Mathematical Population Studies, 1990, Vol. 2(4), pp. 269–287
Reprints available directly from the publisher
Photocopying permitted by license only

Printed in the United States of America

MATHEMATICAL INVESTIGATIONS OF THE ESCAPE FROM THE MALTHUSIAN TRAP

JOHN KOMLOS

Departments of History and Economics
University of Pittsburgh
Pittsburgh, PA 15260

MARC ARTZROUNI

Department of Mathematical Sciences
Loyola University
New Orleans, LA 70118

November 28, 1989, revised May 20, 1990

We present a simulation model that synthesizes Malthusian and Boserupian notions of the way population growth and economic development were intertwined. The non-linear stochastic model consists of a system of equations whose dynamics culminate in an industrial revolution after hundreds of iterations. The Industrial Revolution can thus be conceptualized as a permanent "escape" from the Malthusian trap that occurs once the economy is capable of permanently sustaining an ever growing population. We investigate the conditions for such an escape and their sensitivity to the parameters of the model. This is done in an attempt to understand why some economies might have had difficulties escaping from the Malthusian trap (in contrast to the European experience in the eighteenth and nineteenth centuries). Our results show that the likelihood of an escape is sensitive to the savings rate and to the output elasticities of the two sectors of the economy. When not in a subsistence crisis, the chances that an escape will occur increase for larger values of the ratio of the savings rate to the growth rate of the population. The chances of an escape also increase substantially for larger values of the output elasticities of labor.

KEY WORDS: Demographic economics; Malthusian trap; Industrial Revolution; Boserup.

INTRODUCTION

We propose a simulation model which synthesizes Malthusian and Boserupian notions of economic growth with endogenous population[1] (Boserup, 1981). The model captures the salient features of the demographic and economic experience of Europe between the Neolithic Agricultural Revolution and the Industrial Revolution of the eighteenth century. Our conceptualization describes the "incessant contest"

[1] For other models exploring the Malthusian-Boserupian synthesis see Lee, 1986, 1988; Pryor and Maurer, 1982; Robinson and Schutjer, 1984. For other models with endogenous population growth see Steinmann, 1984; Krelle, 1988; Niehans, 1963; Nelson, 1956.

between population growth and the means of subsistence by formalizing the mechanism of the "Malthusian trap": when the per capita output of nutrients falls below a biologically determined minimum, the population is subject to random mortality crises that can take on disastrous proportions. As the population decreases, its nutritional status improves, thereby enabling the population to grow unhindered until it once again falls below subsistence level.[2] These are long-run cycles in population and nutritional status during the course of which both capital accumulation and technological progress take place.[3] Consequently, at the beginning of each phase of demographic expansion the society has a greater likelihood of possessing sufficient wealth and knowledge—broadly defined—to break out of the homeostatic Malthusian equilibrium. Because in the past accumulation tended to accelerate during periods of population expansion, we incorporate into the model such non-linear processes. We refer to these as "Boserupian" episodes, taking the concept of the positive economic effects of population growth more broadly than is usually conceived, i.e., narrowly as population-induced technological change. We do this inasmuch as we believe that we remain within the essence of the Boserupian notion that the effect of population growth can be other than capital diluting (Simon, 1986; Steinmann, 1984).

This homeostasis, we believe, prevailed until the Industrial Revolution, during which the world experienced an explosion in both population and per capita output of such magnitude that an escape from the Malthusian trap resulted: the per-capita output of nutrients remained above the minimum needed for the human population to grow unhindered. We are thereby able to conceptualize the Industrial Revolution as the culmination of a slow process begun millennia before and thus resolve the apparent inconsistency between the continuity of economic processes and the discontinuity of the Industrial Revolution.[4] Our model does not, however, encompass the period after the Industrial Revolution.[5]

The model is specified with the European experience in mind, and its applicability to other regions of the world remains an open question. For example, could our model shed some light on the failure of the Asian economies to experience an industrial revolution in the eighteenth century?[6] (Jones, 1981, Goldstone, 1985) China, after all, did undergo economic processes similar to those of the European countries. It also had a growing stock of capital, a growing population, and experienced agricultural improvements. Moreover, for a while at least, China enjoyed technological superiority relative to Western Europe. Why was Europe, and not an Asian society the first to escape from the Malthusian trap? Therefore, one of our main goals is to explore the parameter spaces of our model in order to provide some insights into the reasons why some economies might have diverged from the European path.

---

[2]For an examination of cycles in nutritional status using anthropometric evidence see Komlos (1985, 1987).

[3]For an examination of technological change using models from evolutionary biology see Mokyr (1990).

[4]For an overview of the recent scholarship on the Industrial Revolution see Mokyr (1985).

[5]For further details of the model see Artzrouni and Komlos (1985).

[6]For a sensible discussion of the failures and successes of the diffusion of the Industrial Revolution to the rest of the world see Landes (1990).

The paper is divided into five sections. In the first part we describe briefly the salient features of the model and specify the parameter values for which the model accurately replicates the growth pattern of the world's population between 8000 B.C. and 1700 A.D. Our aim here is not to provide nuanced interpretations of all aspects of economic and demographic processes and their interaction. Rather, it is to provide a framework that adheres to historically valid stylized facts.[7] In the second part we investigate mathematically the conditions under which a population can be expected to escape the Malthusian trap. In the third section we explore the sensitivity of these conditions to the parameter values. In the fourth we look at the actual dynamics of the process and examine the conditions that help or hinder an escape. In the conclusion we summarize and discuss the main findings.

THE MODEL

Our specification assumes that the economy is composed of two sectors: one producing only nutrients and another producing everything else including capital (this all-other-goods sector is denoted as AOG). The outputs $Q_A(t)$ of the nutrient-producing sector and $Q_I(t)$ of the AOG sector are described by production functions of the Cobb-Douglas type:

$$Q_I(t) = C_1 K(t)^{\alpha_1} P_I(t)^{\alpha_2} \tag{1}$$

$$Q_A(t) = C_2 K(t)^{\beta_1} P_A(t)^{\beta_2} \tag{2}$$

where $K(t)$ is the aggregate capital stock in period t, and $P_A(t)$ and $P_I(t)$ are the workers engaged in the two sectors in period t; $P(t) = P_A(t) + P_I(t)$ is the total population. The labor force participation rate is therefore one. The unit of time is the decade and the model is implemented for the period extending from about 8000 B.C. to the nineteenth century.

The quantity of nutrients available per capita is given by

$$S(t) = Q_A(t)/P(t). \tag{3}$$

The capital $K(t)$ is accumulated through the following process:

$$K(t+1) = K(t) + \lambda(t) Q_I(t) \tag{4}$$

where the savings rate $\lambda(t)$ grows slowly from one percent per decade in 8000 B.C., to four percent in 1700 A.D.[8] saving are assumed to equal investments. Capital is defined broadly to include not only land, as well as human and physical capital, but also technological and scientific knowledge broadly conceived. Thus the capital stock is not partitioned between the two sectors. We resort to this simplification

[7]For a historical justification of the assumptions see Jones (1981) and Komlos (1986).

[8]We experimented with different specifications of the savings function with quite similar results to the ones reported here. Instead of the above specification we also tried a step function for the savings rate which fluctuates back and forth between 2 and 5 percent of the output of the AOG sector. In addition, we also tried a logistic function whose slope depended on $S(t) - S^*$. The point is that, true to the historical record accumulation does have to take place, somehow for the escape to occur. However, there is hardly a historical warrant for preferring one savings mechanism over another in such a long-run view of economic development. The higher the savings rate relative to population growth, the faster capital accumulated, and the sooner the escape will occur.

because of the uncertainty concerning the relative size or the relative growth rates of the capital stock in the two sectors during these millennia. Allocating the capital stock at an arbitrary rate between the two sectors would have added nothing of significance to the model's analytic power. In any case, the accumulation of knowledge, the increased sophistication of social organization, and many of the great scientific and geographic discoveries of the past often advanced production in both sectors. These intangibles as well as social overhead capital cannot be partitioned in principle.

