 - 2\xi_0{}^2 + \xi_0{}^3 + \sigma(\xi_1{}^2 - \xi_0\xi_1{}^2) \end{bmatrix},$$

$$p_1' = -\left(\frac{\sigma\sqrt{2\sigma H_0\mu}}{(1-\xi_0)^3}\right)[\xi_1\eta_0\eta_1 - \xi_0\eta_1{}^2 + \xi_0 - 2\xi_0{}^2 + \eta_1{}^2 + \xi_0{}^3 + \sigma\xi_1{}^2 - \sigma\xi_0\xi_1{}^2].$$

$$p_2' = -\sigma\sqrt{\frac{\mu}{2}}(\sigma H_0)^{-3/2}\left[\left(\frac{1}{2}\right)(A_1)(-q_2)\left(\frac{1}{\|\boldsymbol{q}\|^3}\right) + (-H_0)(q_1q_2)\left(\frac{1}{\|\boldsymbol{q}\|^3}\right) + (-H_0)\left(\frac{1}{\mu}\right)(-p_1p_2)\right],$$

from (VIII.1.),

$$p_2' = -\sigma\sqrt{\frac{\mu}{2}}(\sigma H_0)^{-3/2}\begin{bmatrix} \left(\frac{1}{2}\right)\sigma(\xi_0\eta_1 - \xi_1\eta_0)(-1)\left(\frac{1}{\sigma 2H_0}\right)(\eta_2 - \xi_0\eta_2 + \xi_2\eta_0)\left(\frac{-2H_0}{1-\xi_0}\right)^3 \\ +(-H_0)\left(\frac{1}{\sigma 2H_0}\right)(\eta_1 - \xi_0\eta_1 + \xi_1\eta_0)\left(\frac{1}{\sigma 2H_0}\right)(\eta_2 - \xi_0\eta_2 + \xi_2\eta_0)\left(\frac{-2H_0}{1-\xi_0}\right)^3 \\ +(-H_0)\left(\frac{1}{\mu}\right)(-1)\left(\sigma\sqrt{\sigma 2H_0\mu}\frac{\xi_1}{1-\xi_0}\right)\left(\sigma\sqrt{\sigma 2H_0\mu}\frac{\xi_2}{1-\xi_0}\right) \end{bmatrix},$$

because of (IX.2.) and (IX.3.),

$$p_2' = -\left(\frac{\sigma\sqrt{2\sigma H_0\mu}}{(1-\xi_0)^3}\right)\begin{bmatrix} (\xi_0\eta_1 - \xi_1\eta_0)(\eta_2 - \xi_0\eta_2 + \xi_2\eta_0) + (\eta_1 - \xi_0\eta_1 + \xi_1\eta_0)(\eta_2 - \xi_0\eta_2 + \xi_2\eta_0) \\ +\sigma\xi_1\xi_2(1-\xi_0) \end{bmatrix},$$

$$p_2' = -\left(\frac{\sigma\sqrt{2\sigma H_0\mu}}{(1-\xi_0)^3}\right)\begin{bmatrix} -\xi_1\eta_0\eta_2 + \xi_0\xi_1\eta_0\eta_2 - \xi_1\xi_2\eta_0{}^2 + \xi_0\eta_1\eta_2 - \xi_0{}^2\eta_1\eta_2 + \xi_0\xi_2\eta_0\eta_1 \\ +\eta_1\eta_2 - \xi_0\eta_1\eta_2 + \xi_2\eta_0\eta_1 - \xi_0\eta_1\eta_2 + \xi_0{}^2\eta_1\eta_2 - \xi_0\xi_2\eta_0\eta_1 + \xi_1\eta_0\eta_2 \\ -\xi_0\xi_1\eta_0\eta_2 + \xi_1\xi_2\eta_0{}^2 + \sigma(\xi_1\xi_2 - \xi_0\xi_1\xi_2) \end{bmatrix},$$

$$p_2' = -\left(\frac{\sigma\sqrt{2\sigma H_0\mu}}{(1-\xi_0)^3}\right)[+\eta_1\eta_2 + \xi_2\eta_0\eta_1 - \xi_0\eta_1\eta_2 + \sigma\xi_1\xi_2 - \sigma\xi_0\xi_1\xi_2].$$

$$p_3' = -\sigma\sqrt{\frac{\mu}{2}}(\sigma H_0)^{-3/2}\left[\left(\frac{1}{2}\right)(A_1)(-q_3)\left(\frac{1}{\|\boldsymbol{q}\|^3}\right) + (-H_0)(q_1q_3)\left(\frac{1}{\|\boldsymbol{q}\|^3}\right) + (-H_0)\left(\frac{1}{\mu}\right)(-p_1p_3)\right],$$

from (VIII.1.),

$$p_3' = -\sigma\sqrt{\frac{\mu}{2}}(\sigma H_0)^{-3/2}\begin{bmatrix} \left(\frac{1}{2}\right)\sigma(\xi_0\eta_1 - \xi_1\eta_0)(-1)\left(\frac{1}{\sigma 2H_0}\right)(\eta_3 - \xi_0\eta_3 + \xi_3\eta_0)\left(\frac{-2H_0}{1-\xi_0}\right)^3 \\ +(-H_0)\left(\frac{1}{\sigma 2H_0}\right)(\eta_1 - \xi_0\eta_1 + \xi_1\eta_0)\left(\frac{1}{\sigma 2H_0}\right)(\eta_3 - \xi_0\eta_3 + \xi_3\eta_0)\left(\frac{-2H_0}{1-\xi_0}\right)^3 \\ +(-H_0)\left(\frac{1}{\mu}\right)(-1)\left(\sigma\sqrt{\sigma 2H_0\mu}\frac{\xi_1}{1-\xi_0}\right)\left(\sigma\sqrt{\sigma 2H_0\mu}\frac{\xi_3}{1-\xi_0}\right) \end{bmatrix},$$

because of (IX.2.) and (IX.3.),

$$p_3' = -\left(\frac{\sigma\sqrt{2\sigma H_0\mu}}{(1-\xi_0)^3}\right)\begin{bmatrix} (-1)\sigma(\xi_0\eta_1 - \xi_1\eta_0)(\eta_3 - \xi_0\eta_3 + \xi_3\eta_0) \\ +(\eta_1 - \xi_0\eta_1 + \xi_1\eta_0)(\eta_3 - \xi_0\eta_3 + \xi_3\eta_0) + \sigma\xi_1\xi_3(1-\xi_0) \end{bmatrix},$$

$$p_3' = -\left(\frac{\sigma\sqrt{2\sigma H_0\mu}}{(1-\xi_0)^3}\right)\begin{bmatrix} -\xi_1\eta_0\eta_3 + \xi_0\xi_1\eta_0\eta_3 - \xi_1\xi_3\eta_0{}^2 + \xi_0\eta_1\eta_3 - \xi_0{}^2\eta_1\eta_3 + \xi_0\xi_3\eta_0\eta_1 \\ +\eta_1\eta_3 - \xi_0\eta_1\eta_3 + \xi_2\eta_0\eta_1 - \xi_0\eta_1\eta_3 + \xi_0{}^2\eta_1\eta_3 - \xi_0\xi_3\eta_0\eta_1 \\ +\xi_1\eta_0\eta_3 - \xi_0\xi_1\eta_0\eta_3 + \xi_1\xi_3\eta_0{}^2 + \sigma(\xi_1\xi_3 - \xi_0\xi_1\xi_3) \end{bmatrix},$$

$$p_3' = -\left(\frac{\sigma\sqrt{2\sigma H_0\mu}}{(1-\xi_0)^3}\right)[+\eta_1\eta_3 + \xi_3\eta_0\eta_1 - \xi_0\eta_1\eta_3 + \sigma\xi_1\xi_3 - \sigma\xi_0\xi_1\xi_3].$$

$$(\boldsymbol{p}\cdot\boldsymbol{p}') = -\sigma\sqrt{\frac{\mu}{2}}(\sigma H_0)^{-3/2}\begin{bmatrix}\frac{A_1}{2\|\boldsymbol{q}\|^3}(-\boldsymbol{q}\cdot\boldsymbol{p}) + (-H_0)\left(\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|^3} - \frac{p_1}{\|\boldsymbol{q}\|}\right)\\ +(-H_0)\left(\frac{1}{\mu}\right)(p_1\|\boldsymbol{p}\|^2 - p_1\|\boldsymbol{p}\|^2)\end{bmatrix},$$

from (VIII.1.),

$$(\boldsymbol{p}\cdot\boldsymbol{p}') = -\sigma\sqrt{\frac{\mu}{2}}(\sigma H_0)^{-3/2}\begin{bmatrix}\left(\frac{1}{2}\right)\sigma(\xi_0\eta_1 - \xi_1\eta_0)\left(\frac{-2H_0}{1-\xi_0}\right)^3\left(\frac{\mu}{\sqrt{\sigma 2H_0\mu}}\right)\eta_0\\ +(-H_0)\left(\frac{1}{-2H_0}\right)(\eta_1 - \xi_0\eta_1 + \xi_1\eta_0)\left(\frac{-2H_0}{1-\xi_0}\right)^3\left(\frac{-\mu}{\sqrt{-2H_0\mu}}\right)\eta_0\\ +(-H_0)\left(\frac{-2H_0}{1-\xi_0}\right)\left(-\sigma\sqrt{\sigma 2H_0\mu}\frac{\xi_1}{1-\xi_0}\right)\end{bmatrix},$$

because of (IX.2.) and (IX.3.),

$$(\boldsymbol{p}\cdot\boldsymbol{p}') = -2H_0\mu\left(\frac{-\xi_1{\eta_0}^2 + \xi_0\eta_0\eta_1 + \eta_0\eta_1 - \xi_0\eta_0\eta_1 + \xi_1{\eta_0}^2 - \sigma(\xi_1 - \xi_0\xi_1)}{(1-\xi_0)^3}\right),$$

$$(\boldsymbol{p}\cdot\boldsymbol{p}') = 2H_0\mu\frac{\sigma(\xi_1 - \xi_0\xi_1) + \eta_0\eta_1}{(1-\xi_0)^3}.$$

$$(\boldsymbol{q}\cdot\boldsymbol{p}') = -\sigma\sqrt{\frac{\mu}{2}}(\sigma H_0)^{-3/2}\left[\left(\frac{-A_1}{2\|\boldsymbol{q}\|}\right) + (-H_0)\left(\frac{1}{\mu}\right)(q_1\|\boldsymbol{p}\|^2) + (-H_0)\left(\frac{1}{\mu}\right)(-p_1)(\boldsymbol{q}\cdot\boldsymbol{p})\right],$$

from (VIII.1.),

$$(\boldsymbol{q}\cdot\boldsymbol{p}') = -\sigma\sqrt{\frac{\mu}{2}}(\sigma H_0)^{-3/2}\begin{bmatrix}\left(\frac{-1}{2}\right)\sigma(\xi_0\eta_1 - \xi_1\eta_0)\left(\frac{-2H_0}{1-\xi_0}\right)\\ +(-H_0)\left(\frac{1}{\mu}\right)\left(\frac{1}{2\sigma H_0}\right)(\eta_1 - \xi_0\eta_1 + \xi_1\eta_0)(-2H_0\mu)\left(\frac{1+\xi_0}{1-\xi_0}\right)\\ +(-H_0)\left(\frac{1}{\mu}\right)\left(-\sigma\sqrt{\sigma 2H_0\mu}\frac{\xi_1}{1-\xi_0}\right)\left(\frac{-\mu}{\sqrt{\sigma 2H_0\mu}}\right)\eta_0\end{bmatrix},$$

because of (IX.2.) and (IX.3.),

$$(\boldsymbol{q}\cdot\boldsymbol{p}') = -\left(\frac{\sigma\mu}{\sqrt{2\sigma H_0\mu}}\right)\frac{(-\xi_1\eta_0 + \xi_0\eta_1 + \eta_1 + \xi_0\eta_1 - \xi_0\eta_1 - {\xi_0}^2\eta_1 + \xi_1\eta_0 + \xi_0\xi_1\eta_0 - \xi_1\eta_0)}{(1-\xi_0)},$$

$$(\boldsymbol{q}\cdot\boldsymbol{p}') = -\left(\frac{\sigma\mu}{\sqrt{2\sigma H_0\mu}}\right)\frac{(\xi_0\eta_1 + \eta_1 - {\xi_0}^2\eta_1 + \xi_0\xi_1\eta_0 - \xi_1\eta_0)}{(1-\xi_0)}.$$

$$(\boldsymbol{p}\cdot\boldsymbol{q}') = -\sigma\sqrt{\frac{\mu}{2}}(\sigma H_0)^{-3/2}\begin{bmatrix}\left(\frac{1}{2\mu}\right)(A_1)\|\boldsymbol{p}\|^2 + (-H_0)\left(\frac{2}{\mu}\right)(p_1)(\boldsymbol{q}\cdot\boldsymbol{p})\\ +(-H_0)\left(\frac{2}{\mu}\right)(-q_1)\|\boldsymbol{p}\|^2\end{bmatrix},$$

from (VIII.1.),

$$(\boldsymbol{p}\cdot\boldsymbol{q}') = -\sigma\sqrt{\frac{\mu}{2}}(\sigma H_0)^{-3/2}\begin{bmatrix}\left(\frac{1}{2\mu}\right)\sigma(\xi_0\eta_1 - \xi_1\eta_0)(-2H_0\mu)\left(\frac{1+\xi_0}{1-\xi_0}\right)\\ +(-H_0)\left(\frac{2}{\mu}\right)\left(\sigma\sqrt{\sigma 2H_0\mu}\frac{\xi_1}{1-\xi_0}\right)\left(\frac{-\mu}{\sqrt{\sigma 2H_0\mu}}\right)\eta_0\\ +(-H_0)\left(\frac{2}{\mu}\right)\left(\frac{1}{\sigma 2H_0}\right)(-1)(\eta_1 - \xi_0\eta_1 + \xi_1\eta_0)(-2H_0\mu)\left(\frac{1+\xi_0}{1-\xi_0}\right)\end{bmatrix},$$

because of (IX.2.) and (IX.3.),

$$(\boldsymbol{p}\cdot\boldsymbol{q}') = \left(\frac{\sigma\mu}{\sqrt{2\sigma H_0\mu}}\right)\frac{[(\xi_0\eta_1 - \xi_1\eta_0)(1+\xi_0) + 2\xi_1\eta_0 - 2(\eta_1 - \xi_0\eta_1 + \xi_1\eta_0)(1+\xi_0)]}{(1-\xi_0)},$$

$$(\boldsymbol{p}\cdot\boldsymbol{q}') = -\left(\frac{\sigma\mu}{\sqrt{2\sigma H_0\mu}}\right)\left(\frac{(-2\eta_1 + \xi_0\eta_1 - \xi_1\eta_0)(1+\xi_0) - 2\xi_1\eta_0}{(1-\xi_0)}\right),$$

$$(\boldsymbol{p}\cdot\boldsymbol{q}') = -\left(\frac{\sigma\mu}{\sqrt{2\sigma H_0\mu}}\right)\left(\frac{-2\eta_1-2\xi_0\eta_1+\xi_0\eta_1+\xi_0{}^2\eta_1-\xi_1\eta_0-\xi_0\xi_1\eta_0+2\xi_1\eta_0}{(1-\xi_0)}\right),$$

$$(\boldsymbol{p}\cdot\boldsymbol{q}') = -\left(\frac{\sigma\mu}{\sqrt{2\sigma H_0\mu}}\right)\left(\frac{-2\eta_1-\xi_0\eta_1+\xi_0{}^2\eta_1+\xi_1\eta_0-\xi_0\xi_1\eta_0}{(1-\xi_0)}\right).$$

Main results for $H<0$:

$$\eta_0' = -\left(\frac{\sqrt{2\sigma H_0\mu}}{\mu}\right)(\boldsymbol{q}\cdot\boldsymbol{p}'+\boldsymbol{p}\cdot\boldsymbol{q}'),\text{ from (IX.1.)},$$

$$\eta_0' = \frac{\sigma(\xi_0\eta_1+\eta_1-\xi_0{}^2\eta_1+\xi_0\xi_1\eta_0-\xi_1\eta_0-2\eta_1-\xi_0\eta_1+\xi_0{}^2\eta_1+\xi_1\eta_0-\xi_0\xi_1\eta_0)}{(1-\xi_0)},$$

from above,

$$\eta_0' = \sigma\frac{\eta_1}{1-\xi_0} = \sigma\frac{\eta_1-\xi_0\eta_1+\xi_0\eta_1}{1-\xi_0} = \sigma\eta_1+\sigma\frac{\xi_0\eta_1}{1-\xi_0} = \sigma\eta_1+\xi_0\eta_0'.$$

$$\eta_1' = \left(\frac{-\sigma}{2\mu}\right)(\|\boldsymbol{p}\|^2-2H_0\mu)(q_1')+\left(\frac{-\sigma}{2\mu}\right)2(\boldsymbol{p}\cdot\boldsymbol{p}')(q_1)$$
$$+\left(\frac{\sigma}{\mu}\right)[(\boldsymbol{q}\cdot\boldsymbol{p}')(p_1)+(\boldsymbol{p}\cdot\boldsymbol{q}')(p_1)+(\boldsymbol{q}\cdot\boldsymbol{p})(p_1')],$$

from (IX.1.),

$$\eta_1' = \left(\frac{-\sigma}{2\mu}\right)(-4\mathrm{H}_0\mu)\left(\frac{1}{1-\xi_0}\right)\left(\frac{1}{-2H_0}\right)\left(\sigma\eta_0+\frac{\xi_1\eta_1}{1-\xi_0}\right)$$
$$+\left(\frac{-\sigma}{\mu}\right)(2H_0\mu)\frac{\sigma(\xi_1-\xi_0\xi_1)+\eta_0\eta_1}{(1-\xi_0)^3}\left(\frac{1}{\sigma 2H_0}\right)(\eta_1-\xi_0\eta_1+\xi_1\eta_0)$$
$$+\left(\frac{\sigma}{\mu}\right)\left(\frac{-\sigma\mu}{\sqrt{2\sigma H_0\mu}}\right)\frac{(\xi_0\eta_1+\eta_1-\xi_0{}^2\eta_1+\xi_0\xi_1\eta_0-\xi_1\eta_0)}{(1-\xi_0)}\left(\sigma\sqrt{\sigma 2H_0\mu}\frac{\xi_1}{1-\xi_0}\right)$$
$$+\left(\frac{\sigma}{\mu}\right)\left(\frac{-\sigma\mu}{\sqrt{2\sigma H_0\mu}}\right)\left(\frac{-2\eta_1-\xi_0\eta_1+\xi_0{}^2\eta_1+\xi_1\eta_0-\xi_0\xi_1\eta_0}{(1-\xi_0)}\right)\left(\sigma\sqrt{\sigma 2H_0\mu}\frac{\xi_1}{1-\xi_0}\right)$$
$$+\left(\frac{\sigma}{\mu}\right)\left(\frac{-\mu}{\sqrt{\sigma 2H_0\mu}}\right)\eta_0\left(\frac{-\sigma\sqrt{2\sigma H_0\mu}}{(1-\xi_0)^3}\right)[\xi_1\eta_0\eta_1-\xi_0\eta_1{}^2+\xi_0-2\xi_0{}^2+\eta_1{}^2+\xi_0{}^3+\sigma(\xi_1{}^2-\xi_0\xi_1{}^2)],$$

from above, (IX.2.), (IX.3.), and (IX.3.) proof,

$$\eta_1' = \frac{-1}{(1-\xi_0)^3}\begin{bmatrix}\sigma\left(\sigma\eta_0+\frac{\xi_1\eta_1}{1-\xi_0}\right)(1-\xi_0)^2+(\sigma\xi_1-\sigma\xi_0\xi_1+\eta_0\eta_1)(\eta_1-\xi_0\eta_1+\xi_1\eta_0)\\ +\sigma(\xi_0\eta_1+\eta_1-\xi_0{}^2\eta_1+\xi_0\xi_1\eta_0-\xi_1\eta_0)\xi_1(1-\xi_0)\\ +\sigma(-2\eta_1-\xi_0\eta_1+\xi_0{}^2\eta_1+\xi_1\eta_0-\xi_0\xi_1\eta_0)\xi_1(1-\xi_0)\\ +(\eta_0)(\xi_1\eta_0\eta_1-\xi_0\eta_1{}^2+\xi_0-2\xi_0{}^2+\eta_1{}^2+\xi_0{}^3+\sigma\xi_1{}^2-\sigma\xi_0\xi_1{}^2)\end{bmatrix},$$

$$\eta_1' = \frac{-1}{(1-\xi_0)^3}\begin{bmatrix}\eta_0(1-\xi_0)^2+\sigma(\xi_1\eta_1)(1-\xi_0)+(\sigma\xi_1-\sigma\xi_0\xi_1+\eta_0\eta_1)(\eta_1-\xi_0\eta_1+\xi_1\eta_0)\\ +\sigma(-\eta_1)\xi_1(1-\xi_0)\\ -(\eta_0)(\xi_1\eta_0\eta_1-\xi_0\eta_1{}^2+\xi_0-2\xi_0{}^2+\eta_1{}^2+\xi_0{}^3+\sigma\xi_1{}^2-\sigma\xi_0\xi_1{}^2)\end{bmatrix},$$

$$\eta_1' = \frac{-1}{(1-\xi_0)^3}\begin{bmatrix}\eta_0-2\xi_0\eta_0+\xi_0{}^2\eta_0+\sigma\xi_1\eta_1-\sigma\xi_0\xi_1\eta_1+\sigma\xi_1\eta_1-\sigma\xi_0\xi_1\eta_1+\sigma\xi_1{}^2\eta_0-\sigma\xi_0\xi_1\eta_1\\ +\sigma\xi_0{}^2\xi_1\eta_1-\sigma\xi_0\xi_1{}^2\eta_0+\eta_0\eta_1{}^2-\xi_0\eta_0\eta_1{}^2+\xi_1\eta_0{}^2\eta_1-\sigma\xi_1\eta_1+\sigma\xi_0\xi_1\eta_1-\xi_1\eta_0{}^2\eta_1\\ +\xi_0\eta_0\eta_1{}^2-\xi_0\eta_0+2\xi_0{}^2\eta_0-\eta_0\eta_1{}^2-\xi_0{}^3\eta_0-\sigma\xi_1{}^2\eta_0+\sigma\xi_0\xi_1{}^2\eta_0\end{bmatrix},$$

$$\eta_1' = \frac{-1}{(1-\xi_0)^3}[\eta_0-3\xi_0\eta_0+3\xi_0{}^2\eta_0-\xi_0{}^3\eta_0+\sigma\xi_1\eta_1-2\sigma\xi_0\xi_1\eta_1+\sigma\xi_0{}^2\xi_1\eta_1],$$

$$\eta_1' = \frac{-1}{(1-\xi_0)^3}[\eta_0(1-3\xi_0+3\xi_0{}^2-\xi_0{}^3)+\sigma\xi_1\eta_1(1-2\xi_0+\xi_0{}^2)],$$

$$\eta_1' = \frac{1}{(1-\xi_0)^3}\left[-\eta_0(1-\xi_0)^3 - \sigma\xi_1\eta_1(1-\xi_0)^2\right],$$
$$\eta_1' = -\eta_0 - \frac{\sigma\xi_1\eta_1}{1-\xi_0} = -\eta_0 + \xi_1\eta_0'.$$

$$\eta_2' = \left(\frac{-\sigma}{2\mu}\right)(\|\boldsymbol{p}\|^2 - 2H_0\mu)(q_2') + \left(\frac{-\sigma}{2\mu}\right)2(\boldsymbol{p}\cdot\boldsymbol{p}')(q_2)$$
$$+\left(\frac{\sigma}{\mu}\right)\left[(\boldsymbol{q}\cdot\boldsymbol{p}')(p_2) + (\boldsymbol{p}\cdot\boldsymbol{q}')(p_2) + (\boldsymbol{q}\cdot\boldsymbol{p})(p_2')\right],$$

from (IX.1.),

$$\eta_2' = \left(\frac{-\sigma}{2\mu}\right)(-4\mathrm{H}_0\mu)\left(\frac{1}{1-\xi_0}\right)\left(\frac{1}{-2H_0}\right)\left(\frac{-\xi_0\xi_2\eta_1-\xi_1\eta_2+\xi_0\xi_1\eta_2+2\xi_2\eta_1}{1-\xi_0}\right)$$
$$+\left(\frac{-\sigma}{\mu}\right)(2H_0\mu)\frac{(\sigma\xi_1-\sigma\xi_0\xi_1+\eta_0\eta_1)}{(1-\xi_0)^3}\left(\frac{1}{\sigma 2H_0}\right)(\eta_2-\xi_0\eta_2+\xi_2\eta_0)$$
$$+\left(\frac{\sigma}{\mu}\right)\left(\frac{-\sigma\mu}{\sqrt{2\sigma H_0\mu}}\right)\frac{(\xi_0\eta_1+\eta_1-{\xi_0}^2\eta_1+\xi_0\xi_1\eta_0-\xi_1\eta_0)}{(1-\xi_0)}\left(\sigma\sqrt{\sigma 2H_0\mu}\frac{\xi_2}{1-\xi_0}\right)$$
$$+\left(\frac{\sigma}{\mu}\right)\left(\frac{-\sigma\mu}{\sqrt{2\sigma H_0\mu}}\right)\left(\frac{-2\eta_1-\xi_0\eta_1+{\xi_0}^2\eta_1+\xi_1\eta_0-\xi_0\xi_1\eta_0}{(1-\xi_0)}\right)\left(\sigma\sqrt{\sigma 2H_0\mu}\frac{\xi_2}{1-\xi_0}\right)$$
$$+\left(\frac{\sigma}{\mu}\right)\left(\frac{-\mu}{\sqrt{\sigma 2H_0\mu}}\right)\eta_0\left(\frac{-\sigma\sqrt{2\sigma H_0\mu}}{(1-\xi_0)^3}\right)\left[\eta_1\eta_2+\xi_2\eta_0\eta_1-\xi_0\eta_1\eta_2+\sigma\xi_1\xi_2-\sigma\xi_0\xi_1\xi_2\right],$$

from above, (IX.2.), (IX.3.), and (IX.3.) proof,

$$\eta_2' = \frac{-1}{(1-\xi_0)^3}\begin{bmatrix}\sigma(-\xi_0\xi_2\eta_1-\xi_1\eta_2+\xi_0\xi_1\eta_2+2\xi_2\eta_1)(1-\xi_0)\\ +(\sigma\xi_1-\sigma\xi_0\xi_1+\eta_0\eta_1)(\eta_2-\xi_0\eta_2+\xi_2\eta_0)\\ +\sigma\left(\xi_0\eta_1+\eta_1-{\xi_0}^2\eta_1+\xi_0\xi_1\eta_0-\xi_1\eta_0\right)\xi_2(1-\xi_0)\\ +\sigma\left(-2\eta_1-\xi_0\eta_1+{\xi_0}^2\eta_1+\xi_1\eta_0-\xi_0\xi_1\eta_0\right)\xi_2(1-\xi_0)\\ -(\eta_0)(\eta_1\eta_2+\xi_2\eta_0\eta_1-\xi_0\eta_1\eta_2+\sigma\xi_1\xi_2-\sigma\xi_0\xi_1\xi_2)\end{bmatrix},$$

$$\eta_2' = \frac{-1}{(1-\xi_0)^3}\begin{bmatrix}\sigma(-\xi_0\xi_2\eta_1-\xi_1\eta_2+\xi_0\xi_1\eta_2+2\xi_2\eta_1)(1-\xi_0)\\ +(\sigma\xi_1-\sigma\xi_0\xi_1+\eta_0\eta_1)(\eta_2-\xi_0\eta_2+\xi_2\eta_0)\\ +\sigma(-\eta_1)\xi_2(1-\xi_0)\\ -(\eta_0)(\eta_1\eta_2+\xi_2\eta_0\eta_1-\xi_0\eta_1\eta_2+\sigma\xi_1\xi_2-\sigma\xi_0\xi_1\xi_2)\end{bmatrix},$$

$$\eta_2' = \frac{-1}{(1-\xi_0)^3}\begin{bmatrix}-\sigma\xi_0\xi_2\eta_1-\sigma\xi_1\eta_2+\sigma\xi_0\xi_1\eta_2+2\sigma\xi_2\eta_1\\ +\sigma{\xi_0}^2\xi_2\eta_1+\sigma\xi_0\xi_1\eta_2-\sigma{\xi_0}^2\xi_1\eta_2-2\sigma\xi_0\xi_2\eta_1\\ +\sigma\xi_1\eta_2-\sigma\xi_0\xi_1\eta_2+\sigma\xi_1\xi_2\eta_0\\ -\sigma\xi_0\xi_1\eta_2+\sigma{\xi_0}^2\xi_1\eta_2-\sigma\xi_0\xi_1\xi_2\eta_0\\ +\eta_0\eta_1\eta_2-\xi_0\eta_0\eta_1\eta_2+\xi_2{\eta_0}^2\eta_1\\ -\sigma\xi_2\eta_1+\sigma\xi_0\xi_2\eta_1\\ -\eta_0\eta_1\eta_2-\xi_2{\eta_0}^2\eta_1+\xi_0\eta_0\eta_1\eta_2-\sigma\xi_1\xi_2\eta_0+\sigma\xi_0\xi_1\xi_2\eta_0\end{bmatrix},$$

$$\eta_2' = \frac{-1}{(1-\xi_0)^3}\left[+\sigma\xi_2\eta_1-2\sigma\xi_0\xi_2\eta_1+\sigma{\xi_0}^2\xi_2\eta_1\right],$$
$$\eta_2' = \frac{-1}{(1-\xi_0)^3}\left[\sigma\xi_2\eta_1\left(1-2\xi_0+{\xi_0}^2\right)\right],$$
$$\eta_2' = \frac{-\sigma\xi_2\eta_1}{1-\xi_0} = \xi_2\eta_0'.$$

$$\eta_3' = \left(\frac{-\sigma}{2\mu}\right)(\|\boldsymbol{p}\|^2 - 2H_0\mu)(q_3') + \left(\frac{-\sigma}{2\mu}\right)2(\boldsymbol{p}\cdot\boldsymbol{p}')(q_3)$$

$$+\left(\frac{\sigma}{\mu}\right)[(\boldsymbol{q}\cdot\boldsymbol{p}')(p_3)+(\boldsymbol{p}\cdot\boldsymbol{q}')(p_3)+(\boldsymbol{q}\cdot\boldsymbol{p})(p_3')],$$

from (IX.1.),

$$\eta_3' = \left(\frac{-\sigma}{2\mu}\right)(-4H_0\mu)\left(\frac{1}{1-\xi_0}\right)\left(\frac{1}{-2H_0}\right)\left(\frac{-\xi_0\xi_3\eta_1-\xi_1\eta_3+\xi_0\xi_1\eta_3+2\xi_3\eta_1}{1-\xi_0}\right)$$
$$+\left(\frac{-\sigma}{\mu}\right)(2H_0\mu)\frac{(\sigma\xi_1-\sigma\xi_0\xi_1+\eta_0\eta_1)}{(1-\xi_0)^3}\left(\frac{1}{\sigma 2H_0}\right)(\eta_3-\xi_0\eta_3+\xi_3\eta_0)$$
$$+\left(\frac{\sigma}{\mu}\right)\left(\frac{-\sigma\mu}{\sqrt{2\sigma H_0\mu}}\right)\frac{(\xi_0\eta_1+\eta_1-\xi_0{}^2\eta_1+\xi_0\xi_1\eta_0-\xi_1\eta_0)}{(1-\xi_0)}\left(\sigma\sqrt{\sigma 2H_0\mu}\frac{\xi_3}{1-\xi_0}\right)$$
$$+\left(\frac{\sigma}{\mu}\right)\left(\frac{-\sigma\mu}{\sqrt{2\sigma H_0\mu}}\right)\left(\frac{-2\eta_1-\xi_0\eta_1+\xi_0{}^2\eta_1+\xi_1\eta_0-\xi_0\xi_1\eta_0}{(1-\xi_0)}\right)\left(\sigma\sqrt{\sigma 2H_0\mu}\frac{\xi_3}{1-\xi_0}\right)$$
$$+\left(\frac{\sigma}{\mu}\right)\left(\frac{-\mu}{\sqrt{\sigma 2H_0\mu}}\right)\eta_0\left(\frac{-\sigma\sqrt{2\sigma H_0\mu}}{(1-\xi_0)^3}\right)[\eta_1\eta_3+\xi_3\eta_0\eta_1-\xi_0\eta_1\eta_3+\sigma\xi_1\xi_3-\sigma\xi_0\xi_1\xi_3],$$

from above, (IX.2.), (IX.3.), and (IX.3.) proof,

$$\eta_3' = \frac{-1}{(1-\xi_0)^3}\begin{bmatrix}\sigma(-\xi_0\xi_3\eta_1-\xi_1\eta_3+\xi_0\xi_1\eta_3+2\xi_3\eta_1)(1-\xi_0)\\ +(\sigma\xi_1-\sigma\xi_0\xi_1+\eta_0\eta_1)(\eta_3-\xi_0\eta_3+\xi_3\eta_0)\\ +\sigma(\xi_0\eta_1+\eta_1-\xi_0{}^2\eta_1+\xi_0\xi_1\eta_0-\xi_1\eta_0)\xi_3(1-\xi_0)\\ +\sigma(-2\eta_1-\xi_0\eta_1+\xi_0{}^2\eta_1+\xi_1\eta_0-\xi_0\xi_1\eta_0)\xi_3(1-\xi_0)\\ -(\eta_0)(\eta_1\eta_3+\xi_3\eta_0\eta_1-\xi_0\eta_1\eta_3+\sigma\xi_1\xi_3-\sigma\xi_0\xi_1\xi_3)\end{bmatrix},$$

$$\eta_3' = \frac{-1}{(1-\xi_0)^3}\begin{bmatrix}\sigma(-\xi_0\xi_3\eta_1-\xi_1\eta_3+\xi_0\xi_1\eta_3+2\xi_3\eta_1)(1-\xi_0)\\ +(\sigma\xi_1-\sigma\xi_0\xi_1+\eta_0\eta_1)(\eta_3-\xi_0\eta_3+\xi_3\eta_0)\\ +\sigma(-\eta_1)\xi_3(1-\xi_0)\\ -(\eta_0)(\eta_1\eta_3+\xi_3\eta_0\eta_1-\xi_0\eta_1\eta_3+\sigma\xi_1\xi_3-\sigma\xi_0\xi_1\xi_3)\end{bmatrix},$$

$$\eta_3' = \frac{-1}{(1-\xi_0)^3}\begin{bmatrix}-\sigma\xi_0\xi_3\eta_1-\sigma\xi_1\eta_3+\sigma\xi_0\xi_1\eta_3+2\sigma\xi_3\eta_1\\ +\sigma\xi_0{}^2\xi_3\eta_1+\sigma\xi_0\xi_1\eta_3-\sigma\xi_0{}^2\xi_1\eta_3-2\sigma\xi_0\xi_3\eta_1\\ +\sigma\xi_1\eta_3-\sigma\xi_0\xi_1\eta_3+\sigma\xi_1\xi_3\eta_0\\ -\sigma\xi_0\xi_1\eta_3+\sigma\xi_0{}^2\xi_1\eta_3-\sigma\xi_0\xi_1\xi_3\eta_0\\ +\eta_0\eta_1\eta_3-\xi_0\eta_0\eta_1\eta_3+\xi_3\eta_0{}^2\eta_1\\ -\sigma\xi_3\eta_1+\sigma\xi_0\xi_3\eta_1\\ -\eta_0\eta_1\eta_3-\xi_3\eta_0{}^2\eta_1+\xi_0\eta_0\eta_1\eta_3-\sigma\xi_1\xi_3\eta_0+\sigma\xi_0\xi_1\xi_3\eta_0\end{bmatrix},$$

$$\eta_3' = \frac{-1}{(1-\xi_0)^3}[+\sigma\xi_3\eta_1-2\sigma\xi_0\xi_3\eta_1+\sigma\xi_0{}^2\xi_3\eta_1],$$

$$\eta_3' = \frac{-1}{(1-\xi_0)^3}[\sigma\xi_3\eta_1(1-2\xi_0+\xi_0{}^2)],$$

$$\eta_3' = \frac{-\sigma\xi_3\eta_1}{1-\xi_0} = \xi_3\eta_0'.$$

$$\xi_0' = \frac{2(\boldsymbol{p}\cdot\boldsymbol{p}')}{\|\boldsymbol{p}\|^2-2H_0\mu}-\frac{2(\boldsymbol{p}\cdot\boldsymbol{p}')(\|\boldsymbol{p}\|^2+2H_0\mu)}{(\|\boldsymbol{p}\|^2-2H_0\mu)^2} = \frac{2(\boldsymbol{p}\cdot\boldsymbol{p}')(-4H_0\mu)}{(\|\boldsymbol{p}\|^2-2H_0\mu)^2},$$

from (IX.1.),

$$\xi_0' = (-4H_0\mu)(4H_0\mu)\frac{(\sigma\xi_1-\sigma\xi_0\xi_1+\eta_0\eta_1)}{(1-\xi_0)^3}\left(\frac{1-\xi_0}{-4H_0\mu}\right)^2,$$

from above and (IX.3.) proof,

$$\xi_0' = \frac{(-\sigma\xi_1+\sigma\xi_0\xi_1-\eta_0\eta_1)}{(1-\xi_0)},$$

$\xi_0' = -\sigma\xi_1 - \eta_0\left(\frac{\eta_1}{1-\xi_0}\right) = -\sigma\xi_1 + \sigma\eta_0\eta_0'.$

