

Black hole formation and classicalization in ultra-Planckian $2 \rightarrow N$ scattering

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Abstract

We establish a connection between the ultra-Planckian scattering amplitudes in field and string theory and unitarization by black hole formation in these scattering processes. Using as a guideline an explicit microscopic theory in which the black hole represents a bound-state of many soft gravitons at the quantum critical point, we were able to identify and compute a set of perturbative amplitudes relevant for black hole formation. These are the tree-level N -graviton scattering S -matrix elements in a kinematical regime (called classicalization limit) where the two incoming ultra-Planckian gravitons produce a large number N of soft gravitons. We compute these amplitudes by using the Kawai–Lewellen–Tye relations, as well as scattering equations and string theory techniques. We discover that this limit reveals the key features of the microscopic corpuscular black hole N -portrait. In particular, the perturbative suppression factor of a N -graviton final state, derived from the amplitude, matches the non-perturbative black hole entropy when N reaches the quantum criticality value, whereas final states with different value of N are either suppressed or excluded by non-perturbative corpuscular physics. Thus we identify the microscopic reason behind the black hole dominance over other final states including non-black hole classical object. In the parameterization of the classicalization limit the scattering equations can be solved exactly allowing us to obtain closed expressions for the high-energy limit of the open and closed superstring tree-level scattering amplitudes for a generic number N of external legs. We demonstrate matching and complementarity between the string theory and field theory in different large- s and large- N regimes.

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1. Introduction and summary

The formulation of a microscopic picture of black hole production in high-energy particle scattering is crucial for understanding the nature of quantum gravity at ultra-Planckian energies. In particular, this issue is central to the idea that gravity is UV-complete in a non-Wilsonian sense [1], based on the concept of classicalization [2].

The standard (Wilsonian) approach to UV-completion implies that interactions at higher and higher energies are regulated by integrating-in-weakly-coupled degrees of freedom of shorter and shorter wavelengths. When applied to gravity, the Wilsonian picture would imply that at energies exceeding the Planck mass, $\sqrt{s} \gg M_P$, the UV-completion must be achieved by new quantum degrees of freedom of wavelength much shorter than the Planck length, $R \sim \frac{1}{\sqrt{s}} \ll L_P$. In the classicalization approach, instead of introducing new hard quanta, the UV-completion is accomplished by means of *collective* states composed of a large number $N \sim s/M_P^2$ of soft gravitons of wavelength $R \sim \sqrt{N}L_P$ [3] that, in the mean-field approximation, recover the semi-classical behavior of macroscopic black holes [4]. To put it shortly, classicalization replaces the hard quanta by a multiplicity of soft ones, which in mean-field (large N) approximation acquire some properties of classical objects.

In the conventional semi-classical approach, the current understanding of black hole production is rather unsettling. On one hand, it is widely accepted that scattering of very highly energetic particles results into a black hole formation. This acceptance is based on the following argument: according to classical gravity any source of center of mass energy \sqrt{s} when localized within its gravitational (Schwarzschild) radius $R = \sqrt{s}L_P^2$ must form a black hole. This argument is insensitive to the precise nature of the source and in particular should be applicable to elementary particle sources. Thus, it is reasonable to expect that, for example, a two-particle scattering with center of mass energy of the order of the solar mass for an impact parameter less than 3 km, should result into the formation of a solar mass black hole.

On the other hand, we have to admit that this way of thinking challenges the view about black holes as *classical* macroscopic objects, since production of usual macroscopic objects in two-particle collisions is expected to be exponentially-suppressed. For example, in the above thought experiment of two-particle collision at solar mass energy it is exponentially-unlikely for a sun-like object to be produced in the final state instead of a black hole.

What makes black holes so different from ordinary classical objects from the point of view of their microscopic structure?

Of course, one can certainly say that what makes black holes very special is their Bekenstein–Hawking entropy. However, without a microscopic explanation of entropy creation in two-particle collision, this invocation of the entropy is only making the puzzle more complicated. Indeed, it is totally mysterious how an initial two-particle state with zero entropy gains such an enormous entropy in the process of the collision.

The above questions are impossible to answer without having a microscopic theory of the black hole and the corresponding microscopic mechanism of black hole formation in particle scattering processes. This is why the above questions have not been settled although the study of black hole formation in particle collisions at ultra-Planckian energies has been pioneered long

ago [5–7] and since then has even been taken as far as predicting production of micro black holes at LHC [8]. The reason is the lack of a quantum corpuscular picture of black holes which subsequently makes it impossible to figure out how the quantum gravity amplitude translates into the formation of a black hole final state.