Another important task is to model the dynamics of population growth. We postulate that the growth rate of the total population $P(t)$ is r^* (per decade) as long as $S(t) \geq S^*$ (i.e., $P(t+1) = P(t)[1+r^*]$). It appears to be a reasonable approximation of reality that as long as subsistence was above a critical level, the population tended to increase.

We specify the homeostatic stabilizing mechanism by postulating that whenever the per capita nutritional intake $S(t)$ drops below S^*, the total population $P(t)$ is subject in a random fashion to lowered growth rates, which can become negative (i.e., $P(t+1) = P(t)[1+r(t)]$, where $r(t)$ is a random variable that is smaller than r^*—possibly negative—and is generated by a Monte Carlo-type simulation process). The population is thus susceptible to small fluctuations as well as to those catastrophic collapses which have characterized populations of the past. As the population declines, $S(t)$ increases and eventually exceeds S^*. A cycle is completed and the growth rate is again r^* per decade until the next crisis. Hence, the model yields oscillations of $S(t)$ about S^*, and captures the "incessant contest" between population growth and available resources. (For a detailed description of the model see (Artzrouni and Komlos, 1985)). The society has escaped the Malthusian trap only if $S(t)$ remains permanently above S^*, in which case the population grows unhindered with a constant growth rate r^*.

We next specify the allocation of the population between the two sectors. If the population is not in a crisis (i.e. $S(t) \geq S^*$), then we postulate that the population in both sectors grows at the same rate r^* (i.e., $P_I(t+1) = P_I(t)[1+r^*]$ and $P_A(t+1) = P_A(t)[1+r^*]$). If the population is in a Malthusian crisis, we assume that the AOG sector absorbs any change in the total population (i.e., $P_A(t+1) = P_A(t)$ and $P_I(t+1) = P(t+1) - P_A(t)$; if $P_I(t+1)$ becomes negative in this last equation then $P_I(t+1)$ is reset to 0 and $P_A(t+1)$ is set equal to $P(t+1)$). The simplified flowchart below describes the essential features of the model (Figure 1).

The initial value $P(0)$ of the population in 8000 B.C. was taken equal to six million (i.e., $P(0) = 6$; $P_I(0) = 1$ and $P_A(0) = 5$). The "escape rate" r^* was set at 5 percent per decade. The exponents α_1, α_2, β_1, β_2, were set at 0.5. The values of C_1, C_2, S^*, and $K(0)$ were chosen by trial and error in such a way that the model would generate a time series $P(t)$ that would approximate the actual values of the world's population from 8000 B.C. to 1700 A.D. ($C_1 = 1.3$; $C_2 = .213$; $S^* = 0.08$; $K(0) = 4$) (Figure 2).

The model was specified with the European experience in mind, and its applicability to other regions of the world is left unexplored. For example, could our model shed light on the failure of the Chinese economy to experience an industrial revolution in the eighteenth century? After all, in the Middle Ages at least China was by

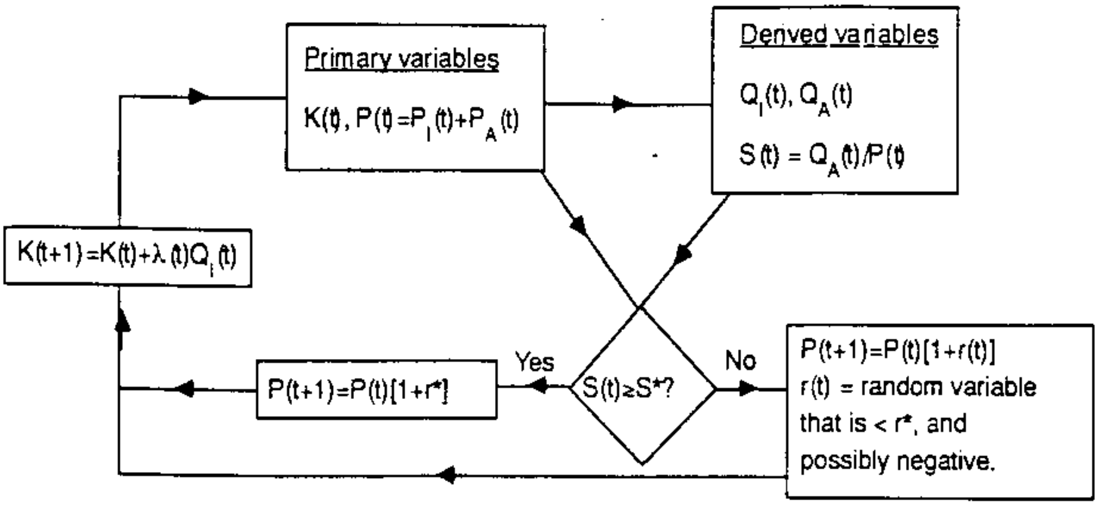

FIGURE 1. Simplified flowchart describing the mechanism of the model.

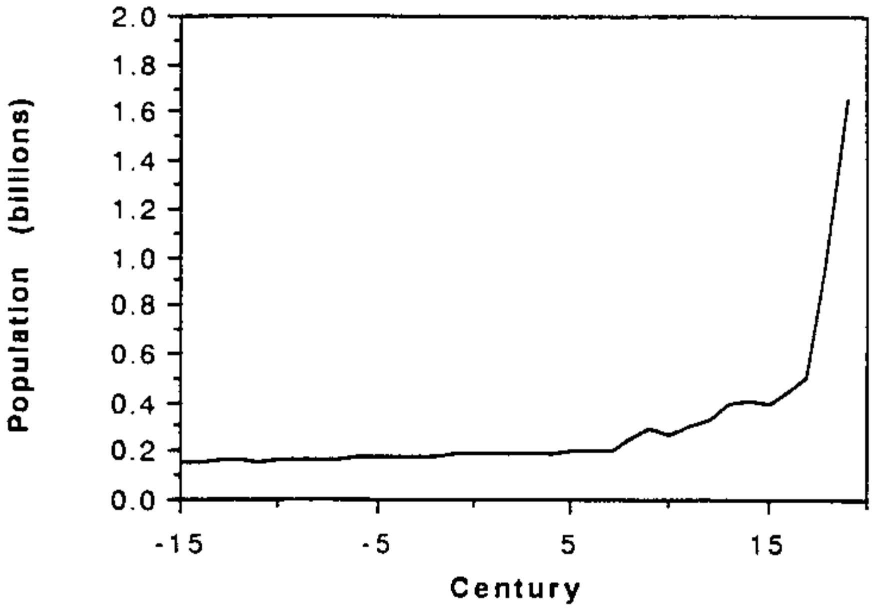

FIGURE 2. Simulation of the world's population growth (only the period from 1500 B.C. to 1700 A.D. is shown).

no means a backward economy relative to Western Europe. Why was Europe, and not China, or India, or the Ottoman Empire the first to escape from the Malthusian trap? We believe that a detailed analysis of the escape conditions, coupled with a sensitivity analysis of the model to the various parameters might illuminate such issues.