$\xi_1' = \left(-2\sqrt{2\sigma H_0\mu}\right)(\|\boldsymbol{p}\|^2 - 2H_0\mu)^{(-1)}(p_1') + \left(-2\sqrt{2\sigma H_0\mu}\right)(-1)(\|\boldsymbol{p}\|^2 - 2H_0\mu)^{(-2)}2(\boldsymbol{p}\cdot\boldsymbol{p}')(p_1),$

from (IX.1.),

$\xi_1' = \left(-2\sqrt{2\sigma H_0\mu}\right)\left(\frac{1-\xi_0}{-4H_0\mu}\right)\left(\frac{-\sigma\sqrt{2\sigma H_0\mu}}{(1-\xi_0)^3}\right)\begin{bmatrix}\xi_1\eta_0\eta_1 - \xi_0{\eta_1}^2 + \xi_0 - 2{\xi_0}^2 \\ +{\eta_1}^2 + {\xi_0}^3 + \sigma{\xi_1}^2 - \sigma\xi_0{\xi_1}^2\end{bmatrix}$

$+\left(-2\sqrt{2\sigma H_0\mu}\right)(-1)\left(\frac{1-\xi_0}{-4H_0\mu}\right)^2 2(2H_0\mu)\frac{(\sigma\xi_1 - \sigma\xi_0\xi_1 - \eta_0\eta_1)}{(1-\xi_0)^3}\left(\sigma\sqrt{2\sigma H_0\mu}\frac{\xi_1}{1-\xi_0}\right),$

from above and (IX.3.) proof,

$\xi_1' = \frac{-1}{(1-\xi_0)^2}\left[\xi_1\eta_0\eta_1 - \xi_0{\eta_1}^2 + \xi_0 - 2{\xi_0}^2 + {\eta_1}^2 + {\xi_0}^3 - \sigma{\xi_1}^2 + \sigma\xi_0{\xi_1}^2 + (\sigma\xi_1 - \sigma\xi_0\xi_1 - \eta_0\eta_1)\xi_1\right],$

$\xi_1' = \frac{-1}{(1-\xi_0)^2}\begin{pmatrix}\xi_1\eta_0\eta_1 - \xi_0{\eta_1}^2 + \xi_0 - 2{\xi_0}^2 + {\eta_1}^2 + {\xi_0}^3 - \sigma{\xi_1}^2 + \sigma\xi_0{\xi_1}^2 \\ +\sigma{\xi_1}^2 - \sigma\xi_0{\xi_1}^2 - \xi_1\eta_0\eta_1\end{pmatrix},$

$\xi_1' = \frac{-1}{(1-\xi_0)^2}\left(-\xi_0{\eta_1}^2 + \xi_0 - 2{\xi_0}^2 + {\eta_1}^2 + {\xi_0}^3\right),$

$\xi_1' = \frac{-1}{(1-\xi_0)^2}\left(+{\eta_1}^2 - \xi_0{\eta_1}^2 + \xi_0 - {\xi_0}^2 - {\xi_0}^2 + {\xi_0}^3\right),$

$\xi_1' = \frac{-{\eta_1}^2 - \xi_0 + {\xi_0}^2}{1-\xi_0},$

$\xi_1' = -\xi_0 - \frac{{\eta_1}^2}{1-\xi_0} = -\xi_0 + \sigma\eta_1\eta_0'.$

$\xi_2' = \left(-2\sqrt{2\sigma H_0\mu}\right)(\|\boldsymbol{p}\|^2 - 2H_0\mu)^{(-1)}(p_2') + \left(-2\sqrt{2\sigma H_0\mu}\right)(-1)(\|\boldsymbol{p}\|^2 - 2H_0\mu)^{(-2)}2(\boldsymbol{p}\cdot\boldsymbol{p}')(p_2),$

from (IX.1.),

$\xi_2' = \left(-2\sqrt{2\sigma H_0\mu}\right)\left(\frac{1-\xi_0}{-4H_0\mu}\right)\left(\frac{-\sigma\sqrt{2\sigma H_0\mu}}{(1-\xi_0)^3}\right)[+\eta_1\eta_2 + \xi_2\eta_0\eta_1 - \xi_0\eta_1\eta_2 + \sigma\xi_1\xi_2 - \sigma\xi_0\xi_1\xi_2]$

$+\left(-2\sqrt{2\sigma H_0\mu}\right)(-1)\left(\frac{1-\xi_0}{-4H_0\mu}\right)^2 2(2H_0\mu)\frac{(\sigma\xi_1 - \sigma\xi_0\xi_1 - \eta_0\eta_1)}{(1-\xi_0)^3}\left(\sigma\sqrt{2\sigma H_0\mu}\frac{\xi_2}{1-\xi_0}\right),$

from above and (IX.3.) proof,

$\xi_2' = \frac{-1}{(1-\xi_0)^2}[+\eta_1\eta_2 + \xi_2\eta_0\eta_1 - \xi_0\eta_1\eta_2 + \sigma\xi_1\xi_2 - \sigma\xi_0\xi_1\xi_2 - (\sigma\xi_1 - \sigma\xi_0\xi_1 + \eta_0\eta_1)\xi_2],$

$\xi_2' = \frac{-1}{(1-\xi_0)^2}(+\eta_1\eta_2 + \xi_2\eta_0\eta_1 - \xi_0\eta_1\eta_2 + \sigma\xi_1\xi_2 - \sigma\xi_0\xi_1\xi_2 - \sigma\xi_1\xi_2 + \sigma\xi_0\xi_1\xi_2 - \xi_2\eta_0\eta_1),$

$\xi_2' = \frac{-1}{(1-\xi_0)^2}(+\eta_1\eta_2 - \xi_0\eta_1\eta_2),$

$\xi_2' = \frac{-\eta_1\eta_2}{1-\xi_0} = -\eta_2\eta_0'.$

$\xi_3' = \left(-2\sqrt{-2H_0\mu}\right)(\|\boldsymbol{p}\|^2 - 2H_0\mu)^{(-1)}(p_3') + \left(-2\sqrt{-2H_0\mu}\right)(-1)(\|\boldsymbol{p}\|^2 - 2H_0\mu)^{(-2)}2(\boldsymbol{p}\cdot\boldsymbol{p}')(p_3),$

from (IX.1.),

$\xi_3' = \left(-2\sqrt{-2H_0\mu}\right)\left(\frac{1-\xi_0}{-4H_0\mu}\right)\left(\frac{\sqrt{-2H_0\mu}}{(1-\xi_0)^3}\right)[+\eta_1\eta_3 + \xi_3\eta_0\eta_1 - \xi_0\eta_1\eta_3 - \xi_1\xi_3 + \xi_0\xi_1\xi_3]$

$+\left(-2\sqrt{-2H_0\mu}\right)(-1)\left(\frac{1-\xi_0}{-4H_0\mu}\right)^2 2(-2H_0\mu)\frac{(\xi_1-\xi_0\xi_1-\eta_0\eta_1)}{(1-\xi_0)^3}\left(\sigma\sqrt{\sigma 2H_0\mu}\frac{\xi_3}{1-\xi_0}\right),$

from above and (IX.3.) proof,

$\xi_3' = \frac{-1}{(1-\xi_0)^2}[+\eta_1\eta_3 + \xi_3\eta_0\eta_1 - \xi_0\eta_1\eta_3 - \xi_1\xi_3 + \xi_0\xi_1\xi_3 + (\xi_1 - \xi_0\xi_1 - \eta_0\eta_1)\xi_3],$

$\xi_3' = \frac{-1}{(1-\xi_0)^2}(+\eta_1\eta_3 + \xi_3\eta_0\eta_1 - \xi_0\eta_1\eta_3 - \xi_1\xi_3 + \xi_0\xi_1\xi_3 + \xi_1\xi_3 - \xi_0\xi_1\xi_3 - \xi_3\eta_0\eta_1),$

$\xi_3' = \frac{-1}{(1-\xi_0)^2}(+\eta_1\eta_3 - \xi_0\eta_1\eta_3),$

$\xi_3' = \frac{-\eta_1\eta_3}{1-\xi_0} = -\eta_3\eta_0'.$■

(IX.5.) Proof. The formulas for ξ_i, η_i are the solutions of the differential equations given in (IX.4.), and the additional conditions on the integration constants follow from the additional conditions in (IX.3.).

(i) For $H < 0$:

First, we observe

$[\xi_0 + i\xi_1 + i(\eta_0 + i\eta_1)]' = [\xi_0 + i\xi_1 + i\eta_0 - \eta_1]'$

$= \xi_1 - \eta_0\eta_0' - i\xi_0 - \eta_1\eta_0' + i\eta_1 + i\xi_0\eta_0' + \eta_0 - \xi_1\eta_0'$

$= [\xi_0 + i\xi_1 + i\eta_0 - \eta_1](1 - \eta_0').$

Then we check to see if the solution holds

$Be^{-is}e^{i(\eta_0+\alpha)} = Be^{-is}[cos(\eta_0 + \alpha) + isin(\eta_0 + \alpha)],$

$\left[Be^{-is}e^{i(\eta_0+\alpha)}\right]'$

$= -iBe^{-is}e^{i(\eta_0+\alpha)} + iBe^{-is}e^{i(\eta_0+\alpha)}\eta_0'$

$= -i\left[Be^{-is}e^{i(\eta_0+\alpha)}\right](1 - \eta_0').$

This gives us

$\xi_0 = \frac{1}{2}\left[(\xi_0 + i\xi_1) + \overline{(\xi_0 + i\xi_1)}\right],$

$\xi_0 = \frac{1}{2}\left[Be^{-is}cos(\eta_0 + \alpha) + \bar{B}e^{is}cos(\eta_0 + \bar{\alpha})\right].$

$\xi_1 = \frac{1}{2i}\left[(\xi_0 + i\xi_1) - \overline{(\xi_0 + i\xi_1)}\right],$

$\xi_1 = \frac{1}{2i}\left[Be^{-is}cos(\eta_0 + \alpha) - \bar{B}e^{is}cos(\eta_0 + \bar{\alpha})\right].$

$\eta_0 = \frac{1}{2}\left[(\eta_0 + i\eta_1) + \overline{(\eta_0 + i\eta_1)}\right],$

$\eta_0 = \frac{1}{2}\left[Be^{-is}sin(\eta_0 + \alpha) + \bar{B}e^{is}sin(\eta_0 + \bar{\alpha})\right].$

$\eta_1 = \frac{1}{2i}\left[(\eta_0 + i\eta_1) + \overline{(\eta_0 + i\eta_1)}\right],$

$\eta_1 = \frac{1}{2i}\left[Be^{-is}sin(\eta_0 + \alpha) - \bar{B}e^{is}sin(\eta_0 + \bar{\alpha})\right].$

Now we observe

$(\xi_2 + i\eta_2)' = -\eta_2\eta_0' + i\xi_2\eta_0' = i(\xi_2 + i\eta_2)\eta_0',$

Then we check to see if the solution holds

$(\xi_2 + i\eta_2)' = B_2[cos(\eta_0 + \alpha_2) + i\ sin(\eta_0 + \alpha_2)]'$

$(\xi_2 + i\eta_2)' = \left[B_2 e^{i(\eta_0+\alpha_2)}\right]' = iB_2 e^{i(\eta_0+\alpha_2)}\eta_0'$

$(\xi_2 + i\eta_2)' = i(\xi_2 + i\eta_2)\eta_0'.$

$(\xi_3 + i\eta_3)$ is analogous.

$\xi_2 = B_2 cos(\eta_0 + \alpha_2).$

$\xi_3 = B_3 cos(\eta_0 + \alpha_3).$

$\eta_2 = B_2 sin(\eta_0 + \alpha_2).$

$\eta_3 = B_3 sin(\eta_0 + \alpha_3).$

$1 = {\xi_0}^2 + {\xi_1}^2 + {\xi_2}^2 + {\xi_3}^2,$

from (IX.3.),

$1 = (\xi_0 + i\xi_1)(\xi_0 - i\xi_1) + {\xi_2}^2 + {\xi_3}^2,$

$1 = Be^{-is}cos(\eta_0 + \alpha)\bar{B}e^{is}cos(\eta_0 + \bar{\alpha}) + {B_2}^2 cos^2(\eta_0 + \alpha_2) + {B_3}^2 cos^2(\eta_0 + \alpha_3).$

$1 = B\bar{B}cos(\eta_0 + \alpha)cos(\eta_0 + \bar{\alpha}) + {B_2}^2 cos^2(\eta_0 + \alpha_2) + {B_3}^2 cos^2(\eta_0 + \alpha_3).$

$1 = {\eta_0}^2 + {\eta_1}^2 + {\eta_2}^2 + {\eta_3}^2,$

from (IX.3.),

$1 = (\eta_0 + i\eta_1)(\eta_0 - i\eta_1) + {\eta_2}^2 + {\eta_3}^2,$

$1 = Be^{-is}sin(\eta_0 + \alpha)\bar{B}e^{is}sin(\eta_0 + \bar{\alpha}) + {B_2}^2 sin^2(\eta_0 + \alpha_2) + {B_3}^2 sin^2(\eta_0 + \alpha_3).$

$1 = B\bar{B}sin(\eta_0 + \alpha)sin(\eta_0 + \bar{\alpha}) + {B_2}^2 sin^2(\eta_0 + \alpha_2) + {B_3}^2 sin^2(\eta_0 + \alpha_3).$

$2 = B\bar{B}[cos(\eta_0 + \alpha)cos(\eta_0 + \bar{\alpha}) + sin(\eta_0 + \alpha)sin(\eta_0 + \bar{\alpha})]$

$+{B_2}^2[cos^2(\eta_0 + \alpha_2) + sin^2(\eta_0 + \alpha_2)] + {B_3}^2[cos^2(\eta_0 + \alpha_3) + sin^2(\eta_0 + \alpha_3)],$

$2 = B\bar{B}\ cos(\eta_0 + \alpha - \eta_0 - \bar{\alpha}) + {B_2}^2 + {B_3}^2,$

because $cosxcosy + sinxsiny = cos(x - y)$ and $cos^2x + sin^2x = 1,$

$2 = B\bar{B}\ cos(\eta_0 + \alpha - \eta_0 - \bar{\alpha}) + {B_2}^2 + {B_3}^2,$

$2 = \|B\|^2 cosh(2Im\alpha) + {B_2}^2 + {B_3}^2,$

because $cos(ix) = cosh(x)$ for real x.

$0 = \xi_0\eta_0 + \xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3$

$0 = \frac{1}{2}\left[Be^{-is}cos(\eta_0 + \alpha) + \bar{B}e^{is}cos(\eta_0 + \bar{\alpha})\right]\frac{1}{2}\left[Be^{-is}sin(\eta_0 + \alpha) + \bar{B}e^{is}sin(\eta_0 + \bar{\alpha})\right]$

$+\frac{1}{2i}\left[Be^{-is}cos(\eta_0 + \alpha) - \bar{B}e^{is}cos(\eta_0 + \bar{\alpha})\right]\frac{1}{2i}\left[Be^{-is}sin(\eta_0 + \alpha) - \bar{B}e^{is}sin(\eta_0 + \bar{\alpha})\right]$

$+B_2 cos(\eta_0 + \alpha_2)B_2\ sin(\eta_0 + \alpha_2) + B_3 cos(\eta_0 + \alpha_3)B_3 sin(\eta_0 + \alpha_3),$

$0 = \frac{1}{4}B^2 e^{-2is}\ sin(\eta_0 + \alpha)cos(\eta_0 + \alpha) + \frac{1}{4}B\bar{B}sin(\eta_0 + \bar{\alpha})cos(\eta_0 + \alpha)$

$+\frac{1}{4}B\bar{B}sin(\eta_0+\alpha)cos(\eta_0+\bar{\alpha})+\frac{1}{4}\bar{B}^2e^{2is}sin(\eta_0+\bar{\alpha})cos(\eta_0+\bar{\alpha})$

$-\frac{1}{4}B^2e^{-2is}\,sin(\eta_0+\alpha)cos(\eta_0+\alpha)+\frac{1}{4}B\bar{B}sin(\eta_0+\bar{\alpha})cos(\eta_0+\alpha)$

$+\frac{1}{4}B\bar{B}sin(\eta_0+\alpha)cos(\eta_0+\bar{\alpha})-\frac{1}{4}\bar{B}^2e^{2is}sin(\eta_0+\bar{\alpha})cos(\eta_0+\bar{\alpha})$

$+{B_2}^2\,sin(\eta_0+\alpha_2)cos(\eta_0+\alpha_2)+{B_3}^2\,sin(\eta_0+\alpha_3)cos(\eta_0+\alpha_3),$

$0=\frac{1}{2}B\bar{B}sin(\eta_0+\bar{\alpha})cos(\eta_0+\alpha)+\frac{1}{2}B\bar{B}sin(\eta_0+\alpha)cos(\eta_0+\bar{\alpha})$

$+{B_2}^2\,sin(\eta_0+\alpha_2)cos(\eta_0+\alpha_2)+{B_3}^2\,sin(\eta_0+\alpha_3)cos(\eta_0+\alpha_3),$

$0=\frac{\|B\|^2}{2}sin(2\eta_0+2Re\alpha)+\frac{{B_2}^2}{2}sin(2\eta_0+2\alpha_2)+\frac{{B_3}^2}{2}sin(2\eta_0+2\alpha_3),$

because $sin(2a)=2sin(a)cos(a)$ and $sin(a+b)=sin(a)cos(b)+cos(a)sin(b)$.

Multiplying by 2 and taking the square, we get:

$(*)\colon 0=\|B\|^4\,sin^2(2\eta_0+2Re\alpha)+{B_2}^4\,sin^2(2\eta_0+2\alpha_2)+{B_3}^4\,sin^2(2\eta_0+2\alpha_3)$

$+2\|B\|^2{B_2}^2sin(2\eta_0+2Re\alpha)\,sin(2\eta_0+2\alpha_2)+2\|B\|^2{B_3}^2sin(2\eta_0+2Re\alpha)\,sin(2\eta_0+2\alpha_3)$

$+2{B_2}^2{B_3}^2\,sin(2\eta_0+2\alpha_2)\,sin(2\eta_0+2\alpha_3).$

Now we'll save this equation for the moment, and create another equation for zero by subtracting the two expressions for one from each other:

$0=\|B\|^2[cos(\eta_0+\alpha)cos(\eta_0+\bar{\alpha})-sin(\eta_0+\alpha)sin(\eta_0+\bar{\alpha})]$

$+{B_2}^2[cos^2(\eta_0+\alpha_2)-sin^2(\eta_0+\alpha_2)]+{B_3}^2[cos^2(\eta_0+\alpha_3)-sin^2(\eta_0+\alpha_3)],$

$0=\|B\|^2cos(2\eta_0+2Re\alpha)+{B_2}^2cos(2\eta_0+2\alpha_2)+{B_3}^2cos(2\eta_0+2\alpha_3),$

because $cos(a+b)=cos(a)cos(b)-sin(a)sin(b)$ and $cos(2a)=cos^2(a)-sin^2(a)$.

By taking the square, we get:

$(**)\colon 0=\|B\|^4cos^2(2\eta_0+2Re\alpha)+{B_2}^4cos^2(2\eta_0+2\alpha_2)+{B_3}^4cos^2(2\eta_0+2\alpha_3)$

$+2\|B\|^2{B_2}^2cos(2\eta_0+2Re\alpha)\,cos(2\eta_0+2\alpha_2)+2\|B\|^2{B_3}^2cos(2\eta_0+2Re\alpha)\,cos(2\eta_0+2\alpha_3)$

$+2{B_2}^2{B_3}^2\,cos(2\eta_0+2\alpha_2)\,cos(2\eta_0+2\alpha_3).$

Now we can add the expressions (*) and (**) and use the identity $sin^2(a)+cos^2(a)=1$ to get:

$0=\|B\|^4\;+{B_2}^4+{B_3}^4$

$+2\|B\|^2{B_2}^2[sin(2\eta_0+2Re\alpha)\,sin(2\eta_0+2\alpha_2)+cos(2\eta_0+2Re\alpha)\,cos(2\eta_0+2\alpha_2)]$

$+2\|B\|^2{B_3}^2[sin(2\eta_0+2Re\alpha)\,sin(2\eta_0+2\alpha_3)+cos(2\eta_0+2Re\alpha)\,cos(2\eta_0+2\alpha_3)]$

$+2{B_2}^2{B_3}^2[sin(2\eta_0+2\alpha_2)\,sin(2\eta_0+2\alpha_3)+cos(2\eta_0+2\alpha_2)\,cos(2\eta_0+2\alpha_3)].$

$0=\|B\|^4\;+{B_2}^4+{B_3}^4+2\|B\|^2{B_2}^2[cos(2Re\alpha-2\alpha_2)]+2\|B\|^2{B_3}^2[cos(2Re\alpha-2\alpha_3)]$

$+2{B_2}^2{B_3}^2[cos(2\alpha_2-2\alpha_3)],$

because $cos(a-b)=cos(a)cos(b)+sin(a)sin(b)$.

$A_1=\xi_1\eta_0-\xi_0\eta_1,$

from (IX.3.),

$A_1=\frac{1}{2i}\left[Be^{-is}cos(\eta_0+\alpha)-\bar{B}e^{is}cos(\eta_0+\bar{\alpha})\right]\frac{1}{2}\left[Be^{-is}sin(\eta_0+\alpha)+\bar{B}e^{is}sin(\eta_0+\bar{\alpha})\right]$

$-\frac{1}{2}\left[Be^{-is}cos(\eta_0+\alpha)+\bar{B}e^{is}cos(\eta_0+\bar{\alpha})\right]\frac{1}{2i}\left[Be^{-is}sin(\eta_0+\alpha)-\bar{B}e^{is}sin(\eta_0+\bar{\alpha})\right],$

$A_1=\frac{1}{4i}\left[2Be^{-is}cos(\eta_0+\alpha)\bar{B}e^{is}sin(\eta_0+\bar{\alpha})-2Be^{-is}sin(\eta_0+\alpha)\bar{B}e^{is}cos(\eta_0+\bar{\alpha})\right],$

$A_1=\frac{1}{2i}[B\bar{B}sin(\eta_0+\bar{\alpha}-\eta_0-\alpha)],$

because $sinxcosy-cosxsiny=sin(x-y)$,

$A_1=\frac{1}{2i}[B\bar{B}sin(-i2Im\alpha)],$

$A_1=\frac{-1}{2}\|B\|^2 sinh(2Im\alpha),$

because $sin(ix)=isinh(x)$ for real x.

(ii) For $H>0$:

First, we observe

$[\xi_0+\xi_1+\eta_0+\eta_1]'=-\xi_1+\eta_0\eta_0'-\xi_0+\eta_1\eta_0'-\eta_1+\xi_0\eta_0'-\eta_0+\xi_1\eta_0'$

$[\xi_0+\xi_1+\eta_0+\eta_1]'=(\xi_0+\xi_1+\eta_0+\eta_1)(-1+\eta_0').$

Then we check to see if the solution holds

$[\xi_0+\xi_1+\eta_0+\eta_1]'=B_0e^{-s}[cosh(\eta_0+\alpha_0)+sinh(\eta_0+\alpha_0)],$

$[\xi_0+\xi_1+\eta_0+\eta_1]'=\left[B_0e^{-s}e^{(\eta_0+\alpha_0)}\right]',$

$[\xi_0+\xi_1+\eta_0+\eta_1]'=-B_0e^{-s}e^{(\eta_0+\alpha_0)}+B_0e^{-s}e^{(\eta_0+\alpha_0)}\eta_0',$

$[\xi_0+\xi_1+\eta_0+\eta_1]'=(\xi_0+\xi_1+\eta_0+\eta_1)(-1+\eta_0').$

Then we observe

$[\xi_0+\xi_1-(\eta_0+\eta_1)]'=-\xi_1+\eta_0\eta_0'-\xi_0+\eta_1\eta_0'+\eta_1-\xi_0\eta_0'+\eta_0-\xi_1\eta_0'$

$[\xi_0+\xi_1-(\eta_0+\eta_1)]'=(\xi_0+\xi_1-\eta_0-\eta_1)(-1-\eta_0').$

Then we check to see if the solution holds

$[\xi_0+\xi_1-(\eta_0+\eta_1)]'=B_0e^{-s}[cosh(\eta_0+\alpha_0)-sinh(\eta_0+\alpha_0)],$

$[\xi_0+\xi_1-(\eta_0+\eta_1)]'=\left[B_0e^{-s}e^{-(\eta_0+\alpha_0)}\right]',$

$[\xi_0+\xi_1-(\eta_0+\eta_1)]'=-B_0e^{-s}e^{-(\eta_0+\alpha_0)}-B_0e^{-s}e^{-(\eta_0+\alpha_0)}\eta_0',$

$[\xi_0+\xi_1-(\eta_0+\eta_1)]'=(\xi_0+\xi_1-\eta_0-\eta_1)(-1-\eta_0').$

Then we observe

$[\xi_0-\xi_1+\eta_0-\eta_1]'=-\xi_1+\eta_0\eta_0'+\xi_0-\eta_1\eta_0'-\eta_1+\xi_0\eta_0'+\eta_0-\xi_1\eta_0'$

$[\xi_0-\xi_1+\eta_0-\eta_1]'=(\xi_0-\xi_1+\eta_0-\eta_1)(1+\eta_0').$

Then we check to see if the solution holds

$[\xi_0-\xi_1+\eta_0-\eta_1]'=B_1e^{+s}[cosh(\eta_0+\alpha_1)+sinh(\eta_0+\alpha_1)],$

$[\xi_0-\xi_1+\eta_0-\eta_1]'=\left[B_1e^{+s}e^{(\eta_0+\alpha_1)}\right]',$

$[\xi_0-\xi_1+\eta_0-\eta_1]'=B_1e^{+s}e^{(\eta_0+\alpha_1)}+B_1e^{+s}e^{(\eta_0+\alpha_1)}\eta_0',$

$[\xi_0-\xi_1+\eta_0-\eta_1]'=(\xi_0-\xi_1+\eta_0-\eta_1)(1+\eta_0').$

Then we observe

$[\xi_0-\xi_1-(\eta_0-\eta_1)]'=-\xi_1+\eta_0\eta_0'+\xi_0-\eta_1\eta_0'+\eta_1-\xi_0\eta_0'-\eta_0+\xi_1\eta_0',$

$[\xi_0 - \xi_1 - (\eta_0 - \eta_1)]' = \big(\xi_0 - \xi_1 - (\eta_0 - \eta_1)\big)(1 - \eta_0')$.

Then we check to see if the solution holds

$[\xi_0 - \xi_1 - (\eta_0 - \eta_1)]' = B_1 e^{+s}[cosh(\eta_0 + \alpha_1) - sinh(\eta_0 + \alpha_1)]$,

$[\xi_0 - \xi_1 - (\eta_0 - \eta_1)]' = \big[B_1 e^{+s} e^{-(\eta_0 + \alpha_1)}\big]'$,

$[\xi_0 - \xi_1 - (\eta_0 - \eta_1)]' = B_1 e^{+s} e^{-(\eta_0 + \alpha_1)} - B_1 e^{+s} e^{-(\eta_0 + \alpha_1)} \eta_0'$,

$[\xi_0 - \xi_1 - (\eta_0 - \eta_1)]' = \big(\xi_0 - \xi_1 - (\eta_0 - \eta_1)\big)(1 - \eta_0')$.

This gives us

$\xi_0 = \frac{1}{2}[(\xi_0 + \xi_1) + (\xi_0 - \xi_1)]$,

$\xi_0 = \frac{1}{2}[B_0 e^{-s} cosh(\eta_0 + \alpha_0) + B_1 e^{+s} cosh(\eta_0 + \alpha_1)]$.

$\xi_1 = \frac{1}{2}[(\xi_0 + \xi_1) - (\xi_0 - \xi_1)]$,

$\xi_1 = \frac{1}{2}[B_0 e^{-s} cosh(\eta_0 + \alpha_0) - B_1 e^{+s} cosh(\eta_0 + \alpha_1)]$.

$\eta_0 = \frac{1}{2}[(\eta_0 + \eta_1) + (\eta_0 - \eta_1)]$,

$\eta_0 = \frac{1}{2}[B_0 e^{-s} sinh(\eta_0 + \alpha_0) + B_1 e^{+s} sinh(\eta_0 + \alpha_1)]$.

$\eta_1 = \frac{1}{2}[(\eta_0 + \eta_1) - (\eta_0 - \eta_1)]$,

$\eta_1 = \frac{1}{2}[B_0 e^{-s} sinh(\eta_0 + \alpha_0) - B_1 e^{+s} sinh(\eta_0 + \alpha_1)]$.

Then we observe

$[\xi_2 + \eta_2]' = \eta_2 \eta_0' + \xi_2 \eta_0'$,

$[\xi_2 + \eta_2]' = (\xi_2 + \eta_2)\eta_0'$.

Then we check to see if the solution holds

$[\xi_2 + \eta_2]' = B_2[cosh(\eta_0 + \alpha_2) + sinh(\eta_0 + \alpha_2)]'$,

$[\xi_2 + \eta_2]' = \big[B_2 e^{(\eta_0 + \alpha_2)}\big]'$,

$[\xi_2 + \eta_2]' = B_2 e^{(\eta_0 + \alpha_2)} \eta_0'$,

$[\xi_2 + \eta_2]' = (\xi_2 + \eta_2)\eta_0'$.

Then we observe

$[\xi_2 - \eta_2]' = \eta_2 \eta_0' - \xi_2 \eta_0'$,

$[\xi_2 - \eta_2]' = -(\xi_2 - \eta_2)\eta_0'$.

Then we check to see if the solution holds

$[\xi_2 - \eta_2]' = B_2[cosh(\eta_0 + \alpha_2) - sinh(\eta_0 + \alpha_2)]'$,

$[\xi_2 - \eta_2]' = \big[B_2 e^{-(\eta_0 + \alpha_2)}\big]'$,

$[\xi_2 - \eta_2]' = -B_2 e^{-(\eta_0 + \alpha_2)} \eta_0'$,

$[\xi_2 - \eta_2]' = -(\xi_2 - \eta_2)\eta_0'$.

ξ_3 and η_3 are analogous.

$\xi_2 = B_2 cosh(\eta_0 + \alpha_2)$.

$\xi_3 = B_3 cosh(\eta_0 + \alpha_3)$.

$\eta_2 = B_2 sinh(\eta_0 + \alpha_2).$

$\eta_3 = B_3 sinh(\eta_0 + \alpha_3).$

$1 = {\xi_0}^2 - {\xi_1}^2 - {\xi_2}^2 - {\xi_3}^2,$

from (IX.3.),

$1 = (\xi_0 + \xi_1)(\xi_0 - \xi_1) - {\xi_2}^2 - {\xi_3}^2,$

$1 = B_0 e^{-s} cosh(\eta_0 + \alpha_0) B_1 e^s cosh(\eta_0 + \alpha_1) - {B_2}^2 cosh^2(\eta_0 + \alpha_2) - {B_3}^2 cosh^2(\eta_0 + \alpha_3),$

$1 = B_0 B_1 cosh(\eta_0 + \alpha_0) cosh(\eta_0 + \alpha_1) - {B_2}^2 cosh^2(\eta_0 + \alpha_2) - {B_3}^2 cosh^2(\eta_0 + \alpha_3),$

$-1 = {\eta_0}^2 - {\eta_1}^2 - {\eta_2}^2 - {\eta_3}^2,$

from (IX.3.),

$-1 = (\eta_0 + \eta_1)(\eta_0 - \eta_1) - {\eta_2}^2 - {\eta_3}^2,$

$-1 = B_0 e^{-s} sinh(\eta_0 + \alpha_0) B_1 e^s sinh(\eta_0 + \alpha_1) - {B_2}^2 sinh^2(\eta_0 + \alpha_2) - {B_3}^2 sinh^2(\eta_0 + \alpha_3)$

$1 = -B_0 B_1 sinh(\eta_0 + \alpha_0) sinh(\eta_0 + \alpha_1) + {B_2}^2 sinh^2(\eta_0 + \alpha_2) + {B_3}^2 sinh^2(\eta_0 + \alpha_3)$

$2 = B_0 B_1 [cosh(\eta_0 + \alpha_0) cosh(\eta_0 + \alpha_1) - sinh(\eta_0 + \alpha_0) sinh(\eta_0 + \alpha_1)]$

$-{B_2}^2 [cosh^2(\eta_0 + \alpha_2) - sinh^2(\eta_0 + \alpha_2)] - {B_3}^2 [cosh^2(\eta_0 + \alpha_3) - sinh^2(\eta_0 + \alpha_3)],$

$2 = B_0 B_1 \cosh(\alpha_0 - \alpha_1) - {B_2}^2 - {B_3}^2,$

because $coshxcoshy - sinhxsinhy = cosh(x - y)$ and $cosh^2 x - sinh^2 x = 1$.

