State Prices Implicit in Valuation Formulae for Derivative Securities*

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Abstract

Derivative assets analysis usually takes a model of the underlying price process as given and attempts to value derivative securities relative to that model. This paper studies the following "inverse" problem: given a valuation formula for a derivative asset, what can be inferred about the underlying asset price process? Assuming continuous sample paths, we show that a sufficiently regular pricing formula for some derivative asset completely determines the risk-neutral law of the underlying price. In particular, such a valuation formula implies a unique set of state prices for payoffs contingent on the price path of the underlying security. As an illustration of our main result, we analyse certain pricing formulae for European options on zero-coupon bonds.

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Introduction

Derivative assets analysis usually takes a model of the underlying price processes as given and attempts to value derivatives relative to that model. The recent literature on "implied trees"[1] has renewed the interest in the "inverse" problem: given some set of derivatives prices, what can we say about the price processes of the underlying securities? More precisely, this literature tries to construct models which, in contrast to the Black-Scholes model, are consistent with the observed market prices of standard European options and thus provide a better framework for the hedging and pricing of "exotic" over-the-counter derivatives.

This paper studies a different variant of the "inverse" problem: given a *valuation formula* for a derivative asset, what can be inferred about the underlying asset prices? We assume that the price of a derivative asset is a deterministic function of the underlying security prices and time, and investigate the restrictions such a pricing relationship imposes on the underlying price dynamics.[2]

We restrict ourselves to the simplest possible setting with a riskless cash account, one risky security, and one derivative. Assuming that asset prices are continuous semimartingales, we consider pricing formulae that satisfy a variant of the fundamental valuation equation which is familiar from derivative asset pricing in a diffusion setting. In fact, such a formula will hold whenever the risk-neutralised price of the underlying asset follows a diffusion process.[3] We show that this condition is also necessary: a pricing formula of this type can hold only if the risk-neutralised price process of the underlying asset is a diffusion; moreover, the diffusion coefficient is uniquely determined by the valuation formula. In other words, the formula completely determines the risk-neutral law of the underlying asset price, thus implying a unique system of state prices for payoffs contingent on the price path of the underlying security.

While similar in spirit to Breeden and Litzenberger's (1978) calculation of state prices implicit in option prices, the approach of this paper relies on rather different mathematical tools, based mainly on semimartingale calculus. The main result follows directly from a characterisation theorem for continuous local martingales which extends work by McGill, Rajeev and Rao (1988) on Brownian motion.

Of course, this result is based on a purely theoretical assumption – knowledge of the price of a derivative asset at all dates and in all states of the world. However, it could have some practical relevance for nonparametric approaches to derivative asset pricing via learning networks.[4] In principle, once a pricing formula has been learnt by a network, the techniques presented here could be used to identify the risk-neutral law of the underlying asset prices.

As an illustration of these techniques, we analyse pricing formulae for options on zero-coupon bonds which have been proposed in the literature. We consider option

[1] Cf. Rubinstein (1994) and the references therein, in particular Dupire (1994).

[2] This can be seen as an analogue to the problem of identifying necessary conditions which equilibrium asset price processes must satisfy in a given exchange economy; see Bick (1990) and He and Leland (1993).

[3] Provided the diffusion coefficient is almost always non-zero; see Duffie (1991).

[4] See Hutchinson, Lo and Poggio (1994).

price formulae of two types. The first one goes back to Merton's (1973) paper on the valuation of stock options, and has been obtained in a variety of bond price or interest rate models.[5] These models have in common that they allow for negative interest rates. A direct application of our result confirms that such formulae are indeed inconsistent with non-negative interest rates.

The second type of pricing formula is closely related to the first. Schöbel (1986) and, more recently, Briys, Crouhy and Schöbel (1991) proposed this type of formula for the valuation of discount bond options in an environment where interest rates do remain non-negative. Their formula is obtained by solving Mertons's fundamental valuation equation with an additional boundary condition that follows from the non-negativity of interest rates. Applying our martingale technique, we show that this pricing formula implies a positive probability for a certain process of implied forward rates to be absorbed at its lower bound 0 during the life of the option. Thus, while these authors obtain a pricing formula that is formally consistent with non-negative interest rates, they implicitly accept an implausible bond price and interest rate behaviour.

The rest of the paper is organised as follows. After introducing the setup, Section 1 states and interprets the main result. Section 2 analyses pricing formulae for options on zero-coupon bonds. Section 3 concludes the paper. All proofs are given in an appendix.

1 Martingale Measures and Pricing Formulae

We fix a finite time interval $\mathcal{T} = [0, T]$, a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $(\mathcal{F}_t)_{t\in\mathcal{T}}$ satisfying the usual conditions. $\mathcal{F}_0$ is assumed to be almost trivial, and $\mathcal{F}_T = \mathcal{F}$.

Consider three securities, labelled 0, 1 and 2. We make the following assumptions:

- Trade in these securities is continuous and frictionless.
- The securities pay no dividends.
- Security 0 has a constant price $X_t^0 \equiv 1$.
- The price processes of securities 1 and 2, denoted by X^1 and X^2, are positive continuous semimartingales.

