In:American Economic Review,1977,Vol.67(2),pp.188-193

A Note on the Arrow-Lind Theorem

By L. P. Foldes and R. Rees*

Kenneth Arrow and Robert Lind have recently proved a theorem on risky public projects, stating that under certain conditions the social cost of the risk tends to zero as the population tends to infinity, so that projects can be evaluated on the basis of expected net benefit alone. The present note gives an alternative formulation and a short new proof of the theorem, and uses these to examine the role of certain assumptions concerning the operation of the public sector which in the original were left implicit or received inadequate attention. Some general critical comments on the applicability of the theorem are also offered.

The conditions stated by Arrow and Lind as sufficient for the validity of their result include the following: (i) the government initially appropriates all benefits and pays all costs, distributing the net returns subsequently "through changes in the level of taxes" (p. 371); (ii) the net returns are statistically independent of each person's disposable income in the absence of the project; and (iii) each person's share of the net returns tends to zero as the number n of persons tends to infinity. The result is proved formally only for the case where "all taxpayers [are] identical in that they [have] the same utility function, their incomes [are] represented by identically distributed variables, and they [are] subject to the same tax rates"; but the authors state that ". . . the basic theorem still holds, provided that as n becomes larger the share of the public investment borne by any individual becomes arbitrarily smaller" (p. 373).

This theorem, if generally applicable, would have important practical consequences. It would tend to support an extension of public sector investment by justifying the use of a riskless discount rate applied to expected returns. It would also argue in favor of state participation in private investments where this allows risks to be spread over a larger number of persons. A review of the explicit assumptions alone must cast doubt on the general validity of such applications. The assumption of independence is unrealistic for many investments, for example in infrastructure and "basic" industries whose returns are highly correlated with national income. The assumption that the share of the net benefits of an investment accruing to any person becomes negligible as population tends to infinity is unacceptable in at least three cases: for public goods, where the benefit is not "shared" but increases with the population; for projects whose scale must be adjusted roughly in proportion to the size of population (such as the construction of a grid system of electricity distribution); and for projects whose benefits accrue wholly or in part to a section of the population which is "small" in the sense of the theorem. The last reservation applies not merely to those projects which are specifically designed to benefit only a small part of the population, but also to those special benefits and costs from any project which happen to accrue unavoidably to limited groups. Arrow and Lind avoid this problem in their formal discussion by assuming that the government taxes all benefits and compensates all losses, although they acknowledge that this is unrealistic.

Be that as it may, the present note accepts the Arrow-Lind approach more or less on its own terms, and considers more fully the role of certain implicit assumptions concerning the fiscal system and public expenditure. Specifically, it will be recalled that Arrow and Lind work with only two random variables, the disposable income of a typical individual and the income from distribution of project returns by the government. Although the latter is referred to in-

*London School of Economics, and Queen Mary College, London, respectively. We would like to thank members of the Public Sector Economics Seminar at Queen Mary College for helpful discussion of this paper.

formally as representing "changes in the level of taxes," the actual model makes no mention of ordinary taxes or government spending. Project returns are simply appropriated by government, presumably through lump sum compensating taxes, and then distributed to taxpayers by way of a "100 percent dividend," i.e., by lump sum transfers in fixed proportions absorbing the whole of the return. Suppose that this method of distribution were replaced by a more realistic system, for example a variation in the rate of a proportional income tax (gross incomes being regarded as random variables unaffected by the rate of tax). It can be shown that this change would make no essential difference in the Arrow-Lind model *as it stands* because individuals are statistically identical, so that in terms of expected utility each would gain on the swings what he lost on the roundabouts. When this assumption is abandoned, the change will benefit some taxpayers and hurt others. More significantly, a person's disposable income without the project and the effect of the project on that income may become dependent random variables even though the gross income and the total return on the project are independent.[1] Such dependence would, of course, vitiate application of the Arrow-Lind theorem. The point at issue here is not only that the repercussions of the project revenue through the system of taxation may create statistical dependence which would not otherwise have existed, but also that the Arrow-Lind assumption of independence owes some of its apparent appeal to the "unnecessary" condition that people are identical. The analysis can indeed be extended if this condition is discarded, but the substantive content of the assumption of independence must be considerably strengthened. Further ramifications arise if allowance is made (a) for the possibility of using project returns to finance changes in public expenditure as well as in rates of taxation; (b) for the necessity to treat expenditure or tax rates or both as random variables if the government is to balance its budget; (c) for the possibility that individuals obtain differential benefits from public spending; and (d) for the possibility that individuals derive some direct benefits from projects which are not offset by lump sum taxes, and that these benefits or the public "dividends" or both are subjected to ordinary income taxes.