$0 = \xi_0 \eta_0 - \xi_1 \eta_1 - \xi_2 \eta_2 - \xi_3 \eta_3,$

from (IX.3.),

$0 = \frac{1}{2}[B_0 e^{-s} cosh(\eta_0 + \alpha_0) + B_1 e^{+s} cosh(\eta_0 + \alpha_1)]$

$* \frac{1}{2}[B_0 e^{-s} sinh(\eta_0 + \alpha_0) + B_1 e^{+s} sinh(\eta_0 + \alpha_1)]$

$-\frac{1}{2}[B_0 e^{-s} cosh(\eta_0 + \alpha_0) - B_1 e^{+s} cosh(\eta_0 + \alpha_1)]$

$* \frac{1}{2}[B_0 e^{-s} sinh(\eta_0 + \alpha_0) - B_1 e^{+s} sinh(\eta_0 + \alpha_1)]$

$-B_2 cosh(\eta_0 + \alpha_2) B_2 sinh(\eta_0 + \alpha_2) - B_3 cosh(\eta_0 + \alpha_3) B_3 sinh(\eta_0 + \alpha_3),$

$0 = \frac{1}{4}{B_0}^2 e^{-2s} sinh(\eta_0 + \alpha_0) cosh(\eta_0 + \alpha_0)$

$+\frac{1}{4}{B_1}^2 e^{2s} sinh(\eta_0 + \alpha_1) cosh(\eta_0 + \alpha_1)$

$+\frac{1}{4} B_0 B_1 [sinh(\eta_0 + \alpha_1) cosh(\eta_0 + \alpha_0) + sinh(\eta_0 + \alpha_0) cosh(\eta_0 + \alpha_1)]$

$-{B_2}^2 sinh(\eta_0 + \alpha_2) cosh(\eta_0 + \alpha_2) - {B_3}^2 sinh(\eta_0 + \alpha_3) cosh(\eta_0 + \alpha_3),$

$-\frac{1}{4}{B_0}^2 e^{-2s} sinh(\eta_0 + \alpha_0) cosh(\eta_0 + \alpha_0)$

$-\frac{1}{4}{B_1}^2 e^{2s} sinh(\eta_0 + \alpha_1) cosh(\eta_0 + \alpha_1)$

$+\frac{1}{4}B_0B_1[sinh(\eta_0+\alpha_1)cosh(\eta_0+\alpha_0)+sinh(\eta_0+\alpha_0)cosh(\eta_0+\alpha_1)],$

$0=\frac{1}{2}B_0B_1[sinh(\eta_0+\alpha_1)cosh(\eta_0+\alpha_0)+sinh(\eta_0+\alpha_0)cosh(\eta_0+\alpha_1)]$

$-\frac{1}{4}B_2{}^2 sinh(\eta_0+\alpha_2)cosh(\eta_0+\alpha_2)-\frac{1}{4}B_3{}^2 sinh(\eta_0+\alpha_3)cosh(\eta_0+\alpha_3),$

$0=\frac{1}{2}B_0B_1 sinh(2\eta_0+\alpha_0+\alpha_1)-\frac{1}{2}B_2{}^2 sinh(2\eta_0+2\alpha_2)-\frac{1}{2}B_3{}^2 sinh(2\eta_0+2\alpha_3),$

because $sinh(a+b)=sinh(a)cosh(b)+sinh(b)cosh(a)$and $sinh(2a)=2sinh(a)cosh(a),$

Multiplying by 2 and taking the square, we get:

$(*)$: $0=B_0{}^2B_1{}^2 sinh^2(2\eta_0+\alpha_0+\alpha_1)+B_2{}^4 sinh^2(2\eta_0+2\alpha_2)+B_3{}^4 sinh^2(2\eta_0+2\alpha_3)$

$-2B_0B_1B_2{}^2 sinh(2\eta_0+\alpha_0+\alpha_1)sinh(2\eta_0+2\alpha_2)$

$-2B_0B_1B_3{}^2 sinh(2\eta_0+\alpha_0+\alpha_1)sinh(2\eta_0+2\alpha_3)$

$+2B_2{}^2B_3{}^2 sinh(2\eta_0+2\alpha_2)sinh(2\eta_0+2\alpha_3),$

Now we'll save this equation for the moment, and create another equation for zero by subtracting the two expressions for one from each other:

$0=B_0B_1[cosh(\eta_0+\alpha_0)cosh(\eta_0+\alpha_1)+sinh(\eta_0+\alpha_0)sinh(\eta_0+\alpha_1)]$

$-B_2{}^2[cosh^2(\eta_0+\alpha_2)+sinh^2(\eta_0+\alpha_2)]-B_3{}^2[cosh^2(\eta_0+\alpha_3)+sinh^2(\eta_0+\alpha_3)],$

$0=B_0B_1 cosh(2\eta_0+\alpha_0+\alpha_1)-B_2{}^2 cosh(2\eta_0+2\alpha_2)-B_2{}^2 cosh(2\eta_0+2\alpha_2),$

because $cosh(a+b)=cosh(a)cosh(b)+sinh(a)sinh(b)$

and $cosh(2a)=cosh^2(a)+sinh^2(a)$.

By taking the square, we get:

$(**)$: $0=B_0{}^2B_1{}^2 cosh^2(2\eta_0+\alpha_0+\alpha_1)+B_2{}^4 cosh^2(2\eta_0+2\alpha_2)+B_3{}^4 cosh^2(2\eta_0+2\alpha_3)$

$-2B_0B_1B_2{}^2 cosh(2\eta_0+\alpha_0+\alpha_1)cosh(2\eta_0+2\alpha_2)$

$-2B_0B_1B_3{}^2 cosh(2\eta_0+\alpha_0+\alpha_1)cosh(2\eta_0+2\alpha_3)$

$-2B_2{}^2B_3{}^2 cosh(2\eta_0+2\alpha_2)cosh(2\eta_0+2\alpha_3).$

Now we can subtract the expression (*) from (**) to get:

$0=B_0{}^2B_1{}^2[-sinh^2(2\eta_0+\alpha_0+\alpha_1)+cosh^2(2\eta_0+\alpha_0+\alpha_1)]$

$+B_2{}^4[-sinh^2(2\eta_0+2\alpha_2)+cosh^2(2\eta_0+2\alpha_2)]$

$+B_3{}^4[-sinh^2(2\eta_0+2\alpha_3)+cosh^2(2\eta_0+2\alpha_3)]$

$-2B_0B_1B_2{}^2[-sinh(2\eta_0+\alpha_0+\alpha_1)sinh(2\eta_0+2\alpha_2)]$

$-2B_0B_1B_2{}^2[cosh(2\eta_0+\alpha_0+\alpha_1)cosh(2\eta_0+2\alpha_2)]$

$-2B_0B_1B_3{}^2[-sinh(2\eta_0+\alpha_0+\alpha_1)sinh(2\eta_0+2\alpha_3)]$

$-2B_0B_1B_3{}^2[cosh(2\eta_0+\alpha_0+\alpha_1)cosh(2\eta_0+2\alpha_3)]$

$+2B_2{}^2B_3{}^2[-sinh(2\eta_0+2\alpha_2)sinh(2\eta_0+2\alpha_3)+cosh(2\eta_0+2\alpha_2)cosh(2\eta_0+2\alpha_3)],$

$0=B_0{}^2B_1{}^2+B_2{}^4+B_3{}^4-2B_0B_1B_2{}^2 cosh(\alpha_0+\alpha_1-2\alpha_2)-2B_0B_1B_3{}^2 cosh(\alpha_0+\alpha_1-2\alpha_3)$

$+2B_2{}^2B_3{}^2 cosh(2\alpha_2-2\alpha_3),$

because $cosh^2(a)-sinh^2(a)=1$ and $cosh(a-b)=cosh(a)cosh(b)-sinh(a)sinh(b)$.

$A_1 = \xi_0\eta_1 - \xi_1\eta_0$,

from (IX.3.),

$$A_1 = \frac{1}{2}[B_0e^{-s}cosh(\eta_0 + \alpha_0) + B_1e^{+s}cosh(\eta_0 + \alpha_1)]$$
$$* \frac{1}{2}[B_0e^{-s}sinh(\eta_0 + \alpha_0) - B_1e^{+s}sinh(\eta_0 + \alpha_1)]$$
$$-\frac{1}{2}[B_0e^{-s}cosh(\eta_0 + \alpha_0) - B_1e^{+s}cosh(\eta_0 + \alpha_1)]$$
$$* \frac{1}{2}[B_0e^{-s}sinh(\eta_0 + \alpha_0) + B_1e^{+s}sinh(\eta_0 + \alpha_1)],$$

$$A_1 = \frac{1}{4}B_0B_1[-2cosh(\eta_0 + \alpha_0)e^{+s}sinh(\eta_0 + \alpha_1) + 2cosh(\eta_0 + \alpha_1)e^{+s}sinh(\eta_0 + \alpha_0)],$$

$$A_1 = \frac{1}{2}sinh(\alpha_0 - \alpha_1),$$

because $sinhxcoshy - sinhycoshx = sinh(x - y)$. ■

Remark. Because of $\xi_0 \geq 1$ and $\xi_0^2 - \xi_1^2 \geq 1$, we have $\xi_0 + \xi_1 \geq 0$. Then (IX.5.) (ii) leads to $B_0 \geq 0$ and $B_1 \geq 0$.

(IX.6.) Proof.

For $H < 0$, the equation is:

$$\eta_0 = (1/2)\big[Be^{-is}\sin(\eta_0 + \alpha) + \bar{B}e^{+is}\sin(\eta_0 + \bar{\alpha})\big].$$

For fixed *s*, we will search for the intersection of the two functions that are on the left and the right of the equality sign. The left side is the straight line through zero with a slope of $+1$, and the right side is a periodic, bounded function of η_0. Therefore, there is at least one intersection. The uniqueness stems from the fact that the slope of the function on the right side is always ≤ 1. If we denote this slope with *S*, the we have:

$$S = (1/2)\big[Be^{-is}cos(\eta_0 + \alpha) + \bar{B}e^{+is}cos(\eta_0 + \bar{\alpha})\big] \Rightarrow$$

$$S^2 = (1/4)\begin{bmatrix} \|B\|^2cos(\eta_0 + \alpha)cos(\eta_0 + \bar{\alpha}) + \|B\|^2cos(\eta_0 + \bar{\alpha})cos(\eta_0 + \alpha) \\ +B^2e^{-2is}cos^2(\eta_0 + \alpha) + \bar{B}^2e^{+2is}cos^2(\eta_0 + \bar{\alpha}) \end{bmatrix}$$

$$S^2 = (1/4)[2(\xi_0 + i\xi_1)(\xi_0 - \xi_1) + (\xi_0 + i\xi_1)^2 + (\xi_0 - i\xi_1)^2] = \xi_0^2 \leq 1.$$

(ii) For $H > 0$, the equation is:

$$\eta_0 = (1/2)[B_0e^{-s}\sinh(\eta_0 + \alpha_0) + B_1e^{+s}\sinh(\eta_0 + \alpha_1)].$$

This time, the right side is a function that behaves like

$e^{+\eta_0}$ (resp. -$e^{-\eta_0}$) for $\eta_0 \gg 0$ (resp. $\eta_0 \ll 0$), since $B_0 \geq 0$ and $B_1 \geq 0$.

As a result, there is at least one intersection. The uniqueness stems from the fact that the slope of the function on the right side is always ≥ 1. If we denote this slope with *S*, the we have:

$$S = (1/2)[B_0e^{-s}\cosh(\eta_0 + \alpha_0) + B_1e^{+s}\cosh(\eta_0 + \alpha_1)].$$

$$S^2 = (1/4)\begin{bmatrix} B_0^2e^{-2s}\cosh^2(\eta_0 + \alpha_0) + B_1^2e^{+2s}\cosh^2(\eta_0 + \alpha_1) \\ +2B_0B_1\cosh(\eta_0 + \alpha_0)\cosh(\eta_0 + \alpha_1) \end{bmatrix}$$

$$S^2 = (1/4)[(\xi_0 + \xi_1)^2 + (\xi_0 - \xi_1)^2 + 2(\xi_0 + \xi_1)(\xi_0 - \xi_1)] = \xi_0^2 \geq 1.$$

In both cases, the slope can only have a value of 1 at isolated points, since the function is analytical. ■

(IX.8.) Proof.

For $H < 0$:

$$\sum_{i=0}^{3} x_i^2 = \left[\left(\frac{\|\mathbf{q}\|\|\mathbf{p}\|^2}{\mu} - 1\right)\cos\psi + \sqrt{\frac{-2H}{\mu}}(\mathbf{q}\cdot\mathbf{p})\sin\psi\right]^2 + \left[\sqrt{\frac{-2H}{\mu}}\|\boldsymbol{q}\|\boldsymbol{p}\,cos\psi + \left(\frac{\boldsymbol{q}}{\|\boldsymbol{q}\|} - \frac{(\boldsymbol{q}\cdot\boldsymbol{p})\boldsymbol{p}}{\mu}\right)sin\psi\right]^2,$$

$$\sum_{i=0}^{3} x_i^2 = \left[\frac{\|\mathbf{q}\|^2\|\mathbf{p}\|^4}{\mu^2} - \frac{2\|\mathbf{q}\|\|\mathbf{p}\|^2}{\mu} + 1 + \left(\frac{-2H}{\mu}\right)\|\boldsymbol{q}\|^2\|\boldsymbol{p}\|^2\right]cos^2\psi$$

$$+2\sqrt{\frac{-2H}{\mu}}\left[\frac{\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu} - (\boldsymbol{q}\cdot\boldsymbol{p}) + (\boldsymbol{q}\cdot\boldsymbol{p}) - \frac{\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right]cos\psi sin\psi$$

$$+\left[\left(\frac{-2H}{\mu}\right)(\boldsymbol{q}\cdot\boldsymbol{p})^2 + 1 - \frac{2(\boldsymbol{q}\cdot\boldsymbol{p})^2}{\|\boldsymbol{q}\|\mu} + \frac{(\boldsymbol{q}\cdot\boldsymbol{p})^2\|\boldsymbol{p}\|^2}{\mu^2}\right]sin^2\psi,$$

from (IX.7.),

$$\sum_{i=0}^{3} x_i^2 = \left[\left(\frac{2H}{\mu}\right)\|\boldsymbol{q}\|^2\|\boldsymbol{p}\|^2 + 1 + \left(\frac{-2H}{\mu}\right)\|\boldsymbol{q}\|^2\|\boldsymbol{p}\|^2\right]cos^2\psi$$

$$+\left[\left(\frac{-2H}{\mu}\right)(\boldsymbol{q}\cdot\boldsymbol{p})^2 + 1 + \left(\frac{2H}{\mu}\right)(\boldsymbol{q}\cdot\boldsymbol{p})^2\right]sin^2\psi,$$

from the definition of H,

$$\sum_{i=0}^{3} x_i^2 = cos^2\psi + sin^2\psi,$$

$$\sum_{i=0}^{3} x_i^2 = 1.$$

$$\sum_{i=0}^{3} x_i y_i = \left[\left(\frac{\|\boldsymbol{q}\|\cdot\|\boldsymbol{p}\|^2}{\mu} - 1\right)cos\psi + \sqrt{\frac{-2H}{\mu}}(\boldsymbol{q}\cdot\boldsymbol{p})sin\psi\right]\left[(\boldsymbol{q}\cdot\boldsymbol{p})cos\psi - \sqrt{\frac{\mu}{-2H}}\left(\frac{\|\boldsymbol{q}\|\cdot\|\boldsymbol{p}\|^2}{\mu} - 1\right)sin\psi\right]$$

$$+\left[\sqrt{\frac{-2H}{\mu}}\|\boldsymbol{q}\|\boldsymbol{p}\,cos\psi + \left(\frac{\boldsymbol{q}}{\|\boldsymbol{q}\|} - \frac{(\boldsymbol{q}\cdot\boldsymbol{p})\boldsymbol{p}}{\mu}\right)sin\psi\right]\left[\sqrt{\frac{\mu}{-2H}}\left(\frac{\boldsymbol{q}}{\|\boldsymbol{q}\|} - \frac{(\boldsymbol{q}\cdot\boldsymbol{p})\boldsymbol{p}}{\mu}\right)cos\psi - (\|\boldsymbol{q}\|\boldsymbol{p})\,sin\psi\right],$$

$$\sum_{i=0}^{3} x_i y_i = \left[\frac{\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu} - (\boldsymbol{q}\cdot\boldsymbol{p}) + (\boldsymbol{q}\cdot\boldsymbol{p}) - \frac{\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right]cos^2\psi$$

$$+\begin{bmatrix} -\sqrt{\frac{\mu}{-2H}}\left(\frac{\|\boldsymbol{q}\|\cdot\|\boldsymbol{p}\|^2}{\mu} - 1\right)^2 + \sqrt{\frac{-2H}{\mu}}(\boldsymbol{q}\cdot\boldsymbol{p})^2 \\ -\sqrt{\frac{-2H}{\mu}}\|\boldsymbol{q}\|^2\|\boldsymbol{p}\|^2 + \sqrt{\frac{\mu}{-2H}}\left(\frac{\boldsymbol{q}}{\|\boldsymbol{q}\|} - \frac{(\boldsymbol{q}\cdot\boldsymbol{p})\boldsymbol{p}}{\mu}\right)^2 \end{bmatrix} cos\psi sin\psi$$

$$+\left[-\frac{\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu} + (\boldsymbol{q}\cdot\boldsymbol{p}) - (\boldsymbol{q}\cdot\boldsymbol{p}) + \frac{\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right]sin^2\psi,$$

$$\sum_{i=0}^{3} x_i y_i = \begin{bmatrix} -\sqrt{\frac{\mu}{-2H}}\left(\frac{\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2}{\mu} - 1\right)^2 + \sqrt{\frac{-2H}{\mu}}(\boldsymbol{q}\cdot\boldsymbol{p})^2 \\ -\sqrt{\frac{-2H}{\mu}}\|\boldsymbol{q}\|^2\|\boldsymbol{p}\|^2 + \sqrt{\frac{\mu}{-2H}}\left(\frac{\boldsymbol{q}}{\|\boldsymbol{q}\|} - \frac{(\boldsymbol{q}\cdot\boldsymbol{p})\boldsymbol{p}}{\mu}\right)^2 \end{bmatrix} cos\psi sin\psi,$$

$$\sum_{i=0}^{3} x_i y_i = \begin{bmatrix} \sqrt{\frac{\mu}{-2H}}\left(-\frac{\|\boldsymbol{q}\|^2\|\boldsymbol{p}\|^4}{\mu^2} + \frac{2\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2}{\mu} - 1 + 1 - \frac{2(\boldsymbol{q}\cdot\boldsymbol{p})^2}{\|\boldsymbol{q}\|\mu} + \frac{(\boldsymbol{q}\cdot\boldsymbol{p})^2\|\boldsymbol{p}\|^2}{\mu^2}\right) \\ +\sqrt{\frac{-2H}{\mu}}\left((\boldsymbol{q}\cdot\boldsymbol{p})^2 - \|\boldsymbol{q}\|^2\|\boldsymbol{p}\|^2\right) \end{bmatrix} cos\psi sin\psi,$$

$$\sum_{i=0}^{3} x_i y_i = \begin{bmatrix} \sqrt{\frac{\mu}{-2H}}\left(\left(\frac{-2H}{\mu}\right)\|\boldsymbol{q}\|^2\|\boldsymbol{p}\|^2 - \left(\frac{-2H}{\mu}\right)(\boldsymbol{q}\cdot\boldsymbol{p})^2\right) \\ +\sqrt{\frac{-2H}{\mu}}\left((\boldsymbol{q}\cdot\boldsymbol{p})^2 - \|\boldsymbol{q}\|^2\|\boldsymbol{p}\|^2\right) \end{bmatrix} cos\psi sin\psi,$$

$$\sum_{i=0}^{3} x_i y_i = \left[\sqrt{\frac{-2H}{\mu}}\left(\|\boldsymbol{q}\|^2\|\boldsymbol{p}\|^2 - (\boldsymbol{q}\cdot\boldsymbol{p})^2\right) + \sqrt{\frac{-2H}{\mu}}\left((\boldsymbol{q}\cdot\boldsymbol{p})^2 - \|\boldsymbol{q}\|^2\|\boldsymbol{p}\|^2\right)\right] cos\psi sin\psi,$$

$$\sum_{i=0}^{3} x_i y_i = 0.$$

$$\sum_{i=0}^{3} y_i^2 = \left[(\mathbf{q}\cdot\mathbf{p})\cos\psi - \sqrt{\frac{\mu}{-2H}}\left(\frac{\|\mathbf{q}\|\|\mathbf{p}\|^2}{\mu} - 1\right)\sin\psi\right]^2$$

$$+\left[\sqrt{\frac{\mu}{-2H}}\left(\frac{\boldsymbol{q}}{\|\boldsymbol{q}\|} - \frac{(\boldsymbol{q}\cdot\boldsymbol{p})\boldsymbol{p}}{\mu}\right)cos\psi - (\|\boldsymbol{q}\|\boldsymbol{p})\, sin\psi\right]^2,$$

$$\sum_{i=0}^{3} y_i^2 = \left[(\boldsymbol{q}\cdot\boldsymbol{p})^2 + \left(\frac{\mu}{-2H}\right)\left(1 - \frac{2(\boldsymbol{q}\cdot\boldsymbol{p})^2}{\|\boldsymbol{q}\|\mu} + \frac{(\boldsymbol{q}\cdot\boldsymbol{p})^2\|\boldsymbol{p}\|^2}{\mu^2}\right)\right] cos^2\psi$$

$$+\left[-2\sqrt{\frac{\mu}{-2H}}\left(\frac{\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu} - (\boldsymbol{q}\cdot\boldsymbol{p}) + (\boldsymbol{q}\cdot\boldsymbol{p}) - \frac{\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right)\right] cos\psi sin\psi$$

$$+\left[\left(\frac{\mu}{-2H}\right)\left(\frac{\|\boldsymbol{q}\|^2\|\boldsymbol{p}\|^4}{\mu^2} - \frac{2\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2}{\mu} + 1\right) + \|\boldsymbol{q}\|^2\|\boldsymbol{p}\|^2\right] sin^2\psi,$$

$$\sum_{i=0}^{3} y_i^2 = \left[(\boldsymbol{q}\cdot\boldsymbol{p})^2 + \left(\frac{\mu}{-2H}\right)\left(1 + \left(\frac{2H}{\mu}\right)(\boldsymbol{q}\cdot\boldsymbol{p})^2\right)\right] cos^2\psi$$

$$+\left[\left(\frac{\mu}{-2H}\right)\left(\frac{2H}{\mu}\|\boldsymbol{q}\|^2\|\boldsymbol{p}\|^2 + 1\right) + \|\boldsymbol{q}\|^2\|\boldsymbol{p}\|^2\right] sin^2\psi,$$

$$\sum_{i=0}^{3} y_i^2 = \left[\left(\frac{\mu}{-2H}\right)\right] cos^2\psi + \left[\left(\frac{\mu}{-2H}\right)\right] sin^2\psi,$$

$$\sum_{i=0}^{3} y_i^2 = \frac{\mu}{-2H}.$$

For $H > 0$:

$$x_0^2 - \sum_{i=1}^{3} x_i^2 = \left[\left(\frac{\|\mathbf{q}\|\|\mathbf{p}\|^2}{\mu} - 1\right)\cosh\psi - \sqrt{\frac{2H}{\mu}}(\mathbf{q}\cdot\mathbf{p})\sinh\psi\right]^2$$

$$-\left[\sqrt{\frac{2H}{\mu}}\|\boldsymbol{q}\|\boldsymbol{p}\, cosh\psi + \left(\frac{\boldsymbol{q}}{\|\boldsymbol{q}\|} - \frac{(\boldsymbol{q}\cdot\boldsymbol{p})\boldsymbol{p}}{\mu}\right) sinh\psi\right]^2,$$

$$x_0^2 - \sum_{i=1}^{3} x_i^2 = \left[\frac{\|\mathbf{q}\|^2\|\mathbf{p}\|^4}{\mu^2} - \frac{2\|\mathbf{q}\|\|\mathbf{p}\|^2}{\mu} + 1 - \left(\frac{2H}{\mu}\right)\|\boldsymbol{q}\|^2\|\boldsymbol{p}\|^2\right] cosh^2\psi$$

$$+2\sqrt{\frac{2H}{\mu}}\left[-\frac{\|\mathbf{q}\|\|\mathbf{p}\|^2(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu} + (\boldsymbol{q}\cdot\boldsymbol{p}) - (\boldsymbol{q}\cdot\boldsymbol{p}) + \frac{\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right] cosh\psi sinh\psi$$

$$+\left[\left(\frac{2H}{\mu}\right)(\boldsymbol{q}\cdot\boldsymbol{p})^2 - 1 + \frac{2(\boldsymbol{q}\cdot\boldsymbol{p})^2}{\|\boldsymbol{q}\|\mu} - \frac{(\boldsymbol{q}\cdot\boldsymbol{p})^2\|\boldsymbol{p}\|^2}{\mu^2}\right] sinh^2\psi,$$

from (IX.7.),

$$x_0^2 - \sum_{i=1}^{3} x_i^2 = \left[\left(\frac{2H}{\mu}\right)\|\boldsymbol{q}\|^2\|\boldsymbol{p}\|^2 + 1 - \left(\frac{2H}{\mu}\right)\|\boldsymbol{q}\|^2\|\boldsymbol{p}\|^2\right] cosh^2\psi$$

$$+\left[\left(\frac{2H}{\mu}\right)(\boldsymbol{q}\cdot\boldsymbol{p})^2 - 1 - \left(\frac{2H}{\mu}\right)(\boldsymbol{q}\cdot\boldsymbol{p})^2\right] sinh^2\psi,$$

from the definition of H,

$$x_0^2 - \sum_{i=1}^{3} x_i^2 = cosh^2\psi - sinh^2\psi,$$
$$x_0^2 - \sum_{i=1}^{3} x_i^2 = 1.$$

$$x_0 y_0 - \sum_{i=1}^{3} x_i y_i = \left[\left(\frac{\|\boldsymbol{q}\|\cdot\|\boldsymbol{p}\|^2}{\mu} - 1\right) cosh\psi - \sqrt{\frac{2H}{\mu}}(\boldsymbol{q}\cdot\boldsymbol{p}) sinh\psi\right]$$
$$*\left[(\boldsymbol{q}\cdot\boldsymbol{p}) cosh\psi - \sqrt{\frac{\mu}{2H}}\left(\frac{\|\boldsymbol{q}\|\cdot\|\boldsymbol{p}\|^2}{\mu} - 1\right) sinh\psi\right]$$
$$-\left[\sqrt{\frac{2H}{\mu}}\|\boldsymbol{q}\|\boldsymbol{p}\ cosh\psi + \left(\frac{\boldsymbol{q}}{\|\boldsymbol{q}\|} - \frac{(\boldsymbol{q}\cdot\boldsymbol{p})\boldsymbol{p}}{\mu}\right) sinh\psi\right]$$
$$*\left[-\sqrt{\frac{\mu}{2H}}\left(\frac{\boldsymbol{q}}{\|\boldsymbol{q}\|} - \frac{(\boldsymbol{q}\cdot\boldsymbol{p})\boldsymbol{p}}{\mu}\right) cosh\psi - (\|\boldsymbol{q}\|\boldsymbol{p})\ sinh\psi\right],$$
$$x_0 y_0 - \sum_{i=1}^{3} x_i y_i = \left[\frac{\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu} - (\boldsymbol{q}\cdot\boldsymbol{p}) + (\boldsymbol{q}\cdot\boldsymbol{p}) - \frac{\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right] cosh^2\psi$$
$$+\left[-\sqrt{\frac{\mu}{2H}}\left(\frac{\|\boldsymbol{q}\|\cdot\|\boldsymbol{p}\|^2}{\mu} - 1\right)^2 - \sqrt{\frac{2H}{\mu}}(\boldsymbol{q}\cdot\boldsymbol{p})^2 + \sqrt{\frac{2H}{\mu}}\|\boldsymbol{q}\|^2\|\boldsymbol{p}\|^2 + \sqrt{\frac{\mu}{2H}}\left(\frac{\boldsymbol{q}}{\|\boldsymbol{q}\|} - \frac{(\boldsymbol{q}\cdot\boldsymbol{p})\boldsymbol{p}}{\mu}\right)^2\right] cosh\psi sinh\psi$$
$$+\left[+\frac{\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu} - (\boldsymbol{q}\cdot\boldsymbol{p}) + (\boldsymbol{q}\cdot\boldsymbol{p}) - \frac{\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right] sinh^2\psi,$$
$$x_0 y_0 - \sum_{i=1}^{3} x_i y_i = \begin{bmatrix} -\sqrt{\frac{\mu}{2H}}\left(\frac{\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2}{\mu} - 1\right)^2 - \sqrt{\frac{2H}{\mu}}(\boldsymbol{q}\cdot\boldsymbol{p})^2 \\ +\sqrt{\frac{2H}{\mu}}\|\boldsymbol{q}\|^2\|\boldsymbol{p}\|^2 + \sqrt{\frac{\mu}{2H}}\left(\frac{\boldsymbol{q}}{\|\boldsymbol{q}\|} - \frac{(\boldsymbol{q}\cdot\boldsymbol{p})\boldsymbol{p}}{\mu}\right)^2 \end{bmatrix} cosh\psi sinh\psi,$$
$$x_0 y_0 - \sum_{i=1}^{3} x_i y_i = \begin{bmatrix} \sqrt{\frac{\mu}{2H}}\left(-\frac{\|\boldsymbol{q}\|^2\|\boldsymbol{p}\|^4}{\mu^2} + \frac{2\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2}{\mu} - 1 + 1 - \frac{2(\boldsymbol{q}\cdot\boldsymbol{p})^2}{\|\boldsymbol{q}\|\mu} + \frac{(\boldsymbol{q}\cdot\boldsymbol{p})^2\|\boldsymbol{p}\|^2}{\mu^2}\right) \\ +\sqrt{\frac{2H}{\mu}}(-(\boldsymbol{q}\cdot\boldsymbol{p})^2 + \|\boldsymbol{q}\|^2\|\boldsymbol{p}\|^2) \end{bmatrix} cosh\psi sinh\psi,$$
$$x_0 y_0 - \sum_{i=1}^{3} x_i y_i = \begin{bmatrix} \sqrt{\frac{\mu}{2H}}\left(-\left(\frac{2H}{\mu}\right)\|\boldsymbol{q}\|^2\|\boldsymbol{p}\|^2 + \left(\frac{2H}{\mu}\right)(\boldsymbol{q}\cdot\boldsymbol{p})^2\right) \\ +\sqrt{\frac{2H}{\mu}}(-(\boldsymbol{q}\cdot\boldsymbol{p})^2 + \|\boldsymbol{q}\|^2\|\boldsymbol{p}\|^2) \end{bmatrix} cosh\psi sinh\psi,$$
$$x_0 y_0 - \sum_{i=1}^{3} x_i y_i = \left[\sqrt{\frac{2H}{\mu}}(-\|\boldsymbol{q}\|^2\|\boldsymbol{p}\|^2 + (\boldsymbol{q}\cdot\boldsymbol{p})^2) + \sqrt{\frac{2H}{\mu}}(-(\boldsymbol{q}\cdot\boldsymbol{p})^2 + \|\boldsymbol{q}\|^2\|\boldsymbol{p}\|^2)\right] cosh\psi sinh\psi,$$
$$x_0 y_0 - \sum_{i=1}^{3} x_i y_i = 0.$$

$$\mathrm{y}_0^2 - \sum_{\mathrm{i}=1}^{3} \mathrm{y}_\mathrm{i}^2 = \left[(\mathbf{q}\cdot\mathbf{p})\cosh\psi - \sqrt{\frac{\mu}{2\mathrm{H}}}\left(\frac{\|\mathbf{q}\|\|\mathbf{p}\|^2}{\mu} - 1\right)\sinh\psi\right]^2$$
$$-\left[-\sqrt{\frac{\mu}{2H}}\left(\frac{\boldsymbol{q}}{\|\boldsymbol{q}\|} - \frac{(\boldsymbol{q}\cdot\boldsymbol{p})\boldsymbol{p}}{\mu}\right) cosh\psi - (\|\boldsymbol{q}\|\boldsymbol{p})\ sinh\psi\right]^2,$$
$$\mathrm{y}_0^2 - \sum_{\mathrm{i}=1}^{3} \mathrm{y}_\mathrm{i}^2 = \left[(\boldsymbol{q}\cdot\boldsymbol{p})^2 - \left(\frac{\mu}{2H}\right)\left(1 - \frac{2(\boldsymbol{q}\cdot\boldsymbol{p})^2}{\|\boldsymbol{q}\|\mu} + \frac{(\boldsymbol{q}\cdot\boldsymbol{p})^2\|\boldsymbol{p}\|^2}{\mu^2}\right)\right] cosh^2\psi$$
$$+\left[-2\sqrt{\frac{\mu}{2\mathrm{H}}}\left(\frac{\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu} - (\boldsymbol{q}\cdot\boldsymbol{p}) + (\boldsymbol{q}\cdot\boldsymbol{p}) - \frac{\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right)\right] cosh\psi sinh\psi$$
$$+\left[\left(\frac{\mu}{2H}\right)\left(\frac{\|\boldsymbol{q}\|^2\|\boldsymbol{p}\|^4}{\mu^2} - \frac{2\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2}{\mu} + 1\right) - \|\boldsymbol{q}\|^2\|\boldsymbol{p}\|^2\right] sinh^2\psi,$$

$$y_0^2 - \sum_{i=1}^{3} y_i^2 = \left[(\boldsymbol{q}\cdot\boldsymbol{p})^2 - \left(\frac{\mu}{2H}\right)\left(1 + \left(\frac{2H}{\mu}\right)(\boldsymbol{q}\cdot\boldsymbol{p})^2\right)\right] cosh^2\psi$$
$$+\left[\left(\frac{\mu}{2H}\right)\left(\frac{2H}{\mu}\|\boldsymbol{q}\|^2\|\boldsymbol{p}\|^2 + 1\right) - \|\boldsymbol{q}\|^2\|\boldsymbol{p}\|^2\right] sinh^2\psi,$$
$$y_0^2 - \sum_{i=1}^{3} y_i^2 = \left[-\left(\frac{\mu}{2H}\right)\right] cosh^2\psi + \left[\left(\frac{\mu}{2H}\right)\right] sinh^2\psi,$$
$$y_0^2 - \sum_{i=1}^{3} y_i^2 = -\frac{\mu}{2H}. \blacksquare$$

(IX.9.) Proof. This is largely analogous to (IX.4.), but there are two significant differences. First, we have no inverse to the mapping, so we will not express $\boldsymbol{q}$ and $\boldsymbol{p}$ in terms of ξ and η, and, second, the differences between $H<0$ and $H>0$ are more than just signs, so we will not use the variable σ in the proof.