Security 0 can be thought of as a riskless cash account with zero interest. Alternatively, we can interpret (X^0, X^1, X^2) as a normalised price system, expressed in units of some numeraire asset.

Note that we do not assume that the filtration $(\mathcal{F}_t)_{t\in\mathcal{T}}$ is generated by the price processes of assets 0, 1 and 2. We allow the filtration to contain more information than just past prices of these three assets. This additional information could be the price history of other securities or, more generally, non-price information about arbitrary economic variables. For later use, let $(\mathcal{G}_t^i)_{t\in\mathcal{T}}$ be the completion of the filtration generated by X^i, and write $\mathcal{G}^i = \mathcal{G}_T^i$.

[5] See for example Ball and Torous (1983), El Karoui and Rochet (1989), Kemna, de Munnik and Vorst (1989), Hull and White (1990), Jamshidian (1991), Heath, Jarrow and Morton (1992).

A *martingale measure* for the price system (X^0, X^1, X^2) is defined as a probability measure Q equivalent to P such that both processes X^i $(i = 1, 2)$ are Q-martingales. The existence of a martingale measure ensures absence of arbitrage opportunities in a suitably chosen space of admissible trading strategies. Such a measure, if it exists, is in general not unique.[6]

We say that the price of asset 2 is given by a *pricing formula* if there is some function $u(t, x)$ such that

$$X_t^2 = u(t, X_t^1)$$

for all $t \in \mathcal{T}$. The literature on the valuation of derivative assets has calculated pricing formulae for a variety of securities. Adopting for a moment the perspective of derivative assets analysis, think of assets 0 and 1 as primitive securities, and of asset 2 as a derivative with payoff depending on the price of asset 1 at the terminal date. Given the price processes of the primitive assets, the task is to determine the fair price of asset 2. Typically, this involves the following steps.[7] First, one establishes the existence of a martingale measure for the system (X^0, X^1) of primitive asset prices. Next, one proves that the derivative claim is attainable, i.e., that it can be replicated by a dynamically adjusted self-financing trading strategy in the primitive assets. The price of the derivative asset must then be equal to the value of the replicating portfolio; any deviation would lead to arbitrage opportunities. Moreover, the price of the derivative is again a martingale under the given martingale measure, so it can be calculated without reference to a replicating strategy, just by taking expectations of the final payoff under the martingale measure. Finally, if the primitive asset prices have the Markov property, then the solution of the valuation problem indeed takes the form of a pricing formula.

Assume for example that (X^0, X^1) has a martingale measure Q such that X^1 solves the stochastic differential equation

$$dX_t^1 = \sigma(t, X_t^1)\, dW_t \tag{SDE}$$

with $\sigma(t, x)$ sufficiently regular and W a Wiener process under Q. Then one has the following well-known result. Asset 2 is attainable, and its price process is of the form $X_t^2 = u(t, X_t^1)$ with $u(t, x)$ being a solution of the partial differential equation

$$u_t + \tfrac{1}{2}\sigma^2 u_{xx} = 0. \tag{PDE}$$

Thus, if X^1 has a martingale measure under which it is a diffusion satisfying (SDE), we get pricing formulae for derivatives involving solutions to the valuation equation (PDE). Our aim is to prove a converse to this statement.

Returning to the general setup, let us assume that (X^0, X^1, X^2) has a martingale measure, and let the price of asset 2 be given by a pricing formula $X_t^2 = u(t, X_t^1)$ where u is once continuously differentiable with respect to t and twice with respect to x. Fix a time t and a realisation x of the random variable X_t^1. Suppose that $u_{xx}(t, x) > 0$, say, so u is strictly convex in its second argument around (t, x). By Jensen's inequality, the

[6]Uniqueness of the martingale measure corresponds to completeness of the securities market. See Harrison and Pliska (1981, 1983).

[7]Cf. Harrison and Piska (1981).

holder of asset 2 can therefore expect a gain from the random movements of X^1 over a short time interval. The existence of a martingale measure, however, precludes such a gain. To balance the Jensen effect, the passing of time must therefore have a tendency to reduce the value of asset 2, in other words, $u_t(t,x) < 0$. By the same argument, $u_{xx}(t,x) < 0$ implies $u_t(t,x) > 0$.[8] Thus, whenever $u_{xx}(t,x) \neq 0$, we can define

$$\sigma(t,x) = \sqrt{-\frac{2\,u_t(t,x)}{u_{xx}(t,x)}}$$

and thereby satisfy (PDE) at the given point (t,x). In this sense, (PDE) is just a consequence of a simple "no expected gain" argument, and does not impose restrictions on the underlying process X^1. In the theorem below, we shall therefore make the additional assumption that the above function $\sigma(t,x)$ which we defined point by point on a subset of the domain of u has in fact a *continuous* extension to the whole of that domain.