THE ESCAPE ANALYSIS

We recall that the population is said to have escaped from the Malthusian trap when $S(t)$ remains larger than S^* ($P_I(t)$ and $P_A(t)$ then grow at a constant rate

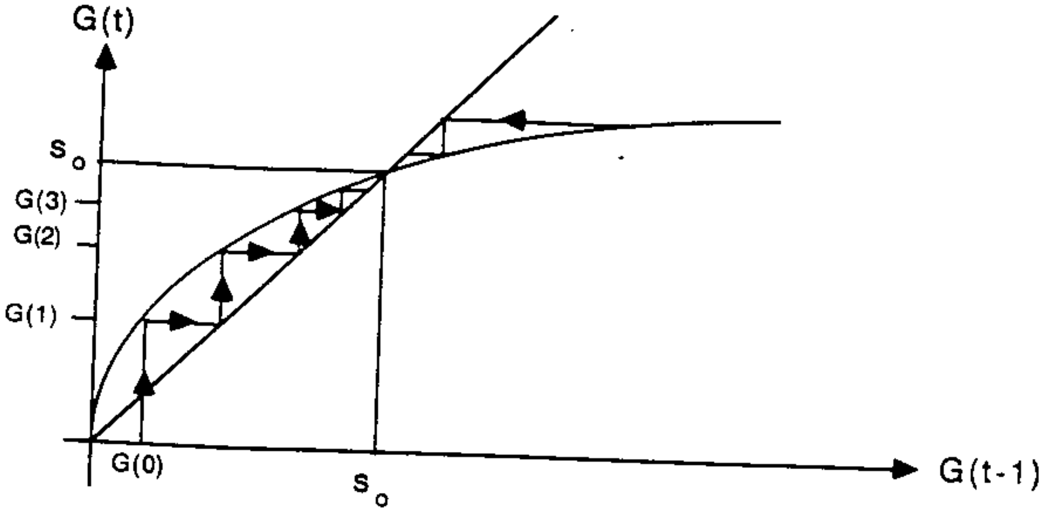

FIGURE 3. Phase-plane representation of the dynamics of $G(t)$ in an escape.

r^*.) We begin by examining necessary conditions for the escape, i.e., conditions that are satisfied when the escape occurs. In order to simplify the analysis we assume that the savings rate $\lambda(t)$ remains constant once the escape occurs. (If the escape occurs at some period t with a fixed value of the savings rate, the escape will occur *a fortiori* if the savings rate continues to increase). It can be seen from Eqs. (1)–(4) that the growth rate $G(t)$ of the capital stock satisfies

$$G(t) = \frac{K(t+1)}{K(t)} - 1 = \frac{s^{\alpha_2} G(t-1)}{[G(t-1)+1]^{1-\alpha_1}} \tag{5}$$

where $s = 1 + r^*$. (For simplicity of notations we assume that $t = 0$ is the time at which the population has escaped). The iterative process of Eq. (5) defines $G(t)$ recursively as a function of $G(t-1)$. The fixed point of this process is

$$s_0 = s^{\alpha_2/(1-\alpha_1)} - 1. \tag{6}$$

This fixed point is the point of equilibrium of the process generating $G(t)$. Indeed, it can be seen that if $G(t_0) = s_0$ for some t_0, then $G(t) = s_0$ for all $t \geq t_0$. Furthermore, s_0 is an attractive fixed point: if the value $G(0)$ of the growth rate at the first epoch of the escape is less than s_0, then $G(t)$ ($t = 1, 2, \ldots$) will increase monotonically and converge to s_0. If $G(0)$ is larger than s_0, then $G(t)$ will decrease monotonically to s_0. The process is graphically represented in Figure 3.

We now define the level of subsistence as the dimensionless quantity $D(t) = S(t)/S^*$. We observe that $S(t) \geq S^*$ is equivalent to $D(t) \geq 1$. From Eq. (5) it can be seen that

$$D(t)G(t)^{\delta} = D(0)G(0)^{\delta}u^{t} \qquad t = 0, 1, \ldots \tag{7}$$

where $\delta = \beta_1/(1-\alpha_1)$ and

$$u = s^{\beta_2 - 1 + \alpha_2\delta} \tag{8}$$

From Eq. (7) it follows that

$$D(t) = D(0)\left[\frac{G(0)}{G(t)}\right]^{\delta} u^{t} \qquad t = 0, 1, \ldots \tag{9}$$

Given that $G(t)$ tends to s_0 when t becomes large, Eq. (9) shows that u is necessarily larger than or equal to 1. Indeed, if $u < 1$ then $D(t)$ of Eq. (9) would tend to 0, which contradicts the assumption that the population has escaped. The condition $u < 1$ under which an escape cannot occur is equivalent to

$$\alpha_2/(1-\alpha_1) < (1-\beta_2)/\beta_1 \tag{10}$$

This inequality may help understand why some economies did not escape the Malthusian trap. If for example the Oriental economies were plagued by decreasing returns to scale, their output elasticities may have satisfied inequality (10), thus precluding the possibility of an escape. Perhaps both output elasticities of labor α_2 and β_2 were small, in which case inequality (10) is also satisfied. In such a case, as population grows, the output $Q_I(t)$ will not grow quickly enough in Eq. (1), and consequently the growth of the capital stock $K(t)$ will be insufficient to enable $Q_A(t)$ to keep up with population growth; $S(t)$ will then eventually fall below S^*. This result is worthy of emphasis: in an overpopulated economy which does not have constant returns to scale the output elasticity of labor can be crucial. If it is small then the escape is impossible. We now distinguish between the cases $u = 1$ and $u > 1$.

1. $u = 1$:

When $u = 1$ inequality (10) becomes an equality and is

$$\alpha_2/(1-\alpha_1) = (1-\beta_2)/\beta_1 \tag{11}$$

We observe that Eq. (11) is satisfied, in particular, when both sectors have constant returns to scale. (Both sides of Eq. (11) are then equal to 1.) If $u = 1$ Eq. (9) shows that when the population has escaped $D(t)$ approaches the limit

$$L = D(0)[G(0)/s_0]^{\delta} \tag{12}$$

where L is necessarily larger than 1. When $u = 1$ sufficient conditions for the escape follow directly from the previous discussion. In the space of initial values $[G(0), D(0)]$ there are two regions E_1 and E_2 for which the escape will occur. If $G(0) \geq s_0$ then Eq. (9) shows that $D(t)$ will always be larger than $D(0)$ since $G(t)$ decreases to s_0. Hence $D(0) \geq 1$ is sufficient to ensure an escape and E_1 is

$$E_1 = \begin{cases} D(0) \geq 1 \\ G(0) \geq s_0 \end{cases}. \tag{13}$$

If $G(0) \leq s_0$ then $G(t)$ increases to s_0 and therefore E_2 is

$$E_2 = \begin{cases} D(0) \geq [s_0/G(0)]^{\delta} \\ G(0) \leq s_0 \end{cases}. \tag{14}$$

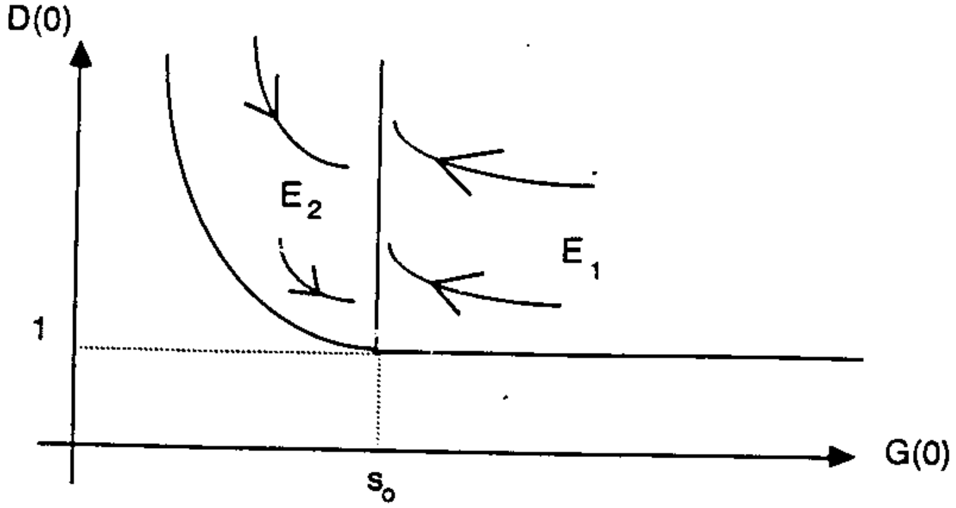

FIGURE 4. Escape region for $[G(0),D(0)]$; $(u = 1)$.