(i) Along an integral curve of M_1:

Intermediate results for $H < 0$:

$$A_1 = \frac{q_1}{\|\boldsymbol{q}\|} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p}) - q_1\|\boldsymbol{p}\|^2}{\mu},$$

from (VIII.1.),

$$q_1' = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[\left(\frac{1}{2\mu}\right)(A_1)(p_1) + (-H_0)\left(\frac{1}{\mu}\right)(q_2p_2 + q_3p_3)\right],$$

from (VIII.1.),

$$q_1' = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[\left(\frac{1}{2\mu}\right)\left(\frac{q_1}{\|\boldsymbol{q}\|} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p}) - q_1\|\boldsymbol{p}\|^2}{\mu}\right)(p_1) + (-H_0)\left(\frac{1}{\mu}\right)\left((\boldsymbol{q}\cdot\boldsymbol{p}) - q_1p_1\right)\right],$$

$$q_1' = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[\frac{q_1p_1}{2\mu\|\boldsymbol{q}\|} + \frac{{p_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2} - \frac{q_1p_1\|\boldsymbol{p}\|^2}{2\mu^2} - \frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu} + \frac{H_0q_1p_1}{\mu}\right],$$

$$q_1' = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[\frac{q_1p_1}{2\mu\|\boldsymbol{q}\|} + \frac{{p_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2} - \frac{q_1p_1\|\boldsymbol{p}\|^2}{2\mu^2} - \frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu} + \frac{q_1p_1}{\mu}\left(\frac{\|\boldsymbol{p}\|^2}{2\mu} - \frac{1}{\|\boldsymbol{q}\|}\right)\right],$$

$$q_1' = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[\frac{q_1p_1}{2\mu\|\boldsymbol{q}\|} + \frac{{p_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2} - \frac{q_1p_1\|\boldsymbol{p}\|^2}{2\mu^2} - \frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu} + \frac{q_1p_1\|\boldsymbol{p}\|^2}{2\mu^2} - \frac{q_1p_1}{\mu\|\boldsymbol{q}\|}\right],$$

$$q_1' = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[-\frac{q_1p_1}{2\mu\|\boldsymbol{q}\|} + \frac{{p_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2} - \frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right],$$

$$q_2' = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[\left(\frac{1}{2\mu}\right)(A_1)(p_2) + (-H_0)\left(\frac{1}{\mu}\right)(q_2p_1 - 2q_1p_2)\right],$$

from (VIII.1.),

$$q_2' = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[\left(\frac{1}{2\mu}\right)\left(\frac{q_1}{\|\boldsymbol{q}\|} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p}) - q_1\|\boldsymbol{p}\|^2}{\mu}\right)(p_2) + (-H_0)\left(\frac{1}{\mu}\right)(q_2p_1 - 2q_1p_2)\right],$$

$$q_2' = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[\frac{q_1p_2}{2\mu\|\boldsymbol{q}\|} + \frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2} - \frac{q_1p_2\|\boldsymbol{p}\|^2}{2\mu^2} - \frac{H_0q_2p_1}{\mu} + \frac{2H_0q_1p_2}{\mu}\right],$$

$$q_2' = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[\frac{q_1p_2}{2\mu\|\boldsymbol{q}\|} + \frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2} - \frac{q_1p_2\|\boldsymbol{p}\|^2}{2\mu^2} - \frac{H_0q_2p_1}{\mu} + \frac{2q_1p_2}{\mu}\left(\frac{\|\boldsymbol{p}\|^2}{2\mu} - \frac{1}{\|\boldsymbol{q}\|}\right)\right],$$

$$q_2' = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[\frac{q_1p_2}{2\mu\|\boldsymbol{q}\|} + \frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2} - \frac{q_1p_2\|\boldsymbol{p}\|^2}{2\mu^2} - \frac{H_0q_2p_1}{\mu} + \frac{q_1p_2\|\boldsymbol{p}\|^2}{\mu^2} - \frac{2q_1p_2}{\mu\|\boldsymbol{q}\|}\right],$$

$$q_2' = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[-\frac{3q_1p_2}{2\mu\|\boldsymbol{q}\|} + \frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2} + \frac{q_1p_2\|\boldsymbol{p}\|^2}{2\mu^2} - \frac{H_0q_2p_1}{\mu}\right],$$

$$q_2' = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[-\frac{3q_1p_2}{2\mu\|\boldsymbol{q}\|} + \frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2} + \frac{q_1p_2\|\boldsymbol{p}\|^2}{2\mu^2} - \frac{q_1p_2}{\mu\|\boldsymbol{q}\|} + \frac{q_1p_2}{\mu\|\boldsymbol{q}\|} - \frac{H_0q_2p_1}{\mu}\right],$$

$$q_2' = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[-\frac{q_1p_2}{2\mu\|\boldsymbol{q}\|} + \frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2} + \frac{H_0q_1p_2}{\mu} - \frac{H_0q_2p_1}{\mu}\right],$$

$$q_3' = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[-\frac{q_1p_3}{2\mu\|\boldsymbol{q}\|} + \frac{p_1p_3(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2} + \frac{H_0q_1p_3}{\mu} - \frac{H_0q_3p_1}{\mu}\right],$$

analogous to the last case,

$$p_1' = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\begin{bmatrix} \left(\frac{1}{2}\right)(A_1)(-q_1)\left(\frac{1}{\|\boldsymbol{q}\|^3}\right) \\ +(-H_0)(-\|\boldsymbol{q}\|^2 + {q_1}^2)\left(\frac{1}{\|\boldsymbol{q}\|^3}\right) \\ +(-H_0)\left(\frac{1}{\mu}\right)(\|\boldsymbol{p}\|^2 - {p_1}^2) \end{bmatrix},$$

from (VIII.1.),

$$p_1' = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\begin{bmatrix} \left(\frac{1}{2}\right)\left(\frac{q_1}{\|\boldsymbol{q}\|} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p}) - q_1\|\boldsymbol{p}\|^2}{\mu}\right)(-q_1)\left(\frac{1}{\|\boldsymbol{q}\|^3}\right) \\ +(-H_0)(-\|\boldsymbol{q}\|^2 + {q_1}^2)\left(\frac{1}{\|\boldsymbol{q}\|^3}\right) \\ +(-H_0)\left(\frac{1}{\mu}\right)(\|\boldsymbol{p}\|^2 - {p_1}^2) \end{bmatrix},$$

$$p_1' = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[-\frac{{q_1}^2}{2\|\boldsymbol{q}\|^4} - \frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^3} + \frac{{q_1}^2\|\boldsymbol{p}\|^2}{2\mu\|\boldsymbol{q}\|^3} + \frac{H_0}{\|\boldsymbol{q}\|} - \frac{H_0{q_1}^2}{\|\boldsymbol{q}\|^3} - \frac{H_0\|\boldsymbol{p}\|^2}{\mu} + \frac{H_0{p_1}^2}{\mu}\right],$$

$$p_1' = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[-\frac{{q_1}^2}{2\|\boldsymbol{q}\|^4} - \frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^3} + \frac{{q_1}^2\|\boldsymbol{p}\|^2}{2\mu\|\boldsymbol{q}\|^3} - \frac{{q_1}^2}{\|\boldsymbol{q}\|^4} + \frac{{q_1}^2}{\|\boldsymbol{q}\|^4} + \frac{H_0}{\|\boldsymbol{q}\|} - \frac{H_0{q_1}^2}{\|\boldsymbol{q}\|^3} - \frac{H_0\|\boldsymbol{p}\|^2}{\mu} + \frac{H_0{p_1}^2}{\mu}\right],$$

$$p_1' = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[-\frac{{q_1}^2}{2\|\boldsymbol{q}\|^4} - \frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^3} + \frac{H_0{q_1}^2}{\|\boldsymbol{q}\|^3} + \frac{{q_1}^2}{\|\boldsymbol{q}\|^4} + \frac{H_0}{\|\boldsymbol{q}\|} - \frac{H_0{q_1}^2}{\|\boldsymbol{q}\|^3} - \frac{H_0\|\boldsymbol{p}\|^2}{\mu} + \frac{H_0{p_1}^2}{\mu}\right],$$

$$p_1' = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[\frac{{q_1}^2}{2\|\boldsymbol{q}\|^4} - \frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^3} + \frac{H_0}{\|\boldsymbol{q}\|} - \frac{H_0\|\boldsymbol{p}\|^2}{\mu} + \frac{H_0{p_1}^2}{\mu}\right],$$

$$p_2' = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\begin{bmatrix} \left(\frac{1}{2}\right)(A_1)(-q_2)\left(\frac{1}{\|\boldsymbol{q}\|^3}\right) \\ +(-H_0)(q_1q_2)\left(\frac{1}{\|\boldsymbol{q}\|^3}\right) \\ +(-H_0)\left(\frac{1}{\mu}\right)(-p_1p_2) \end{bmatrix},$$

from (VIII.1.),

$$p_2' = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\begin{bmatrix} \left(\frac{1}{2}\right)\left(\frac{q_1}{\|\boldsymbol{q}\|} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p}) - q_1\|\boldsymbol{p}\|^2}{\mu}\right)(-q_2)\left(\frac{1}{\|\boldsymbol{q}\|^3}\right) \\ +(-H_0)(q_1q_2)\left(\frac{1}{\|\boldsymbol{q}\|^3}\right) \\ +(-H_0)\left(\frac{1}{\mu}\right)(-p_1p_2) \end{bmatrix},$$

from (VIII.1.),

$$p_2' = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[-\frac{q_1q_2}{2\|\boldsymbol{q}\|^4} - \frac{q_2p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^3} + \frac{q_1q_2\|\boldsymbol{p}\|^2}{2\mu\|\boldsymbol{q}\|^3} - \frac{q_1q_2}{\|\boldsymbol{q}\|^4} + \frac{q_1q_2}{\|\boldsymbol{q}\|^4} - \frac{H_0q_1q_2}{\|\boldsymbol{q}\|^3} + \frac{H_0p_1p_2}{\mu}\right],$$

$$p_2' = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[\frac{q_1q_2}{2\|\boldsymbol{q}\|^4} - \frac{q_2p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^3} + \frac{H_0q_1q_2}{\|\boldsymbol{q}\|^3} - \frac{H_0q_1q_2}{\|\boldsymbol{q}\|^3} + \frac{H_0p_1p_2}{\mu}\right],$$

$$p_2' = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[\frac{q_1q_2}{2\|\boldsymbol{q}\|^4} - \frac{q_2p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^3} + \frac{H_0p_1p_2}{\mu}\right],$$

$$p_3' = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[\frac{q_1q_3}{2\|\boldsymbol{q}\|^4} - \frac{q_3p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^3} + \frac{H_0p_1p_3}{\mu}\right],$$

analogous to the last case.

$$(\boldsymbol{p}\cdot\boldsymbol{p}') = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[\frac{A_1}{2\|\boldsymbol{q}\|^3}(-\boldsymbol{q}\cdot\boldsymbol{p}) + (-H_0)\left(\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|^3} - \frac{p_1}{\|\boldsymbol{q}\|}\right) + (-H_0)\left(\frac{1}{\mu}\right)(p_1\|\boldsymbol{p}\|^2 - p_1\|\boldsymbol{p}\|^2)\right],$$

from (VIII.1.),

$$(\boldsymbol{p}\cdot\boldsymbol{p}') = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\begin{bmatrix}\left(\frac{1}{2\|\boldsymbol{q}\|^3}\right)\left(\frac{q_1}{\|\boldsymbol{q}\|} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p}) - q_1\|\boldsymbol{p}\|^2}{\mu}\right)(-\boldsymbol{q}\cdot\boldsymbol{p}) \\ +(-H_0)\left(\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|^3} - \frac{p_1}{\|\boldsymbol{q}\|}\right) \\ +(-H_0)\left(\frac{1}{\mu}\right)(p_1\|\boldsymbol{p}\|^2 - p_1\|\boldsymbol{p}\|^2)\end{bmatrix},$$

$$(\boldsymbol{p}\cdot\boldsymbol{p}') = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[-\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\|\boldsymbol{q}\|^4} - \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu\|\boldsymbol{q}\|^3} + \frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{2\mu\|\boldsymbol{q}\|^3} - \frac{H_0q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|^3} + \frac{H_0p_1}{\|\boldsymbol{q}\|}\right],$$

$$(\boldsymbol{p}\cdot\boldsymbol{p}') = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[-\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\|\boldsymbol{q}\|^4} - \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu\|\boldsymbol{q}\|^3} + \frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{2\mu\|\boldsymbol{q}\|^3} - \frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|^4} + \frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|^4} - \frac{H_0q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|^3} + \frac{H_0p_1}{\|\boldsymbol{q}\|}\right],$$

$$(\boldsymbol{p}\cdot\boldsymbol{p}') = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\|\boldsymbol{q}\|^4} - \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu\|\boldsymbol{q}\|^3} + \frac{H_0q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|^3} - \frac{H_0q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|^3} + \frac{H_0p_1}{\|\boldsymbol{q}\|}\right],$$

$$(\boldsymbol{p}\cdot\boldsymbol{p}') = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\|\boldsymbol{q}\|^4} - \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu\|\boldsymbol{q}\|^3} + \frac{H_0p_1}{\|\boldsymbol{q}\|}\right],$$

$$(\boldsymbol{q}\cdot\boldsymbol{p}') = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[\left(\frac{-A_1}{2\|\boldsymbol{q}\|}\right) + (-H_0)\left(\frac{1}{\mu}\right)(q_1\|\boldsymbol{p}\|^2) + (-H_0)\left(\frac{1}{\mu}\right)(-p_1)(\boldsymbol{q}\cdot\boldsymbol{p})\right],$$

from (VIII.1.),

$$(\boldsymbol{q}\cdot\boldsymbol{p}') = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\begin{bmatrix}\left(\frac{-1}{2\|\boldsymbol{q}\|}\right)\left(\frac{q_1}{\|\boldsymbol{q}\|} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p}) - q_1\|\boldsymbol{p}\|^2}{\mu}\right) \\ +(-H_0)\left(\frac{1}{\mu}\right)(q_1\|\boldsymbol{p}\|^2) \\ +(-H_0)\left(\frac{1}{\mu}\right)(-p_1)(\boldsymbol{q}\cdot\boldsymbol{p})\end{bmatrix},$$

$$(\boldsymbol{q}\cdot\boldsymbol{p}') = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[-\frac{q_1}{2\|\boldsymbol{q}\|^2} - \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|} + \frac{q_1\|\boldsymbol{p}\|^2}{2\mu\|\boldsymbol{q}\|} - \frac{H_0q_1\|\boldsymbol{p}\|^2}{\mu} + \frac{H_0p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right],$$

$$(\boldsymbol{q}\cdot\boldsymbol{p}') = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[-\frac{q_1}{2\|\boldsymbol{q}\|^2} - \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|} + \frac{q_1\|\boldsymbol{p}\|^2}{2\mu\|\boldsymbol{q}\|} - \frac{H_0q_1\|\boldsymbol{p}\|^2}{\mu} + \frac{H_0p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right],$$

$$(\boldsymbol{q}\cdot\boldsymbol{p}') = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[-\frac{q_1}{2\|\boldsymbol{q}\|^2} - \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|} + \frac{q_1\|\boldsymbol{p}\|^2}{2\mu\|\boldsymbol{q}\|} - \frac{q_1}{\|\boldsymbol{q}\|^2} + \frac{q_1}{\|\boldsymbol{q}\|^2} - \frac{H_0q_1\|\boldsymbol{p}\|^2}{\mu} + \frac{H_0p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right],$$

$$(\boldsymbol{q}\cdot\boldsymbol{p}') = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[\frac{q_1}{2\|\boldsymbol{q}\|^2} - \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|} + \frac{H_0 q_1}{\|\boldsymbol{q}\|} - \frac{H_0 q_1\|\boldsymbol{p}\|^2}{\mu} + \frac{H_0 p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right],$$

$$(\boldsymbol{p}\cdot\boldsymbol{q}') = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[\left(\frac{1}{2\mu}\right)(A_1)\|\boldsymbol{p}\|^2 + (-H_0)\left(\frac{2}{\mu}\right)(p_1)(\boldsymbol{q}\cdot\boldsymbol{p}) + (-H_0)\left(\frac{2}{\mu}\right)(-q_1)\|\boldsymbol{p}\|^2\right],$$

from (VIII.1.),

$$(\boldsymbol{p}\cdot\boldsymbol{q}') = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\begin{bmatrix}\left(\frac{1}{2\mu}\right)\left(\frac{q_1}{\|\boldsymbol{q}\|} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p}) - q_1\|\boldsymbol{p}\|^2}{\mu}\right)\|\boldsymbol{p}\|^2 \\ +(-H_0)\left(\frac{2}{\mu}\right)(p_1)(\boldsymbol{q}\cdot\boldsymbol{p}) \\ +(-H_0)\left(\frac{2}{\mu}\right)(-q_1)\|\boldsymbol{p}\|^2\end{bmatrix},$$

$$(\boldsymbol{p}\cdot\boldsymbol{q}') = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[\frac{q_1\|\boldsymbol{p}\|^2}{2\mu\|\boldsymbol{q}\|} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{2\mu^2} - \frac{q_1\|\boldsymbol{p}\|^4}{2\mu^2} - \frac{2H_0 p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu} + \frac{2H_0 q_1\|\boldsymbol{p}\|^2}{\mu}\right],$$

$$(\boldsymbol{p}\cdot\boldsymbol{q}') = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[\frac{q_1\|\boldsymbol{p}\|^2}{2\mu\|\boldsymbol{q}\|} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{2\mu^2} - \frac{q_1\|\boldsymbol{p}\|^4}{2\mu^2} + \frac{q_1\|\boldsymbol{p}\|^2}{\mu\|\boldsymbol{q}\|} - \frac{q_1\|\boldsymbol{p}\|^2}{\mu\|\boldsymbol{q}\|} - \frac{2H_0 p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu} + \frac{2H_0 q_1\|\boldsymbol{p}\|^2}{\mu}\right],$$

$$(\boldsymbol{p}\cdot\boldsymbol{q}') = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[-\frac{q_1\|\boldsymbol{p}\|^2}{2\mu\|\boldsymbol{q}\|} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{2\mu^2} - \frac{H_0 q_1\|\boldsymbol{p}\|^2}{\mu} - \frac{2H_0 p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu} + \frac{2H_0 q_1\|\boldsymbol{p}\|^2}{\mu}\right],$$

$$(\boldsymbol{p}\cdot\boldsymbol{q}') = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\begin{bmatrix}-\frac{q_1\|\boldsymbol{p}\|^2}{2\mu\|\boldsymbol{q}\|} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{2\mu^2} - \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu\|\boldsymbol{q}\|} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu\|\boldsymbol{q}\|} \\ -\frac{H_0 q_1\|\boldsymbol{p}\|^2}{\mu} - \frac{2H_0 p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu} + \frac{2H_0 q_1\|\boldsymbol{p}\|^2}{\mu}\end{bmatrix},$$

$$(\boldsymbol{p}\cdot\boldsymbol{q}') = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[-\frac{q_1\|\boldsymbol{p}\|^2}{2\mu\|\boldsymbol{q}\|} - \frac{H_0 p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu\|\boldsymbol{q}\|} + \frac{H_0 q_1\|\boldsymbol{p}\|^2}{\mu}\right],$$

$$(\boldsymbol{p}\cdot\boldsymbol{q}') = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[-\frac{q_1\|\boldsymbol{p}\|^2}{2\mu\|\boldsymbol{q}\|} + \frac{q_1}{\|\boldsymbol{q}\|^2} - \frac{q_1}{\|\boldsymbol{q}\|^2} - \frac{H_0 p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu\|\boldsymbol{q}\|} + \frac{H_0 q_1\|\boldsymbol{p}\|^2}{\mu}\right],$$

$$(\boldsymbol{p}\cdot\boldsymbol{q}') = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[-\frac{H_0 q_1}{\|\boldsymbol{q}\|} - \frac{q_1}{\|\boldsymbol{q}\|^2} - \frac{H_0 p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu\|\boldsymbol{q}\|} + \frac{H_0 q_1\|\boldsymbol{p}\|^2}{\mu}\right],$$

$$(\boldsymbol{q}\cdot\boldsymbol{p})' = (\boldsymbol{q}\cdot\boldsymbol{p}') + (\boldsymbol{p}\cdot\boldsymbol{q}'),$$

$$(\boldsymbol{q}\cdot\boldsymbol{p})' = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\begin{bmatrix}\frac{q_1}{2\|\boldsymbol{q}\|^2} - \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|} + \frac{H_0 q_1}{\|\boldsymbol{q}\|} - \frac{H_0 q_1\|\boldsymbol{p}\|^2}{\mu} + \frac{H_0 p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu} \\ -\frac{H_0 q_1}{\|\boldsymbol{q}\|} - \frac{q_1}{\|\boldsymbol{q}\|^2} - \frac{H_0 p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu\|\boldsymbol{q}\|} + \frac{H_0 q_1\|\boldsymbol{p}\|^2}{\mu}\end{bmatrix},$$

$$(\boldsymbol{q}\cdot\boldsymbol{p})' = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[-\frac{q_1}{2\|\boldsymbol{q}\|^2} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right].$$

$$(\boldsymbol{q}\cdot\boldsymbol{q}') = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\begin{bmatrix}\left(\frac{1}{2\mu}\right)(A_1)(\boldsymbol{q}\cdot\boldsymbol{p}) \\ -\left(\frac{H_0}{\mu}\right)(q_1q_2p_2 + q_1q_2p_2 + q_2q_2p_1 - 2q_1q_2p_2 + q_3q_3p_1 - 2q_1q_3p_3)\end{bmatrix},$$

from (VIII.1.),

$$(\boldsymbol{q}\cdot\boldsymbol{q}') = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\begin{bmatrix}\left(\frac{1}{2\mu}\right)(A_1)(\boldsymbol{q}\cdot\boldsymbol{p}) \\ -\left(\frac{H_0}{\mu}\right)(-q_1q_2p_2 - q_1q_3p_3 + q_2q_2p_1 + q_3q_3p_1)\end{bmatrix},$$

$$(\boldsymbol{q}\cdot\boldsymbol{q}') = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\begin{bmatrix} \left(\frac{1}{2\mu}\right)(A_1)(\boldsymbol{q}\cdot\boldsymbol{p}) \\ -\left(\frac{H_0}{\mu}\right)(-q_1q_1p_1 - q_1q_2p_2 - q_1q_3p_3 + q_1q_1p_1 + q_2q_2p_1 + q_3q_3p_1) \end{bmatrix},$$

$$(\boldsymbol{q}\cdot\boldsymbol{q}') = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[\left(\frac{1}{2\mu}\right)\left(\frac{q_1}{\|\boldsymbol{q}\|} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p}) - q_1\|\boldsymbol{p}\|^2}{\mu}\right)(\boldsymbol{q}\cdot\boldsymbol{p}) - \left(\frac{H_0}{\mu}\right)(-q_1(\boldsymbol{q}\cdot\boldsymbol{p}) + \|\boldsymbol{q}\|^2 p_1)\right],$$

$$(\boldsymbol{q}\cdot\boldsymbol{q}') = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2} - \frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{2\mu^2} + \frac{H_0 q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu} - \frac{H_0\|\boldsymbol{q}\|^2 p_1}{\mu}\right],$$

$$(\boldsymbol{q}\cdot\boldsymbol{q}') = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2} - \frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{2\mu^2} + \frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\left(\frac{\|\boldsymbol{p}\|^2}{2\mu} - \frac{1}{\|\boldsymbol{q}\|}\right) - \frac{H_0\|\boldsymbol{q}\|^2 p_1}{\mu}\right],$$

$$(\boldsymbol{q}\cdot\boldsymbol{q}') = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[-\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2} - \frac{H_0\|\boldsymbol{q}\|^2 p_1}{\mu}\right],$$

$$\|\boldsymbol{q}\|' = \sqrt{(\boldsymbol{q}\cdot\boldsymbol{q})}' = \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{(\boldsymbol{q}\cdot\boldsymbol{q})}}\right)2(\boldsymbol{q}\cdot\boldsymbol{q}') = \frac{(\boldsymbol{q}\cdot\boldsymbol{q}')}{\|\boldsymbol{q}\|},$$

$$\|\boldsymbol{q}\|' = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[-\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|} - \frac{H_0\|\boldsymbol{q}\|p_1}{\mu}\right].$$

$$\psi' = \sqrt{\frac{-2H_0}{\mu}}(\boldsymbol{q}\cdot\boldsymbol{p})',$$

$$\psi' = \sqrt{\frac{-2H_0}{\mu}}\sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[-\frac{q_1}{2\|\boldsymbol{q}\|^2} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right],$$

$$\psi' = (-H_0)^{-1}\left[-\frac{q_1}{2\|\boldsymbol{q}\|^2} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right].$$

Main results for $H < 0$:

$$\begin{aligned} x_0' = &\left(\frac{1}{\mu}\right)[\|\boldsymbol{q}\|'\|\boldsymbol{p}\|^2 + \|\boldsymbol{q}\|(2)(\boldsymbol{p}\cdot\boldsymbol{p}')]cos\psi \\ &+ \left[\left(\frac{\|\boldsymbol{q}\|\cdot\|\boldsymbol{p}\|^2}{\mu} - 1\right)\right](-1)sin\psi(\psi') \\ &+ \sqrt{\frac{-2H_0}{\mu}}[(\boldsymbol{q}\cdot\boldsymbol{p})']sin\psi \\ &+ \sqrt{\frac{-2H_0}{\mu}}(\boldsymbol{q}\cdot\boldsymbol{p})cos\psi(\psi'), \end{aligned}$$

$$\begin{aligned} x_0' = &\left(\frac{1}{\mu}\right)\sqrt{\frac{\mu}{2}}(-H_0)^{-\frac{3}{2}}\left[-\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|} - \frac{H_0\|\boldsymbol{q}\|p_1}{\mu}\right]\|\boldsymbol{p}\|^2\ cos\psi \\ &+ \left(\frac{2}{\mu}\right)\|\boldsymbol{q}\|\sqrt{\frac{\mu}{2}}(-H_0)^{-\frac{3}{2}}\left[\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\|\boldsymbol{q}\|^4} - \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu\|\boldsymbol{q}\|^3} + \frac{H_0 p_1}{\|\boldsymbol{q}\|}\right]cos\psi \\ &+ \sqrt{\frac{-2H_0}{\mu}}(\boldsymbol{q}\cdot\boldsymbol{p})(-H_0)^{-1}\left[-\frac{q_1}{2\|\boldsymbol{q}\|^2} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right]cos\psi \\ &- \left(\frac{\|\boldsymbol{q}\|\cdot\|\boldsymbol{p}\|^2}{\mu} - 1\right)(-H_0)^{-1}\left[-\frac{q_1}{2\|\boldsymbol{q}\|^2} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right]sin\psi \\ &+ \sqrt{\frac{-2H_0}{\mu}}\sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[-\frac{q_1}{2\|\boldsymbol{q}\|^2} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right]sin\psi, \end{aligned}$$

$$
\begin{aligned}
x_0' = {} & \left(\frac{1}{\mu}\right)\sqrt{\frac{\mu}{2}}(-H_0)^{-\frac{3}{2}}\left[-\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{2\mu\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2\|\boldsymbol{p}\|^2}{2\mu^2\|\boldsymbol{q}\|}-\frac{H_0\|\boldsymbol{q}\|p_1\|\boldsymbol{p}\|^2}{\mu}\right] cos\psi \\
& +\left(\frac{2}{\mu}\right)\sqrt{\frac{\mu}{2}}(-H_0)^{-\frac{3}{2}}\left[\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\|\boldsymbol{q}\|^3}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu\|\boldsymbol{q}\|^2}+\frac{H_0p_1}{1}\right] cos\psi \\
& +\sqrt{\frac{2}{-\mu H_0}}\left[-\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu\|\boldsymbol{q}\|}\right] cos\psi \\
& -(-H_0)^{-1}\left[-\frac{q_1\|\boldsymbol{q}\|\cdot\|\boldsymbol{p}\|^2}{2\mu\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{q}\|\cdot\|\boldsymbol{p}\|^2}{2\mu^2\|\boldsymbol{q}\|}+\frac{q_1}{2\|\boldsymbol{q}\|^2}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right] sin\psi \\
& +(-H_0)^{-1}\left[-\frac{q_1}{2\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right] sin\psi,
\end{aligned}
$$

$$
\begin{aligned}
x_0' = {} & \sqrt{\frac{1}{2\mu}}(-H_0)^{-\frac{3}{2}}\begin{bmatrix} -\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{2\mu\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2\|\boldsymbol{p}\|^2}{2\mu^2\|\boldsymbol{q}\|}-\frac{H_0p_1\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2}{\mu} \\ +\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|^3}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{\mu\|\boldsymbol{q}\|^2}+\frac{2H_0p_1}{1}+\frac{H_0q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|^2}-\frac{H_0p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{\mu\|\boldsymbol{q}\|}\end{bmatrix} cos\psi \\
& -(-H_0)^{-1}\left[-\frac{q_1\|\boldsymbol{p}\|^2}{2\mu\|\boldsymbol{q}\|}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{2\mu^2}+\frac{q_1}{\|\boldsymbol{q}\|^2}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu\|\boldsymbol{q}\|}\right] sin\psi,
\end{aligned}
$$

$$
\begin{aligned}
x_0' = {} & \sqrt{\frac{1}{2\mu}}(-H_0)^{-\frac{3}{2}}\begin{bmatrix} \left(-\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{\mu\|\boldsymbol{q}\|}\right)\left(\frac{\|\boldsymbol{p}\|^2}{2\mu}-\frac{1}{\|\boldsymbol{q}\|}\right)-\frac{H_0p_1\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2}{\mu} \\ +\frac{2H_0p_1}{1}+\frac{H_0q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|^2}-\frac{H_0p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{\mu\|\boldsymbol{q}\|}\end{bmatrix} cos\psi \\
& -(-H_0)^{-1}\left[\left(-\frac{q_1}{\|\boldsymbol{q}\|}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right)\left(\frac{\|\boldsymbol{p}\|^2}{2\mu}-\frac{1}{\|\boldsymbol{q}\|}\right)\right] sin\psi,
\end{aligned}
$$

$$
\begin{aligned}
x_0' = {} & \sqrt{\frac{1}{2\mu}}(-H_0)^{-\frac{3}{2}}\left[-\frac{H_0q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|^2}+\frac{H_0p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{\mu\|\boldsymbol{q}\|}-\frac{H_0p_1\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2}{\mu}+\frac{2H_0p_1}{1}+\frac{H_0q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|^2}-\frac{H_0p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{\mu\|\boldsymbol{q}\|}\right] cos\psi \\
& -(-H_0)^{-1}\left[-\frac{H_0q_1}{\|\boldsymbol{q}\|}+\frac{H_0p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right] sin\psi,
\end{aligned}
$$

$$
x_0' = \sqrt{\frac{1}{2\mu}}(-H_0)^{-3/2}\left[-\frac{H_0p_1\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2}{\mu}+\frac{2H_0p_1}{1}\right] cos\psi-(-H_0)^{-1}\left[-\frac{H_0q_1}{\|\boldsymbol{q}\|}+\frac{H_0p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right] sin\psi,
$$

$$
x_0' = \sqrt{\frac{2}{-\mu H_0}}\left[\frac{p_1\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2}{2\mu}-\frac{p_1}{1}\right] cos\psi-\left[\frac{q_1}{\|\boldsymbol{q}\|}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right] sin\psi,
$$

$$
x_0' = \sqrt{\frac{2}{-\mu H_0}}\left[\frac{p_1\|\boldsymbol{q}\|}{1}\left(H_0+\frac{1}{\|\boldsymbol{q}\|}\right)-\frac{p_1}{1}\right] cos\psi-\left[\frac{q_1}{\|\boldsymbol{q}\|}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right] sin\psi,
$$

$$
x_0' = -p_1\|\boldsymbol{q}\| \, cos\psi-\left[\frac{q_1}{\|\boldsymbol{q}\|}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right] sin\psi,
$$

$$
x_0' = -x_1.
$$

$$
\begin{aligned}
x_1' = {} & \sqrt{\frac{-2\mathrm{H}_0}{\mu}}[\|\boldsymbol{q}\|'p_1+\|\boldsymbol{q}\|p_1{}']cos\psi \\
& +\sqrt{\frac{-2H_0}{\mu}}[\|\boldsymbol{q}\|p_1](-1)sin\psi(\psi') \\
& +\left[\frac{q_1{}'}{\|\boldsymbol{q}\|}+q_1(-1)\left(\frac{1}{\|\boldsymbol{q}\|^2}\right)\|\boldsymbol{q}\|'-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})'}{\mu}-\frac{(\boldsymbol{q}\cdot\boldsymbol{p})p_1'}{\mu}\right] sin\psi \\
& +\left[\frac{q_1}{\|\boldsymbol{q}\|}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right] cos\psi(\psi'),
\end{aligned}
$$