For a similar reason, we shall also stipulate that u be sufficiently non-linear, i.e., that u_{xx} does not vanish too often. Clearly, a linear pricing formula will not restrict the underlying process at all – a statement like "two shares cost twice the price of one share" will not tell us anything about the underlying stock price model.

We are now ready to formulate the main result of this paper.

Theorem 1.1 *Let the price system (X^0, X^1, X^2) on $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in\mathcal{T}})$ satisfy*

$$X_t^2 = u(t, X_t^1)$$

where $u(t,x)$ is a solution of (PDE) *with a continuous function $\sigma(t,x)$. Assume that $\{t \in \mathcal{T} :\ \sigma(t, X_t^1) = 0\}$ has Lebesgue measure zero almost surely.*[9] *In addition, suppose that at least one of the following two conditions holds:*

$$u_{xx}(t, X_t^1) \neq 0 \text{ for all } t \in \mathcal{T} \text{ almost surely,}$$

or

$$\{t \in \mathcal{T} :\ u_{xx}(t, X_t^1) = 0\} \text{ is almost surely a Lebesgue null set,}$$
$$u(t,x) \text{ is analytic, and } \sigma^2(t,x) \text{ has partial derivatives of all order.}$$[10]

[8] A mathematically precise argument runs as follows. For $T > t$, Itô's lemma implies

$$X_T^2 - X_t^2 - \int_t^T u_x(s, X_s^1)\, dX_s^1 = \int_t^T u_t(s, X_s^1)\, ds + \tfrac{1}{2} \int_t^T u_{xx}(s, X_s^1)\, d\langle X^1\rangle_s\,.$$

Under a martingale measure, the left hand side is a continuous local martingale while the right hand side is of finite variation, so both must vanish identically. This requires $u_t(t, X_t^1)$ and $u_{xx}(t, X_t^1)$ to be of opposite sign whenever the latter expression is non-zero.

[9] This assumption is inessential. See Remark A.5 in the Appendix.

[10] This condition allows for derivatives whose payoff profile has both concave and convex sections, such as option spreads.

Finally, let Q be a martingale measure for this price system. Then there is a Wiener process W under Q such that the price process X^1 satisfies (SDE) *with the given function $\sigma(t,x)$.*

The theorem is a direct consequence of a somewhat more general mathematical result which we prove in the Appendix.

According to Theorem 1.1, a pricing formula satisfying (PDE) under the stated conditions completely characterises the behaviour of the price of asset 1 under the martingale measure Q.[11] Indeed, (SDE) and the fact that W is a Wiener process completely determine the law of X^1 under Q. As a first consequence, note that the pricing formula implies the Markov property for X^1 under Q.[12] (PDE) is then just the associated backward equation.

More important, the law of X^1 is the same under *all* martingale measures. In other words, all martingale measures coincide on $\mathcal{G}^1$. By a theorem of Jacka (1992), this implies that all $\mathcal{G}^1$-measurable contingent claims are attainable, hence priced by arbitrage.[13] This holds in particular for the Arrow-Debreu security with time T payoff 1_A where $A \in \mathcal{G}^1$. The pricing formula thus implies a unique system of Arrow-Debreu or state prices for events in $\mathcal{G}^1$. As usual, these prices are obtained by taking the expectation of the corresponding Arrow-Debreu payoffs under any martingale measure.

The idea of extracting state prices from derivative prices goes back at least to Breeden and Litzenberger (1978).[14] In the present setting, their argument can be rendered as follows. Assume that we have a securities market with assets 0 and 1 as before but, instead of asset 2, European call options written on asset 1 for any strike price and exercise date. Let $C_0^{T,K}$ denote today's call price for exercise date T and strike price K. Assume that there exists a martingale measure for this securities market, and let F_T be the corresponding distribution function for the random variable X_T^1. Call prices must satisfy

$$C_0^{T,K} = \int_K^\infty (x-K)\; dF_T(x)$$

by definition of a martingale measure. Integration by parts yields

$$C_0^{T,K} = \int_K^\infty (1 - F_T(x))\; dx$$

and hence

$$F_T(K) = 1 + \frac{\partial}{\partial K} C_0^{T,K}.$$

Thus the distribution function F_T is uniquely determined by the given option prices.[15]

[11] We assume in the following that σ satisfies the regularity conditions for uniqueness of weak solutions of (SDE). See for example Karatzas and Shreve (1988).

[12] Recall that we merely assumed this price to be a continuous semimartingale.

[13] Alternatively, this result follows from the martingale representation property of X^1 with respect to the smaller filtration $(\mathcal{G}_t^1)_{t\in\mathcal{T}}$; cf. Ikeda and Watanabe (1989).

[14] See also Ross (1976).

[15] The value $F_T(K)$ is the price of the Arrow-Debreu claim $1_{X_T^1 \le K}$. Differentiating once more, where possible, we get the *state price density* $f_T(K) = \frac{\partial^2}{\partial K^2} C_0^{T,K}$. This is Breeden and Litzenberger's original result.