The escape regions E_1 and E_2 are represented graphically in Figure 4.

Given an initial point $[G(0),D(0)]$ in the escape region, the previous analysis shows that the subsequent points $[G(t),D(t)]$ will approach the vertical axis $G(0) = s_0$; $G(t)$ tends to s_0 and $D(t)$ tends to the limit L. Four possible trajectories of the points $[G(t),D(t)]$ are shown. When $[G(0),D(0)]$ is in $E_2 D(t)$ decreases to its limit and $G(t)$ increases to s_0; when $[G(0),D(0)]$ is in $E_1 D(t)$ increases to its limit and $G(t)$ decreases to s_0.

To summarize this analysis, the escape hinges on sufficiently large initial values of $[G(0),D(0)]$. Of course $D(t)$ must be larger than 1. If $G(t) \geq s_0$ then the escape occurs, regardless of the value of $D(t)$, as long as $D(t) \geq 1$ (region E_1); $G(t)$ may be less than s_0, but the smaller value of $G(t)$ must then be compensated by a larger value of $D(0)$ (region E_2). This is intuitively plausible. Indeed, this shows that an escape hinges on a compromise between a sufficiently high growth rate of the capital, and a sufficiently high nutritional lèvel. Hence, the growth rate of the capital stock might be lower if the nutritional status of the population is higher, because the population will have time to build up its capital stock before its nutritional status deteriorates below the level of subsistence. This is an important insight. Whereas in prior decades scholars were intent on measuring the rate of growth of capital stock as a crucial component of the Industrial Revolution they failed to emphasize the importance of nutritional status in the process. In other words, two societies in which capital grows at the same rate may not both have an industrial revolution escape if the nutritional status of one of the populations is lower than that of the other.

2. $u > 1$:

The analysis for $u > 1$ is similar to the previous one. If $G(0) \geq s_0$ and $D(0) \geq 1$ then $G(t)$ will decrease to s_0 and $D(t)$ of Eq. (9) will increase monotonically since u^t will also increase with t. Hence the escape is guaranteed for $G(0) \geq s_0$ and $D(0) \geq 1$. If $G(0) < s_0$ the situation is slightly more complex since $G(t)$ on the right-hand side of Eq. (9) then increases monotonically to s_0. Given the term u^t in Eq. (9) it can be seen that $D(t)$ will increase monotonically and approach infinity if

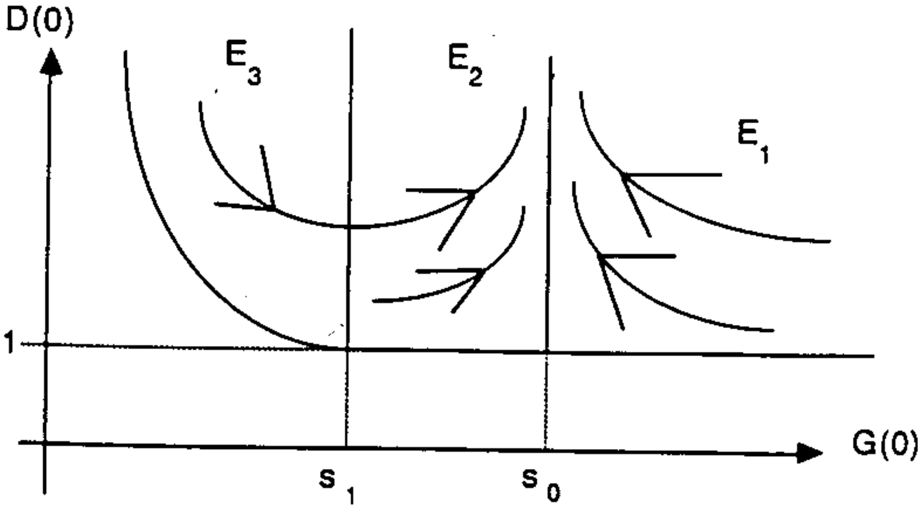

FIGURE 5. Escape region for $[G(0), D(0)]$; $(u > 1)$.

$G(0)$ is larger than the quantity

$$s_1 = s^{(1-\beta_2)/\beta_1} - 1 \tag{15}$$

which is itself less than s_0. Therefore if $s_1 < G(0) < s_0$ an escape will occur as long as $D(0) > 1$, since $D(t)$ then increases monotonically as $G(t)$ approaches s_0. If $G(0) < s_1$, $D(t)$ first decreases, then increases as soon as $G(t)$ becomes larger than s_1. In order for $D(t)$ to remain larger than 1 while $D(t)$ decreases (i.e., as long as $G(t) \leq s_1$) the initial value $D(0)$ must satisfy, for a given $G(0)$,

$$D(0) \geq \min_{G(t) \leq s_1} \left[\frac{G(t)}{G(0)}\right]^{\delta} \frac{1}{u^t} \tag{16}$$

The results are illustrated in Figure 5 which depicts the three components E_1, E_2 and E_3 of the escape region.

If $G(0) > s_0$ and $D(0) \geq 1$ (region E_1) then as before $G(t)$ decreases monotonically to s_0 while $D(t)$ increases to infinity. If $s_1 \leq G(0) \leq s_0$ and $D(0) \geq 1$ (Region E_2) then $G(t)$ increases to s_0 and $D(t)$ increases to infinity. If $G(0) < s_1$ and $D(0)$ satisfies (16) (Region E_3) then the escape will also occur, with $D(t)$ first decreasing, until $G(t)$ reaches s_1, and then increasing to infinity, once the point $[G(0), D(0)]$ enters E_2.

We will now examine in more detail the escape condition in terms of the demographic variables $P_I(t)$ and $P_A(t)$. In order to simplify the analysis we will assume that the economy has constant returns to scale, i.e., $\alpha_1 + \alpha_2 = 1 = \beta_1 + \beta_2$. In such a case $u = 1$ which implies that if the population escapes the Malthusian trap then $D(t)$ will converge to an equilibrium value (and not grow indefinitely, which would occur if $u > 1$).