$$x_1' = \sqrt{\frac{-2H_0}{\mu}}\begin{bmatrix} \sqrt{\frac{\mu}{2}}(-H_0)^{-\frac{3}{2}}\left(-\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|}-\frac{H_0\|\boldsymbol{q}\|p_1}{\mu}\right)p_1 \\ +\|\boldsymbol{q}\|\sqrt{\frac{\mu}{2}}(-H_0)^{-\frac{3}{2}}\left(\frac{{q_1}^2}{2\|\boldsymbol{q}\|^4}-\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^3}+\frac{H_0}{\|\boldsymbol{q}\|}-\frac{H_0\|\boldsymbol{p}\|^2}{\mu}+\frac{H_0{p_1}^2}{\mu}\right)\end{bmatrix}cos\psi$$

$$+\sqrt{\frac{-2H_0}{\mu}}[\|\boldsymbol{q}\|p_1](-1)(-H_0)^{-1}\left[-\frac{q_1}{2\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right]sin\psi$$

$$+\begin{bmatrix} \frac{1}{\|\boldsymbol{q}\|}\sqrt{\frac{\mu}{2}}(-H_0)^{-\frac{3}{2}}\left(-\frac{q_1p_1}{2\mu\|\boldsymbol{q}\|}+\frac{{p_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2}-\frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right) \\ +q_1(-1)\left(\frac{1}{\|\boldsymbol{q}\|^2}\right)\sqrt{\frac{\mu}{2}}(-H_0)^{-\frac{3}{2}}\left(-\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|}-\frac{H_0\|\boldsymbol{q}\|p_1}{\mu}\right) \\ -\frac{p_1}{\mu}\sqrt{\frac{\mu}{2}}(-H_0)^{-\frac{3}{2}}\left(-\frac{q_1}{2\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right) \\ -\frac{(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\sqrt{\frac{\mu}{2}}(-H_0)^{-\frac{3}{2}}\left(\frac{{q_1}^2}{2\|\boldsymbol{q}\|^4}-\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^3}+\frac{H_0}{\|\boldsymbol{q}\|}-\frac{H_0\|\boldsymbol{p}\|^2}{\mu}+\frac{H_0{p_1}^2}{\mu}\right)\end{bmatrix}sin\psi$$

$$+\left[\frac{q_1}{\|\boldsymbol{q}\|}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right](-H_0)^{-1}\left[-\frac{q_1}{2\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right]cos\psi,$$

$$x_1' = (-H_0)^{-1}\begin{bmatrix} -\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}+\frac{{p_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|}-\frac{H_0\|\boldsymbol{q}\|{p_1}^2}{\mu}+\frac{{q_1}^2}{2\|\boldsymbol{q}\|^3}-\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}+\frac{H_0}{1} \\ -\frac{H_0\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2}{\mu}+\frac{H_0\|\boldsymbol{q}\|{p_1}^2}{\mu}-\frac{{q_1}^2}{2\|\boldsymbol{q}\|^3}+\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}+\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}-\frac{{p_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|}\end{bmatrix}cos\psi$$

$$+\sqrt{\frac{-2H_0}{\mu}}(-H_0)^{-1}\left[+\frac{q_1p_1}{2\|\boldsymbol{q}\|}-\frac{{p_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu}\right]sin\psi$$

$$+\sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\begin{bmatrix} -\frac{q_1p_1}{2\mu\|\boldsymbol{q}\|^2}+\frac{{p_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2\|\boldsymbol{q}\|}-\frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu\|\boldsymbol{q}\|}+\frac{{q_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^4}-\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|^3}+\frac{H_0\|\boldsymbol{q}\|q_1p_1}{\mu\|\boldsymbol{q}\|^2}+\frac{q_1p_1}{2\mu\|\boldsymbol{q}\|^2} \\ -\frac{{p_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2\|\boldsymbol{q}\|}-\frac{{q_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^4}+\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|^3}-\frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu\|\boldsymbol{q}\|}+\frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{\mu^2}-\frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p}){p_1}^2}{\mu^2}\end{bmatrix}sin\psi,$$

$$x_1' = (-H_0)^{-1}\left[+\frac{H_0}{1}-\frac{H_0\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2}{\mu}\right]cos\psi$$

$$+\sqrt{\frac{-2H_0}{\mu}}(-H_0)^{-1}\left[+\frac{q_1p_1}{2\|\boldsymbol{q}\|}-\frac{{p_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu}\right]sin\psi$$

$$+\sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[-\frac{2H_0(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu\|\boldsymbol{q}\|}+\frac{H_0q_1p_1}{\mu\|\boldsymbol{q}\|}+\frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{\mu^2}-\frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p}){p_1}^2}{\mu^2}\right]sin\psi,$$

$$x_1' = \left[-1+\frac{\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2}{\mu}\right]cos\psi$$

$$+\sqrt{\frac{-2H_0}{\mu}}(-H_0)^{-1}\left[+\frac{q_1p_1}{2\|\boldsymbol{q}\|}-\frac{{p_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu}\right]sin\psi$$

$$+\sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[-\frac{2H_0(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu\|\boldsymbol{q}\|}+\frac{H_0q_1p_1}{\mu\|\boldsymbol{q}\|}+\frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{\mu^2}-\frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p}){p_1}^2}{\mu^2}\right]sin\psi,$$

$$x_1' = \left[\frac{\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2}{\mu}-1\right]cos\psi+\sqrt{\frac{2}{-H_0\mu}}\left[+\frac{q_1p_1}{2\|\boldsymbol{q}\|}-\frac{{p_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu}+\frac{(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|}-\frac{q_1p_1}{2\|\boldsymbol{q}\|}-\frac{(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{2\mu}+\frac{(\boldsymbol{q}\cdot\boldsymbol{p}){p_1}^2}{2\mu}\right]sin\psi,$$

$$x_1' = \left[\frac{\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2}{\mu}-1\right]cos\psi+\sqrt{\frac{2}{-H_0\mu}}\left[+\frac{(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|}-\frac{(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{2\mu}\right]sin\psi,$$

$$x_1' = \left[\frac{\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2}{\mu}-1\right]cos\psi+\sqrt{\frac{-2H_0}{\mu}}(\boldsymbol{q}\cdot\boldsymbol{p})\,sin\psi,$$

$$x_1' = x_0.$$

$$
\begin{aligned}
x_2' = {} & \sqrt{\frac{-2H_0}{\mu}}\,[\|\boldsymbol{q}\|' p_2 + \|\boldsymbol{q}\| p_2{}']cos\psi \\
& + \sqrt{\frac{-2H_0}{\mu}}\,[\|\boldsymbol{q}\| p_2](-1)sin\psi(\psi') \\
& + \left[\frac{q_2{}'}{\|\boldsymbol{q}\|} + q_2(-1)\left(\frac{1}{\|\boldsymbol{q}\|^2}\right)\|\boldsymbol{q}\|' - \frac{p_2(\boldsymbol{q}\cdot\boldsymbol{p})'}{\mu} - \frac{(\boldsymbol{q}\cdot\boldsymbol{p})p_2'}{\mu}\right] sin\psi \\
& + \left[\frac{q_2}{\|\boldsymbol{q}\|} - \frac{p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right] cos\psi(\psi'),
\end{aligned}
$$

$$
\begin{aligned}
x_2' = {} & \sqrt{\frac{-2H_0}{\mu}}\begin{bmatrix} \sqrt{\frac{\mu}{2}}(-H_0)^{-\frac{3}{2}}\left(-\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|} - \frac{H_0\|\boldsymbol{q}\|p_1}{\mu}\right)p_2 \\ +\|\boldsymbol{q}\|\sqrt{\frac{\mu}{2}}(-H_0)^{-\frac{3}{2}}\left(\frac{q_1q_2}{2\|\boldsymbol{q}\|^4} - \frac{q_2p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^3} + \frac{H_0p_1p_2}{\mu}\right) \end{bmatrix} cos\psi \\
& + \sqrt{\frac{-2H_0}{\mu}}\,[\|\boldsymbol{q}\|p_2](-1)(-H_0)^{-1}\left[-\frac{q_1}{2\|\boldsymbol{q}\|^2} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right] sin\psi \\
& + \begin{bmatrix} \frac{1}{\|\boldsymbol{q}\|}\sqrt{\frac{\mu}{2}}(-H_0)^{-\frac{3}{2}}\left(-\frac{q_1p_2}{2\mu\|\boldsymbol{q}\|} + \frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2} + \frac{H_0q_1p_2}{\mu} - \frac{H_0q_2p_1}{\mu}\right) \\ +q_2(-1)\left(\frac{1}{\|\boldsymbol{q}\|^2}\right)\sqrt{\frac{\mu}{2}}(-H_0)^{-\frac{3}{2}}\left(-\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|} - \frac{H_0\|\boldsymbol{q}\|p_1}{\mu}\right) \\ -\frac{p_2}{\mu}\sqrt{\frac{\mu}{2}}(-H_0)^{-\frac{3}{2}}\left(-\frac{q_1}{2\|\boldsymbol{q}\|^2} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right) \\ -\frac{(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\sqrt{\frac{\mu}{2}}(-H_0)^{-\frac{3}{2}}\left(\frac{q_1q_2}{2\|\boldsymbol{q}\|^4} - \frac{q_2p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^3} + \frac{H_0p_1p_2}{\mu}\right) \end{bmatrix} sin\psi \\
& + \left[\frac{q_2}{\|\boldsymbol{q}\|} - \frac{p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right](-H_0)^{-1}\left[-\frac{q_1}{2\|\boldsymbol{q}\|^2} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right] cos\psi,
\end{aligned}
$$

$$
\begin{aligned}
x_2' = {} & (-H_0)^{-1}\begin{bmatrix} -\frac{q_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2} + \frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|} - \frac{H_0\|\boldsymbol{q}\|p_1p_2}{\mu} + \frac{q_1q_2}{2\|\boldsymbol{q}\|^3} - \frac{q_2p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2} \\ +\frac{H_0p_1p_{2\|\boldsymbol{q}\|}}{\mu} - \frac{q_1q_2}{2\|\boldsymbol{q}\|^3} + \frac{q_2p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2} + \frac{q_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2} - \frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|} \end{bmatrix} cos\psi \\
& + \sqrt{\frac{-2H_0}{\mu}}(-H_0)^{-1}\left[+\frac{q_1p_2}{2\|\boldsymbol{q}\|} - \frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu}\right] sin\psi \\
& + \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\begin{bmatrix} -\frac{q_1p_2}{2\mu\|\boldsymbol{q}\|^2} + \frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2\|\boldsymbol{q}\|} + \frac{H_0q_1p_2}{\mu\|\boldsymbol{q}\|} - \frac{H_0q_2p_1}{\mu\|\boldsymbol{q}\|} + \frac{q_1q_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^4} - \frac{q_2p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|^3} \\ +\frac{H_0q_2p_1}{\mu\|\boldsymbol{q}\|} + \frac{q_1p_2}{2\mu\|\boldsymbol{q}\|^2} - \frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2\|\boldsymbol{q}\|} - \frac{q_1q_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^4} + \frac{q_2p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|^3} - \frac{H_0p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu^2} \end{bmatrix} sin\psi,
\end{aligned}
$$

$$
\begin{aligned}
x_2' = {} & (-H_0)^{-1}\left[-\frac{H_0\|\boldsymbol{q}\|p_1p_2}{\mu} + \frac{H_0p_1p_2\|\boldsymbol{q}\|}{\mu}\right] cos\psi \\
& + \sqrt{\frac{-2\mathrm{H}_0}{\mu}}(-H_0)^{-1}\left[+\frac{q_1p_2}{2\|\boldsymbol{q}\|} - \frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu}\right] sin\psi \\
& + \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[+\frac{H_0q_1p_2}{\mu\|\boldsymbol{q}\|} - \frac{H_0p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu^2}\right] sin\psi,
\end{aligned}
$$

$$
x_2' = (-H_0)^{-1}[0]cos\psi + \sqrt{\frac{1}{-2\mu H_0}}\left[+\frac{q_1p_2}{\|\boldsymbol{q}\|} - \frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right] sin\psi + \sqrt{\frac{1}{-2\mu H_0}}\left[-\frac{q_1p_2}{\|\boldsymbol{q}\|} + \frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right] sin\psi,
$$

$$
x_2' = 0.
$$

$x_3' = 0,$

analogous to the last case.

$$y_0' = [(\boldsymbol{q}\cdot\boldsymbol{p})']cos\psi$$
$$+(\boldsymbol{q}\cdot\boldsymbol{p})(-1)sin\psi(\psi')$$
$$-\sqrt{\frac{\mu}{-2H_0}}\left[\frac{\|\boldsymbol{q}\|'\|\boldsymbol{p}\|^2}{\mu}+\frac{\|\boldsymbol{q}\|(2)(\boldsymbol{p}\cdot\boldsymbol{p}')}{\mu}\right]sin\psi$$
$$-\sqrt{\frac{\mu}{-2H_0}}\left[\left(\frac{\|\boldsymbol{q}\|\cdot\|\boldsymbol{p}\|^2}{\mu}-1\right)\right]cos\psi(\psi'),$$

$$y_0' = +\sqrt{\frac{\mu}{2}}(-H_0)^{-\frac{3}{2}}\left[-\frac{q_1}{2\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right]cos\psi$$
$$-\sqrt{\frac{\mu}{-2H_0}}\left(\frac{\|\boldsymbol{q}\|\cdot\|\boldsymbol{p}\|^2}{\mu}-1\right)(-H_0)^{-1}\left[-\frac{q_1}{2\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right]cos\psi$$
$$-(\boldsymbol{q}\cdot\boldsymbol{p})(-H_0)^{-1}\left[-\frac{q_1}{2\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right]sin\psi$$
$$-\sqrt{\frac{\mu}{-2H_0}}\left(\frac{1}{\mu}\right)\sqrt{\frac{\mu}{2}}(-H_0)^{-\frac{3}{2}}\left[-\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|}-\frac{H_0\|\boldsymbol{q}\|p_1}{\mu}\right]\|\boldsymbol{p}\|^2\ sin\psi$$
$$-\sqrt{\frac{\mu}{-2H_0}}\left(\frac{1}{\mu}\right)2\|\boldsymbol{q}\|\sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\|\boldsymbol{q}\|^4}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu\|\boldsymbol{q}\|^3}+\frac{H_0p_1}{\|\boldsymbol{q}\|}\right]sin\psi,$$

$$y_0' = +\sqrt{\frac{\mu}{2}}(-H_0)^{-\frac{3}{2}}\left[-\frac{q_1}{2\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right]cos\psi$$
$$+\sqrt{\frac{\mu}{-2H_0}}(-H_0)^{-1}\left[+\frac{q_1\|\boldsymbol{p}\|^2}{2\mu\|\boldsymbol{q}\|}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{2\mu^2}-\frac{q_1}{2\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right]cos\psi$$
$$+(-H_0)^{-1}\left[+\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\|\boldsymbol{q}\|^2}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu\|\boldsymbol{q}\|}\right]sin\psi$$
$$-\sqrt{\frac{\mu}{-2H_0}}\left(\frac{1}{\mu}\right)\sqrt{\frac{\mu}{2}}(-H_0)^{-\frac{3}{2}}\left[-\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{2\mu\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2\|\boldsymbol{p}\|^2}{2\mu^2\|\boldsymbol{q}\|}-\frac{H_0\|\boldsymbol{q}\|p_1\|\boldsymbol{p}\|^2}{\mu}\right]\ sin\psi$$
$$-\sqrt{\frac{\mu}{-2H_0}}\left(\frac{1}{\mu}\right)\sqrt{\frac{\mu}{2}}(-H_0)^{-\frac{3}{2}}\left[\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|^3}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{\mu\|\boldsymbol{q}\|^2}+\frac{2H_0p_1}{1}\right]sin\psi,$$

$$y_0' = +\sqrt{\frac{\mu}{2}}(-H_0)^{-\frac{3}{2}}\left[-\frac{q_1}{2\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}+\frac{q_1\|\boldsymbol{p}\|^2}{2\mu\|\boldsymbol{q}\|}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{2\mu^2}-\frac{q_1}{2\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right]cos\psi$$
$$+(-H_0)^{-1}\left[+\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\|\boldsymbol{q}\|^2}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu\|\boldsymbol{q}\|}\right]sin\psi$$
$$-\frac{1}{2}(-H_0)^{-2}\left[-\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{2\mu\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2\|\boldsymbol{p}\|^2}{2\mu^2\|\boldsymbol{q}\|}-\frac{H_0\|\boldsymbol{q}\|p_1\|\boldsymbol{p}\|^2}{\mu}+\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|^3}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{\mu\|\boldsymbol{q}\|^2}+\frac{2H_0p_1}{1}\right]\ sin\psi,$$

$$y_0' = +\sqrt{\frac{\mu}{2}}(-H_0)^{-\frac{3}{2}}\left[+\frac{q_1}{\|\boldsymbol{q}\|}\left(\frac{\|\boldsymbol{p}\|^2}{2\mu}-\frac{1}{\|\boldsymbol{q}\|}\right)-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\left(\frac{\|\boldsymbol{p}\|^2}{2\mu}-\frac{1}{\|\boldsymbol{q}\|}\right)\right]cos\psi$$
$$+(-H_0)^{-1}\left[+\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\|\boldsymbol{q}\|^2}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu\|\boldsymbol{q}\|}\right]sin\psi$$
$$-\frac{1}{2}(-H_0)^{-2}\left[-\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|^2}\left(\frac{\|\boldsymbol{p}\|^2}{2\mu}-\frac{1}{\|\boldsymbol{q}\|}\right)+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{\mu\|\boldsymbol{q}\|}\left(\frac{\|\boldsymbol{p}\|^2}{2\mu}-\frac{1}{\|\boldsymbol{q}\|}\right)-\frac{H_0\|\boldsymbol{q}\|p_1\|\boldsymbol{p}\|^2}{\mu}+\frac{2H_0p_1}{1}\right]\ sin\psi,$$

$$y_0' = +\sqrt{\frac{\mu}{2}}(-H_0)^{-\frac{3}{2}}\left[+\frac{H_0q_1}{\|\boldsymbol{q}\|}-\frac{H_0p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right]cos\psi$$

$$+(-H_0)^{-1}\left[+\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\|\boldsymbol{q}\|^2}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu\|\boldsymbol{q}\|}\right]sin\psi$$

$$-(-H_0)^{-2}\left[-\frac{H_0 q_1(\boldsymbol{q}\cdot\mathrm{p})}{2\|\boldsymbol{q}\|^2}+\frac{H_0 p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu\|\boldsymbol{q}\|}-\frac{H_0\|\boldsymbol{q}\|p_1\|\boldsymbol{p}\|^2}{2\mu}+\frac{H_0 p_1}{1}\right]\ sin\psi,$$

$$y_0' = +\sqrt{\frac{\mu}{-2H_0}}\left[-\frac{q_1}{\|\boldsymbol{q}\|}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right]cos\psi$$

$$+(-H_0)^{-1}\left[+\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\|\boldsymbol{q}\|^2}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu\|\boldsymbol{q}\|}\right]sin\psi$$

$$+(-H_0)^{-1}\left[-\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu\|\boldsymbol{q}\|}-\frac{\|\boldsymbol{q}\|p_1\|\boldsymbol{p}\|^2}{2\mu}+\frac{p_1}{1}\right]\ sin\psi,$$

$$y_0' = +\sqrt{\frac{\mu}{-2H_0}}\left[-\frac{q_1}{\|\boldsymbol{q}\|}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right]cos\psi+(-H_0)^{-1}\left[-\frac{\|\boldsymbol{q}\|p_1\|\boldsymbol{p}\|^2}{2\mu}+\frac{p_1}{1}\right]\ sin\psi,$$

$$y_0' = +\sqrt{\frac{\mu}{-2H_0}}\left[-\frac{q_1}{\|\boldsymbol{q}\|}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right]cos\psi+(-H_0)^{-1}\left[-\frac{\|\boldsymbol{q}\|p_1}{1}\left(\frac{\|\boldsymbol{p}\|^2}{2\mu}-\frac{1}{\|\boldsymbol{q}\|}\right)\right]\ sin\psi,$$

$$y_0' = +\sqrt{\frac{\mu}{-2H_0}}\left[-\frac{q_1}{\|\boldsymbol{q}\|}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right]cos\psi+\|\boldsymbol{q}\|p_1\ sin\psi,$$

$$y_0' = -y_1.$$

$$y_1' = \sqrt{\frac{\mu}{-2H_0}}\left[\frac{q_1{}'}{\|\boldsymbol{q}\|}+q_1(-1)\left(\frac{1}{\|\boldsymbol{q}\|^2}\right)\|\boldsymbol{q}\|'-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})'}{\mu}-\frac{(\boldsymbol{q}\cdot\boldsymbol{p})p_1'}{\mu}\right]cos\psi$$

$$+\sqrt{\frac{\mu}{-2H_0}}\left[\frac{q_1}{\|\boldsymbol{q}\|}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right](-1)sin\psi(\psi')$$

$$-[p_1\|\boldsymbol{q}\|'+\|\boldsymbol{q}\|p_1{}']sin\psi-[\|\boldsymbol{q}\|p_1]cos\psi(\psi'),$$

$$y_1' = \sqrt{\frac{\mu}{-2H_0}}\begin{bmatrix} \frac{1}{\|\boldsymbol{q}\|}\sqrt{\frac{\mu}{2}}(-H_0)^{-\frac{3}{2}}\left(-\frac{q_1p_1}{2\mu\|\boldsymbol{q}\|}+\frac{p_1{}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2}-\frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right) \\ +q_1(-1)\left(\frac{1}{\|\boldsymbol{q}\|^2}\right)\sqrt{\frac{\mu}{2}}(-H_0)^{-\frac{3}{2}}\left(-\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|}-\frac{H_0\|\boldsymbol{q}\|p_1}{\mu}\right) \\ -\frac{p_1}{\mu}\sqrt{\frac{\mu}{2}}(-H_0)^{-\frac{3}{2}}\left(-\frac{q_1}{2\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right) \\ -\frac{(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\sqrt{\frac{\mu}{2}}(-H_0)^{-\frac{3}{2}}\left(\frac{q_1{}^2}{2\|\boldsymbol{q}\|^4}-\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^3}+\frac{H_0}{\|\boldsymbol{q}\|}-\frac{H_0\|\boldsymbol{p}\|^2}{\mu}+\frac{H_0p_1{}^2}{\mu}\right) \end{bmatrix}cos\psi$$

$$+\sqrt{\frac{\mu}{-2H_0}}\left[\frac{q_1}{\|\boldsymbol{q}\|}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right](-1)(-H_0)^{-1}\left[-\frac{q_1}{2\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right]sin\psi$$

$$-\begin{bmatrix} p_1\sqrt{\frac{\mu}{2}}(-H_0)^{-\frac{3}{2}}\left(-\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|}-\frac{H_0\|\boldsymbol{q}\|p_1}{\mu}\right) \\ +\|\boldsymbol{q}\|\sqrt{\frac{\mu}{2}}(-H_0)^{-\frac{3}{2}}\left(\frac{q_1{}^2}{2\|\boldsymbol{q}\|^4}-\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^3}+\frac{H_0}{\|\boldsymbol{q}\|}-\frac{H_0\|\boldsymbol{p}\|^2}{\mu}+\frac{H_0p_1{}^2}{\mu}\right) \end{bmatrix}sin\psi$$

$$-[\|\boldsymbol{q}\|p_1](-H_0)^{-1}\left[-\frac{q_1}{2\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right]cos\psi,$$

$$y_1' = \sqrt{\frac{\mu}{-2H_0}}\left[\begin{array}{c} \sqrt{\frac{\mu}{2}}(-H_0)^{-\frac{3}{2}}\left(-\frac{q_1p_1}{2\mu\|\boldsymbol{q}\|^2}+\frac{{p_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2\|\boldsymbol{q}\|}-\frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu\|\boldsymbol{q}\|}\right) \\ +\sqrt{\frac{\mu}{2}}(-H_0)^{-\frac{3}{2}}\left(+\frac{{q_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^4}-\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|^3}+\frac{H_0q_1p_1}{\mu\|\boldsymbol{q}\|}\right) \\ +\sqrt{\frac{\mu}{2}}(-H_0)^{-\frac{3}{2}}\left(+\frac{q_1p_1}{2\mu\|\boldsymbol{q}\|^2}-\frac{{p_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2\|\boldsymbol{q}\|}\right) \\ +\sqrt{\frac{\mu}{2}}(-H_0)^{-\frac{3}{2}}\left(-\frac{{q_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^4}+\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|^3}-\frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|\mu}+\frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{\mu^2}-\frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p}){p_1}^2}{\mu^2}\right) \end{array}\right]cos\psi$$

$$+(-H_0)^{-1}\left[+\frac{q_1p_1}{2\|\boldsymbol{q}\|}-\frac{{p_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu}\right]cos\psi$$

$$+\sqrt{\frac{\mu}{-2H_0}}(-H_0)^{-1}\left[+\frac{{q_1}^2}{2\|\boldsymbol{q}\|^3}-\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}-\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}+\frac{{p_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|}\right]sin\psi$$

$$+\left[\begin{array}{c} \sqrt{\frac{\mu}{2}}(-H_0)^{-\frac{3}{2}}\left(+\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}-\frac{{p_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|}+\frac{H_0\|\boldsymbol{q}\|{p_1}^2}{\mu}\right) \\ +\sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left(-\frac{{q_1}^2}{2\|\boldsymbol{q}\|^3}+\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}-\frac{H_0}{1}+\frac{H_0\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2}{\mu}-\frac{H_0\|\boldsymbol{q}\|{p_1}^2}{\mu}\right) \end{array}\right]sin\psi,$$

$$y_1' = \frac{\mu}{2}(-H_0)^{-2}\left[\begin{array}{c} -\frac{q_1p_1}{2\mu\|\boldsymbol{q}\|^2}+\frac{{p_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2\|\boldsymbol{q}\|}-\frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu\|\boldsymbol{q}\|}+\frac{{q_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^4}-\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|^3}+\frac{H_0q_1p_1}{\mu\|\boldsymbol{q}\|}+\frac{q_1p_1}{2\mu\|\boldsymbol{q}\|^2} \\ -\frac{{p_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2\|\boldsymbol{q}\|}-\frac{{q_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^4}+\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|^3}-\frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|\mu}+\frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{\mu^2}-\frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p}){p_1}^2}{\mu^2} \end{array}\right]cos\psi$$

$$+(-H_0)^{-1}\left[+\frac{q_1p_1}{2\|\boldsymbol{q}\|}-\frac{{p_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu}\right]cos\psi$$

$$+\sqrt{\frac{\mu}{-2H_0}}(-H_0)^{-1}\left[\begin{array}{c} +\frac{{q_1}^2}{2\|\boldsymbol{q}\|^3}-\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}-\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}+\frac{{p_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|}+\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}-\frac{{p_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|} \\ +\frac{H_0\|\boldsymbol{q}\|{p_1}^2}{\mu}-\frac{{q_1}^2}{2\|\boldsymbol{q}\|^3}+\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}-\frac{H_0}{1}+\frac{H_0\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2}{\mu}-\frac{H_0\|\boldsymbol{q}\|{p_1}^2}{\mu} \end{array}\right]sin\psi,$$

$$y_1' = \frac{\mu}{2}(-H_0)^{-2}\left[-\frac{2H_0(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu\|\boldsymbol{q}\|}+\frac{H_0q_1p_1}{\mu\|\boldsymbol{q}\|}+\frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{\mu^2}-\frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p}){p_1}^2}{\mu^2}\right]cos\psi$$

$$+(-H_0)^{-1}\left[+\frac{q_1p_1}{2\|\boldsymbol{q}\|}-\frac{{p_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu}\right]cos\psi$$

$$+\sqrt{\frac{\mu}{-2H_0}}(-H_0)^{-1}\left[\begin{array}{c} +\frac{{q_1}^2}{2\|\boldsymbol{q}\|^3}-\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}-\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}+\frac{{p_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|}+\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}-\frac{{p_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|} \\ +\frac{H_0\|\boldsymbol{q}\|{p_1}^2}{\mu}-\frac{{q_1}^2}{2\|\boldsymbol{q}\|^3}+\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}-\frac{H_0}{1}+\frac{H_0\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2}{\mu}-\frac{H_0\|\boldsymbol{q}\|{p_1}^2}{\mu} \end{array}\right]sin\psi,$$

$$y_1' = (-H_0)^{-1}\left[\frac{(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|}-\frac{q_1p_1}{2\|\boldsymbol{q}\|}-\frac{(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{2\mu}+\frac{(\boldsymbol{q}\cdot\boldsymbol{p}){p_1}^2}{2\mu}\right]cos\psi$$

$$+(-H_0)^{-1}\left[+\frac{q_1p_1}{2\|\boldsymbol{q}\|}-\frac{{p_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu}\right]cos\psi$$

$$+\sqrt{\frac{\mu}{-2H_0}}(-H_0)^{-1}\left[-\frac{H_0}{1}+\frac{H_0\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2}{\mu}\right]sin\psi,$$

$$y_1' = (-H_0)^{-1}\left[\frac{(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|}-\frac{(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{2\mu}\right]cos\psi+\sqrt{\frac{\mu}{-2H_0}}\left[+\frac{1}{1}-\frac{\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2}{\mu}\right]sin\psi,$$

$$y_1' = [(\boldsymbol{q}\cdot\boldsymbol{p})\ ]cos\psi+\sqrt{\frac{\mu}{-2H_0}}\left[+1-\frac{\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2}{\mu}\right]sin\psi,$$

$$y_1' = y_0.$$

$$y_2' = \sqrt{\frac{\mu}{-2H_0}}\left[\frac{q_2{}'}{\|\boldsymbol{q}\|} + q_2(-1)\left(\frac{1}{\|\boldsymbol{q}\|^2}\right)\|\boldsymbol{q}\|' - \frac{p_2(\boldsymbol{q}\cdot\boldsymbol{p})'}{\mu} - \frac{(\boldsymbol{q}\cdot\boldsymbol{p})p_2'}{\mu}\right]cos\psi$$
$$+\sqrt{\frac{\mu}{-2H_0}}\left[\frac{q_2}{\|\boldsymbol{q}\|} - \frac{p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right](-1)sin\psi(\psi')$$
$$-[p_2\|\boldsymbol{q}\|' + \|\boldsymbol{q}\|p_2{}']sin\psi$$
$$-[\|\boldsymbol{q}\|p_2]cos\psi(\psi'),$$

$$y_2' = \sqrt{\frac{\mu}{-2H_0}}\begin{bmatrix} \frac{1}{\|\boldsymbol{q}\|}\sqrt{\frac{\mu}{2}}(-H_0)^{-\frac{3}{2}}\left(-\frac{q_1p_2}{2\mu\|\boldsymbol{q}\|} + \frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2} + \frac{H_0q_1p_2}{\mu} - \frac{H_0q_2p_1}{\mu}\right) \\ +q_2(-1)\left(\frac{1}{\|\boldsymbol{q}\|^2}\right)\sqrt{\frac{\mu}{2}}(-H_0)^{-\frac{3}{2}}\left(-\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|} - \frac{H_0\|\boldsymbol{q}\|p_1}{\mu}\right) \\ -\frac{p_2}{\mu}\sqrt{\frac{\mu}{2}}(-H_0)^{-\frac{3}{2}}\left(-\frac{q_1}{2\|\boldsymbol{q}\|^2} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right) \\ -\frac{(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\sqrt{\frac{\mu}{2}}(-H_0)^{-\frac{3}{2}}\left(\frac{q_1q_2}{2\|\boldsymbol{q}\|^4} - \frac{q_2p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^3} + \frac{H_0p_1p_2}{\mu}\right) \end{bmatrix}cos\psi$$
$$+\sqrt{\frac{\mu}{-2H_0}}\left[\frac{q_2}{\|\boldsymbol{q}\|} - \frac{p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right](-1)(-H_0)^{-1}\left[-\frac{q_1}{2\|\boldsymbol{q}\|^2} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right]sin\psi$$
$$+\begin{bmatrix} -p_2\sqrt{\frac{\mu}{2}}(-H_0)^{-\frac{3}{2}}\left(-\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|} - \frac{H_0\|\boldsymbol{q}\|p_1}{\mu}\right) \\ -\|\boldsymbol{q}\|\sqrt{\frac{\mu}{2}}(-H_0)^{-\frac{3}{2}}\left(\frac{q_1q_2}{2\|\boldsymbol{q}\|^4} - \frac{q_2p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^3} + \frac{H_0p_1p_2}{\mu}\right) \end{bmatrix}sin\psi$$
$$-[\|\boldsymbol{q}\|p_2](-H_0)^{-1}\left[-\frac{q_1}{2\|\boldsymbol{q}\|^2} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right]cos\psi,$$

$$y_2' = \sqrt{\frac{\mu}{-2H_0}}\begin{bmatrix} \sqrt{\frac{\mu}{2}}(-H_0)^{-\frac{3}{2}}\left(-\frac{q_1p_2}{2\mu\|\boldsymbol{q}\|^2} + \frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2\|\boldsymbol{q}\|} + \frac{H_0q_1p_2}{\mu\|\boldsymbol{q}\|} - \frac{H_0q_2p_1}{\mu\|\boldsymbol{q}\|}\right) \\ +\sqrt{\frac{\mu}{2}}(-H_0)^{-\frac{3}{2}}\left(+\frac{q_1q_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^4} - \frac{q_2p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|^3} + \frac{H_0q_2p_1}{\mu\|\boldsymbol{q}\|}\right) \\ +\sqrt{\frac{\mu}{2}}(-H_0)^{-\frac{3}{2}}\left(+\frac{q_1p_2}{2\mu\|\boldsymbol{q}\|^2} - \frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2\|\boldsymbol{q}\|}\right) \\ +\sqrt{\frac{\mu}{2}}(-H_0)^{-\frac{3}{2}}\left(-\frac{q_1q_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^4} + \frac{q_2p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|^3} - \frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p})p_1p_2}{\mu^2}\right) \end{bmatrix}cos\psi$$
$$+(-H_0)^{-1}\left[+\frac{q_1p_2}{2\|\boldsymbol{q}\|} - \frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu}\right]cos\psi$$
$$+\sqrt{\frac{\mu}{-2H_0}}(-H_0)^{-1}\left[+\frac{q_1q_2}{2\|\boldsymbol{q}\|^3} - \frac{q_2p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2} - \frac{q_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2} + \frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|}\right]sin\psi$$
$$+\begin{bmatrix} \sqrt{\frac{\mu}{2}}(-H_0)^{-\frac{3}{2}}\left(+\frac{q_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2} - \frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|} + \frac{H_0\|\boldsymbol{q}\|p_1p_2}{\mu}\right) \\ +\sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left(-\frac{q_1q_2}{2\|\boldsymbol{q}\|^3} + \frac{q_2p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2} - \frac{H_0\|\boldsymbol{q}\|p_1p_2}{\mu}\right) \end{bmatrix}sin\psi,$$

$$y_2' = \frac{\mu}{2}(-H_0)^{-2}\begin{bmatrix} -\frac{q_1p_2}{2\mu\|\boldsymbol{q}\|^2} + \frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2\|\boldsymbol{q}\|} + \frac{H_0q_1p_2}{\mu\|\boldsymbol{q}\|} - \frac{H_0q_2p_1}{\mu\|\boldsymbol{q}\|} + \frac{q_1q_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^4} - \frac{q_2p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|^3} \\ +\frac{H_0q_2p_1}{\mu\|\boldsymbol{q}\|} + +\frac{q_1p_2}{2\mu\|\boldsymbol{q}\|^2} - \frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2\|\boldsymbol{q}\|} - \frac{q_1q_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^4} + \frac{q_2p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|^3} - \frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p})p_1p_2}{\mu^2} \end{bmatrix}cos\psi$$
$$+(-H_0)^{-1}\left[+\frac{q_1p_2}{2\|\boldsymbol{q}\|} - \frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu}\right]cos\psi$$

$$+\sqrt{\frac{\mu}{-2H_0}}(-H_0)^{-1}\begin{bmatrix}+\frac{q_1q_2}{2\|\boldsymbol{q}\|^3}-\frac{q_2p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}-\frac{q_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}+\frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|}+\frac{q_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}\\-\frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|}+\frac{H_0\|\boldsymbol{q}\|p_1p_2}{\mu}-\frac{q_1q_2}{2\|\boldsymbol{q}\|^3}+\frac{q_2p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}-\frac{H_0\|\boldsymbol{q}\|p_1p_2}{\mu}\end{bmatrix}sin\psi,$$

$$y_2' = (-H_0)^{-1}\left[-\frac{q_1p_2}{2\|\boldsymbol{q}\|}+\frac{(\boldsymbol{q}\cdot\boldsymbol{p})p_1p_2}{2\mu}\right]cos\psi$$

$$+(-H_0)^{-1}\left[+\frac{q_1p_2}{2\|\boldsymbol{q}\|}-\frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu}\right]cos\psi$$

$$+\sqrt{\frac{\mu}{-2H_0}}(-H_0)^{-1}[0]sin\psi,$$

$$y_2' = 0.$$

$$y_3' = 0,$$

analogous to the last case.