Dupire (1994) takes this analysis one step further. He assumes the existence of a martingale measure under which X^1 is a diffusion process satisfying (SDE) for some unknown function $\sigma(t,x)$. Using the forward equation associated with such a diffusion, he shows that given the call price $C_0^{T,K}$ for all T and K, it is possible, under certain regularity conditions, to back out the function σ from the distribution functions F_T. Therefore, the law of the process is completely determined by these call prices, and we have again a unique set of Arrow-Debreu prices for events in $\mathcal{G}^1$.[16]

Our approach, as summarised in Theorem 1.1, and Dupire's approach can be regarded as "dual" to each other. This feature appears most clearly in the analysis of call option prices. Suppose that time t call prices are given by some function $u(t,x;T,K)$ where x is the concurrent price of the underlying asset, T the exercise date, and K the exercise price. Dupire's result means that a unique set of state prices can be extracted from the values $u(0,x_0;T,K)$ where the initial price of the underlying asset is fixed, while T and K are variable. Theorem 1.1, on the other hand, determines state prices on the basis of the values $u(t,x;\overline{T},\overline{K})$ for fixed option characteristics, but variable t and x.[17] Thus, Dupire's result and Theorem 1.1 are "dual" in the sense that the former varies the "forward variables" (T,K), and the latter the "backward variables" (t,x).

Finally, note that under the stated conditions, our theorem also allows us to check the consistency of pricing formulae for different derivatives written on the same underlying asset. In fact, these can only be consistent if the implied diffusion coefficient

$$\sigma(t,x) = \sqrt{-\frac{2\,u_t(t,x)}{u_{xx}(t,x)}}$$

is the same for all derivatives.

2 Bond Options and Implied Forward Yields

In this section, we use our result to analyse pricing formulae for European options on zero-coupon bonds. We fix a time interval $\mathcal{T} = [0,T]$ and a filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in\mathcal{T}})$ as in Section 1. Let S^0 be the price process of a default-free discount bond maturing at T, i.e., satisfying $S^0_T = 1$ almost surely. This bond, which we call the reference bond, will serve as numeraire. Consider a second bond, called the underlying bond, that matures at a time $T' > T$. Let S^1 be its price process up to time T. The third security is a European call option written on the underlying bond with exercise date T and strike price K. Its price process is denoted by S^2. By definition, the terminal value of the option is $S^2_T = [S^1_T - K]^+$. We make the following assumptions:

- Trade in the bonds and the option is continuous and frictionless.
- The price processes of the bonds and the option are positive continuous semimartingales.

[16] Dupire's work is one of the first contributions to the recent literature on "implied trees"; see Rubinstein (1994) and the references therein.

[17] The choice of $\overline{T}$ and $\overline{K}$ is irrelevant, of course, since Theorem 1.1 does not depend on the particular form of the derivative's terminal value.

In order to obtain the setting studied in Section 1, we define the processes

$$X^i = \frac{S^i}{S^0} \qquad (i = 0,\, 1,\, 2)$$

which describe asset prices in units of the reference bond. In less abstract terms, X_t^1 and X_t^2 are just the time t forward prices of the underlying bond and the option for delivery at T. These processes are again positive continuous semimartingales.

The *forward yield* Y_t implied by the bond prices S_t^0 and S_t^1 is

$$Y_t = -\frac{\log S_t^1 - \log S_t^0}{T' - T} = -\frac{\log X_t^1}{T' - T}.$$

This is the continuously compounded interest rate as seen at time t for a loan which starts at T and is repaid at T'. The bond price model (S^0, S^1) is said to *generate negative forward yields* if $P\left(\{\omega \in \Omega:\ \exists t \in \mathcal{T}\ \ Y_t(\omega) < 0\}\right) > 0$; otherwise, the bond price model satisfies $P\left(\{\omega \in \Omega:\ \forall t \in \mathcal{T}\ \ Y_t(\omega) \geq 0\}\right) = 1$ and is said to have *non-negative forward yields*. Finally, we say that the bond price model has *positive forward yields* if $P\left(\{\omega \in \Omega:\ \forall t \in \mathcal{T}\ \ Y_t(\omega) > 0\}\right) = 1$.

2.1 Merton Type Option Prices

This section deals with the type of valuation formulae going back to Black and Scholes (1973) and Merton (1973). Let a positive continuous function $\nu : \mathcal{T} \to I\!\!R_{++}$ be given. We say that the price system (S^0, S^1, S_t^2) satisfies the *Merton call price formula for volatility function* ν if

$$S_t^2 = S_t^0\, u(t, X_t^1)$$

or, equivalently,

$$X_t^2 = u(t, X_t^1)$$

with u defined as follows:

$$u(t,x) = x\,\Phi\left(\frac{1}{\sqrt{s(t)}}\left[\log\frac{x}{K} + \frac{s(t)}{2}\right]\right) - K\,\Phi\left(\frac{1}{\sqrt{s(t)}}\left[\log\frac{x}{K} - \frac{s(t)}{2}\right]\right)$$