We drop the functional notations $P_I(t)$, $P_A(t)$, $K(t)$, and $\lambda(t)$; we let P_I, P_A, K, and λ denote the values of the variables at the outset of the escape. In view of Eqs. (3)–(5) the escape regions E_1 and E_2 of Eqs. (13) and (14) may be equivalently

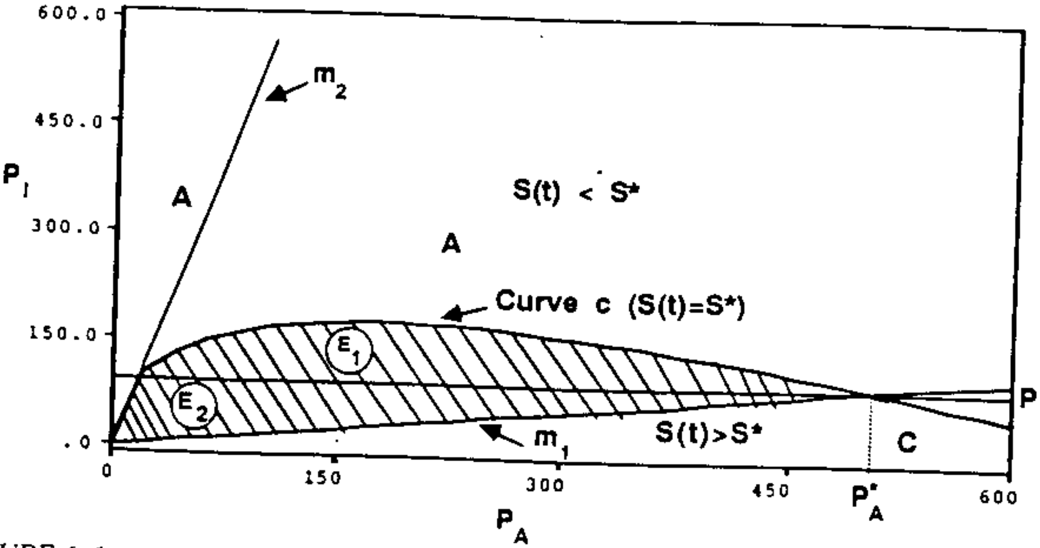

FIGURE 6. Locus of escape conditions for (P_A, P_I). $K = 100$; $r^* = 0.05$; $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = .5$.

expressed as

$$E_1 = \begin{cases} P_I \geq P_I^* & (17) \\ P_I \leq \Theta_2 P_A^{\beta_2} - P_A & (18) \end{cases}$$

$$E_2 = \begin{cases} P_I \leq P_I^* & (19) \\ P_A^{\beta_2} P_I^{\beta_1} \geq (P_A + P_I)\theta_4/\theta_3 & (20) \end{cases}$$

where

$$\theta_1 = \lambda C_1 K^{\alpha_1 - 1} \qquad P_I^* = (s_0/\theta_1)^{1/\alpha_2} \tag{21}$$

$$\theta_2 = C_2 K^{\beta_1}/S^* \qquad \theta_3 = C_2(\lambda C_1)^{\delta}/S^* \qquad \theta_4 = s_0^{\delta} \tag{22}$$

A typical locus of escape conditions on P_I and P_A is represented in Figure 6.

The escape region $E_1 + E_2$ is defined by three equations. First there is the equation $P_I = \Theta_2 P_A^{\beta_2} - P_A$ (curve c in Figure 6) which separates the plane (P_A, P_I) into two regions: if the point (P_A, P_I) is above c (i.e., in region A) then $S(t) < S^*$, i.e., the population is in a nutritional crisis. The population P_I is then subjected to mortality crises until the population falls below the curve c (into region C) where $S(t) > S^*$, i.e., where the population is at least temporarily above the subsistence level. (It can easily be seen from the definition of $S(t)$ that the points on the curve c are such that $S(t) = S^*$). In order to escape permanently $S(t)$ must *remain* above S^*. This constraint (which is expressed by (20)) implies two more equations, namely the straight lines m_1 and m_2. The two straight lines determine a cone, and the escape region $E_1 + E_2$ is inside that cone, and under the curve c.

We note that the escape region $E = E_1 + E_2$ expresses two constraints on P_A and P_I. First, the fact that any point (P_A, P_I) must be below the curve c in order for the population to be above its subsistence level implies that for a given value of the capital $K(t)$, the total population must not be too large in order to sustain itself. This is an intuitively obvious constraint.

A second, more subtle constraint, is implied by the straight lines m_1 and m_2. The fact that (P_A, P_I) must be in the cone defined by these two lines indicates that there must be a compromise between the two sectors of the population: neither sector can be much larger or much smaller than the other. This result is again intuitively plausible because if the nutrition-producing population P_A is large at the time of escape, there will be enough nutrients produced, but not enough capital. Hence $P(t) = P_A(t) + P_I(t)$ will grow more quickly than $Q_A(t)$, which will eventually result in a value of $S(t)$ falling below S^*. Conversely, if the capital producing sector P_I is large, the production of nutrients will be insufficient to sustain a large population, and will rapidly lead to a subsistence crisis.

We now proceed to explore the sensitivity of the escape region to the parameters of the model.

SENSITIVITY ANALYSIS

It can be seen from (20) that the slopes w_1 and w_2 of the lines m_1 and m_2 are the two positive solutions (when they exist) of the equation

$$x^{\beta_1} = \mu(1 + x) \tag{23}$$

where

$$\mu = \frac{S^*}{C_2}\left(\frac{r^*}{\lambda C_1}\right)^{\beta_1/\alpha_2}. \tag{24}$$

The smaller slope (w_1) is less than β_1/β_2 and the larger one (w_2) is larger than β_1/β_2. In order for the two solutions w_1 and w_2 to exist μ must be smaller than a critical value μ_c equal to

$$\mu_c = \beta_1^{\beta_1}\beta_2^{\beta_2}. \tag{25}$$

The smaller μ is, the smaller w_1 will be, the larger w_2 will be, and therefore the more "open" the cone defined by the lines m_1 and m_2 will be. If μ is close to μ_c the cone closes up. At the limit, if $\mu = \mu_c$ the two lines are confounded and the escape region disappears. The two lines then intersect the curve c at the point where the curve c reaches its maximum. (The point (P_A^*, P_I^*) also approaches this maximum if μ is close to μ_c).

The condition $\mu < \mu_c$ is equivalent to

$$\frac{\lambda}{r^*} > \frac{1}{C_1}\left(\frac{S^*}{C_2\mu_c}\right)^{\alpha_2/\beta_1} \stackrel{\text{def}}{=} R^*. \tag{26}$$

This inequality shows that for given output elasticities, and given values of S^*, C_1, C_2 the ratio of the savings rate λ to the escape rate r^* must be larger than R^*. In other words the smaller R^*, the smaller λ/r^* needs to be in order for an escape region to exist.

The value of R^* has a natural interpretation. For example if S^* is large, this means that the nutritional level needed to escape is relatively high, and R^* will also be larger. If R^* is larger, the saving rate λ must be larger or the escape rate r^* of the population must be smaller in order to bring the ratio λ/r^* above R^*. An increase in either C_1 or C_2 also makes an escape easier since R^* then becomes smaller.

These simple observations help to explain why certain economies may not have escaped from the Malthusian trap. Perhaps the ratio of the saving rate to the population growth rate was not large enough, either because λ was too small, or because r^* was too large; or perhaps because the coefficients C_1 and C_2 were not large enough.

A sensitivity analysis of R^* as a function of the output elasticities appears in Table 1. The calculations in panel A of Table 1 show that at any level of α_1 (three examples are provided: $\alpha_1 = 0.2$, 0.5 and 0.8) R^* increases rapidly and consistently as β_1 increases from 0.2 to 0.6. At values of β_1 greater than 0.6, R^* decreases slightly. Also, the values of R^* are larger for larger values of α_1. These results show that for a fixed value of the output elasticity of capital (α_1) in the AOG sector the likelihood of an escape decrease as β_1 increases (since the requirement on the ratio λ/r^* is greater—i.e., λ/r^* must be larger for an escape region to exist). Hence, technological changes that increases β_2 (even with a corresponding decline in β_1) will increase the likelihood of an escape.