Intermediate results for $H > 0$:

$$A_1 = \frac{q_1}{\|\boldsymbol{q}\|}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})-q_1\|\boldsymbol{p}\|^2}{\mu},$$

from (VIII.1.),

$$q_1' = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[\left(\frac{1}{2\mu}\right)(A_1)(p_1)+(-H_0)\left(\frac{1}{\mu}\right)(q_2p_2+q_3p_3)\right],$$

from (VIII.1.),

$$q_1' = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[\left(\frac{1}{2\mu}\right)\left(\frac{q_1}{\|\boldsymbol{q}\|}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})-q_1\|\boldsymbol{p}\|^2}{\mu}\right)(p_1)+(-H_0)\left(\frac{1}{\mu}\right)\left((\boldsymbol{q}\cdot\boldsymbol{p})-q_1p_1\right)\right],$$

$$q_1' = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[\frac{q_1p_1}{2\mu\|\boldsymbol{q}\|}+\frac{{p_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2}-\frac{q_1p_1\|\boldsymbol{p}\|^2}{2\mu^2}-\frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}+\frac{H_0q_1p_1}{\mu}\right],$$

$$q_1' = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[\frac{q_1p_1}{2\mu\|\boldsymbol{q}\|}+\frac{{p_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2}-\frac{q_1p_1\|\boldsymbol{p}\|^2}{2\mu^2}-\frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}+\frac{q_1p_1}{\mu}\left(\frac{\|\boldsymbol{p}\|^2}{2\mu}-\frac{1}{\|\boldsymbol{q}\|}\right)\right],$$

$$q_1' = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[\frac{q_1p_1}{2\mu\|\boldsymbol{q}\|}+\frac{{p_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2}-\frac{q_1p_1\|\boldsymbol{p}\|^2}{2\mu^2}-\frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}+\frac{q_1p_1\|\boldsymbol{p}\|^2}{2\mu^2}-\frac{q_1p_1}{\mu\|\boldsymbol{q}\|}\right],$$

$$q_1' = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[-\frac{q_1p_1}{2\mu\|\boldsymbol{q}\|}+\frac{{p_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2}-\frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right],$$

$$q_2' = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[\left(\frac{1}{2\mu}\right)(A_1)(p_2)+(-H_0)\left(\frac{1}{\mu}\right)(q_2p_1-2q_1p_2)\right],$$

from (VIII.1.),

$$q_2' = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[\left(\frac{1}{2\mu}\right)\left(\frac{q_1}{\|\boldsymbol{q}\|}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})-q_1\|\boldsymbol{p}\|^2}{\mu}\right)(p_2)+(-H_0)\left(\frac{1}{\mu}\right)(q_2p_1-2q_1p_2)\right],$$

$$q_2' = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[\frac{q_1p_2}{2\mu\|\boldsymbol{q}\|}+\frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2}-\frac{q_1p_2\|\boldsymbol{p}\|^2}{2\mu^2}-\frac{H_0q_2p_1}{\mu}+\frac{2H_0q_1p_2}{\mu}\right],$$

$$q_2' = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[\frac{q_1p_2}{2\mu\|\boldsymbol{q}\|} + \frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2} - \frac{q_1p_2\|\boldsymbol{p}\|^2}{2\mu^2} - \frac{H_0q_2p_1}{\mu} + \frac{2q_1p_2}{\mu}\left(\frac{\|\boldsymbol{p}\|^2}{2\mu} - \frac{1}{\|\boldsymbol{q}\|}\right)\right],$$

$$q_2' = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[\frac{q_1p_2}{2\mu\|\boldsymbol{q}\|} + \frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2} - \frac{q_1p_2\|\boldsymbol{p}\|^2}{2\mu^2} - \frac{H_0q_2p_1}{\mu} + \frac{q_1p_2\|\boldsymbol{p}\|^2}{\mu^2} - \frac{2q_1p_2}{\mu\|\boldsymbol{q}\|}\right],$$

$$q_2' = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[-\frac{3q_1p_2}{2\mu\|\boldsymbol{q}\|} + \frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2} + \frac{q_1p_2\|\boldsymbol{p}\|^2}{2\mu^2} - \frac{H_0q_2p_1}{\mu}\right],$$

$$q_2' = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[-\frac{3q_1p_2}{2\mu\|\boldsymbol{q}\|} + \frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2} + \frac{q_1p_2\|\boldsymbol{p}\|^2}{2\mu^2} - \frac{q_1p_2}{\mu\|\boldsymbol{q}\|} + \frac{q_1p_2}{\mu\|\boldsymbol{q}\|} - \frac{H_0q_2p_1}{\mu}\right],$$

$$q_2' = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[-\frac{q_1p_2}{2\mu\|\boldsymbol{q}\|} + \frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2} + \frac{H_0q_1p_2}{\mu} - \frac{H_0q_2p_1}{\mu}\right],$$

$$q_3' = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[-\frac{q_1p_3}{2\mu\|\boldsymbol{q}\|} + \frac{p_1p_3(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2} + \frac{H_0q_1p_3}{\mu} - \frac{H_0q_3p_1}{\mu}\right],$$

analogous to the last case,

$$p_1' = \sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\begin{bmatrix}\left(\frac{1}{2}\right)(A_1)(-q_1)\left(\frac{1}{\|\boldsymbol{q}\|^3}\right)\\ +(-H_0)(-\|\boldsymbol{q}\|^2 + {q_1}^2)\left(\frac{1}{\|\boldsymbol{q}\|^3}\right)\\ +(-H_0)\left(\frac{1}{\mu}\right)(\|\boldsymbol{p}\|^2 - {p_1}^2)\end{bmatrix},$$

from (VIII.1.),

$$p_1' = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\begin{bmatrix}\left(\frac{1}{2}\right)\left(\frac{q_1}{\|\boldsymbol{q}\|} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p}) - q_1\|\boldsymbol{p}\|^2}{\mu}\right)(-q_1)\left(\frac{1}{\|\boldsymbol{q}\|^3}\right)\\ +(-H_0)(-\|\boldsymbol{q}\|^2 + {q_1}^2)\left(\frac{1}{\|\boldsymbol{q}\|^3}\right)\\ +(-H_0)\left(\frac{1}{\mu}\right)(\|\boldsymbol{p}\|^2 - {p_1}^2)\end{bmatrix},$$

$$p_1' = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[-\frac{{q_1}^2}{2\|\boldsymbol{q}\|^4} - \frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^3} + \frac{{q_1}^2\|\boldsymbol{p}\|^2}{2\mu\|\boldsymbol{q}\|^3} + \frac{H_0}{\|\boldsymbol{q}\|} - \frac{H_0{q_1}^2}{\|\boldsymbol{q}\|^3} - \frac{H_0\|\boldsymbol{p}\|^2}{\mu} + \frac{H_0{p_1}^2}{\mu}\right],$$

$$p_1' = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[-\frac{{q_1}^2}{2\|\boldsymbol{q}\|^4} - \frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^3} + \frac{{q_1}^2\|\boldsymbol{p}\|^2}{2\mu\|\boldsymbol{q}\|^3} - \frac{{q_1}^2}{\|\boldsymbol{q}\|^4} + \frac{{q_1}^2}{\|\boldsymbol{q}\|^4} + \frac{H_0}{\|\boldsymbol{q}\|} - \frac{H_0{q_1}^2}{\|\boldsymbol{q}\|^3} - \frac{H_0\|\boldsymbol{p}\|^2}{\mu} + \frac{H_0{p_1}^2}{\mu}\right],$$

$$p_1' = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[-\frac{{q_1}^2}{2\|\boldsymbol{q}\|^4} - \frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^3} + \frac{H_0{q_1}^2}{\|\boldsymbol{q}\|^3} + \frac{{q_1}^2}{\|\boldsymbol{q}\|^4} + \frac{H_0}{\|\boldsymbol{q}\|} - \frac{H_0{q_1}^2}{\|\boldsymbol{q}\|^3} - \frac{H_0\|\boldsymbol{p}\|^2}{\mu} + \frac{H_0{p_1}^2}{\mu}\right],$$

$$p_1' = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[\frac{{q_1}^2}{2\|\boldsymbol{q}\|^4} - \frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^3} + \frac{H_0}{\|\boldsymbol{q}\|} - \frac{H_0\|\boldsymbol{p}\|^2}{\mu} + \frac{H_0{p_1}^2}{\mu}\right],$$

$$p_2' = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[\left(\frac{1}{2}\right)(A_1)(-q_2)\left(\frac{1}{\|\boldsymbol{q}\|^3}\right) + (-H_0)(q_1q_2)\left(\frac{1}{\|\boldsymbol{q}\|^3}\right) + (-H_0)\left(\frac{1}{\mu}\right)(-p_1p_2)\right],$$

from (VIII.1.),

$$p_2' = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\begin{bmatrix}\left(\frac{1}{2}\right)\left(\frac{q_1}{\|\boldsymbol{q}\|} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p}) - q_1\|\boldsymbol{p}\|^2}{\mu}\right)(-q_2)\left(\frac{1}{\|\boldsymbol{q}\|^3}\right)\\ +(-H_0)(q_1q_2)\left(\frac{1}{\|\boldsymbol{q}\|^3}\right)\\ +(-H_0)\left(\frac{1}{\mu}\right)(-p_1p_2)\end{bmatrix},$$

$$p_2' = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[-\frac{q_1q_2}{2\|\boldsymbol{q}\|^4} - \frac{q_2p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^3} + \frac{q_1q_2\|\boldsymbol{p}\|^2}{2\mu\|\boldsymbol{q}\|^3} - \frac{q_1q_2}{\|\boldsymbol{q}\|^4} + \frac{q_1q_2}{\|\boldsymbol{q}\|^4} - \frac{H_0q_1q_2}{\|\boldsymbol{q}\|^3} + \frac{H_0p_1p_2}{\mu}\right],$$

$$p_2' = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[\frac{q_1q_2}{2\|\boldsymbol{q}\|^4} - \frac{q_2p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^3} + \frac{H_0q_1q_2}{\|\boldsymbol{q}\|^3} - \frac{H_0q_1q_2}{\|\boldsymbol{q}\|^3} + \frac{H_0p_1p_2}{\mu}\right],$$

$$p_2' = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[\frac{q_1q_2}{2\|\boldsymbol{q}\|^4} - \frac{q_2p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^3} + \frac{H_0p_1p_2}{\mu}\right],$$

$$p_3' = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[\frac{q_1q_3}{2\|\boldsymbol{q}\|^4} - \frac{q_3p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^3} + \frac{H_0p_1p_3}{\mu}\right],$$

analogous to the last case.

$$(\boldsymbol{p}\cdot\boldsymbol{p}') = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[\frac{A_1}{2\|\boldsymbol{q}\|^3}(-\boldsymbol{q}\cdot\boldsymbol{p}) + (-H_0)\left(\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|^3} - \frac{p_1}{\|\boldsymbol{q}\|}\right) + (-H_0)\left(\frac{1}{\mu}\right)(p_1\|\boldsymbol{p}\|^2 - p_1\|\boldsymbol{p}\|^2)\right],$$

from (VIII.1.),

$$(\boldsymbol{p}\cdot\boldsymbol{p}') = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\begin{bmatrix}\left(\frac{1}{2\|\boldsymbol{q}\|^3}\right)\left(\frac{q_1}{\|\boldsymbol{q}\|} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p}) - q_1\|\boldsymbol{p}\|^2}{\mu}\right)(-\boldsymbol{q}\cdot\boldsymbol{p}) \\ +(-H_0)\left(\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|^3} - \frac{p_1}{\|\boldsymbol{q}\|}\right) \\ +(-H_0)\left(\frac{1}{\mu}\right)(p_1\|\boldsymbol{p}\|^2 - p_1\|\boldsymbol{p}\|^2)\end{bmatrix},$$

$$(\boldsymbol{p}\cdot\boldsymbol{p}') = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[-\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\|\boldsymbol{q}\|^4} - \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu\|\boldsymbol{q}\|^3} + \frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{2\mu\|\boldsymbol{q}\|^3} - \frac{H_0q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|^3} + \frac{H_0p_1}{\|\boldsymbol{q}\|}\right],$$

$$(\boldsymbol{p}\cdot\boldsymbol{p}') = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[-\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\|\boldsymbol{q}\|^4} - \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu\|\boldsymbol{q}\|^3} + \frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{2\mu\|\boldsymbol{q}\|^3} - \frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|^4} + \frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|^4} - \frac{H_0q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|^3} + \frac{H_0p_1}{\|\boldsymbol{q}\|}\right],$$

$$(\boldsymbol{p}\cdot\boldsymbol{p}') = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\|\boldsymbol{q}\|^4} - \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu\|\boldsymbol{q}\|^3} + \frac{H_0q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|^3} - \frac{H_0q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|^3} + \frac{H_0p_1}{\|\boldsymbol{q}\|}\right],$$

$$(\boldsymbol{p}\cdot\boldsymbol{p}') = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\|\boldsymbol{q}\|^4} - \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu\|\boldsymbol{q}\|^3} + \frac{H_0p_1}{\|\boldsymbol{q}\|}\right],$$

$$(\boldsymbol{q}\cdot\boldsymbol{p}') = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[\left(\frac{-A_1}{2\|\boldsymbol{q}\|}\right) + (-H_0)\left(\frac{1}{\mu}\right)(q_1\|\boldsymbol{p}\|^2) + (-H_0)\left(\frac{1}{\mu}\right)(-p_1)(\boldsymbol{q}\cdot\boldsymbol{p})\right],$$

from (VIII.1.),

$$(\boldsymbol{q}\cdot\boldsymbol{p}') = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\begin{bmatrix}\left(\frac{-1}{2\|\boldsymbol{q}\|}\right)\left(\frac{q_1}{\|\boldsymbol{q}\|} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p}) - q_1\|\boldsymbol{p}\|^2}{\mu}\right) \\ +(-H_0)\left(\frac{1}{\mu}\right)(q_1\|\boldsymbol{p}\|^2) \\ +(-H_0)\left(\frac{1}{\mu}\right)(-p_1)(\boldsymbol{q}\cdot\boldsymbol{p})\end{bmatrix},$$

$$(\boldsymbol{q}\cdot\boldsymbol{p}') = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[-\frac{q_1}{2\|\boldsymbol{q}\|^2} - \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|} + \frac{q_1\|\boldsymbol{p}\|^2}{2\mu\|\boldsymbol{q}\|} - \frac{H_0q_1\|\boldsymbol{p}\|^2}{\mu} + \frac{H_0p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right],$$

$$(\boldsymbol{q}\cdot\boldsymbol{p}') = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[-\frac{q_1}{2\|\boldsymbol{q}\|^2} - \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|} + \frac{q_1\|\boldsymbol{p}\|^2}{2\mu\|\boldsymbol{q}\|} - \frac{H_0q_1\|\boldsymbol{p}\|^2}{\mu} + \frac{H_0p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right],$$

$$(\boldsymbol{q}\cdot\boldsymbol{p}') = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[-\frac{q_1}{2\|\boldsymbol{q}\|^2} - \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|} + \frac{q_1\|\boldsymbol{p}\|^2}{2\mu\|\boldsymbol{q}\|} - \frac{q_1}{\|\boldsymbol{q}\|^2} + \frac{q_1}{\|\boldsymbol{q}\|^2} - \frac{H_0q_1\|\boldsymbol{p}\|^2}{\mu} + \frac{H_0p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right],$$

$$(\boldsymbol{q}\cdot\boldsymbol{p}') = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[\frac{q_1}{2\|\boldsymbol{q}\|^2} - \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|} + \frac{H_0 q_1}{\|\boldsymbol{q}\|} - \frac{H_0 q_1\|\boldsymbol{p}\|^2}{\mu} + \frac{H_0 p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right],$$

$$(\boldsymbol{p}\cdot\boldsymbol{q}') = -\sqrt{\frac{\mu}{2}}(-H_0)^{-3/2}\left[\left(\frac{1}{2\mu}\right)(A_1)\|\boldsymbol{p}\|^2 + (-H_0)\left(\frac{2}{\mu}\right)(p_1)(\boldsymbol{q}\cdot\boldsymbol{p}) + (-H_0)\left(\frac{2}{\mu}\right)(-q_1)\|\boldsymbol{p}\|^2\right],$$

from (VIII.1.),

$$(\boldsymbol{p}\cdot\boldsymbol{q}') = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\begin{bmatrix}\left(\frac{1}{2\mu}\right)\left(\frac{q_1}{\|\boldsymbol{q}\|} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p}) - q_1\|\boldsymbol{p}\|^2}{\mu}\right)\|\boldsymbol{p}\|^2 \\ +(-H_0)\left(\frac{2}{\mu}\right)(p_1)(\boldsymbol{q}\cdot\boldsymbol{p}) \\ +(-H_0)\left(\frac{2}{\mu}\right)(-q_1)\|\boldsymbol{p}\|^2\end{bmatrix},$$

$$(\boldsymbol{p}\cdot\boldsymbol{q}') = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[\frac{q_1\|\boldsymbol{p}\|^2}{2\mu\|\boldsymbol{q}\|} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{2\mu^2} - \frac{q_1\|\boldsymbol{p}\|^4}{2\mu^2} - \frac{2H_0 p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu} + \frac{2H_0 q_1\|\boldsymbol{p}\|^2}{\mu}\right],$$

$$(\boldsymbol{p}\cdot\boldsymbol{q}') = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[\frac{q_1\|\boldsymbol{p}\|^2}{2\mu\|\boldsymbol{q}\|} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{2\mu^2} - \frac{q_1\|\boldsymbol{p}\|^4}{2\mu^2} + \frac{q_1\|\boldsymbol{p}\|^2}{\mu\|\boldsymbol{q}\|} - \frac{q_1\|\boldsymbol{p}\|^2}{\mu\|\boldsymbol{q}\|} - \frac{2H_0 p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu} + \frac{2H_0 q_1\|\boldsymbol{p}\|^2}{\mu}\right],$$

$$(\boldsymbol{p}\cdot\boldsymbol{q}') = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[-\frac{q_1\|\boldsymbol{p}\|^2}{2\mu\|\boldsymbol{q}\|} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{2\mu^2} - \frac{H_0 q_1\|\boldsymbol{p}\|^2}{\mu} - \frac{2H_0 p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu} + \frac{2H_0 q_1\|\boldsymbol{p}\|^2}{\mu}\right],$$

$$(\boldsymbol{p}\cdot\boldsymbol{q}') = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\begin{bmatrix}-\frac{q_1\|\boldsymbol{p}\|^2}{2\mu\|\boldsymbol{q}\|} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{2\mu^2} - \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu\|\boldsymbol{q}\|} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu\|\boldsymbol{q}\|} \\ -\frac{H_0 q_1\|\boldsymbol{p}\|^2}{\mu} - \frac{2H_0 p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu} + \frac{2H_0 q_1\|\boldsymbol{p}\|^2}{\mu}\end{bmatrix},$$

$$(\boldsymbol{p}\cdot\boldsymbol{q}') = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[-\frac{q_1\|\boldsymbol{p}\|^2}{2\mu\|\boldsymbol{q}\|} - \frac{H_0 p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu\|\boldsymbol{q}\|} + \frac{H_0 q_1\|\boldsymbol{p}\|^2}{\mu}\right],$$

$$(\boldsymbol{p}\cdot\boldsymbol{q}') = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[-\frac{q_1\|\boldsymbol{p}\|^2}{2\mu\|\boldsymbol{q}\|} + \frac{q_1}{\|\boldsymbol{q}\|^2} - \frac{q_1}{\|\boldsymbol{q}\|^2} - \frac{H_0 p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu\|\boldsymbol{q}\|} + \frac{H_0 q_1\|\boldsymbol{p}\|^2}{\mu}\right],$$

$$(\boldsymbol{p}\cdot\boldsymbol{q}') = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[-\frac{H_0 q_1}{\|\boldsymbol{q}\|} - \frac{q_1}{\|\boldsymbol{q}\|^2} - \frac{H_0 p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu\|\boldsymbol{q}\|} + \frac{H_0 q_1\|\boldsymbol{p}\|^2}{\mu}\right],$$

$$(\boldsymbol{q}\cdot\boldsymbol{p})' = (\boldsymbol{q}\cdot\boldsymbol{p}') + (\boldsymbol{p}\cdot\boldsymbol{q}'),$$

$$(\boldsymbol{q}\cdot\boldsymbol{p})' = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\begin{bmatrix}\frac{q_1}{2\|\boldsymbol{q}\|^2} - \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|} + \frac{H_0 q_1}{\|\boldsymbol{q}\|} - \frac{H_0 q_1\|\boldsymbol{p}\|^2}{\mu} + \frac{H_0 p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu} \\ -\frac{H_0 q_1}{\|\boldsymbol{q}\|} - \frac{q_1}{\|\boldsymbol{q}\|^2} - \frac{H_0 p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu\|\boldsymbol{q}\|} + \frac{H_0 q_1\|\boldsymbol{p}\|^2}{\mu}\end{bmatrix},$$

$$(\boldsymbol{q}\cdot\boldsymbol{p})' = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[-\frac{q_1}{2\|\boldsymbol{q}\|^2} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right].$$

$$(\boldsymbol{q}\cdot\boldsymbol{q}') = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\begin{bmatrix}\left(\frac{1}{2\mu}\right)(A_1)(\boldsymbol{q}\cdot\boldsymbol{p}) \\ -\left(\frac{H_0}{\mu}\right)(q_1q_2p_2 + q_1q_2p_2 + q_2q_2p_1 - 2q_1q_2p_2 + q_3q_3p_1 - 2q_1q_3p_3)\end{bmatrix},$$

from (VIII.1.),

$$(\boldsymbol{q}\cdot\boldsymbol{q}') = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[\left(\frac{1}{2\mu}\right)(A_1)(\boldsymbol{q}\cdot\boldsymbol{p}) - \left(\frac{H_0}{\mu}\right)(-q_1q_2p_2 - q_1q_3p_3 + q_2q_2p_1 + q_3q_3p_1)\right],$$

$$(\boldsymbol{q}\cdot\boldsymbol{q}') = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\begin{bmatrix} \left(\frac{1}{2\mu}\right)(A_1)(\boldsymbol{q}\cdot\boldsymbol{p}) \\ -\left(\frac{H_0}{\mu}\right)(-q_1q_1p_1 - q_1q_2p_2 - q_1q_3p_3 + q_1q_1p_1 + q_2q_2p_1 + q_3q_3p_1)\end{bmatrix},$$

$$(\boldsymbol{q}\cdot\boldsymbol{q}') = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\begin{bmatrix} \left(\frac{1}{2\mu}\right)\left(\frac{q_1}{\|\boldsymbol{q}\|} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p}) - q_1\|\boldsymbol{p}\|^2}{\mu}\right)(\boldsymbol{q}\cdot\boldsymbol{p}) \\ -\left(\frac{H_0}{\mu}\right)(-q_1(\boldsymbol{q}\cdot\boldsymbol{p}) + \|\boldsymbol{q}\|^2 p_1)\end{bmatrix},$$

$$(\boldsymbol{q}\cdot\boldsymbol{q}') = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2} - \frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{2\mu^2} + \frac{H_0 q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu} - \frac{H_0\|\boldsymbol{q}\|^2 p_1}{\mu}\right],$$

$$(\boldsymbol{q}\cdot\boldsymbol{q}') = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2} - \frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{2\mu^2} + \frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\left(\frac{\|\boldsymbol{p}\|^2}{2\mu} - \frac{1}{\|\boldsymbol{q}\|}\right) - \frac{H_0\|\boldsymbol{q}\|^2 p_1}{\mu}\right],$$

$$(\boldsymbol{q}\cdot\boldsymbol{q}') = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[-\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2} - \frac{H_0\|\boldsymbol{q}\|^2 p_1}{\mu}\right],$$

$$\|\boldsymbol{q}\|' = \sqrt{(\boldsymbol{q}\cdot\boldsymbol{q})}' = \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{(\boldsymbol{q}\cdot\boldsymbol{q})}}\right)2(\boldsymbol{q}\cdot\boldsymbol{q}') = \frac{(\boldsymbol{q}\cdot\boldsymbol{q}')}{\|\boldsymbol{q}\|},$$

$$\|\boldsymbol{q}\|' = -\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[-\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|} - \frac{H_0\|\boldsymbol{q}\|p_1}{\mu}\right].$$

$$\psi' = \sqrt{\frac{2H_0}{\mu}}(\boldsymbol{q}\cdot\boldsymbol{p})',$$

$$\psi' = \sqrt{\frac{2H_0}{\mu}}(-1)\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[-\frac{q_1}{2\|\boldsymbol{q}\|^2} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right],$$

$$\psi' = -(H_0)^{-1}\left[-\frac{q_1}{2\|\boldsymbol{q}\|^2} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right].$$

Main results for $H > 0$:

$$\begin{aligned} x_0' = {} & \left(\frac{1}{\mu}\right)[\|\boldsymbol{q}\|'\|\boldsymbol{p}\|^2 + \|\boldsymbol{q}\|(2)(\boldsymbol{p}\cdot\boldsymbol{p}')]cosh\psi \\ & + \left[\left(\frac{\|\boldsymbol{q}\|\cdot\|\boldsymbol{p}\|^2}{\mu} - 1\right)\right]sinh\psi(\psi') \\ & - \sqrt{\frac{2H_0}{\mu}}[(\boldsymbol{q}\cdot\boldsymbol{p})']sinh\psi \\ & - \sqrt{\frac{2H_0}{\mu}}(\boldsymbol{q}\cdot\boldsymbol{p})cosh\psi(\psi'), \end{aligned}$$

$$\begin{aligned} x_0' = {} & \left(\frac{1}{\mu}\right)(-1)\sqrt{\frac{\mu}{2}}(H_0)^{-\frac{3}{2}}\left[-\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|} - \frac{H_0\|\boldsymbol{q}\|p_1}{\mu}\right]\|\boldsymbol{p}\|^2 cosh\psi \\ & + \left(\frac{2}{\mu}\right)\|\boldsymbol{q}\|(-1)\sqrt{\frac{\mu}{2}}(H_0)^{-\frac{3}{2}}\left[\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\|\boldsymbol{q}\|^4} - \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu\|\boldsymbol{q}\|^3} + \frac{H_0 p_1}{\|\boldsymbol{q}\|}\right]cosh\psi \\ & - \sqrt{\frac{2H_0}{\mu}}(\boldsymbol{q}\cdot\boldsymbol{p})(-1)(H_0)^{-1}\left[-\frac{q_1}{2\|\boldsymbol{q}\|^2} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right]cosh\psi \\ & + \left(\frac{\|\boldsymbol{q}\|\cdot\|\boldsymbol{p}\|^2}{\mu} - 1\right)(-1)(H_0)^{-1}\left[-\frac{q_1}{2\|\boldsymbol{q}\|^2} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right]sinh\psi \end{aligned}$$

$$-\sqrt{\frac{2H_0}{\mu}}(-1)\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[-\frac{q_1}{2\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right]sinh\psi,$$

$$x_0'=\left(\frac{1}{\mu}\right)\sqrt{\frac{\mu}{2}}(H_0)^{-\frac{3}{2}}\left[+\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{2\mu\|\boldsymbol{q}\|^2}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2\|\boldsymbol{p}\|^2}{2\mu^2\|\boldsymbol{q}\|}+\frac{H_0\|\boldsymbol{q}\|p_1\|\boldsymbol{p}\|^2}{\mu}\right]cosh\psi$$

$$+\left(\frac{2}{\mu}\right)\sqrt{\frac{\mu}{2}}(H_0)^{-\frac{3}{2}}\left[-\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\|\boldsymbol{q}\|^3}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu\|\boldsymbol{q}\|^2}-\frac{H_0p_1}{1}\right]cosh\psi$$

$$+\sqrt{\frac{2}{\mu H_0}}\left[-\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu\|\boldsymbol{q}\|}\right]cosh\psi$$

$$+(H_0)^{-1}\left[+\frac{q_1\|\boldsymbol{p}\|^2}{2\mu\|\boldsymbol{q}\|}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{q}\|\cdot\|\boldsymbol{p}\|^2}{2\mu^2\|\boldsymbol{q}\|}-\frac{q_1}{2\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right]sinh\psi$$

$$+(H_0)^{-1}\left[-\frac{q_1}{2\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right]sinh\psi,$$

$$x_0'=\sqrt{\frac{1}{2\mu}}(H_0)^{-\frac{3}{2}}\begin{bmatrix}+\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{2\mu\|\boldsymbol{q}\|^2}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2\|\boldsymbol{p}\|^2}{2\mu^2\|\boldsymbol{q}\|}+\frac{H_0p_1\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2}{\mu}-\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|^3}\\ +\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{\mu\|\boldsymbol{q}\|^2}-\frac{2H_0p_1}{1}-\frac{H_0q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|^2}+\frac{H_0p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{\mu\|\boldsymbol{q}\|}\end{bmatrix}cosh\psi$$

$$+(\mathrm{H}_0)^{-1}\left[+\frac{q_1\|\boldsymbol{p}\|^2}{2\mu\|\boldsymbol{q}\|}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{2\mu^2}-\frac{q_1}{\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu\|\boldsymbol{q}\|}\right]sinh\psi,$$

$$x_0'=\sqrt{\frac{1}{2\mu}}(H_0)^{-\frac{3}{2}}\begin{bmatrix}\left(+\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|^2}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{\mu\|\boldsymbol{q}\|}\right)\left(\frac{\|\boldsymbol{p}\|^2}{2\mu}-\frac{1}{\|\boldsymbol{q}\|}\right)\\ +\frac{H_0p_1\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2}{\mu}-\frac{2H_0p_1}{1}-\frac{H_0q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|^2}+\frac{H_0p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{\mu\|\boldsymbol{q}\|}\end{bmatrix}cosh\psi$$

$$+(H_0)^{-1}\left[\left(+\frac{q_1}{\|\boldsymbol{q}\|}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right)\left(\frac{\|\boldsymbol{p}\|^2}{2\mu}-\frac{1}{\|\boldsymbol{q}\|}\right)\right]sinh\psi,$$

$$x_0'=\sqrt{\frac{1}{2\mu}}(H_0)^{-\frac{3}{2}}\left[+\frac{H_0q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|^2}-\frac{H_0p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{\mu\|\boldsymbol{q}\|}+\frac{H_0p_1\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2}{\mu}-\frac{2H_0p_1}{1}-\frac{H_0q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|^2}+\frac{H_0p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{\mu\|\boldsymbol{q}\|}\right]cosh\psi$$

$$+(H_0)^{-1}\left[+\frac{H_0q_1}{\|\boldsymbol{q}\|}-\frac{H_0p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right]sinh\psi,$$

$$x_0'=\sqrt{\frac{1}{2\mu}}(H_0)^{-3/2}\left[+\frac{H_0p_1\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2}{\mu}-\frac{2H_0p_1}{1}\right]cosh\psi+(H_0)^{-1}\left[+\frac{H_0q_1}{\|\boldsymbol{q}\|}-\frac{H_0p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right]sinh\psi,$$

$$x_0'=\sqrt{\frac{2}{\mu H_0}}\left[\frac{p_1\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2}{2\mu}-\frac{p_1}{1}\right]cosh\psi+\left[\frac{q_1}{\|\boldsymbol{q}\|}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right]sinh\psi,$$

$$x_0'=\sqrt{\frac{2}{\mu H_0}}\left[\frac{p_1\|\boldsymbol{q}\|}{1}\left(H_0+\frac{1}{\|\boldsymbol{q}\|}\right)-\frac{p_1}{1}\right]cosh\psi+\left[\frac{q_1}{\|\boldsymbol{q}\|}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right]sinh\psi,$$

$$x_0'=\sqrt{\frac{2H_0}{\mu}}p_1\|\boldsymbol{q}\|\,cosh\psi+\left[\frac{q_1}{\|\boldsymbol{q}\|}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right]sinh\psi,$$

$$x_0'=x_1.$$

$$x_1'=\sqrt{\frac{2H_0}{\mu}}[\|\boldsymbol{q}\|'p_1+\|\boldsymbol{q}\|p_1{}']cosh\psi$$