The model considered below incorporates these features. The general conclusion is that the conditions for the validity of the Arrow-Lind theorem are considerably more stringent than is apparent from the original exposition, and that the circumstances in which the conclusions of the theorem apply are extremely restricted.

I. An Alternative Proof of the Arrow-Lind Theorem

Turning now to a mathematical discussion, we begin with a formulation which follows Arrow and Lind's "implicit" treatment of the public sector but does not assume that persons are identical or that project benefits are initially appropriated by the state. A proof of the main theorem is given which is in some respects simpler and more general than the original one, though it does require that marginal utility be continuous. Structural assumptions about the public sector are then introduced and some of their implications noted.

The economy contains n persons $i = 1, \ldots, n$, where n may take values n_0, $n_0 + 1, \ldots$ starting with some n_0. Several random variables will be defined, all of which are supposed to have as domain the same probability space; an elementary event ω corresponds to a state of nature influencing the economy. All random variables are assumed to be integrable, i.e., to have finite expectations. The random variable $x_i = x_i(n)$ represents i's income in the absence of a certain public project when population is n, while $x_i(n) + r_i(n)$ represents his income if the project is introduced. These incomes are defined after all taxes, subsidies, and other effects of government are taken into account, so

[1] It should be noted that the impact of the system of public finance may well be to create negative correlation between project benefits and net disposable incomes, so that by ignoring this influence one may understate the value of public projects to a community of risk averters.

that r_i is the total impact of the project upon i's income, including repercussions through the public sector.[2]

One natural interpretation of this setup (though not the only one) is to regard $r_i(n)$ as i's share of the random variable $Z(n)$ representing total returns to the project, whether obtained directly or through changes in taxation; in this case $Z = \Sigma_i r_i$, and all r_i have the same sign as Z. An even more special case would be to assume that all x_i are identically distributed and invariant to n while $r_i(n) = Z(n)/n$ for each i.

Suppose now that the preferences of i among risky incomes are defined by the expected values $Eu_i(.)$ of his utility function, which is assumed to be defined and finite on the real line, strictly increasing, continuously differentiable, and such that $Eu_i(x_i)$ and $Eu_i(x_i + r_i)$ are finite for each n (possibly because u_i is bounded). For given n, i is made better off by the project if and only if,

$$(1) \qquad 0 \leq E\{u_i(x_i + r_i) - u_i(x_i)\}$$

Applying the mean value theorem for derivatives (separately for each elementary event ω) to the expression under the expectation sign, the condition becomes

$$(2) \qquad 0 \leq E\{r_i u_i'(x_i + \theta_i r_i)\}$$

where for each ω the value $\theta_i(\omega)$ lies in $[0, 1]$.[3] On multiplying by n, (2) becomes

$$(3) \qquad 0 \leq E\{nr_i u_i'(x_i + \theta_i r_i)\}$$

Now assume that for a given i as $n \rightarrow \infty$,

(a) there are integrable limiting random variables R_i and $\bar{x}_i$, independent of one another, such that $nr_i(n) \rightarrow R_i$ and $x_i(n) \rightarrow \bar{x}_i$ with probability one; and

(b) passage to the limit under the expectation sign on the right-hand side of (3) is permissible.[4]

For example, (a) obviously holds in the special case mentioned above where $r_i(n) = Z(n)/n$. On the other hand (a) is not satisfied if r_i represents i's untaxed benefit from a pure public good, since then r_i does not vary with n and $|nr_i| \rightarrow \infty$ unless $r_i = 0$.