Panel B of Table 1 shows that for β_1 constant R^* increases monotonically as α_1 increases. (Three examples are provided: $\beta_1 = 0.2$, 0.5, and 0.8). This confirms the results from panel A since an escape will be more likely for a smaller output elasticity α_1 of capital in the AOG sector.

Thus in our model the larger the output elasticities of labor in both sectors, the easier it will be for the population to escape from the Malthusian trap.

We next investigate the dynamics of the model and explore the determinants of an escape from the Malthusian trap.

DYNAMIC ANALYSIS

Figure 6 depicts the compromise that must exist at any given time between P_A and P_I in order for an escape to occur. If the point (P_A, P_I) is in the escape region $E_1 + E_2$ we recall that both $P_A(t)$ and $P_I(t)$ grow at a constant rate r^*. (In what follows the points $(P_A(t), P_I(t))$ represent the sequence of populations in the two sectors following some initial values P_A and P_I).

When the escape occurs the points $(P_A(t), P_I(t))$ will move beyond (P_A, P_I) and along the ray between the origin and (P_A, P_I). The curve c then moves upward with the points $(P_A(t), P_I(t))$ which remain in a continuously expanding escape region. (The straight lines m_1 and m_2 do not depend on $K(t)$ and remain unchanged as $K(t)$ increases; the curve c on the contrary expands as $K(t)$ increases).

We recall that if the population has not escaped the Malthusian trap, it is subject to a regulating mechanism that leaves society always close to the nutritional status $S(t) = S^*$ above which the escape occurs. The resulting homeostasis keeps the point $(P_A(t), P_I(t))$ always close to the curve c of Figure 6 since that curve represents the locus of points for which $S(t)$ is equal to S^*.

To a given level K of the capital stock there corresponds the value P_A^* of the population in the nutrient-producing sector at which the curve c and the straight line m_1 intersect (see Figure 6). The actual value of P_A is then either less than or

TABLE 1
Sensitivity of R^* (the right-hand side of (26)) to the output elasticities α_1 and β_1 ($\beta_2 = 1 - \beta_1$; $\alpha_2 = 1 - \alpha_1$). [C_1, C_2, S^* are fixed at their previous values]. Panel A: sensitivity of R^* to β_1 for fixed α_1. Panel B: sensitivity of R^* to α_1 for fixed β_1.

Panel A			Panel B		
α_1	β_1	R^*	α_1	β_1	R^*
.200	.200	.113	.200	.200	.113
.200	.300	.288	.300	.200	.144
.200	.400	.417	.400	.200	.183
.200	.500	.487	.500	.200	.232
.200	.600	.511	.600	.200	.295
.200	.700	.505	.700	.200	.375
.200	.800	.477	.800	.200	.477
.500	.200	.232	.200	.500	.487
.500	.300	.416	.300	.500	.515
.500	.400	.525	.400	.500	.546
.500	.500	.578	.500	.500	.578
.500	.600	.596	.600	.500	.612
.500	.700	.591	.700	.500	.648
.500	.800	.570	.800	.500	.686
.800	.200	.477	.200	.800	.477
.800	.300	.602	.300	.800	.506
.800	.400	.660	.400	.800	.537
.800	.500	.686	.500	.800	.570
.800	.600	.695	.600	.800	.605
.800	.700	.692	.700	.800	.643
.800	.800	.682	.800	.800	.682

greater than P_A^* with a corresponding value of P_I for which the point (P_A, P_I) is close to the curve c. We now distinguish between these two possibilities for P_A.

If $P_A < P_A^*$, with (P_A, P_I) close to the curve c, then the ratio P_I/P_A will be larger than w_1, the slope of the straight line m_1. If (P_A, P_I) is just below the curve c then the population is in E_1 and has escaped the Malthusian trap. If (P_A, P_I) is just above c, the population is in a nutritional crisis, but as soon as a mortality crisis brings down the population P_I the point (P_A, P_I) will fall back into E_1 and the population will escape the Malthusian trap (unless the mortality crisis is of extreme magnitude and (P_A, P_I) falls below the line m_1).

If $P_A > P_A^*$, with (P_A, P_I) close to the curve c, then P_I/P_A is less than w_1. If (P_A, P_I) is just below the curve c, then the points $(P_A(t), P_I(t))$ grow along the ray between the origin and (P_A, P_I). The slope of that ray is P_I/P_A which is less than w_1, and therefore the population tends to move further away from the escape region. The curve c expands with $K(t)$ but the point $(P_A(t), P_I(t))$ eventually jumps

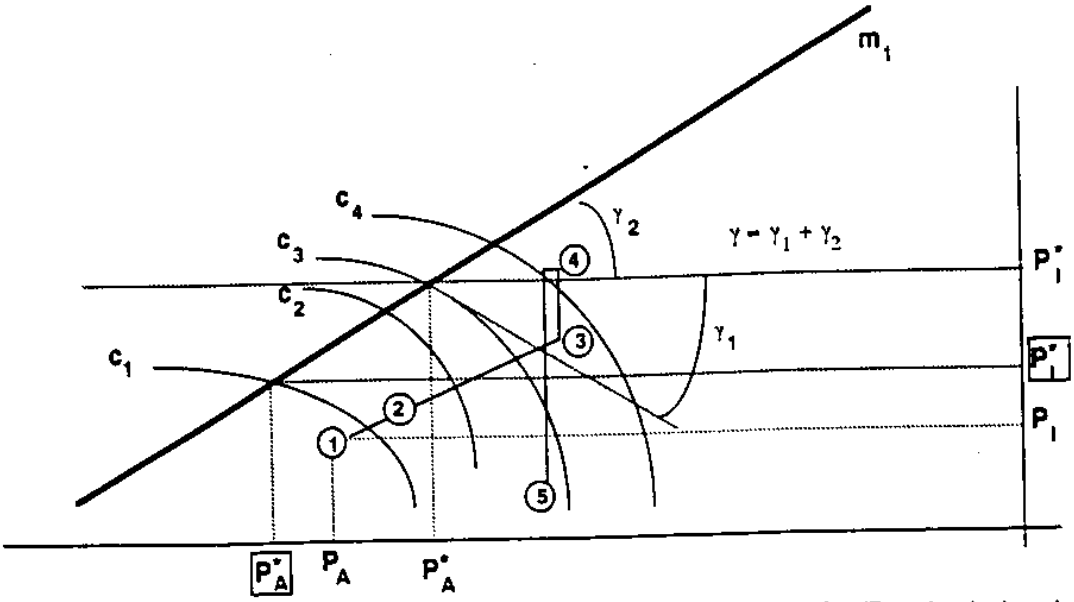

FIGURE 7. Dynamics of the points $(P_A(t), P_I(t))$ when P_A is larger than P_A^* (Boxed value) and just below the curve c (here denoted c_1).

over c, and the population is back in a nutritional crisis. Figure 7 depicts a stylized representation of the dynamics of the points $(P_A(t), P_I(t))$ in this case.

In the example of Figure 7 there corresponds to an initial point (P_A, P_I) a curve c (denoted c_1). (The coordinates of the corresponding point (P_A^*, P_I^*) are boxed). The initial point 1 (a circled 1) is below c_1, so the population is above the subsistence level and grows at rate r^*. The next point (2) gives rise to a new value $K(t)$ of the capital and thus to a new curve c (denoted c_2). In the example the population is still above the subsistence level (since 2 is below the curve c_2). Therefore the population grows along the same ray to point 3. To this point there corresponds a curve c_3 which is now below the point (P_A, P_I) and the population is in a Malthusian crisis. The values (P_A^*, P_I^*) prevailing at that period are also given in Figure 5.