where Φ is the standard normal distribution function and

$$s(t) = \int_t^T \nu^2(\xi)\, d\xi.$$

On $\{T\} \times I\!\!R_+$ and $\mathcal{T} \times \{0\}$, u satisfies the standard boundary conditions for a call option,

$$\begin{aligned} u(T,x) &= (x-K)^+, \\ u(t,0) &= 0. \end{aligned}$$

It is well known that u solves (PDE) with $\sigma(t,x) = \nu(t)\,x$, that is,

$$u_t + \tfrac{1}{2}\nu^2 x^2 u_{xx} = 0. \tag{1}$$

Bond option formulae of this type hold in so-called linear Gaussian models of the term structure of interest rates. Examples are Vasicek (1977) and its extension by Hull and White (1990). A systematic analysis of Gaussian models, as well as derivations of the pricing formulae we are considering here, can be found in El Karoui and Rochet (1989), Jamshidian (1991), and El Karoui, Myneni and Viswanathan (1992). The deterministic volatility examples in Heath, Jarrow and Morton (1992) belong also to this category. Using the framework of Merton (1973), Ball and Torous (1983) and Kemna, de Munnik and Vorst (1989) derived bond option formulae of this type in bond price based models.[18]

The common characteristic of all these models is that they allow for negative interest rates. A direct application of Theorem 1.1 confirms that option price formulae of the Merton type are indeed inconsistent with non-negative interest rates. Of course, the fact that u solves (1) and is strictly convex in x for all $t < T$ is all we need in order to infer properties of the forward price X^1 or the forward yield Y. Applying Theorem 1.1, we get

Proposition 2.1 *Assume that the price system (S^0, S^1, S^2) on $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in\mathcal{T}})$ satisfies the Merton call price formula for volatility function ν. Let Q be a martingale measure for this price system. Then the forward price of the underlying bond solves the stochastic differential equation*

$$dX_t^1 = \nu(t)\, X_t^1\, dW_t \tag{2}$$

where W is a Wiener process under the measure Q.

By the formula for the martingale exponential, (2) is equivalent to

$$X_t^1 = X_0^1 \exp\Big(\int_0^t \nu(s)\, dW_s - \tfrac{1}{2}\int_0^t \nu^2(s)\, ds\Big)\,. \tag{3}$$

Thus, up to the time change

$$t \mapsto \int_0^t \nu^2(s)\, ds,$$

the forward price process X^1 is a driftless geometric Brownian motion under any martingale measure Q, and the forward yield Y is simply a Brownian motion with drift. This implies in particular that Q and, by equivalence of measures, P assign a positive probability to the event $\{\omega : \exists t\ Y_t(\omega) < 0\}$. Thus, we obtain the well-known

Result 2.1 *A bond price model in which an option formula of the Merton type holds necessarily generates negative forward yields.*

2.2 An Upper Bound on the Forward Bond Price

Next consider a price system (S^0, S^1, S^2) where forward yields remain non-negative, i.e., $Y_t \geq 0$ and $X_t^1 \leq 1$ for all t. Assume that the strike price of the call option

[18] See Rady and Sandmann (1994) for a survey of the bond price based approach to debt option pricing.

satisfies $0 < K < 1$; as $S_T^1 = X_T^1 \leq 1$, only these exercise prices are of interest. Using a portfolio dominance argument, Schöbel (1986) derives the following necessary condition for absence of arbitrage:[19]

$$S_t^2 = S_t^0 \, (1 - K) \quad \text{whenever} \quad S_t^1 = S_t^0,$$

that is,

$$X_t^2 = 1 - K \quad \text{whenever} \quad X_t^1 = 1. \tag{4}$$

Thus, the forward call price assumes the deterministic value $1 - K$ when the forward yield Y_t is at its lower bound 0.

Of course, Merton type call prices violate (4). In view of this, Schöbel tries to correct for negative yields by imposing (4) as an additional boundary condition on the Merton valuation equation (1). More precisely, he proposes a modified pricing formula

$$X^2 = u^*(t, X_t^1)$$

where $u^* : \mathcal{T} \times [0,1] \to I\!R_+$ solves (1) subject to the following conditions:

$$\begin{aligned} u^*(T, x) &= (x - K)^+, \\ u^*(t, 0) &= 0, \\ u^*(t, 1) &= 1 - K. \end{aligned}$$

The first and second condition are the usual ones for a call option, while the last condition expresses (4). The solution is

$$u^*(t, x; K) = u(t, x; K) - K \, u(t, x; K^{-1})$$

where $u(t, x; K)$ denotes the Merton call price function for strike price K.

Briys, Crouhy and Schöbel (1991) use a formula of this type to value interest rate caps and floors. They see the second term in u^* as a price correction which ensures consistency of bond option prices with non-negative interest rates. Moreover, they interpret the additional boundary condition as the effect of an absorbing barrier, but do not clarify the nature of the absorption phenomenon. We shall see in a while that the additional boundary condition corresponds in fact to an absorbing barrier for the forward bond price at its upper bound 1.[20]

Before applying the technique underlying Theorem 1.1, let us first point out that absorption of the forward price and the forward yield at their respective boundaries is indeed the only behaviour consistent with the absence of arbitrage.