The present paper is an attempt to establish the missing link between quantum gravity amplitudes and a corpuscular picture of black holes. In particular we will provide the link between the corpuscular black hole portrait [4] on the one hand and the classicalization idea for gravitational scattering amplitudes.¹ By employing the corpuscular black hole picture together with the expressions of graviton scattering amplitudes both in field and string theory we shall uncover some key elements underlying the microscopic origin of black hole formation.

More concretely:

- Guided by non-perturbative input from the corpuscular black hole N -portrait, we identify the black hole formation regime as the regime of multi-particle creation, in form of $2 \rightarrow N$ graviton scattering amplitudes, with number of soft gravitons in the final state being given by the number of black hole constituents, as suggested by classicalization.
- Next, by using powerful field and string-theoretic techniques, in particular scattering equations [9] and Kawai–Lewellen–Tye (KLT) relations [10], we estimate the perturbative part of these N -graviton amplitudes.
- Finally, using the microscopic corpuscular picture of black holes as N -graviton self-bound states at a quantum critical point, we provide the missing non-perturbative information that enables us to translate the N -graviton production processes into the black hole formation, both in field and string theory scatterings.
- We provide a cross-check of perturbative N -graviton amplitudes by applying them to the production of non-black hole type classical configurations described by multi-particle coherent states for which semi-classical estimates must also be valid. We then match the two results and observe that the exponential suppression expected in the semi-classical theory is indeed reproduced by the perturbative $2 \rightarrow N$ gravity amplitudes. Thus, this matching besides of providing an independent information about the multi-graviton amplitudes, also confirms that the microscopic origin of the black hole dominance, relative to other possible multi-particle final states of the same energy, lies in the quantum criticality of the black hole constituents, which is absent for other classical objects.
- One of the outcomes of our analysis is to show the very different large- N behavior of multi-particle amplitudes in gravity in comparison with non-derivatively coupled scalar theories.

The above framework supplies a correct physical picture that among other things explains why the black hole production is the dominant process while the production of other macroscopic multi-particle states is exponentially-suppressed. The perturbative kinematics that we shall identify has just the right suppression to be compensated by the degeneracy of states at the quantum critical point. In other words, in this multi-particle production kinematics, the amplitude itself anticipates what would be the right value for the entropy.

We also observe a nice interplay between the field and string theory amplitudes. In particular, we observe that the string and field theory amplitudes agree whenever the size of the produced

¹ There is another attempt for a synthesis [11] by sewing together two $2 \rightarrow N$ graviton amplitudes into a ladder loop diagram and coherently summing over different N in an eikonal region.

black hole is larger than the string length, or equivalently, when the Reggeization of the amplitude does not take place.

Before moving into the technical part of the paper, to be covered in the following sections, we shall summarize the basic results and their physical meaning. In order to do it we shall briefly review the non-perturbative input coming from the corpuscular black hole portrait, which being a microscopic quantum theory, provides a crucial missing link between the perturbative N -graviton production amplitudes and the unitarization of the theory by black hole formation.

1.1. Non-perturbative input from a microscopic portrait

In order to make the connection explicit let us summarize some non-perturbative input coming from the black hole corpuscular quantum portrait [4,12] (for other aspects of this proposal see [13] and some similarities with this proposal can be found in [14]). This portrait is based on the idea that the black hole is a composite entity. Its corpuscular constituents are gravitons with the characteristic de Broglie wavelength given by the classical size of the black hole, R . That is, the internal (and near-horizon) physics of black holes is fully determined by the quantum interaction of gravitons of wavelength R . We shall be interested in the regimes in which the black hole is much heavier than the Planck mass $M_{BH} \gg M_P$, or equivalently, $R \gg L_P$.

The two crucial properties are:

- For macroscopic black holes, the quantum gravitational coupling α among the individual corpuscles,

$$\alpha \equiv \frac{L_P^2}{R^2} \quad (1.1)$$

is extremely weak.

- The number N of constituents of wavelength R is:

$$N = M_{BH}^2 / M_P^2. \quad (1.2)$$

Thus, quantum-mechanically a black hole represents a self-bound state of soft gravitons, with a very special interplay between the quantum coupling and the number of constituents, $\alpha N = 1$. Or equivalently, the black hole is a state in which the wavelengths of gravitons satisfy, $R = \sqrt{N} L_P$. This property implies that the physics of black holes is similar to that of a graviton Bose–Einstein condensate at a *quantum critical point* [12]. This critical point separates the following two phases. For $\alpha N \ll 1$, the system is in the phase in which collective graviton–graviton attraction is not enough to form a self-bound-state and the graviton Bose-gas is essentially free. For $\alpha N = 1$ the bound-state is formed.