The assumptions are used as follows. It is inferred from (a) that $r_i(n) \rightarrow 0$ with probability one, hence that $u_i'(x_i + \theta_i r_i) \rightarrow u_i'(\bar{x}_i)$ since u_i' is continuous. Then (a) further shows that the expression under the expectation sign in (3) tends to $R_i u_i'(\bar{x}_i)$, so that by virtue of (b) the expectation itself tends to $E\{R_i u_i'(\bar{x}_i)\}$. This in turn equals $ER_i Eu_i'(\bar{x}_i)$ because R_i and $\bar{x}_i$, hence R_i and $u_i'(\bar{x}_i)$, are independent. Thus, as $n \rightarrow \infty$, the condition (3) for an increase in expected utility becomes

$$(4) \qquad 0 \leq ER_i Eu_i'(\bar{x}_i)$$

or simply $0 \leq ER_i$ since $u_i' > 0$. To sum up, *when n is "large," i is made better off by the project if, and only if, the expected change in his income is positive.* If we assume with Arrow and Lind that $R_i = Z$ (invariant to n) for every i, then everyone is made better off, if and only if the total expected return from the project is positive. Clearly this latter result could be obtained from a weaker assumption, for example that $\lim EZ(n)$ exists and that each ER_i has the same sign as this limit.[5]

[2]In allowing x_i as well as r_i to vary with n, we depart from Arrow and Lind's treatment. This change is necessary because the assumption that income after taxes, etc. is invariant to population would impose too many implicit restrictions on the working of the public sector.

[3]Since x_i and r_i are random variables, i.e., measurable functions of events, the same is evidently true of $u_i(x_i + r_i) - u_i(x_i)$ and hence of $r_i u_i'(x_i + \theta_i r_i)$. It can also be shown that $x_i + \theta_i r_i$ and θ_i are random variables, but this fact does not seem to be required.

[4]Various conditions can be invoked to justify this operation. For example, if u_i is a bounded function of income and the random variables $nr_i(n)$ are uniformly bounded, the result follows from the Dominated Convergence Theorem. Weaker assumptions may suffice in particular cases. Incidentally, some condition of this kind is needed to justify the passage to the limit in Arrow and Lind's equation (21).

[5]The discussion in the text derives conditions in which a public project increases the welfare of one person. Similar methods can clearly be used to obtain conditions for an increase in the value of a social welfare function of the form

$$W = \sum_{i=1}^{n} \alpha_i(n) Eu_i$$

where $\alpha_i(n) > 0$ and $\sum_{i=1}^{n} \alpha_i(n) = 1$ for each n

II. Relevance of the Fiscal System

We now specify the model more explicitly in order to answer some of the questions raised in the introduction; various specifications could be chosen, but the discussion which follows is illustrative of the general results. For each n, let the random variable $G^0(n)$ denote government expenditure in the absence of the project, and $G^1(n)$ the corresponding variable if the project is undertaken. (Distributions of project benefits are excluded from G^1, although the line between these payments and general transfers may have to be drawn arbitrarily.) The value of the benefits (free of tax) which i derives from expenditure $G(n)$ is assumed to have the form $c_i(n)G(n)$, where for each n the c_i are nonnegative constants. If all expenditure is devoted to pure public goods, all constants equal unity; whereas they sum to unity if the government distributes purely private goods. Next, let $X_i(n)$ be i's random income before taxes from private sources; $t^0(n)$ the proportional random rate of tax payable on this income in the absence of the project; and $t^1(n)$ the corresponding rate if the project is undertaken. We write $X(n) = \Sigma_i X_i(n)$ for total gross income from private sources: $a_i(n) = X_i(n)/X(n)$ for i's share of this total. For simplicity we ignore borrowing and lending, and assume that personal and government budgets balance. In the absence of the project, the budget identities are

$$(5) \qquad x_i = (1 - t^0)X_i + c_i G^0$$

$$(6) \qquad G^0 = t^0 X$$

These are identities between random variables, holding for $n = n_0, n_0 + 1. \dots$ Note that since X is random, G^0 or t^0 or both must be random.