Lemma 2.1 *Let (S^0, S^1) be a bond price model on $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \in \mathcal{T}})$ with a martingale measure Q. Assume that the model has non-negative forward yields and consider the hitting time $\chi = \inf\{t \in \mathcal{T} : \ X_t^1 = 1\}$. Then $X_t^1 = 1$ on $[\chi, T]$ almost surely under either measure.*

[19] The argument can also be found in Briys, Crouhy and Schöbel (1991) .

[20] See Sondermann (1988) for a similar absorption phenomenon in discrete-time binomial approximations to the Black-Scholes model.

There is a simple intuition behind Lemma 2.1. Assume that in a world of non-negative forward yields, the underlying bond and the reference bond have the same price at some time t_0. A portfolio short one underlying bond and long one reference bond costs nothing at t_0. After t_0, the portfolio value cannot fall below zero since the underlying bond will never cost more than the reference bond. On the other hand, the portfolio cannot rise in value either, otherwise it would certainly trade at a positive price now. Therefore, the two bond prices must coincide for ever, that is, until the shorter lived bond expires. By the same token, forward bond prices and forward yields are absorbed at their upper and lower bound, respectively. Any other boundary behaviour, for example reflection, would lead to arbitrage opportunities.

As a corollary, we get the following simple classification.

Proposition 2.2 *Let (S^0, S^1) be a bond price model admitting a martingale measure. Then exactly one of the following statements holds true:*

(i) *The model generates negative forward yields.*

(ii) *The model has non-negative forward yields, the probability that the forward yield reaches its lower bound 0 is positive, and 0 is an absorbing barrier for the forward yield.*

(iii) *The model has positive forward yields.*

We have seen that bond price models consistent with a Merton type formula belong to category (i). As for models with non-negative yields in which an option formula of the Schöbel type holds, we have to establish which of the two properties (ii) and (iii) is satisfied, that is, whether the bound is reached with positive probability or not. The following proposition does more than that: it gives a complete description of the behaviour of the forward bond price under a martingale measure.

Proposition 2.3 *Assume that the price system (S^0, S^1, S^2) on $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in\mathcal{T}})$ has non-negative forward yields and satisfies the Schöbel call price formula for volatility function ν. Let Q be a martingale measure for this price system. Then there is a Wiener process $\overline{W}$ on an extension $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{Q}, (\overline{\mathcal{F}}_t)_{t\in\mathcal{T}})$ of $(\Omega, \mathcal{F}, Q, (\mathcal{F}_t)_{t\in\mathcal{T}})$ such that the forward price of the underlying bond satisfies*

$$dX_t^1 = 1_{\{t\le\chi\}}\, \nu(t)\, X_t^1\, d\overline{W}_t \tag{5}$$

with $\chi = \inf\{t \in \mathcal{T} :\ X_t^1 = 1\}$.

Let Q and $\overline{W}$ be as in the proposition. By the formula for the martingale exponential, (5) implies that X^1 is the stopped process

$$X_t^1 = \overline{X}_{t\wedge\chi}$$

with

$$\overline{X}_t = X_0^1 \exp\Big(\int_0^t \nu(s)\, d\overline{W}_s - \tfrac{1}{2}\int_0^t \nu^2(s)\, ds\Big).$$

Thus, the forward bond price process implied by Schöbel's option price formula is obtained by imposing an absorbing barrier at 1 on $\overline{X}$, a forward price process of the Merton type. $\overline{Q}$ assigns positive probability to the event that $\overline{X}_t = 1$ for some $t \in \mathcal{T}$. Under both Q and P, the forward bond price therefore reaches its upper boundary with positive probability.

Result 2.2 *A bond price model with non-negative forward yields which satisfies an option price formula of the Schöbel type assigns positive probability to the event that the forward yield reaches its lower bound 0, where it is absorbed.*

With positive probability, therefore, there will be no reward for holding the underlying bond from T to T'. Put differently, the Arrow-Debreu security paying one unit if and only if the forward yield is zero at T commands a positive price.

3 Conclusion

The validity of a valuation formula for a derivative asset has strong implications for the behaviour of the underlying asset price under a martingale measure. In a setting with continuous sample paths, we have studied pricing formulae that depend on a single underlying price and satisfy the fundamental valuation PDE. We have shown that such a formula completely determines the risk-neutral law of the underlying asset price. In particular, there is a unique set of state prices for payoffs contingent on the price path of the underlying asset.

As an illustration of our main result, we have analysed certain pricing formulae for European options on discount bonds. This analysis has shown that the valuation formulae proposed by Schöbel (1986) and Briys, Crouhy and Schöbel (1991) imply an implausible behaviour of the forward yield, involving absorption of this yield at zero.

Appendix

A Characterisation Theorem for Continuous Local Martingales

In this section, we state and prove the mathematical result which underlies Theorem 1.1. We extend the work of McGill, Rajeev and Rao (1988) on Brownian motion to a larger class of continuous local martingales.[21] Throughout the section, we consider a finite time interval $\mathcal{T}$ as before and a filtered probability space satisfying the usual conditions.