At this critical point, order N collective Bogoliubov modes become gapless leading to an exponential degeneracy of states, of order e^{cN} , where c is some positive constant. This exponential degeneracy of states is quickly lifted when we deform the system and move away from the critical point $\alpha N \neq 1$. While for the generic attractive Bose-gas, moving away from the critical point is possible in both directions, ($\alpha N < 1$ or $\alpha N > 1$), for gravitons this is not the case. The gravitons cannot form a sensible state with $\alpha N \gg 1$.² Thus, the viability and the nature of the deformed state depends in which direction we move from the critical point.

² In more than one space dimensions the attractive Bose-gas in the over-critical phase undergoes a quantum collapse. However, for gravity this is impossible since gravitons of a given energy cannot form a configuration smaller than a black

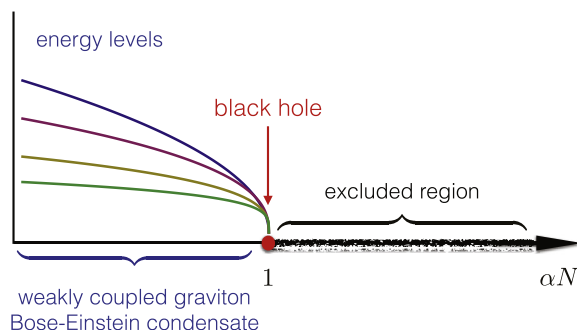


Fig. 1. Bose–Einstein levels and black hole formation.

For $\alpha N < 1$, the system of N gravitons is essentially free. The Bogoliubov frequencies are positive and Bogoliubov levels are separated by a large energy gap ($\sim 1/R$) from the lowest level obtained in the free-graviton approximation. Due to this, the non-perturbative collective quantum effects can be ignored and the system can be well-approximated by an asymptotic N -particle eigenstate of the S -matrix, with no additional non-perturbative information required. Hence, non-perturbative physics gives no additional essential input for states with $\alpha N \ll 1$, and the perturbative approximation can be trusted. In particular, the perturbative amplitudes can be directly applied to the formation of final states with $\alpha N \ll 1$.

However, for the states with $\alpha N > 1$ the situation is very different. The Bogoliubov frequencies of the N -graviton state in this regime are complex, with the Liapunov exponent being much larger than the inverse size of the system. This indicates that such state cannot be treated as a viable asymptotic state of the S -matrix, even approximately. This is remarkable, since translated into the language of N -graviton perturbative amplitudes, this regime would include the region that violates perturbative unitarity. Thus, non-perturbative N -graviton physics provides a selection rule that cuts-out from the Hilbert space those would-be N -particle final states that perturbatively violate unitarity. Non-perturbative corpuscular physics is telling us that such states are not part of the physical Hilbert space.

The situation is schematically depicted in Fig. 1. The critical point $\alpha N = 1$ corresponds to the point of black hole formation. At this point the levels become nearly gapless (up to $1/N$ resolution), and there is a maximal degeneracy of states. For estimating the production rate of N -graviton state at the critical point, the perturbative amplitudes must be supplemented by a non-perturbative factor $\sim e^N$ counting the degeneracy of states, which at the same time represents the black hole entropy factor. The region to the left corresponds to a nearly-free graviton gas. These states are close to asymptotic S -matrix states of N free gravitons and their creation in two graviton collision can be estimated via perturbative matrix elements, with non-perturbative corrections being small. The region to the right is excluded by non-perturbative physics.

The above outline summarizes the non-perturbative information that provides the missing link between the perturbative amplitude and unitarization by classicalization via black hole formation.

hole. As result the graviton bound-state is “stuck” at the critical point slowly loosing the constituent gravitons due to the quantum depletion, reducing N , but maintaining quantum criticality for each N . This is how the corpuscular picture accounts for the Hawking radiation. For the purpose of the present paper, we shall ignore the further evolution of black holes after their formation in the scattering process.

1.2. N -graviton amplitudes and black hole formation

Based on the previous discussion the following picture emerges. In order to estimate the production rate of a N -graviton state we need to supplement the perturbative scattering amplitude, that views a given N -particle state as an asymptotic state of free gravitons, by the non-perturbative information about the viability and quantum degeneracy of this state. This information either will further enhance the rate or will diminish it depending where the given state is in the αN plot.