$$+\sqrt{\frac{2H_0}{\mu}}[\|\boldsymbol{q}\|p_1]sinh\psi(\psi')$$

$$+\left[\frac{q_1{}'}{\|\boldsymbol{q}\|}+q_1(-1)\left(\frac{1}{\|\boldsymbol{q}\|^2}\right)\|\boldsymbol{q}\|'-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})'}{\mu}-\frac{(\boldsymbol{q}\cdot\boldsymbol{p})p_1'}{\mu}\right]sinh\psi$$

$$+\left[\frac{q_1}{\|\boldsymbol{q}\|}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right]cosh\psi(\psi'),$$

$$x_1' = \sqrt{\frac{2H_0}{\mu}}\begin{bmatrix} (-1)\sqrt{\frac{\mu}{2}}(H_0)^{-\frac{3}{2}}\left(-\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|}-\frac{H_0\|\boldsymbol{q}\|p_1}{\mu}\right)p_1 \\ +\|\boldsymbol{q}\|(-1)\sqrt{\frac{\mu}{2}}(H_0)^{-\frac{3}{2}}\left(\frac{q_1{}^2}{2\|\boldsymbol{q}\|^4}-\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^3}+\frac{H_0}{\|\boldsymbol{q}\|}-\frac{H_0\|\boldsymbol{p}\|^2}{\mu}+\frac{H_0p_1{}^2}{\mu}\right)\end{bmatrix}\cosh\psi$$

$$+\sqrt{\frac{2H_0}{\mu}}[\|\boldsymbol{q}\|p_1](-1)(H_0)^{-1}\left[-\frac{q_1}{2\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right]\sinh\psi$$

$$+\begin{bmatrix} \frac{1}{\|\boldsymbol{q}\|}(-1)\sqrt{\frac{\mu}{2}}(H_0)^{-\frac{3}{2}}\left(-\frac{q_1p_1}{2\mu\|\boldsymbol{q}\|}+\frac{p_1{}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2}-\frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right) \\ +q_1(-1)\left(\frac{1}{\|\boldsymbol{q}\|^2}\right)(-1)\sqrt{\frac{\mu}{2}}(H_0)^{-\frac{3}{2}}\left(-\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|}-\frac{H_0\|\boldsymbol{q}\|p_1}{\mu}\right) \\ -\frac{p_1}{\mu}(-1)\sqrt{\frac{\mu}{2}}(H_0)^{-\frac{3}{2}}\left(-\frac{q_1}{2\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right) \\ -\frac{(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}(-1)\sqrt{\frac{\mu}{2}}(H_0)^{-\frac{3}{2}}\left(\frac{q_1{}^2}{2\|\boldsymbol{q}\|^4}-\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^3}+\frac{H_0}{\|\boldsymbol{q}\|}-\frac{H_0\|\boldsymbol{p}\|^2}{\mu}+\frac{H_0p_1{}^2}{\mu}\right)\end{bmatrix}\sinh\psi$$

$$+\left[\frac{q_1}{\|\boldsymbol{q}\|}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right](-1)(H_0)^{-1}\left[-\frac{q_1}{2\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right]\cosh\psi,$$

$$x_1' = (H_0)^{-1}\begin{bmatrix} +\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}-\frac{p_1{}^2(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|}+\frac{H_0\|\boldsymbol{q}\|p_1{}^2}{\mu}-\frac{q_1{}^2}{2\|\boldsymbol{q}\|^3}+\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}-\frac{H_0}{1} \\ +\frac{H_0\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2}{\mu}-\frac{H_0\|\boldsymbol{q}\|p_1{}^2}{\mu}+\frac{q_1{}^2}{2\|\boldsymbol{q}\|^3}-\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}-\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}+\frac{p_1{}^2(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|}\end{bmatrix}\cosh\psi$$

$$+\sqrt{\frac{2H_0}{\mu}}(H_0)^{-1}\left[+\frac{q_1p_1}{2\|\boldsymbol{q}\|}-\frac{p_1{}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu}\right]\sinh\psi$$

$$+\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\begin{bmatrix} +\frac{q_1p_1}{2\mu\|\boldsymbol{q}\|^2}-\frac{p_1{}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2\|\boldsymbol{q}\|}+\frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu\|\boldsymbol{q}\|}-\frac{q_1{}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^4}+\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|^3}-\frac{H_0\|\boldsymbol{q}\|q_1p_1}{\mu\|\boldsymbol{q}\|^2}-\frac{q_1p_1}{2\mu\|\boldsymbol{q}\|^2} \\ +\frac{p_1{}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2\|\boldsymbol{q}\|}+\frac{q_1{}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^4}-\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|^3}+\frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu\|\boldsymbol{q}\|}-\frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{\mu^2}+\frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p})p_1{}^2}{\mu^2}\end{bmatrix}\sinh\psi,$$

$$x_1' = (H_0)^{-1}\left[-\frac{H_0}{1}+\frac{H_0\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2}{\mu}\right]\cosh\psi$$

$$+\sqrt{\frac{2H_0}{\mu}}(H_0)^{-1}\left[+\frac{q_1p_1}{2\|\boldsymbol{q}\|}-\frac{p_1{}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu}\right]\sinh\psi$$

$$+\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[+\frac{2H_0(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu\|\boldsymbol{q}\|}-\frac{H_0\|\boldsymbol{q}\|q_1p_1}{\mu\|\boldsymbol{q}\|^2}-\frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{\mu^2}+\frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p})p_1{}^2}{\mu^2}\right]\sinh\psi,$$

$$x_1' = \left[-1+\frac{\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2}{\mu}\right]\cosh\psi$$

$$+\sqrt{\frac{2H_0}{\mu}}(H_0)^{-1}\left[+\frac{q_1p_1}{2\|\boldsymbol{q}\|}-\frac{p_1{}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu}\right]\sinh\psi$$

$$+\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[+\frac{2H_0(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu\|\boldsymbol{q}\|}-\frac{H_0q_1p_1}{\mu\|\boldsymbol{q}\|}-\frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{\mu^2}+\frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p})p_1{}^2}{\mu^2}\right]\sinh\psi,$$

$$x_1' = \left[\frac{\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2}{\mu}-1\right]\cosh\psi+\sqrt{\frac{2}{H_0\mu}}\left[+\frac{q_1p_1}{2\|\boldsymbol{q}\|}-\frac{p_1{}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu}+\frac{(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|}-\frac{q_1p_1}{2\|\boldsymbol{q}\|}-\frac{(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{2\mu}+\frac{(\boldsymbol{q}\cdot\boldsymbol{p})p_1{}^2}{2\mu}\right]\sin\psi,$$

$$x_1' = \left[\frac{\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2}{\mu}-1\right]\cosh\psi+\sqrt{\frac{2}{H_0\mu}}\left[+\frac{(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|}-\frac{(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{2\mu}\right]\sinh\psi,$$

$$x_1' = \left[\frac{\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2}{\mu}-1\right]\cosh\psi-\sqrt{\frac{2H_0}{\mu}}(\boldsymbol{q}\cdot\boldsymbol{p})\sin\psi,$$

$$x_1' = x_0.$$

$$
\begin{aligned}
x_2' = &\sqrt{\frac{2H_0}{\mu}}\left[\|\boldsymbol{q}\|' p_2 + \|\boldsymbol{q}\| p_2{}'\right] cosh\psi \\
&+\sqrt{\frac{2H_0}{\mu}}\left[\|\boldsymbol{q}\| p_2\right] sinh\psi(\psi') \\
&+\left[\frac{q_2{}'}{\|\boldsymbol{q}\|} + q_2(-1)\left(\frac{1}{\|\boldsymbol{q}\|^2}\right)\|\boldsymbol{q}\|' - \frac{p_2(\boldsymbol{q}\cdot\boldsymbol{p})'}{\mu} - \frac{(\boldsymbol{q}\cdot\boldsymbol{p})p_2'}{\mu}\right] sinh\psi \\
&+\left[\frac{q_2}{\|\boldsymbol{q}\|} - \frac{p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right] cosh\psi(\psi'),
\end{aligned}
$$

$$
\begin{aligned}
x_2' = &\sqrt{\frac{2H_0}{\mu}}\left[\begin{array}{l} (-1)\sqrt{\frac{\mu}{2}}(H_0)^{-\frac{3}{2}}\left(-\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|} - \frac{H_0\|\boldsymbol{q}\|p_1}{\mu}\right)p_2 \\ +\|\boldsymbol{q}\|(-1)\sqrt{\frac{\mu}{2}}(H_0)^{-\frac{3}{2}}\left(\frac{q_1q_2}{2\|\boldsymbol{q}\|^4} - \frac{q_2p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^3} + \frac{H_0p_1p_2}{\mu}\right)\end{array}\right] cosh\psi \\
&+\sqrt{\frac{2H_0}{\mu}}\left[\|\boldsymbol{q}\|p_2\right](-1)(H_0)^{-1}\left[-\frac{q_1}{2\|\boldsymbol{q}\|^2} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right] sinh\psi \\
&+\left[\begin{array}{c} \frac{1}{\|\boldsymbol{q}\|}(-1)\sqrt{\frac{\mu}{2}}(H_0)^{-\frac{3}{2}}\left(-\frac{q_1p_2}{2\mu\|\boldsymbol{q}\|} + \frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2} + \frac{H_0q_1p_2}{\mu} - \frac{H_0q_2p_1}{\mu}\right) \\ +q_2(-1)\left(\frac{1}{\|\boldsymbol{q}\|^2}\right)(-1)\sqrt{\frac{\mu}{2}}(H_0)^{-\frac{3}{2}}\left(-\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|} - \frac{H_0\|\boldsymbol{q}\|p_1}{\mu}\right) \\ -\frac{p_2}{\mu}(-1)\sqrt{\frac{\mu}{2}}(H_0)^{-\frac{3}{2}}\left(-\frac{q_1}{2\|\boldsymbol{q}\|^2} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right) \\ -\frac{(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}(-1)\sqrt{\frac{\mu}{2}}(H_0)^{-\frac{3}{2}}\left(\frac{q_1q_2}{2\|\boldsymbol{q}\|^4} - \frac{q_2p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^3} + \frac{H_0p_1p_2}{\mu}\right)\end{array}\right] sinh\psi \\
&+\left[\frac{q_2}{\|\boldsymbol{q}\|} - \frac{p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right](-1)(H_0)^{-1}\left[-\frac{q_1}{2\|\boldsymbol{q}\|^2} + \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right] cosh\psi,
\end{aligned}
$$

$$
\begin{aligned}
x_2' = &(H_0)^{-1}\left[\begin{array}{l} +\frac{q_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2} - \frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|} + \frac{H_0\|\boldsymbol{q}\|p_1p_2}{\mu} - \frac{q_1q_2}{2\|\boldsymbol{q}\|^3} + \frac{q_2p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2} \\ -\frac{H_0p_1p_{2\|\boldsymbol{q}\|}}{\mu} + \frac{q_1q_2}{2\|\boldsymbol{q}\|^3} - \frac{q_2p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2} - \frac{q_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2} + \frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|}\end{array}\right] cosh\psi \\
&+\sqrt{\frac{2H_0}{\mu}}(H_0)^{-1}\left[+\frac{q_1p_2}{2\|\boldsymbol{q}\|} - \frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu}\right] sinh\psi \\
&+\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[\begin{array}{l} +\frac{q_1p_2}{2\mu\|\boldsymbol{q}\|^2} - \frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2\|\boldsymbol{q}\|} - \frac{H_0q_1p_2}{\mu\|\boldsymbol{q}\|} + \frac{H_0q_2p_1}{\mu\|\boldsymbol{q}\|} - \frac{q_1q_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^4} + \frac{q_2p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|^3} \\ -\frac{H_0q_2p_1}{\mu\|\boldsymbol{q}\|} - \frac{q_1p_2}{2\mu\|\boldsymbol{q}\|^2} + \frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2\|\boldsymbol{q}\|} + \frac{q_1q_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^4} - \frac{q_2p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|^3} + \frac{H_0p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu^2}\end{array}\right] sinh\psi,
\end{aligned}
$$

$$
\begin{aligned}
x_2' = &(H_0)^{-1}\left[+\frac{H_0\|\boldsymbol{q}\|p_1p_2}{\mu} - \frac{H_0p_1p_2\|\boldsymbol{q}\|}{\mu}\right] cosh\psi \\
&+\sqrt{\frac{2H_0}{\mu}}(H_0)^{-1}\left[+\frac{q_1p_2}{2\|\boldsymbol{q}\|} - \frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu}\right] sinh\psi \\
&+\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[-\frac{H_0q_1p_2}{\mu\|\boldsymbol{q}\|} + \frac{H_0p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu^2}\right] sinh\psi,
\end{aligned}
$$

$$
x_2' = (H_0)^{-1}[0]cosh\psi + \sqrt{\frac{1}{2\mu H_0}}\left[+\frac{q_1p_2}{\|\boldsymbol{q}\|} - \frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right] sinh\psi + \sqrt{\frac{1}{2\mu H_0}}\left[-\frac{q_1p_2}{\|\boldsymbol{q}\|} + \frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right] sinh\psi,
$$

$$
x_2' = 0.
$$

$x_3' = 0,$

analogous to the last case.

$$y_0' = [(\boldsymbol{q}\cdot\boldsymbol{p})']cosh\psi$$
$$+(\boldsymbol{q}\cdot\boldsymbol{p})sinh\psi(\psi')$$
$$-\sqrt{\frac{\mu}{2H_0}}\left[\frac{\|\boldsymbol{q}\|'\|\boldsymbol{p}\|^2}{\mu}+\frac{\|\boldsymbol{q}\|(2)(\boldsymbol{p}\cdot\boldsymbol{p}')}{\mu}\right]sinh\psi$$
$$-\sqrt{\frac{\mu}{2H_0}}\left[\left(\frac{\|\boldsymbol{q}\|\cdot\|\boldsymbol{p}\|^2}{\mu}-1\right)\right]cos\psi(\psi'),$$

$$y_0' = -\sqrt{\frac{\mu}{2}}(H_0)^{-\frac{3}{2}}\left[-\frac{q_1}{2\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right]cosh\psi$$
$$-\sqrt{\frac{\mu}{2H_0}}\left(\frac{\|\boldsymbol{q}\|\cdot\|\boldsymbol{p}\|^2}{\mu}-1\right)(-1)(H_0)^{-1}\left[-\frac{q_1}{2\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right]cosh\psi$$
$$+(\boldsymbol{q}\cdot\boldsymbol{p})(-1)(H_0)^{-1}\left[-\frac{q_1}{2\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right]sinh\psi$$
$$-\sqrt{\frac{\mu}{2H_0}}\left(\frac{1}{\mu}\right)(-1)\sqrt{\frac{\mu}{2}}(H_0)^{-\frac{3}{2}}\left[-\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|}-\frac{H_0\|\boldsymbol{q}\|p_1}{\mu}\right]\|\boldsymbol{p}\|^2\, sinh\psi$$
$$-\sqrt{\frac{\mu}{2H_0}}\left(\frac{1}{\mu}\right)2\|\boldsymbol{q}\|(-1)\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left[\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\|\boldsymbol{q}\|^4}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu\|\boldsymbol{q}\|^3}+\frac{H_0p_1}{\|\boldsymbol{q}\|}\right]sinh\psi,$$

$$y_0' = +\sqrt{\frac{\mu}{2}}(H_0)^{-\frac{3}{2}}\left[+\frac{q_1}{2\|\boldsymbol{q}\|^2}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right]cosh\psi$$
$$+\sqrt{\frac{\mu}{2H_0}}(H_0)^{-1}\left[-\frac{q_1\|\boldsymbol{p}\|^2}{2\mu\|\boldsymbol{q}\|}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{2\mu^2}+\frac{q_1}{2\|\boldsymbol{q}\|^2}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right]cosh\psi$$
$$+(H_0)^{-1}\left[+\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\|\boldsymbol{q}\|^2}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu\|\boldsymbol{q}\|}\right]sinh\psi$$
$$+\sqrt{\frac{\mu}{2H_0}}\left(\frac{1}{\mu}\right)\sqrt{\frac{\mu}{2}}(H_0)^{-\frac{3}{2}}\left[-\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{2\mu\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2\|\boldsymbol{p}\|^2}{2\mu^2\|\boldsymbol{q}\|}-\frac{H_0\|\boldsymbol{q}\|p_1\|\boldsymbol{p}\|^2}{\mu}\right]\, sinh\psi$$
$$+\sqrt{\frac{\mu}{2H_0}}\left(\frac{1}{\mu}\right)\sqrt{\frac{\mu}{2}}(H_0)^{-\frac{3}{2}}\left[\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|^3}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{\mu\|\boldsymbol{q}\|^2}+\frac{2H_0p_1}{1}\right]sinh\psi,$$

$$y_0' = +\sqrt{\frac{\mu}{2}}(H_0)^{-\frac{3}{2}}\left[+\frac{q_1}{2\|\boldsymbol{q}\|^2}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}-\frac{q_1\|\boldsymbol{p}\|^2}{2\mu\|\boldsymbol{q}\|}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{2\mu^2}+\frac{q_1}{2\|\boldsymbol{q}\|^2}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right]cosh\psi$$
$$+(H_0)^{-1}\left[+\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\|\boldsymbol{q}\|^2}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu\|\boldsymbol{q}\|}\right]sinh\psi$$
$$+\frac{1}{2}(H_0)^{-2}\left[-\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{2\mu\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2\|\boldsymbol{p}\|^2}{2\mu^2\|\boldsymbol{q}\|}-\frac{H_0\|\boldsymbol{q}\|p_1\|\boldsymbol{p}\|^2}{\mu}+\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|^3}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{\mu\|\boldsymbol{q}\|^2}+\frac{2H_0p_1}{1}\right]\, sinh\psi,$$

$$y_0' = +\sqrt{\frac{\mu}{2}}(H_0)^{-\frac{3}{2}}\left[-\frac{q_1}{\|\boldsymbol{q}\|}\left(\frac{\|\boldsymbol{p}\|^2}{2\mu}-\frac{1}{\|\boldsymbol{q}\|}\right)+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\left(\frac{\|\boldsymbol{p}\|^2}{2\mu}-\frac{1}{\|\boldsymbol{q}\|}\right)\right]cosh\psi$$
$$+(H_0)^{-1}\left[+\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\|\boldsymbol{q}\|^2}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu\|\boldsymbol{q}\|}\right]sinh\psi$$
$$+\frac{1}{2}(H_0)^{-2}\left[-\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|^2}\left(\frac{\|\boldsymbol{p}\|^2}{2\mu}-\frac{1}{\|\boldsymbol{q}\|}\right)+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{\mu\|\boldsymbol{q}\|}\left(\frac{\|\boldsymbol{p}\|^2}{2\mu}-\frac{1}{\|\boldsymbol{q}\|}\right)-\frac{H_0\|\boldsymbol{q}\|p_1\|\boldsymbol{p}\|^2}{\mu}+\frac{2H_0p_1}{1}\right]\, sinh\psi,$$

$$y_0' = +\sqrt{\frac{\mu}{2}}(H_0)^{-\frac{3}{2}}\left[-\frac{H_0q_1}{\|\boldsymbol{q}\|}+\frac{H_0\mathrm{p}_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right]cosh\psi$$

$$+(H_0)^{-1}\left[+\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\|\boldsymbol{q}\|^2}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu\|\boldsymbol{q}\|}\right]sinh\psi$$

$$+(H_0)^{-2}\left[-\frac{H_0q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\|\boldsymbol{q}\|^2}+\frac{H_0p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu\|\boldsymbol{q}\|}-\frac{H_0\|\boldsymbol{q}\|p_1\|\boldsymbol{p}\|^2}{2\mu}+\frac{H_0p_1}{1}\right]\ sinh\psi,$$

$$y_0' = +\sqrt{\frac{\mu}{2H_0}}\left[-\frac{q_1}{\|\boldsymbol{q}\|}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right]cosh\psi$$

$$+(H_0)^{-1}\left[+\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\|\boldsymbol{q}\|^2}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu\|\boldsymbol{q}\|}\right]sinh\psi$$

$$+(H_0)^{-1}\left[-\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu\|\boldsymbol{q}\|}-\frac{\|\boldsymbol{q}\|p_1\|\boldsymbol{p}\|^2}{2\mu}+\frac{p_1}{1}\right]\ sinh\psi,$$

$$y_0' = +\sqrt{\frac{\mu}{2H_0}}\left[-\frac{q_1}{\|\boldsymbol{q}\|}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right]cosh\psi+(H_0)^{-1}\left[-\frac{\|\boldsymbol{q}\|p_1\|\boldsymbol{p}\|^2}{2\mu}+\frac{p_1}{1}\right]\ sinh\psi,$$

$$y_0' = +\sqrt{\frac{\mu}{2H_0}}\left[-\frac{q_1}{\|\boldsymbol{q}\|}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right]cosh\psi+(H_0)^{-1}\left[-\frac{\|\boldsymbol{q}\|p_1}{1}\left(\frac{\|\boldsymbol{p}\|^2}{2\mu}-\frac{1}{\|\boldsymbol{q}\|}\right)\right]\ sinh\psi,$$

$$y_0' = -\sqrt{\frac{\mu}{2H_0}}\left[+\frac{q_1}{\|\boldsymbol{q}\|}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right]cosh\psi-\|\boldsymbol{q}\|p_1\ sinh\psi,$$

$$y_0' = y_1.$$

$$y_1' = -\sqrt{\frac{\mu}{2H_0}}\left[\frac{q_1{}'}{\|\boldsymbol{q}\|}+q_1(-1)\left(\frac{1}{\|\boldsymbol{q}\|^2}\right)\|\boldsymbol{q}\|'-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})'}{\mu}-\frac{(\boldsymbol{q}\cdot\boldsymbol{p})p_1'}{\mu}\right]cosh\psi$$

$$-\sqrt{\frac{\mu}{2H_0}}\left[\frac{q_1}{\|\boldsymbol{q}\|}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right]sinh\psi(\psi')$$

$$-[p_1\|\boldsymbol{q}\|'+\|\boldsymbol{q}\|p_1{}']sinh\psi$$

$$-[\|\boldsymbol{q}\|p_1]cosh\psi(\psi'),$$

$$y_1' = -\sqrt{\frac{\mu}{2H_0}}\begin{bmatrix}\frac{1}{\|\boldsymbol{q}\|}(-1)\sqrt{\frac{\mu}{2}}(H_0)^{-\frac{3}{2}}\left(-\frac{q_1p_1}{2\mu\|\boldsymbol{q}\|}+\frac{p_1{}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2}-\frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right)\\ +q_1(-1)\left(\frac{1}{\|\boldsymbol{q}\|^2}\right)(-1)\sqrt{\frac{\mu}{2}}(H_0)^{-\frac{3}{2}}\left(-\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|}-\frac{H_0\|\boldsymbol{q}\|p_1}{\mu}\right)\\ -\frac{p_1}{\mu}(-1)\sqrt{\frac{\mu}{2}}(H_0)^{-\frac{3}{2}}\left(-\frac{q_1}{2\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right)\\ -\frac{(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\ (-1)\sqrt{\frac{\mu}{2}}(H_0)^{-\frac{3}{2}}\left(\frac{q_1{}^2}{2\|\boldsymbol{q}\|^4}-\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^3}+\frac{H_0}{\|\boldsymbol{q}\|}-\frac{H_0\|\boldsymbol{p}\|^2}{\mu}+\frac{H_0p_1{}^2}{\mu}\right)\end{bmatrix}cosh\psi$$

$$-\sqrt{\frac{\mu}{2H_0}}\left[\frac{q_1}{\|\boldsymbol{q}\|}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right](-1)(H_0)^{-1}\left[-\frac{q_1}{2\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right]sinh\psi$$

$$-\begin{bmatrix}p_1(-1)\sqrt{\frac{\mu}{2}}(H_0)^{-\frac{3}{2}}\left(-\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|}-\frac{H_0\|\boldsymbol{q}\|p_1}{\mu}\right)\\ +\|\boldsymbol{q}\|(-1)\sqrt{\frac{\mu}{2}}(H_0)^{-\frac{3}{2}}\left(\frac{q_1{}^2}{2\|\boldsymbol{q}\|^4}-\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^3}+\frac{H_0}{\|\boldsymbol{q}\|}-\frac{H_0\|\boldsymbol{p}\|^2}{\mu}+\frac{H_0p_1{}^2}{\mu}\right)\end{bmatrix}sinh\psi$$

$$-[\|\boldsymbol{q}\|p_1](-1)(H_0)^{-1}\left[-\frac{q_1}{2\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right]cosh\psi,$$

$$y_1' = \sqrt{\frac{\mu}{2H_0}}\left[\begin{array}{c} \sqrt{\frac{\mu}{2}}(H_0)^{-\frac{3}{2}}\left(-\frac{q_1p_1}{2\mu\|\boldsymbol{q}\|^2}+\frac{{p_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2\|\boldsymbol{q}\|}-\frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu\|\boldsymbol{q}\|}\right) \\ +\sqrt{\frac{\mu}{2}}(H_0)^{-\frac{3}{2}}\left(+\frac{{q_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^4}-\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|^3}+\frac{H_0q_1p_1}{\mu\|\boldsymbol{q}\|}\right) \\ +\sqrt{\frac{\mu}{2}}(H_0)^{-\frac{3}{2}}\left(+\frac{q_1p_1}{2\mu\|\boldsymbol{q}\|^2}-\frac{{p_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2\|\boldsymbol{q}\|}\right) \\ +\sqrt{\frac{\mu}{2}}(H_0)^{-\frac{3}{2}}\left(-\frac{{q_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^4}+\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|^3}-\frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|\mu}+\frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{\mu^2}-\frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p}){p_1}^2}{\mu^2}\right) \end{array}\right] cosh\psi$$

$$+(H_0)^{-1}\left[-\frac{q_1p_1}{2\|\boldsymbol{q}\|}+\frac{{p_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu}\right] cosh\psi$$

$$+\sqrt{\frac{\mu}{2H_0}}(H_0)^{-1}\left[-\frac{{q_1}^2}{2\|\boldsymbol{q}\|^3}+\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}+\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}-\frac{{p_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|}\right] sinh\psi$$

$$+\left[\begin{array}{c} \sqrt{\frac{\mu}{2}}(H_0)^{-\frac{3}{2}}\left(-\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}+\frac{{p_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|}-\frac{H_0\|\boldsymbol{q}\|{p_1}^2}{\mu}\right) \\ +\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left(+\frac{{q_1}^2}{2\|\boldsymbol{q}\|^3}-\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}+\frac{H_0}{1}-\frac{H_0\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2}{\mu}+\frac{H_0\|\boldsymbol{q}\|{p_1}^2}{\mu}\right) \end{array}\right] sinh\psi,$$

$$y_1' = \frac{\mu}{2}(H_0)^{-2}\left[\begin{array}{c} -\frac{q_1p_1}{2\mu\|\boldsymbol{q}\|^2}+\frac{{p_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2\|\boldsymbol{q}\|}-\frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu\|\boldsymbol{q}\|}+\frac{{q_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^4}-\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|^3}+\frac{H_0q_1p_1}{\mu\|\boldsymbol{q}\|}+\frac{q_1p_1}{2\mu\|\boldsymbol{q}\|^2} \\ -\frac{{p_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2\|\boldsymbol{q}\|}-\frac{{q_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^4}+\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|^3}-\frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|\mu}+\frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{\mu^2}-\frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p}){p_1}^2}{\mu^2} \end{array}\right] cosh\psi$$

$$+(H_0)^{-1}\left[-\frac{q_1p_1}{2\|\boldsymbol{q}\|}+\frac{{p_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu}\right] cosh\psi$$

$$+\sqrt{\frac{\mu}{2H_0}}(H_0)^{-1}\left[\begin{array}{c} -\frac{{q_1}^2}{2\|\boldsymbol{q}\|^3}+\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}+\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}-\frac{{p_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|}-\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}+\frac{{p_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|} \\ -\frac{H_0\|\boldsymbol{q}\|{p_1}^2}{\mu}+\frac{{q_1}^2}{2\|\boldsymbol{q}\|^3}-\frac{q_1p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}+\frac{H_0}{1}-\frac{H_0\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2}{\mu}+\frac{H_0\|\boldsymbol{q}\|{p_1}^2}{\mu} \end{array}\right] sinh\psi,$$

$$y_1' = \frac{\mu}{2}(H_0)^{-2}\left[-\frac{2H_0(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu\|\boldsymbol{q}\|}+\frac{H_0q_1p_1}{\mu\|\boldsymbol{q}\|}+\frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{\mu^2}-\frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p}){p_1}^2}{\mu^2}\right] cosh\psi$$

$$+(H_0)^{-1}\left[-\frac{q_1p_1}{2\|\boldsymbol{q}\|}+\frac{{p_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu}\right] cosh\psi$$

$$+\sqrt{\frac{\mu}{2H_0}}(H_0)^{-1}\left[+\frac{H_0}{1}-\frac{H_0\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2}{\mu}\right] sinh\psi,$$

$$y_1' = (H_0)^{-1}\left[-\frac{(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|}+\frac{q_1p_1}{2\|\boldsymbol{q}\|}+\frac{(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{2\mu}-\frac{(\boldsymbol{q}\cdot\boldsymbol{p}){p_1}^2}{2\mu}\right] cosh\psi$$

$$+(H_0)^{-1}\left[-\frac{q_1p_1}{2\|\boldsymbol{q}\|}+\frac{{p_1}^2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu}\right] cosh\psi$$

$$+\sqrt{\frac{\mu}{2H_0}}\left[+1-\frac{\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2}{\mu}\right] sinh\psi,$$

$$y_1' = (H_0)^{-1}\left[\frac{(\boldsymbol{q}\cdot\boldsymbol{p})}{\|\boldsymbol{q}\|}-\frac{(\boldsymbol{q}\cdot\boldsymbol{p})\|\boldsymbol{p}\|^2}{2\mu}\right] cosh\psi+\sqrt{\frac{\mu}{2H_0}}\left[1-\frac{\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2}{\mu}\right] sinh\psi,$$

$$y_1' = -(\boldsymbol{q}\cdot\boldsymbol{p})\ cosh\psi+\sqrt{\frac{\mu}{2H_0}}\left[1-\frac{\|\boldsymbol{q}\|\|\boldsymbol{p}\|^2}{\mu}\right] sinh\psi,$$

$$y_1' = y_0.$$

$$y_2' = -\sqrt{\frac{\mu}{2H_0}}\left[\frac{{q_2}'}{\|\boldsymbol{q}\|}+q_2(-1)\left(\frac{1}{\|\boldsymbol{q}\|^2}\right)\|\boldsymbol{q}\|'-\frac{p_2(\boldsymbol{q}\cdot\boldsymbol{p})'}{\mu}-\frac{(\boldsymbol{q}\cdot\boldsymbol{p})p_2'}{\mu}\right] cosh\psi$$

$$-\sqrt{\frac{\mu}{2H_0}}\left[\frac{q_2}{\|\boldsymbol{q}\|}-\frac{p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right]sinh\psi(\psi')$$
$$-[p_2\|\boldsymbol{q}\|'+\|\boldsymbol{q}\|p_2{}']sinh\psi$$
$$-[\|\boldsymbol{q}\|p_2]cosh\psi(\psi'),$$

$$y_2'=-\sqrt{\frac{\mu}{2H_0}}\begin{bmatrix}\frac{1}{\|\boldsymbol{q}\|}(-1)\sqrt{\frac{\mu}{2}}(H_0)^{-\frac{3}{2}}\left(-\frac{q_1p_2}{2\mu\|\boldsymbol{q}\|}+\frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2}+\frac{H_0q_1p_2}{\mu}-\frac{H_0q_2p_1}{\mu}\right)\\+q_2(-1)\left(\frac{1}{\|\boldsymbol{q}\|^2}\right)(-1)\sqrt{\frac{\mu}{2}}(H_0)^{-\frac{3}{2}}\left(-\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|}-\frac{H_0\|\boldsymbol{q}\|p_1}{\mu}\right)\\-\frac{p_2}{\mu}(-1)\sqrt{\frac{\mu}{2}}(H_0)^{-\frac{3}{2}}\left(-\frac{q_1}{2\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right)\\-\frac{(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}(-1)\sqrt{\frac{\mu}{2}}(H_0)^{-\frac{3}{2}}\left(\frac{q_1q_2}{2\|\boldsymbol{q}\|^4}-\frac{q_2p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^3}+\frac{H_0p_1p_2}{\mu}\right)\end{bmatrix}cosh\psi$$
$$-\sqrt{\frac{\mu}{2H_0}}\left[\frac{q_2}{\|\boldsymbol{q}\|}-\frac{p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right](-1)(H_0)^{-1}\left[-\frac{q_1}{2\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right]sinh\psi$$
$$-\begin{bmatrix}p_2(-1)\sqrt{\frac{\mu}{2}}(H_0)^{-\frac{3}{2}}\left(-\frac{q_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|}-\frac{H_0\|\boldsymbol{q}\|p_1}{\mu}\right)\\+\|\boldsymbol{q}\|(-1)\sqrt{\frac{\mu}{2}}(H_0)^{-\frac{3}{2}}\left(\frac{q_1q_2}{2\|\boldsymbol{q}\|^4}-\frac{q_2p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^3}+\frac{H_0p_1p_2}{\mu}\right)\end{bmatrix}sinh\psi$$
$$-[\|\boldsymbol{q}\|p_2](-1)(H_0)^{-1}\left[-\frac{q_1}{2\|\boldsymbol{q}\|^2}+\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|}\right]cosh\psi,$$

$$y_2'=\sqrt{\frac{\mu}{2H_0}}\begin{bmatrix}\sqrt{\frac{\mu}{2}}(H_0)^{-\frac{3}{2}}\left(-\frac{q_1p_2}{2\mu\|\boldsymbol{q}\|^2}+\frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2\|\boldsymbol{q}\|}+\frac{H_0q_1p_2}{\mu\|\boldsymbol{q}\|}-\frac{H_0q_2p_1}{\mu\|\boldsymbol{q}\|}\right)\\+\sqrt{\frac{\mu}{2}}(H_0)^{-\frac{3}{2}}\left(+\frac{q_1q_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^4}-\frac{q_2p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|^3}+\frac{H_0q_2p_1}{\mu\|\boldsymbol{q}\|}\right)\\+\sqrt{\frac{\mu}{2}}(H_0)^{-\frac{3}{2}}\left(+\frac{q_1p_2}{2\mu\|\boldsymbol{q}\|^2}-\frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2\|\boldsymbol{q}\|}\right)\\+\sqrt{\frac{\mu}{2}}(H_0)^{-\frac{3}{2}}\left(-\frac{q_1q_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^4}+\frac{q_2p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|^3}-\frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p})p_1p_2}{\mu^2}\right)\end{bmatrix}cosh\psi$$
$$+(H_0)^{-1}\left[-\frac{q_1p_2}{2\|\boldsymbol{q}\|}+\frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu}\right]cosh\psi$$
$$+\sqrt{\frac{\mu}{2H_0}}(H_0)^{-1}\left[-\frac{q_1q_2}{2\|\boldsymbol{q}\|^3}+\frac{q_2p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}+\frac{q_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}-\frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|}\right]sinh\psi$$
$$+\begin{bmatrix}\sqrt{\frac{\mu}{2}}(H_0)^{-\frac{3}{2}}\left(-\frac{q_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}+\frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|}-\frac{H_0\|\boldsymbol{q}\|p_1p_2}{\mu}\right)\\+\sqrt{\frac{\mu}{2}}(H_0)^{-3/2}\left(+\frac{q_1q_2}{2\|\boldsymbol{q}\|^3}-\frac{q_2p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}+\frac{H_0\|\boldsymbol{q}\|p_1p_2}{\mu}\right)\end{bmatrix}sinh\psi,$$

$$y_2'=\frac{\mu}{2}(H_0)^{-2}\begin{bmatrix}-\frac{q_1p_2}{2\mu\|\boldsymbol{q}\|^2}+\frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2\|\boldsymbol{q}\|}+\frac{H_0q_1p_2}{\mu\|\boldsymbol{q}\|}-\frac{H_0q_2p_1}{\mu\|\boldsymbol{q}\|}+\frac{q_1q_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^4}-\frac{q_2p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|^3}\\+\frac{H_0q_2p_1}{\mu\|\boldsymbol{q}\|}+\frac{q_1p_2}{2\mu\|\boldsymbol{q}\|^2}-\frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu^2\|\boldsymbol{q}\|}-\frac{q_1q_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^4}+\frac{q_2p_1(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|^3}-\frac{H_0(\boldsymbol{q}\cdot\boldsymbol{p})p_1p_2}{\mu^2}\end{bmatrix}cosh\psi$$
$$+(H_0)^{-1}\left[-\frac{q_1p_2}{2\|\boldsymbol{q}\|}+\frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu}\right]cosh\psi$$
$$+\sqrt{\frac{\mu}{2H_0}}(H_0)^{-1}\begin{bmatrix}-\frac{q_1q_2}{2\|\boldsymbol{q}\|^3}+\frac{q_2p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}+\frac{q_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}-\frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|}-\frac{q_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}\\+\frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})^2}{2\mu^2\|\boldsymbol{q}\|}-\frac{H_0\|\boldsymbol{q}\|p_1p_2}{\mu}+\frac{q_1q_2}{2\|\boldsymbol{q}\|^3}-\frac{q_2p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu\|\boldsymbol{q}\|^2}+\frac{H_0\|\boldsymbol{q}\|p_1p_2}{\mu}\end{bmatrix}sinh\psi,$$

$y_2' = (H_0)^{-1}\left[+\frac{q_1p_2}{\mu\|\boldsymbol{q}\|} - \frac{(\boldsymbol{q}\cdot\boldsymbol{p})p_1p_2}{\mu^2}\right] cosh\psi$

$\quad +(H_0)^{-1}\left[-\frac{q_1p_2}{2\|\boldsymbol{q}\|} + \frac{p_1p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{2\mu}\right] cosh\psi$

$\quad +\sqrt{\frac{\mu}{2H_0}}(H_0)^{-1}[0] sinh\psi,$

$y_2' = 0.$

$y_3' = 0,$

analogous to the last case.