Now let the random variable $Z(n)$ denote total net benefit from the project, $z_i(n)$ the net benefit accruing to i (whether directly or through distributions by government), and $Z_G(n) = Z - \Sigma_i z_i$ the portion retained by the government for the finance of its expenditure. In general the variables z_i need not all have the same sign as Z, but Z_G is assumed to have this sign and not to exceed Z in absolute value. The $z_i(n)$ are supposed to be subject to a random proportional tax rate $\tau(n)$, where $0 \leq \tau(n) \leq 1$. Then the budget identities for persons and government when the project is undertaken have the form

$$(7) \qquad x_i + r_i = (1 - t^1)X_i + c_i G^1 + (1 - \tau)z_i$$

$$(8) \qquad G^1 = t^1 X + \tau(Z - Z_G) + Z_G$$

An expression for r_i can now be obtained from (5) and (7); then (6) and (8) can be used to eliminate either $t^0 - t^1$ or $G^0 - G^1$, yielding respectively,

$$(9) \qquad r_i = (c_i - a_i)(G^1 - G^0) + (1 - \tau)z_i + a_i[\tau Z + (1 - \tau)Z_G]$$

$$(10) \qquad r_i = (c_i - a_i)(t^1 - t^0)X + (1 - \tau)z_i + c_i[\tau Z + (1 - \tau)Z_G]$$

For brevity we now consider two special cases of the model.

A. *Government Expenditure Unaffected by the Project*

In this case $G^1(n) = G^0(n)$ for each n, and so (9) reduces to

$$(11) \qquad r_i = (1 - \tau)z_i + a_i[\tau Z + (1 - \tau)Z_G]$$

while (5) and (6) together with $a_i = X_i/X$ yield

$$(12) \qquad x_i = a_i(X - G_0) + c_i G^0 = (na_i)(X - G^0)/n + (nc_i)(G^0/n)$$

In order to apply condition (a) of our proof of the Arrow-Lind Theorem to the values of nr_i and x_i appearing in (11) and (12), it is necessary to make assumptions about the limits of several variables as $n \rightarrow \infty$.

First, to ensure that $nr_i(n)$ converges to some R_i, it is enough by (11) to assume that nz_i, na_i, τ, Z, and Z_G converge (with probability one, to integrable limiting variables). Note that the convergence of $na_i = nX_i/X$ implies that the income share a_i of a given individual tends to zero; this holds, for example, in the special case where $X_i = X/n$. Note also that the case of pure public goods is ruled out by the assumptions about the project variables Z and nz_i.

Secondly, to ensure that $x_i(n)$ converges to some $\bar{x}_i$, it is enough by (12) to assume that

X/n, G^0/n, and nc_i (as well as na_i) converge. In other words, there must be limits with finite expectations of private sector income and government expenditure per head, and of the product of population with i's benefit per dollar of public expenditure; the first two conditions seem reasonable, while the last may be subject to reservations if the level of spending on public goods is maintained when the population is large.

Thirdly, the limiting variables R_i and $\bar{x}_i$ have to be independent; this presents difficulties since by (11) and (12) the term na_i is common to nr_i and x_i. To rule out dependence, it is enough to adopt one of the following assumptions:

(i) That lim na_i is degenerate (constant across states); this appears too special a condition to be acceptable.
(ii) That lim $(X - G^0)/n = 0$; this would be satisfied if $G^0/X \rightarrow 1$ with X/n bounded, or if $X/n \rightarrow 0$. The former case implies a tax rate of $t^0 = G^0/X$ rising to 100 percent as population increases, which is unreasonable. The latter case, that gross private income per head tends to zero, could well occur for Malthusian reasons; even so, applications of the theorem must be made for finite populations, and it is unlikely that $(X - G^0)/n$ would in practice be close enough to zero.
(iii) That lim $[\tau Z + (1 - \tau Z_G] = 0$; this holds in general only if lim τ = lim $Z_G = 0$ (leaving aside the uninteresting case in which lim $Z = 0$). This means that, in the limit, no tax is imposed on project benefits, and no part of them is retained by the government to finance expenditure; in other words *the benefits of the project accrue in full to persons, without liability to tax.*