The perturbative amplitudes relevant for describing the production of a black hole of mass $M_{BH} = \sqrt{s}$ are the perturbative amplitudes at center of mass energy \sqrt{s} in which N gravitons of momenta $p \sim (\sqrt{s} L_P^2)^{-1}$ are created in the final state. As we shall see, the transition probability of this process obtained from the corresponding S -matrix element scales as

$$|\langle 2|S|N \rangle|_{pert}^2 \sim \alpha^N N! = \left(\frac{L_P^2 s}{N^2} \right)^N N!. \quad (1.3)$$

In order to understand the physical picture, we must superimpose the non-perturbative information that we have distilled from the many-body analysis of the N -graviton state. Namely, the region $s/M_P^2 N \gg 1$ is excluded as physically not viable due to the presence of complex Bogoliubov frequencies and very large Liapunov exponent. It is convenient to rewrite the matrix element in terms of the effective ('t Hooft-like) collective coupling,

$$\lambda \equiv \alpha N = s/M_P^2 N, \quad (1.4)$$

which parameterizes the strength of the collective gravitational interaction of the N -graviton system. In this notation the matrix element becomes

$$|\langle 2|\hat{S}|N \rangle|_{pert}^2 \sim \left(\frac{\lambda}{N} \right)^N N!. \quad (1.5)$$

This form makes the physics point very transparent. As we just explained above, the region $\lambda \gg 1$ is excluded by non-perturbative physics, because in this region the graviton gas is over-critical. This includes the region in which perturbative amplitudes would violate unitarity, but multi-particle physics prevents us from going there.

The point $\lambda = 1$ is a critical point. It is allowed both perturbatively as well as non-perturbatively, but the non-perturbative information is very important. Notice, that for $\lambda = 1$, this amplitude has just the right scaling for being compensated by the black hole entropy factor. In the microscopic picture such a factor is indeed appearing due to the exponential degeneracy of states at the critical point $\lambda = 1$. For values of $\lambda \ll 1$, the system is sub-critical. This means that not only graviton–graviton interaction is weak, $\alpha \ll 1$, but also that the collective non-perturbative effects are negligible. Thus, the gravitons are essentially free. The perturbative suppression of the amplitude cannot be compensated by the multiplicity of states, because for $\lambda \ll 1$ the degeneracy of Bogoliubov levels is lifted and there is no longer an enhancement of the number of states. Therefore, using Stirling's formula and the large N limit, the production rate of such multi-particle configurations, unlike black holes, is exponentially suppressed:

$$|\langle 2|\hat{S}|N \rangle|_{pert}^2 \sim e^{-N} \lambda^N. \quad (1.6)$$

As already mentioned, the exponential suppression factor can be compensated by the black hole entropy.

Hence, in this picture, the microscopic explanation of the black hole dominance over other possible multi-particle final states, is that the latter systems are far away from quantum criticality and one must pay an exponential suppression price for their production. In particular, this explains, why at a given ultra-Planckian center of mass energy \sqrt{s} , the production rate of a non-black hole classical configuration is exponentially-suppressed relative to the production rate of a same-energy black hole. The reason is that a non-black hole classical configuration of a given mass represents a coherent state of constituents that are softer and have larger occupation number than the constituents of the same mass black hole. As a result such states always are at the subcritical value of the collective coupling, $\lambda \ll 1$, and no enhancement is available. We shall discuss this point in more details towards the end of the paper, by estimating a production probability of a particular classical configuration both via the quantum $2 \rightarrow N$ process as well as semi-classically and comparing it to the black hole production rate.

1.3. Various regimes

It is instructive to summarize the various regimes of multi-particle production amplitudes, by superimposing the perturbative and non-perturbative inputs. In doing so we shall use both field theory as well as string theory data, namely, s , L_P , L_s and N .

Let us consider first the perturbative input. Here we can distinguish two regimes.

- The stringy regime is achieved for

$$\frac{sL_s^2}{N^2} > 1. \quad (1.7)$$

This is the regime for which the amplitudes effectively Reggeize.

- The field theoretic regime is achieved for,

$$\frac{sL_s^2}{N^2} < 1. \quad (1.8)$$

In this case, we can use both scattering equations and the KLT prescription and the amplitudes computed within field and string theory agree.

On the above perturbative information we need to superimpose the non-perturbative input coming from the black hole's corpuscular portrait. This gives the following three regimes

- The regime of black hole formation at the critical point $\lambda = 1$,

$$\frac{sL_P^2}{N} = 1. \quad (1.9)$$

In this regime the perturbative amplitudes must be supplemented by the information about the exponential multiplicity of the final states.

- Sub-critical regime $\lambda < 1$,

$$\frac{sL_P^2}{N} < 1. \quad (1.10)$$

In this regime the Bogoliubov degeneracy is lifted and the perturbative amplitudes are better and better applicable as we move towards $\lambda \rightarrow 0$.