Theorem A.1 *Let X_t be a continuous local martingale, $\sigma(t,x)$ a continuous function, and $u(t,x)$ a solution of*

$$u_t + \tfrac{1}{2}\sigma^2 u_{xx} = 0 \tag{PDE}$$

such that

(A1) *the process $u(t,X_t)$ is a local martingale;*

(A2) $\{t \in \mathcal{T} : \ \sigma(t,X_t) = 0\}$ *is almost surely a Lebesgue null set;*

(A3) $u_{xx}(t,X_t) \neq 0$ *for all $t \in \mathcal{T}$ almost surely.*

Then there exists a Wiener process W such that

$$X_t = X_0 + \int_0^t \sigma(s,X_s)\, dW_s. \tag{6}$$

Moreover, this continues to hold if (A3) *is replaced with the two conditions*

(A4) $\{t \in \mathcal{T} : \ u_{xx}(t,X_t) = 0\}$ *is almost surely a Lebesgue null set;*

(A5) *$u(t,x)$ is analytic, and $\sigma^2(t,x)$ has partial derivatives of all orders.*

Remark A.1 Note that if (PDE) holds, conditions (A2) and (A4) together are equivalent to the condition that $\{t \in \mathcal{T} : \ u_t(t,X_t) = 0\}$ is almost surely a Lebesgue null set. This is the condition used in McGill, Rajeev and Rao (1988).

Remark A.2 McGill, Rajeev and Rao (1988) study the case $\sigma(t,x) \equiv 1$ with infinite time horizon, i.e., $\mathcal{T} = I\!R_+$. In this case, (PDE) is just the heat equation, (A5) is automatic, and a continuous local martingale satisfying the above conditions is a Brownian motion in accordance with (6). Lévy's characterisation of Brownian motion is recovered as the special case where the solution of the heat equation is taken to be $u(t,x) = x^2 - t$.

Remark A.3 To obtain Theorem 1.1, let Q be a martingale measure for the price system (X^0, X^1, X^2) and apply Theorem A.1 to the martingales X^1 and $u(t,X^1_t) = X^2_t$ on $(\Omega, \mathcal{F}, Q, (\mathcal{F}_t)_{t\in\mathcal{T}})$.

Remark A.4 Obviously, Theorem 1.1 holds as well for the larger set of *local martingale measures*, that is, for measures Q equivalent to P such that the price processes X^i are local martingales. Such measures have been studied for example by Schweizer (1992) and Babbs and Selby (1993).

The proof of Theorem A.1 is given in a sequence of lemmata.

Lemma A.1 *Let X_t be a continuous local martingale with quadratic variation process*

$$\langle X\rangle_t = \int_0^t \sigma^2(s,X_s)\, ds. \tag{7}$$

Assume that (A2) *holds. Then there exists a Wiener process W satisfying* (6).

[21] I am grateful to Lucien Foldes for having drawn my attention to McGill, Rajeev and Rao (1988) after I had obtained a weaker version of Theorem A.1 independently.

PROOF: If (7) holds, we can define a process W by setting

$$W_t = \int_0^t \phi_s \, dX_s$$

where $\phi_s = \sigma(s, X_s)^{-1}$ if $\sigma(s, X_s) \neq 0$, and $\phi_s = 0$ otherwise. W satisfies (6) and has quadratic variation $\langle W \rangle_t = \int_0^t 1_{\{\sigma(s,X_s)\neq 0\}} \, ds$. (A2) implies $\langle W \rangle_t = t$, and the assertion follows from Lévy's characterisation theorem. ■

Remark A.5 If (A2) is not satisfied, i.e., if $\{t \in \mathcal{T} : \sigma(t, X_t) = 0\}$ is not a null set, (7) still implies the representation (6). However, W is then no longer a Wiener process on the original filtered probability space, but on an extension of it. See Ikeda and Watanabe (1989, Theorem 7.1′ on page 90) for details.

Lemma A.2 *Let X_t be a continuous local martingale. If there exist a continuous function $\sigma(t, x)$ and a solution $u(t, x)$ of* (PDE) *such that* (A1) *and* (A3) *are fulfilled, then X_t has quadratic variation given by* (7)*.*

PROOF: Itô's rule, (PDE) and (A1) imply

$$\int_0^t u_{xx}(s, X_s)[d\langle X \rangle_s - \sigma^2(s, X_s)\, ds] = 0.$$

(7) follows by (A3). ■

This completes the proof of Theorem A.1 for conditions (A1)-(A3). The case where we replace (A3) by (A4) and (A5) is covered in the following lemma. Its proof builds on the arguments in McGill, Rajeev and Rao (1988).