(ii) Along an integral curve of L_1:

Intermediate results for $H \neq 0$:

$L_1 = q_2p_3 - q_3p_2.$

$q_1' = \{q_1, L_1\} = \sum_{i=1}^{3}\left(\frac{\partial q_1}{\partial q_i}\frac{\partial L_1}{\partial p_i} - \frac{\partial q_1}{\partial p_i}\frac{\partial L_1}{\partial q_i}\right) = \frac{\partial L_1}{\partial p_1} = 0,$

$q_1' = 0.$

$q_2' = \{q_2, L_1\} = \frac{\partial L_1}{\partial p_2} = -q_3,$

$q_2' = -q_3.$

$q_3' = \{q_3, L_1\} = \frac{\partial L_1}{\partial p_3} = q_2,$

$q_3' = q_2.$

$p_1' = \{p_1, L_1\} = -\frac{\partial L_1}{\partial q_1} = 0,$

$p_1' = 0.$

$p_2' = \{p_2, L_1\} = -\frac{\partial L_1}{\partial q_2} = -p_3,$

$p_2' = -p_3.$

$p_3' = \{p_3, L_1\} = -\frac{\partial L_1}{\partial q_3} = p_2,$

$p_3' = p_2.$

$(\boldsymbol{p}\cdot\boldsymbol{p}') = -p_2p_3 + p_2p_3 = 0,$

$(\boldsymbol{p}\cdot\boldsymbol{p}') = 0.$

$(\boldsymbol{q}\cdot\boldsymbol{p}') = -q_2p_3 + q_3p_2.$

$(\boldsymbol{p}\cdot\boldsymbol{q}') = q_2p_3 - q_3p_2.$

$(\boldsymbol{q}\cdot\boldsymbol{p})' = (\boldsymbol{q}\cdot\boldsymbol{p}') + (\boldsymbol{p}\cdot\boldsymbol{q}') = 0,$

$(\boldsymbol{q}\cdot\boldsymbol{p})' = 0.$

$(\boldsymbol{q}\cdot\boldsymbol{q}') = -q_2 q_3 + q_2 q_3 = 0,$
$(\boldsymbol{q}\cdot\boldsymbol{q}') = \mathbf{0}.$

$\|\boldsymbol{q}\|' = \frac{(\boldsymbol{q}\cdot\boldsymbol{q}')}{\|\boldsymbol{q}\|} = 0,$
$\|\boldsymbol{q}\|' = 0.$

$\psi' = \sqrt{\frac{-2H_0}{\mu}}(\boldsymbol{q}\cdot\boldsymbol{p})' = 0$, for $H < 0$,
$\psi' = 0.$
$\psi' = \sqrt{\frac{2H_0}{\mu}}(\boldsymbol{q}\cdot\boldsymbol{p})' = 0$, for $H > 0$,
$\psi' = 0.$

Main results for $H < 0$:

$$x_0' = \left(\frac{1}{\mu}\right)[\|\boldsymbol{q}\|'\|\boldsymbol{p}\|^2 + \|\boldsymbol{q}\|(2)(\boldsymbol{p}\cdot\boldsymbol{p}')]cos\psi$$
$$+\left[\left(\frac{\|\boldsymbol{q}\|\cdot\|\boldsymbol{p}\|^2}{\mu} - 1\right)\right](-1)sin\psi(\psi')$$
$$+\sqrt{\frac{-2H_0}{\mu}}[(\boldsymbol{q}\cdot\boldsymbol{p})']sin\psi$$
$$+\sqrt{\frac{-2H_0}{\mu}}(\boldsymbol{q}\cdot\boldsymbol{p})cos\psi(\psi'),$$
$$x_0' = \left(\frac{1}{\mu}\right)[(0)\|\boldsymbol{p}\|^2 + \|\boldsymbol{q}\|(2)(0)]cos\psi$$
$$+\left[\left(\frac{\|\boldsymbol{q}\|\cdot\|\boldsymbol{p}\|^2}{\mu} - 1\right)\right](-1)(0)sin\psi$$
$$+\sqrt{\frac{-2H_0}{\mu}}[(0)]sin\psi$$
$$+\sqrt{\frac{-2H_0}{\mu}}(\boldsymbol{q}\cdot\boldsymbol{p})(0)cos\psi,$$
$$x_0' = 0.$$

$$x_1' = \sqrt{\frac{-2H_0}{\mu}}[\|\boldsymbol{q}\|'p_1 + \|\boldsymbol{q}\|p_1{}']cos\psi$$
$$+\sqrt{\frac{-2H_0}{\mu}}[\|\boldsymbol{q}\|p_1](-1)sin\psi(\psi')$$
$$+\left[\frac{q_1{}'}{\|\boldsymbol{q}\|} + q_1(-1)\left(\frac{1}{\|\boldsymbol{q}\|^2}\right)\|\boldsymbol{q}\|' - \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})'}{\mu} - \frac{(\boldsymbol{q}\cdot\boldsymbol{p})p_1'}{\mu}\right]sin\psi$$
$$+\left[\frac{q_1}{\|\boldsymbol{q}\|} - \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right]cos\psi(\psi'),$$

$$x_1' = \sqrt{\frac{-2H_0}{\mu}}[(0)p_1 + \|\boldsymbol{q}\|(0)]cos\psi$$
$$+\sqrt{\frac{-2H_0}{\mu}}[\|\boldsymbol{q}\|p_1](-1)(0)sin\psi$$
$$+\left[\frac{(0)}{\|\boldsymbol{q}\|} + q_1(-1)\left(\frac{1}{\|\boldsymbol{q}\|^2}\right)(0) - \frac{p_1(0)}{\mu} - \frac{(\boldsymbol{q}\cdot\boldsymbol{p})(0)}{\mu}\right]sin\psi$$
$$+\left[\frac{q_1}{\|\boldsymbol{q}\|} - \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right](0)cos\psi,$$
$$x_1' = 0.$$

$$x_2' = \sqrt{\frac{-2H_0}{\mu}}[\|\boldsymbol{q}\|'p_2 + \|\boldsymbol{q}\|p_2{}']cos\psi$$
$$+\sqrt{\frac{-2H_0}{\mu}}[\|\boldsymbol{q}\|p_2](-1)sin\psi(\psi')$$
$$+\left[\frac{q_2{}'}{\|\boldsymbol{q}\|} + q_2(-1)\left(\frac{1}{\|\boldsymbol{q}\|^2}\right)\|\boldsymbol{q}\|' - \frac{p_2(\boldsymbol{q}\cdot\boldsymbol{p})'}{\mu} - \frac{(\boldsymbol{q}\cdot\boldsymbol{p})p_2'}{\mu}\right]sin\psi$$
$$+\left[\frac{q_2}{\|\boldsymbol{q}\|} - \frac{p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right]cos\psi(\psi'),$$
$$x_2' = \sqrt{\frac{-2H_0}{\mu}}[(0)p_2 + \|\boldsymbol{q}\|(-p_3)]cos\psi$$
$$+\sqrt{\frac{-2H_0}{\mu}}[\|\boldsymbol{q}\|p_2](-1)(0)sin\psi$$
$$+\left[\frac{(-q_3)}{\|\boldsymbol{q}\|} + q_2(-1)\left(\frac{1}{\|\boldsymbol{q}\|^2}\right)(0) - \frac{p_2(0)}{\mu} - \frac{(\boldsymbol{q}\cdot\boldsymbol{p})(-p_3)}{\mu}\right]sin\psi$$
$$+\left[\frac{q_2}{\|\boldsymbol{q}\|} - \frac{p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right](0)cos\psi,$$
$$x_2' = \sqrt{\frac{-2H_0}{\mu}}[\|\boldsymbol{q}\|(-p_3)]cos\psi + \left[\frac{(-q_3)}{\|\boldsymbol{q}\|} - \frac{(\boldsymbol{q}\cdot\boldsymbol{p})(-p_3)}{\mu}\right]sin\psi,$$
$$x_2' = -x_3.$$

$$x_3' = \sqrt{\frac{-2H_0}{\mu}}[\|\boldsymbol{q}\|'p_3 + \|\boldsymbol{q}\|p_3{}']cos\psi$$
$$+\sqrt{\frac{-2H_0}{\mu}}[\|\boldsymbol{q}\|p_3](-1)sin\psi(\psi')$$
$$+\left[\frac{q_3{}'}{\|\boldsymbol{q}\|} + q_3(-1)\left(\frac{1}{\|\boldsymbol{q}\|^2}\right)\|\boldsymbol{q}\|' - \frac{p_3(\boldsymbol{q}\cdot\boldsymbol{p})'}{\mu} - \frac{(\boldsymbol{q}\cdot\boldsymbol{p})p_3'}{\mu}\right]sin\psi$$
$$+\left[\frac{q_3}{\|\boldsymbol{q}\|} - \frac{p_3(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right]cos\psi(\psi'),$$
$$x_3' = \sqrt{\frac{-2H_0}{\mu}}[(0)p_3 + \|\boldsymbol{q}\|(p_2)]cos\psi$$
$$+\sqrt{\frac{-2H_0}{\mu}}[\|\boldsymbol{q}\|p_3](-1)(0)sin\psi$$
$$+\left[\frac{q_2}{\|\boldsymbol{q}\|} + q_3(-1)\left(\frac{1}{\|\boldsymbol{q}\|^2}\right)(0) - \frac{p_3(0)}{\mu} - \frac{(\boldsymbol{q}\cdot\boldsymbol{p})(p_2)}{\mu}\right]sin\psi$$

$$+\left[\frac{q_3}{\|\boldsymbol{q}\|}-\frac{p_3(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right](0)cos\psi,$$

$$x_3'=\sqrt{\frac{-2H_0}{\mu}}[\|\boldsymbol{q}\|(p_2)]cos\psi+\left[\frac{q_2}{\|\boldsymbol{q}\|}-\frac{(\boldsymbol{q}\cdot\boldsymbol{p})(p_2)}{\mu}\right]sin\psi,$$

$$x_3'=x_2.$$

$$y_0'=[(\boldsymbol{q}\cdot\boldsymbol{p})']cos\psi+(\boldsymbol{q}\cdot\boldsymbol{p})(-1)sin\psi(\psi')$$
$$-\sqrt{\frac{\mu}{-2H_0}}\left[\frac{\|\boldsymbol{q}\|'\|\boldsymbol{p}\|^2}{\mu}+\frac{\|\boldsymbol{q}\|(2)(\boldsymbol{p}\cdot\boldsymbol{p}')}{\mu}\right]sin\psi$$
$$-\sqrt{\frac{\mu}{-2H_0}}\left[\left(\frac{\|\boldsymbol{q}\|\cdot\|\boldsymbol{p}\|^2}{\mu}-1\right)\right]cos\psi(\psi'),$$

$$y_0'=[(0)]cos\psi+(\boldsymbol{q}\cdot\boldsymbol{p})(-1)sin\psi(0)$$
$$-\sqrt{\frac{\mu}{-2H_0}}\left[\frac{(0)\|\boldsymbol{p}\|^2}{\mu}+\frac{\|\boldsymbol{q}\|(2)(0)}{\mu}\right]sin\psi$$
$$-\sqrt{\frac{\mu}{-2H_0}}\left[\left(\frac{\|\boldsymbol{q}\|\cdot\|\boldsymbol{p}\|^2}{\mu}-1\right)\right]cos\psi(0),$$

$$\mathrm{y}_0'=0.$$

$$y_1'=\sqrt{\frac{\mu}{-2H_0}}\left[\frac{q_1{}'}{\|\boldsymbol{q}\|}+q_1(-1)\left(\frac{1}{\|\boldsymbol{q}\|^2}\right)\|\boldsymbol{q}\|'-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})'}{\mu}-\frac{(\boldsymbol{q}\cdot\boldsymbol{p})p_1'}{\mu}\right]cos\psi$$
$$+\sqrt{\frac{\mu}{-2H_0}}\left[\frac{q_1}{\|\boldsymbol{q}\|}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right](-1)sin\psi(\psi')$$
$$-[p_1\|\boldsymbol{q}\|'+\|\boldsymbol{q}\|p_1{}']sin\psi$$
$$-[\|\boldsymbol{q}\|p_1]cos\psi(\psi'),$$

$$y_1'=\sqrt{\frac{\mu}{-2H_0}}\left[\frac{(0)}{\|\boldsymbol{q}\|}+q_1(-1)\left(\frac{1}{\|\boldsymbol{q}\|^2}\right)(0)-\frac{p_1(0)}{\mu}-\frac{(\boldsymbol{q}\cdot\boldsymbol{p})(0)}{\mu}\right]cos\psi$$
$$+\sqrt{\frac{\mu}{-2H_0}}\left[\frac{q_1}{\|\boldsymbol{q}\|}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right](-1)sin\psi(0)$$
$$-[p_1(0)+\|\boldsymbol{q}\|(0)]sin\psi$$
$$-[\|\boldsymbol{q}\|p_1]cos\psi(0),$$

$$\mathrm{y}_1'=0.$$

$$y_2'=\sqrt{\frac{\mu}{-2H_0}}\left[\frac{q_2{}'}{\|\boldsymbol{q}\|}+q_2(-1)\left(\frac{1}{\|\boldsymbol{q}\|^2}\right)\|\boldsymbol{q}\|'-\frac{p_2(\boldsymbol{q}\cdot\boldsymbol{p})'}{\mu}-\frac{(\boldsymbol{q}\cdot\boldsymbol{p})p_2'}{\mu}\right]cos\psi$$
$$+\sqrt{\frac{\mu}{-2H_0}}\left[\frac{q_2}{\|\boldsymbol{q}\|}-\frac{p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right](-1)sin\psi(\psi')$$
$$-[p_2\|\boldsymbol{q}\|'+\|\boldsymbol{q}\|p_2{}']sin\psi$$
$$-[\|\boldsymbol{q}\|p_2]cos\psi(\psi'),$$

$$y_2'=\sqrt{\frac{\mu}{-2H_0}}\left[\frac{(-q_3)}{\|\boldsymbol{q}\|}+q_2(-1)\left(\frac{1}{\|\boldsymbol{q}\|^2}\right)(0)-\frac{p_2(0)}{\mu}-\frac{(\boldsymbol{q}\cdot\boldsymbol{p})(-p_3)}{\mu}\right]cos\psi$$

$$+\sqrt{\frac{\mu}{-2H_0}}\left[\frac{q_2}{\|\boldsymbol{q}\|}-\frac{p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right](-1)sin\psi(0)$$
$$-[p_2(0)+\|\boldsymbol{q}\|(-p_3)]sin\psi$$
$$-[\|\boldsymbol{q}\|p_2]cos\psi(0),$$
$$y_2'=\sqrt{\frac{\mu}{-2H_0}}\left[\frac{(-\mathrm{q}_3)}{\|\boldsymbol{q}\|}-\frac{(\boldsymbol{q}\cdot\boldsymbol{p})(-p_3)}{\mu}\right]cos\psi-[\|\boldsymbol{q}\|(-p_3)]sin\psi,$$
$$y_2'=-y_3.$$

$$y_3'=\sqrt{\frac{\mu}{-2H_0}}\left[\frac{q_3{}'}{\|\boldsymbol{q}\|}+q_3(-1)\left(\frac{1}{\|\boldsymbol{q}\|^2}\right)\|\boldsymbol{q}\|'-\frac{p_3(\boldsymbol{q}\cdot\boldsymbol{p})'}{\mu}-\frac{(\boldsymbol{q}\cdot\boldsymbol{p})p_3'}{\mu}\right]cos\psi$$
$$+\sqrt{\frac{\mu}{-2H_0}}\left[\frac{q_3}{\|\boldsymbol{q}\|}-\frac{p_3(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right](-1)sin\psi(\psi')$$
$$-[p_3\|\boldsymbol{q}\|'+\|\boldsymbol{q}\|p_3{}']sin\psi$$
$$-[\|\boldsymbol{q}\|p_3]cos\psi(\psi'),$$
$$y_3'=\sqrt{\frac{\mu}{-2H_0}}\left[\frac{(q_2)}{\|\boldsymbol{q}\|}+q_3(-1)\left(\frac{1}{\|\boldsymbol{q}\|^2}\right)(0)-\frac{p_3(0)}{\mu}-\frac{(\boldsymbol{q}\cdot\boldsymbol{p})(p_2)}{\mu}\right]cos\psi$$
$$+\sqrt{\frac{\mu}{-2H_0}}\left[\frac{q_3}{\|\boldsymbol{q}\|}-\frac{p_3(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right](-1)sin\psi(0)$$
$$-[p_3(0)+\|\boldsymbol{q}\|(p_2)]sin\psi$$
$$-[\|\boldsymbol{q}\|p_3]cos\psi(0),$$
$$y_3'=\sqrt{\frac{\mu}{-2H_0}}\left[\frac{(q_2)}{\|\boldsymbol{q}\|}-\frac{(\boldsymbol{q}\cdot\boldsymbol{p})(p_2)}{\mu}\right]cos\psi-[\|\boldsymbol{q}\|(p_2)]sin\psi,$$
$$y_3'=y_2.$$

Main results for $H>0$:

$$x_0'=\left(\frac{1}{\mu}\right)[\|\boldsymbol{q}\|'\|\boldsymbol{p}\|^2+\|\boldsymbol{q}\|(2)(\boldsymbol{p}\cdot\boldsymbol{p}')]cosh\psi$$
$$+\left[\left(\frac{\|\boldsymbol{q}\|\cdot\|\boldsymbol{p}\|^2}{\mu}-1\right)\right]sinh\psi(\psi')$$
$$-\sqrt{\frac{2H_0}{\mu}}[(\boldsymbol{q}\cdot\boldsymbol{p})']sinh\psi$$
$$-\sqrt{\frac{2H_0}{\mu}}(\boldsymbol{q}\cdot\boldsymbol{p})cosh\psi(\psi'),$$
$$x_0'=\left(\frac{1}{\mu}\right)[(0)\|\boldsymbol{p}\|^2+\|\boldsymbol{q}\|(2)(0)]cosh\psi$$
$$+\left[\left(\frac{\|\boldsymbol{q}\|\cdot\|\boldsymbol{p}\|^2}{\mu}-1\right)\right](0)sinh\psi$$
$$-\sqrt{\frac{2H_0}{\mu}}[(0)]sinh\psi$$
$$-\sqrt{\frac{2H_0}{\mu}}(\boldsymbol{q}\cdot\boldsymbol{p})(0)cos\psi,$$
$$x_0'=0.$$

$$x_1' = \sqrt{\frac{2H_0}{\mu}}[\|\boldsymbol{q}\|' p_1 + \|\boldsymbol{q}\| p_1{}'] cosh\psi$$
$$+ \sqrt{\frac{2H_0}{\mu}}[\|\boldsymbol{q}\| p_1] sinh\psi(\psi')$$
$$+ \left[\frac{q_1{}'}{\|\boldsymbol{q}\|} + q_1(-1)\left(\frac{1}{\|\boldsymbol{q}\|^2}\right)\|\boldsymbol{q}\|' - \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})'}{\mu} - \frac{(\boldsymbol{q}\cdot\boldsymbol{p})p_1'}{\mu}\right] sinh\psi$$
$$+ \left[\frac{q_1}{\|\mathrm{q}\|} - \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right] cosh\psi(\psi'),$$
$$x_1' = \sqrt{\frac{2H_0}{\mu}}[(0)p_1 + \|\boldsymbol{q}\|(0)] cosh\psi$$
$$+ \sqrt{\frac{2H_0}{\mu}}[\|\boldsymbol{q}\| p_1](0) sinh\psi$$
$$+ \left[\frac{(0)}{\|\boldsymbol{q}\|} + q_1(-1)\left(\frac{1}{\|\boldsymbol{q}\|^2}\right)(0) - \frac{p_1(0)}{\mu} - \frac{(\boldsymbol{q}\cdot\boldsymbol{p})(0)}{\mu}\right] sinh\psi$$
$$+ \left[\frac{q_1}{\|\boldsymbol{q}\|} - \frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right](0) cos\psi,$$
$$x_1' = 0.$$

$$x_2' = \sqrt{\frac{2H_0}{\mu}}[\|\boldsymbol{q}\|' p_2 + \|\boldsymbol{q}\| p_2{}'] cosh\psi$$
$$+ \sqrt{\frac{2H_0}{\mu}}[\|\boldsymbol{q}\| p_2] sinh\psi(\psi')$$
$$+ \left[\frac{q_2{}'}{\|\boldsymbol{q}\|} + q_2(-1)\left(\frac{1}{\|\boldsymbol{q}\|^2}\right)\|\boldsymbol{q}\|' - \frac{p_2(\boldsymbol{q}\cdot\boldsymbol{p})'}{\mu} - \frac{(\boldsymbol{q}\cdot\boldsymbol{p})p_2'}{\mu}\right] sinh\psi$$
$$+ \left[\frac{q_2}{\|\boldsymbol{q}\|} - \frac{p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right] cosh\psi(\psi'),$$
$$x_2' = \sqrt{\frac{2H_0}{\mu}}[(0)p_2 + \|\boldsymbol{q}\|(-p_3)] \mathrm{cosh}\psi$$
$$+ \sqrt{\frac{2H_0}{\mu}}[\|\boldsymbol{q}\| p_2](0) sinh\psi$$
$$+ \left[\frac{(-q_3)}{\|\boldsymbol{q}\|} + q_2(-1)\left(\frac{1}{\|\boldsymbol{q}\|^2}\right)(0) - \frac{p_2(0)}{\mu} - \frac{(\boldsymbol{q}\cdot\boldsymbol{p})(-p_3)}{\mu}\right] sinh\psi$$
$$+ \left[\frac{q_2}{\|\boldsymbol{q}\|} - \frac{p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right](0) cosh\psi,$$
$$x_2' = \sqrt{\frac{2H_0}{\mu}}[\|\boldsymbol{q}\|(-p_3)] cosh\psi + \left[\frac{(-q_3)}{\|\boldsymbol{q}\|} - \frac{(\boldsymbol{q}\cdot\boldsymbol{p})(-p_3)}{\mu}\right] sinh\psi,$$
$$x_2' = -x_3.$$

$$x_3' = \sqrt{\frac{2H_0}{\mu}}[\|\boldsymbol{q}\|' p_3 + \|\boldsymbol{q}\| p_3{}'] cosh\psi$$
$$+ \sqrt{\frac{2H_0}{\mu}}[\|\boldsymbol{q}\| p_3] sinh\psi(\psi')$$

$$+\left[\frac{q_3{'}}{\|\boldsymbol{q}\|}+q_3(-1)\left(\frac{1}{\|\boldsymbol{q}\|^2}\right)\|\boldsymbol{q}\|'-\frac{p_3(\boldsymbol{q}\cdot\boldsymbol{p})'}{\mu}-\frac{(\boldsymbol{q}\cdot\boldsymbol{p})p_3'}{\mu}\right]sinh\psi$$

$$+\left[\frac{q_3}{\|\boldsymbol{q}\|}-\frac{p_3(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right]cos\psi(\psi'),$$

$$x_3'=\sqrt{\frac{2H_0}{\mu}}[(0)p_3+\|\boldsymbol{q}\|(p_2)]cosh\psi$$

$$+\sqrt{\frac{2H_0}{\mu}}[\|\boldsymbol{q}\|p_3](0)sinh\psi$$

$$+\left[\frac{q_2}{\|\boldsymbol{q}\|}+q_3(-1)\left(\frac{1}{\|\boldsymbol{q}\|^2}\right)(0)-\frac{p_3(0)}{\mu}-\frac{(\boldsymbol{q}\cdot\boldsymbol{p})(p_2)}{\mu}\right]sinh\psi$$

$$+\left[\frac{q_3}{\|\boldsymbol{q}\|}-\frac{p_3(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right](0)cos\psi,$$

$$x_3'=\sqrt{\frac{2H_0}{\mu}}[\|\boldsymbol{q}\|(p_2)]cosh\psi+\left[\frac{q_2}{\|\boldsymbol{q}\|}-\frac{(\boldsymbol{q}\cdot\boldsymbol{p})(p_2)}{\mu}\right]sinh\psi,$$

$$x_3'=x_2.$$

$$y_0'=[(\boldsymbol{q}\cdot\boldsymbol{p})']cosh\psi$$

$$+(\boldsymbol{q}\cdot\boldsymbol{p})sin\psi(\psi')$$

$$-\sqrt{\frac{\mu}{2H_0}}\left[\frac{\|\boldsymbol{q}\|'\|\boldsymbol{p}\|^2}{\mu}+\frac{\|\boldsymbol{q}\|(2)(\boldsymbol{p}\cdot\boldsymbol{p}')}{\mu}\right]sinh\psi$$

$$-\sqrt{\frac{\mu}{2H_0}}\left[\left(\frac{\|\boldsymbol{q}\|\cdot\|\boldsymbol{p}\|^2}{\mu}-1\right)\right]cosh\psi(\psi'),$$

$$y_0'=[(0)]cosh\psi$$

$$+(\boldsymbol{q}\cdot\boldsymbol{p})sin\psi(0)$$

$$-\sqrt{\frac{\mu}{2H_0}}\left[\frac{(0)\|\boldsymbol{p}\|^2}{\mu}+\frac{\|\boldsymbol{q}\|(2)(0)}{\mu}\right]sinh\psi$$

$$-\sqrt{\frac{\mu}{2H_0}}\left[\left(\frac{\|\boldsymbol{q}\|\cdot\|\boldsymbol{p}\|^2}{\mu}-1\right)\right]cosh\psi(0),$$

$$y_0'=0.$$

$$y_1'=-\sqrt{\frac{\mu}{2H_0}}\left[\frac{q_1{'}}{\|\boldsymbol{q}\|}+q_1(-1)\left(\frac{1}{\|\boldsymbol{q}\|^2}\right)\|\boldsymbol{q}\|'-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})'}{\mu}-\frac{(\boldsymbol{q}\cdot\boldsymbol{p})p_1'}{\mu}\right]cosh\psi$$

$$-\sqrt{\frac{\mu}{2H_0}}\left[\frac{q_1}{\|\boldsymbol{q}\|}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right]sinh\psi(\psi')$$

$$-[p_1\|\boldsymbol{q}\|'+\|\boldsymbol{q}\|p_1{'}]sinh\psi$$

$$-[\|\boldsymbol{q}\|p_1]cosh\psi(\psi'),$$

$$y_1'=-\sqrt{\frac{\mu}{2H_0}}\left[\frac{(0)}{\|\boldsymbol{q}\|}+q_1(-1)\left(\frac{1}{\|\boldsymbol{q}\|^2}\right)(0)-\frac{p_1(0)}{\mu}-\frac{(\boldsymbol{q}\cdot\boldsymbol{p})(0)}{\mu}\right]cosh\psi$$

$$-\sqrt{\frac{\mu}{2H_0}}\left[\frac{q_1}{\|\boldsymbol{q}\|}-\frac{p_1(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right]sinh\psi(0)$$

$$-[p_1(0)+\|\boldsymbol{q}\|(0)]sinh\psi$$

$$-[\|\boldsymbol{q}\|p_1]cos\psi(0),$$

$$y_1' = 0.$$

$$y_2' = -\sqrt{\frac{\mu}{2H_0}}\left[\frac{q_2{}'}{\|\boldsymbol{q}\|} + q_2(-1)\left(\frac{1}{\|\boldsymbol{q}\|^2}\right)\|\boldsymbol{q}\|' - \frac{p_2(\boldsymbol{q}\cdot\boldsymbol{p})'}{\mu} - \frac{(\boldsymbol{q}\cdot\boldsymbol{p})p_2'}{\mu}\right]cosh\psi$$
$$-\sqrt{\frac{\mu}{2H_0}}\left[\frac{q_2}{\|\boldsymbol{q}\|} - \frac{p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right]sinh\psi(\psi')$$
$$-[p_2\|\boldsymbol{q}\|' + \|\boldsymbol{q}\|p_2{}']sinh\psi$$
$$-[\|\boldsymbol{q}\|p_2]cos\psi(\psi'),$$

$$y_2' = -\sqrt{\frac{\mu}{2H_0}}\left[\frac{(-q_3)}{\|\boldsymbol{q}\|} + q_2(-1)\left(\frac{1}{\|\boldsymbol{q}\|^2}\right)(0) - \frac{p_2(0)}{\mu} - \frac{(\boldsymbol{q}\cdot\boldsymbol{p})(-p_3)}{\mu}\right]cosh\psi$$
$$-\sqrt{\frac{\mu}{2H_0}}\left[\frac{q_2}{\|\boldsymbol{q}\|} - \frac{p_2(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right]sinh\psi(0)$$
$$-[p_2(0) + \|\boldsymbol{q}\|(-p_3)]sinh\psi$$
$$-[\|\boldsymbol{q}\|p_2]cos\psi(0),$$

$$y_2' = -\sqrt{\frac{\mu}{2H_0}}\left[\frac{(-q_3)}{\|\boldsymbol{q}\|} - \frac{(\boldsymbol{q}\cdot\boldsymbol{p})(-p_3)}{\mu}\right]cosh\psi - [\|\boldsymbol{q}\|(-p_3)]sinh\psi,$$

$$y_2' = -y_3.$$

$$y_3' = -\sqrt{\frac{\mu}{2H_0}}\left[\frac{q_3{}'}{\|\boldsymbol{q}\|} + q_3(-1)\left(\frac{1}{\|\boldsymbol{q}\|^2}\right)\|\boldsymbol{q}\|' - \frac{p_3(\boldsymbol{q}\cdot\boldsymbol{p})'}{\mu} - \frac{(\boldsymbol{q}\cdot\boldsymbol{p})p_3'}{\mu}\right]cosh\psi$$
$$-\sqrt{\frac{\mu}{2H_0}}\left[\frac{q_3}{\|\boldsymbol{q}\|} - \frac{p_3(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right]sinh\psi(\psi')$$
$$-[p_3\|\boldsymbol{q}\|' + \|\boldsymbol{q}\|p_3{}']sinh\psi$$
$$-[\|\boldsymbol{q}\|p_3]cos\psi(\psi'),$$

$$y_3' = -\sqrt{\frac{\mu}{2H_0}}\left[\frac{(q_2)}{\|\boldsymbol{q}\|} + q_3(-1)\left(\frac{1}{\|\boldsymbol{q}\|^2}\right)(0) - \frac{p_3(0)}{\mu} - \frac{(\boldsymbol{q}\cdot\boldsymbol{p})(p_2)}{\mu}\right]cosh\psi$$
$$-\sqrt{\frac{\mu}{2H_0}}\left[\frac{q_3}{\|\boldsymbol{q}\|} - \frac{p_3(\boldsymbol{q}\cdot\boldsymbol{p})}{\mu}\right]sinh\psi(0)$$
$$-[p_3(0) + \|\boldsymbol{q}\|(p_2)]sinh\psi$$
$$-[\|\boldsymbol{q}\|p_3]cosh\psi(0),$$

$$y_3' = -\sqrt{\frac{\mu}{2H_0}}\left[\frac{(q_2)}{\|\boldsymbol{q}\|} - \frac{(\boldsymbol{q}\cdot\boldsymbol{p})(p_2)}{\mu}\right]cosh\psi - [\|\boldsymbol{q}\|(p_2)]sinh\psi,$$

$$y_3' = y_2. \blacksquare$$

X. Concluding Remarks