We recall that in a nutritional crisis the population $P_A(t)$ in the nutrient-producing sector remains stationary whereas the population $P_I(t)$ is subject to randomly determined mortality shocks; $P_I(t)$ may continue to grow for a while, but eventually is subject to a decreasing growth rate, which will bring the population above the subsistence level. In the example, $P_I(t)$ grows for one period and results in a point 4 which is still below subsistence level (since the point 4 is above the corresponding curve c_4). The population $P_I(t)$ is then subjected to a severe mortality crisis, and collapses to the point 5, thus bringing society back above the level of subsistence. This completes a cycle, which has illustrated the mechanics of the homeostatic mechanism implied by our model.

We have focused on the case when P_I/P_A was less than w_1, because we believe that historically the population P_I in the all-other-goods producing sector was indeed more susceptible to mortality crises than the population P_A in the nutrient-producing sector. The analysis conducted so far suggests that the failure of P_I to become sufficiently large with respect to P_A could be a reason why certain economies may have not escaped the Malthusian trap. One ought not think that the absolute level of the labor force in the AOG sector, or in conventional terms the urban-commercial-industrial sector, is the crucial variable for an economy to have an industrial revolution. Rather, the ratio of the labor force employed in the AOG sector to the labor force employed in the nutrient-producing sector is an important determinant of the likelihood of an escape. Even small variations in this ratio will suffice to make a crucial difference to the outcome of economic processes.

It remains to be seen under what circumstances an escape could occur in the case depicted in Figure 7, since we believe that such an example represents the plausible trajectory of a society with a relatively small population in the all-other-goods producing sector that struggles to emancipate itself from the Malthusian trap.

The potential for an escape exists only at the point 3 when the population is in a crisis. In order for the population to escape, $P_I(t)$ would have to grow enough so that $P_I(t)/P_A(t)$ can become larger than w_1, the slope of m_1 (this means that the point 4 would have to jump over the line m_1). However this condition is not in itself sufficient, since in such a case the point 4 will be above the line m_1 but may also be above the curve c. That is actually the case in Figure 7. If the point 4 had been higher (i.e., if $P_I(t)$ had grown more than it did) the population $(P_A(t), P_I(t))$ could have found itself above the line m_1, but the population would still be in a crisis since $(P_A(t), P_I(t))$ would also be above the curve c_4. (A number of actual simulations have shown however that in this case and after a few mortality crises the population will often fall into the escape region—provided the curve c grows sufficiently and the point $(P_A(t), P_I(t))$ does not fall back below the line m_1).

Numerous simulations have shown that there are two types of behavior for a population that starts below the line m_1, depending on the magnitude of the mortality crises. If the mortality crises are severe then the population $P_I(t)$ is regularly decimated and the points $(P_A(t), P_I(t))$ tend to move further away from the line m_1 without ever escaping.

If the mortality crises are not too severe and $P_I(t)$ is allowed to grow somewhat even during a crisis, then the points $(P_A(t), P_I(t))$ tend to move upward, staying just below the line m_1 and always close to the current value of (P_A^*, P_I^*). In this case, it is apparent that the likelihood of an escape will depend crucially on the two angles γ_1 and γ_2 depicted in Figure 7; γ_1 is the angle between the tangent to the curve c at P_A^* and the horizontal axis; γ_2 is the angle between the line m_1 and the horizontal axis (the tangent of γ_2 is w_1, the slope of m_1).

The larger the angle $\gamma = \gamma_1 + \gamma_2$, the more difficult it will be for the population to escape, since in a crisis the population $P_I(t)$ will have to increase more (in absolute terms) when γ is larger.

The angle γ_2 is equal to $\arctan(w_1)$. The tangent to the curve c at P_A^* is $\beta_2(1+w_1)-1$. This tangent is negative and so the angle γ_1, considered as positive is equal to $\gamma_1 = \arctan(1-\beta_2(1+w_1))$ which for a fixed β_2 is a decreasing function of w_1. Therefore when β_2 is fixed, a change in w_1 has opposite effects on γ_1 and γ_2: if w_1 decreases the angle γ_2 decreases but the angle γ_1 increases. In order to disentangle the relative effects of γ_1 and γ_2 on their sum γ we perform a sensitivity analysis of these angles to the values of β_1 and λ.

Table 2 gives the values of the angles γ, γ_1 and γ_2 for different values of β_1 ($\beta_2 = 1-\beta_1$) and different values of λ. For a fixed β_1 the angle γ decreases as λ

TABLE 2
Sensitivity of the angles γ, γ_1 and γ_2 to β_1 and λ. ($\alpha_1 = 0.5$)

β_1	λ	γ	γ_1	γ_2	$w_1=\tan(\gamma_2)$
.200	.04	11.404	10.995	.409	.007
.200	.05	11.370	11.108	.262	.005
.200	.06	11.351	11.173	.178	.003
.500	.04	32.569	22.237	10.332	.182
.500	.05	29.980	24.198	5.782	.101
.500	.06	28.804	25.036	3.768	.066
.800	.04	58.474	35.586	22.888	.422
.800	.05	49.683	37.061	12.622	.224
.800	.06	45.773	37.648	8.125	.143

increases. The sensitivity of γ to λ however is greater for larger values of β_1. Also the values of γ increase for larger values of β_1. The values of γ, γ_1 and γ_2 are insensitive to the value of α_1.

This analysis complements the one concerning the critical value R^* of the ratio λ/r^*. Indeed, we saw that when β_1 is small, R^* is also small which eases the demand on the ratio λ/r^* in order for an escape region to exist. Here we see that if an escape region exists, the escape is more likely when β_1 is small.

Table 2 clearly shows that γ_2 is the driving force behind any variation in the angle γ. Indeed, γ_2 is much more sensitive than γ_1 to variations in the parameters and therefore when γ increases, it is because γ_2 increases. The fact that γ and γ_2 vary in the same direction (at least for a fixed β_1) suggests that γ_2—or its tangent w_1—is the single most important determinant of an escape. Indeed, when γ_2 (or w_1) is small the likelihood of an escape increases, in two ways.

First, when the slope w_1 is smaller this puts less demand on the relative value of P_I since the ratio P_I/P_A need only be larger than w_1 in order for P_I/P_A to be above the line m_1 (which is a necessary condition for the escape).

Second, since γ decreases with γ_2 it becomes easier for the population to move from a nutritional crisis into the escape region. This is best seen by considering the point 3 in Figure 7. If the angle γ had been smaller the point 4 may have been above the line m_1 and in the escape region.

It is worth emphasizing that the angles γ_1 and γ_2 do not depend on the capital stock $K(t)$. Indeed, as $K(t)$ increases the point (P_A^*, P_I^*) moves up along the line m_1 with angles γ_1 and γ_2 that remain unchanged. This implies that the probability of an escape increases with $K(t)$ since, everything else being equal, the *growth rate* required for $P_I(t)$ to move into the escape region becomes smaller. This can be seen in Figure 7: the *absolute value* of the increase needed in the value of $P_I(t)$ for the point 3 to move above m_1 is independent of the actual value of $P_I(t)$. Therefore, with a larger capital stock $K(t)$, the coordinates $(P_A(t), P_I(t))$ (the point 3) are larger which will put less demand on the *growth rate* of $P_I(t)$ in order to bring

$(P_A(t+1), P_I(t+1))$ (the point 4) above m_1. This shows that, everything else being equal, the escape becomes easier as the stock of capital increases.