The only one of these assumptions which has real economic interest is that in (iii). If we adopt it, we are accepting that the public project in question has no fiscal repercussions, a condition which in many practical cases would not be satisfied. For example, an irrigation project would generate additional incomes for farmers, and these would be subject to tax. Again, the returns from investment in British nationalized industries are viewed as inflows to the public sector, and are regarded for many purposes as a source of finance in much the same way as indirect taxation. Suppose that we accept the assumption nevertheless; then (11) reduces to $r_i = nz_i$, and it remains to postulate that $R_i = \lim nz_i$ is independent of the limits of $(X - G^0)/n$, c_iG^0, and $na_i = nX_i/X$. Thus, in the limit, *project income for each person to whom the theorem is applied is to be independent not only of the per capita difference between gross private sector income and public expenditure, but also of the person's benefit per unit of public expenditure and of the ratio of private sector income to the population average.* While there seems to be no theoretical relationship among the variables which rules out such independence a priori, the stated condition is clearly much more restrictive than the simple requirement that the project return be independent of each person's gross income from private sources.

B. *Tax Rates Unaffected by the Project*

In this case $t^1 = t^0$, so that

$$r_i = (1 - \tau)z_i + c_i[\tau Z + (1 - \tau)Z_G] \tag{13}$$

from (10), while (5) and (6) yield

$$x_i = (na_i)(X - t^0X)/n + (nc_i)(t^0X/n) \tag{14}$$

The analysis now proceeds along much the same lines as under Section IIA, except that government expenditure per head G^0/n is replaced by tax revenue per head t^0X/n. Briefly, to ensure the convergence of nr_i and x_i we assume integrable limits for nz_i, na_i, τ, Z, Z_G, X/n, nc_i, and t^0X/n. As regards the independence of the limits, it is now nc_i which appears in both expressions, and we are led as before to the unattractive assumption that lim τ = lim $Z_G = 0$. It then remains to suppose that R_i = lim nz_i is independent of the limits of $(X - t^0X)/n$, c_it^0X, and na_i; thus *public expenditure per head is replaced by tax revenue per head in the independence condition given above.*

A striking feature of the various italicized conditions in this section is that the aggregative assumption of independence adopted by Arrow and Lind can reasonably be expected to hold only if similar assumptions are satisfied by each of a number of sectoral variables. These assumptions must, of course, hold separately for each individual to whom the theorem is applied. This not only represents a strengthening of assumptions, but also makes it much more difficult to establish that the conditions under which the theorem holds are met in any given situation.

III. Conclusions

Using a proof of the Arrow-Lind theorem which makes the roles of the various assumptions more transparent, we have tried in this note to bring out the implications of a more realistic specification of the fiscal system in which public sector investment is embedded. Taxes and expenditure exist with or without any one project, and the government must balance its budget in each state of the world. We find that in this case the sufficient conditions used in the proof of the theorem become a good deal more restrictive. In particular, to ensure independence between the impact of a project on the individual's income and his marginal utility across states of the world we have to assume—leaving aside some unappealing special cases—that the project income is free from taxation and that none of it is retained to finance public expenditure. The assumption of independence between the project's return and private incomes, which itself is open to question, must be extended to a number of sectoral variables in a way for which there is no obvious empirical justification. These results suggest that there is still a need for an analysis of public sector investment criteria under uncertainty,[6] which will yield more general and robust results.

[6]The papers by Rees (1973, 1976) adopt more general approaches to this problem.

REFERENCES

K. J. Arrow and R. C. Lind, "Uncertainty and the Evaluation of Public Investment Decisions," *Amer. Econ. Rev.*, June 1970, *60*, 364–78.

R. Rees, "Public Sector Resource Allocation Under Risk," in Michael Parkin, ed., *A.U.T.E. Essays in Modern Economics*, London 1973.

———, "Uncertainty, Second Best and Public Policy," disc. pap. no. 28, econ. dept., mimeo, Queen Mary College 1976.