Lemma A.3 *Let X_t be a continuous local martingale. If there exist functions $\sigma(t, x)$ and $u(t, x)$ such that* (PDE), (A1), (A4) *and* (A5) *are fulfilled, then the quadratic variation process of X_t satisfies* (7)*.*

PROOF: We start from the obvious equation

$$\begin{aligned}
\langle X \rangle_t &= \int_0^t \sigma^2(s, X_s)\, ds \\
&\quad + \int_0^t 1_{\{u_{xx}(s,X_s)\neq 0\}} \, [d\langle X \rangle_s - \sigma^2(s, X_s)\, ds] \\
&\quad + \int_0^t 1_{\{u_{xx}(s,X_s)=0\}} \, [d\langle X \rangle_s - \sigma^2(s, X_s)\, ds].
\end{aligned}$$

As in the proof of the previous lemma, one obtains

$$\int_0^t u_{xx}(s, X_s)\, [d\langle X \rangle_s - \sigma^2(s, X_s)\, ds] = 0$$

and hence

$$\int_0^t 1_{\{u_{xx}(s,X_s)\neq 0\}} \, [d\langle X \rangle_s - \sigma^2(s, X_s)\, ds] = 0.$$

On the other hand, (A4) implies

$$\int_0^t 1_{\{u_{xx}(s,X_s)=0\}} \, \sigma^2(s, X_s)\, ds = 0.$$

Thus

$$\langle X \rangle_t = \int_0^t \sigma^2(s, X_s)\, ds + \int_0^t 1_{\{u_{xx}(s,X_s)=0\}} \, d\langle X \rangle_s,$$

and (7) holds if

$$\int_0^T 1_{\{u_{xx}(s,X_s)=0\}}\ d\langle X\rangle_s = 0. \tag{8}$$

Consider a new time variable $\xi \geq 0$ and set $t_\xi = \inf\{t \in \mathcal{T} : \langle X\rangle_t > \xi\}$ if this set is not empty, and $t_\xi = T$ otherwise. Then, after extending the filtered probability space, there is a Brownian motion $(B_\xi)_{\xi\geq 0}$ such that $X_t = B_{\langle X\rangle_t}$; see Ikeda and Watanabe (1989, Theorem 7.2′ on page 91). Write $\overline{\xi} = \langle X\rangle_T$. (8) is equivalent to $\{\xi \leq \overline{\xi}:\ u_{xx}(t_\xi, B_\xi) = 0\}$ being a Lebesgue null set. Consider now the stopped continuous semimartingale $Y_\xi = u_{xx}(t_\xi, B_{\xi\wedge\overline{\xi}})$. As a direct consequence of the occupation density formula for semimartingale local time, we have $\int 1_{\{Y_\xi=0\}}\ d\langle Y\rangle_\xi = 0$ and hence

$$\int_0^{\overline{\xi}} 1_{\{u_{xx}(t_\xi,B_\xi)=0\}}\ u_{xxx}^2(t_\xi, B_\xi)\, d\xi = 0$$

which means

$$\{\xi \leq \overline{\xi}:\ u_{xx}(t_\xi, B_\xi) = 0\} \subseteq \{\xi \leq \overline{\xi}:\ u_{xxx}(t_\xi, B_\xi) = 0\}$$

up to a Lebesgue null set. Assuming that (8) does not hold and arguing inductively, one shows that there exists at least one point (t_0, x_0) where u_{xx} and all its space derivatives vanish. Next, using (PDE) and another induction argument, one can easily show that *all* partial derivatives of u_{xx} vanish at (t_0, x_0). But then, due to the analyticity of u_{xx} postulated in (A5), condition (A4) is violated. Thus (8) must hold. ■

Further Proofs

Proof of Proposition 2.1: Apply Theorem 4.16 of Elliott (1982) to the Q-martingale $1 - X^1$. ■

Proof of Proposition 2.3: Note the following properties of the Schöbel call price function: u^* solves (PDE) with $\sigma(t,x) = \nu(t)\,x$, is strictly convex in x for $x < 1$ and satisfies $u^*_t(t,1) = u^*_{xx}(t,1) = 0$ for all t. Using these properties and Lemma 2.1, one shows as in the proof of Lemma A.2 that

$$\langle X^1\rangle_t = \int_0^t 1_{\{s\leq\chi\}}\, \nu^2(s)\, (X^1_s)^2\, ds.$$

The proposition now follows directly from Ikeda and Watanabe (1989, Theorem 7.1′ on page 90). The process $\overline{W}$ is constructed as

$$\overline{W}_t = \int_0^t \frac{1_{\{s\leq\chi\}}}{\nu(s)\, X^1_s}\, dX^1_s + \int_0^t 1_{\{s>\chi\}}\, dW'_s$$

where W' is a Wiener process on some filtered probability space $(\Omega', \mathcal{F}', Q', (\mathcal{F}'_t)_{t\in\mathcal{T}})$. The extension $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{Q}, (\overline{\mathcal{F}}_t)_{t\in\mathcal{T}})$ is obtained by taking the products $\overline{\Omega} = \Omega \times \Omega'$, $\overline{Q} = Q \otimes Q'$ and $\overline{\mathcal{F}}_t = \mathcal{F}_t \otimes \mathcal{F}'_t$. ■