DISCUSSION

We have described a model that captures the dynamics of a Malthusian homeostasis which has constrained the growth of the European populations until the eighteenth century. With a simulation we were able to replicate the world's demographic history until the eighteenth century and generate an escape from the Malthusian trap that was brought about by the accumulation of capital broadly defined, and an increase in the savings rate $\lambda(t)$.

The subsequent analysis showed that with $\alpha_1 = \beta_1 = 0.5$ the ratio λ/r^* had to be larger than 0.578 (Table 1) in order for an escape region to exist. With an escape rate r^* of 0.05, $\lambda(t)$ had to be larger than $0.05 \times 0.578 = 0.028$ per decade. In our simulation $\lambda(t)$ reached that threshold value in about 1600 A.D. The "cone of escape" opened up rapidly thereafter as the savings rate continued to increase, and the point $(P_A(t), P_I(t))$ then easily fell in the broadening escape region. This example illustrates our first finding, namely that there is a lower bound on the rate of savings below which an economy cannot escape from the Malthusian trap. In other words, anecdotal evidence on the rate of growth of the capital stock does not suffice in explaining European exceptionalism; rather, the rate of growth of the European capital stock must be compared carefully to the ones obtained elsewhere before the role of accumulation can be understood in a comparative perspective.

The minimum value R^* of the ratio λ/r^* below which an escape cannot occur is quite sensitive to the output elasticities. When either one of the output elasticities α_1 and β_1 of the capital stock is small, R^* tends to be small, thus making an escape more likely. With small values of R^* it becomes easier for the ratio λ/r^* to exceed the value below which there cannot be an escape (Table 1). Conversely, small elasticities of labor α_2 and β_2 make an escape more difficult since then λ/r^* must be larger before an escape region even exists. Therefore, if during the eighteenth and nineteenth centuries the elasticities of labor were smaller in the Japanese aggregate production function, say, than in England, this factor could be part of the explanation why an industrial revolution first took place in the latter country rather than in the former.

Of course the mere existence of an escape region does not guarantee that the escape will occur. Our findings concerning the dynamics of the model have shown the critical role played by the slope w_1 of the line m_1. The smaller this slope, the easier it is for the population P_I in the AOG sector to become large enough to allow the population to escape. Table 2 shows the sensitivity of this slope to β_1 and λ. For example with $\beta_1 = 0.5$ an increase of the savings rate from 0.04 to 0.06 brings the slope down from 0.182 to 0.066. In other words, with $\lambda = 0.04$ the ratio P_I/P_A needs to be larger than 0.182 in order for the population to escape; with $\lambda = 0.06$ the ratio need only be larger than 0.066 for an escape to occur (provided of course the point (P_A, P_I) is below the curve c at the time). Table 2 also shows that w_1 is very sensitive to β_1: with $\lambda = 0.05 w_1$ grows from 0.005 to 0.101 when β_1

increases from 0.2 to 0.5. This again demonstrates the crucial role played by the output elasticity of labor in the nutrient-producing sector.

In our simulation, presented in Figure 2, the escape hinged on an increasing value of the savings rate $\lambda(t)$ but the above sensitivity analysis shows that an increase in $\lambda(t)$ is only one of the possible shifts in the parameter values that can bring about an industrial revolution. We thus hope to have shown that the Industrial Revolution may well have resulted from a small shift of one of the model's parameters; such a shift can suddenly "open up" the escape region in which the population $(P_A(t), P_I(t))$ then grows unhindered.

The escape analysis is quite independent of the model's specifications during a nutritional crisis. For example the allocation of the population between the two sectors does not play a crucial role. If the population $P_A(t)$ does not remain constant and is also affected by a nutritional crisis, then point 5 in Figure 7 would be moved further to the left to reflect the fact that both sectors of the population have decreased. In addition, one could conceive of a more sophisticated version of this model in which the capital $K(t)$ would not be strictly increasing, and could be allowed to decrease slightly during periods of crisis. This would mean that the curve c would expand in times of prosperity but would shrink again in a crisis. These variants of the model would not change the escape analysis, which, therefore, remains valid for a wide variety of model specifications that may be worthy of further investigations.

To be sure, the answer to the riddle of why Europe was first to industrialize may well lie beyond the scope of the sensitivity analysis proposed here. The answer might turn on environmental, political, and cultural differences emphasized by E. L. Jones (1981). Furthermore, it is conceivable that Japan, and perhaps even China was merely lagging behind Europe, and if Europe's Industrial Revolution had not occurred they might have had an industrial revolution quite independently of Europe.

In sum, there are numerous reasons why the Asian experience of economic development during the early-modern period diverged from the European pattern. The upshot of this paper is that these differences can be fruitfully explored within the conceptualization of our model. However, much more empirical work is needed on the various parameters such as the output elasticities of the factors of production before one can enumerate with more certainty the major reasons for the differences in the European and Oriental patterns of economic development.

REFERENCES

Arthur, B. (1985) Oral comment on the presentation of this model at Stanford University (1985).

Artzrouni, M., and Komlos, J. (1985) Population growth through history and the escape from the Malthusian trap: a homeostatic simulation model, *Genus* **41**: 21–39.

Boserup, E. (1981) *Population and Technological Change: A Study of Long-Term Trends*. University of Chicago Press, Chicago.

Goldstone, J. (1985) Personal correspondence.

Jones, E. L. (1981) *The European Miracle, Environments, Economies, and Geopolitics in the History of Europe and Asia*. Cambridge University Press, Cambridge.

Jones, E. L. (1985) Personal correspondence.

Komlos, J. (1985) Stature and nutrition in the Habsburg Monarchy: The standard of living and economic development in the eighteenth century, *American Historical Review* **90**: 1149–1161.

Komlos, J. (1989) Thinking of the Industrial Revolution. *Journal of European Economic History* **18:** 191–206.

Komlos, J. (1987) The height and weight of West Point cadets: Dietary change in antebellum America, *Journal of Economic History* **47:** 897–927.

– Krelle, W. (1988) *Theorie des wirtschaftlichen Wachstums*. Springer Verlag, Berlin, 2nd ed. O

Landes, D. (1990) Why are we so rich and they so poor? *American Economic Review* **80:** 1–13.

Lee, R. D. (1988) Induced population growth and induced technological progress: Their interaction in the accelerating stage, *Mathematical Population Studies* **1:** 265–288.

Lee, R. D. (1986) Malthus and Boserup: A dynamic synthesis, in David Coleman and Roger Schofield (ed.), *The State of Population Theory. Forward from Malthus*, Basis Blackwell, Oxford: 96–130.

Mokyr, J. (1990) Punctuated equilibria and technological progress, *American Economic Review* **80:** 350–354.

Mokyr, J., ed. (1985) *The Economics of the Industrial Revolution*, Totowa, NJ: Rowman and Allanheld.

Nelson, R. N. (1956) Theory of the low-level equilibrium trap in underdeveloped countries, *American Economic Review* **46:** 894–908.

– Niehans, J. (1963) Economic growth with two endogenous factors, *Quarterly Journal of Economics* **773:** 49–371. O QA12950

Pryor, F. L., and Maurer, S. B. (1982) On induced economic change in precapitalist societies, *Journal of Development Economics* **10:** 325–53.

Robinson, W., and Schutjer, W. (1984) Agricultural development and demographic change: A generalization of the Boserup model, *Economic Development and Cultural Change* 32: 355–66.

Simon, J. (1986) *Theory of Population and Economic Growth*, Basil Blackwell, Oxford.

Steinmann, G. (1984) A model of the history of demographic-economic growth, in G. Steinmann (ed.), *Economic Consequences of Population Change in Industrialized Countries*, Springer-Verlag, Berlin: 29–49.