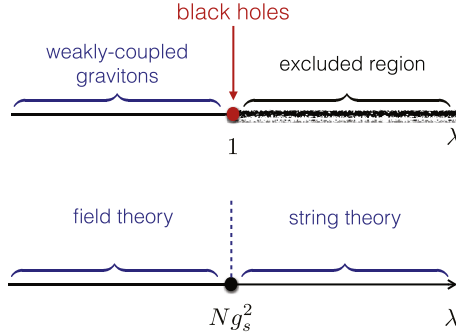


Fig. 2. Graviton physics and interplay between field and string theory as variation of λ .

- Over-critical regime $\lambda > 1$,

$$\frac{sL_P^2}{N} > 1. \quad (1.11)$$

In this regime, the N -graviton states have very high Liapunov exponents and imaginary Bogoliubov frequencies and therefore are not legitimate final states.

The overlap of the perturbative and non-perturbative information is summarized in the plots in Figs. 2 and 3, where various perturbative and non-perturbative regimes are plotted on the λ axis. In order to allow the variation of λ , we vary N while keeping s , L_P and g_s fixed. In this way, we scan all possible multi-graviton final states of the desired kinematical regime for the fixed center of mass energy.

The first plot in Fig. 2 describes various regions on the λ -axis from purely field-theoretic perspective of non-perturbative N -graviton physics. The second plot, describes the interplay between the string-theoretic and field-theoretic domains from purely perturbative perspective. Notice, that after translating (1.7) in terms of the string coupling, the transitional point between the string and field-theory regimes is marked by $\lambda = Ng_s^2$. This makes a nice physical sense. Specifically, the stringy regime becomes important when the gravitational coupling between the constituents becomes weaker than the string coupling. The same plot indicates the obvious point, that for any fixed value of g_s , and for sufficiently large N the field theory regime becomes a good approximation.

Fig. 3 described the result of superimposing the above two perturbative and non-perturbative plots for two different cases, $Ng_s^2 > 1$ and $Ng_s^2 < 1$ respectively. In the first case, there is a region, $1 < \lambda < Ng_s^2$, in which on one hand unitarity is perturbatively violated in field theory and on the other hand perturbative string theory corrections are not effective for restoring it. In this domain unitarity is restored by non-perturbative collective N -graviton physics described above, which excludes this region as unphysical.

In the second plot in Fig. 3, the choice of parameters is such that perturbative string theory and field theory amplitudes crossover without violating unitarity for any λ . Of course, for fixed g_s and L_P , if we allow N to grow, the situation of the first plot is sooner or later achieved. In other words, for sufficiently high s , there is always a window of λ for which some perturbatively-allowed would-be final states violate unitarity. The unitarity in this window is only restored by classicalization, which excludes it from the Hilbert space due to non-perturbative corpuscular physics of the N -graviton system.

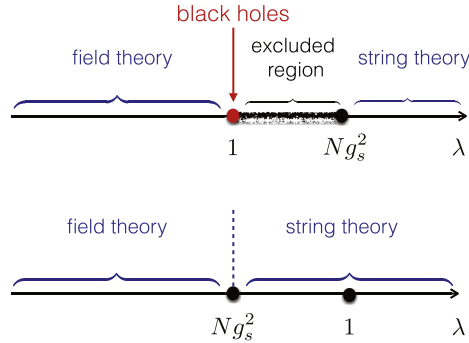


Fig. 3. Perturbative and non-perturbative regimes as variation of λ .

1.4. Outline

This work is organized as follows. In the next section we will specify some of the technical steps concerning the computation of the gravitational scattering amplitudes for a large number of gravitons in the final state in a specific high energy regime called classicalization regime. Together with other high energy limits the classicalization regime will be defined in Section 3. In Section 4 we will present the calculation of the gravitational scattering amplitude exhibiting the details to determine the on-shell scattering amplitude and derive in this way for a large number of N an explicit expression for the transition probability of two particles into $N - 2$. In Section 5 an analogous computation is presented for the case of N -point open and closed string scattering amplitudes in the high energy limit of classicalization. In particular, we show how the relevant combinatorial factor can be derived by using the methods of scattering equations yielding the correct result for the field theory factor appearing in the previous section. In the remaining sections we will provide the interpretation of the results for the scattering amplitudes in the light of the corpuscular picture of black holes together with the idea of classicalization. More concretely, in Section 6 we extend the discussion of Subsections 1.2 and 1.3 by explaining in which way the full gravitational scattering amplitude is built as an overlap between the perturbative N -graviton amplitude, calculated in Sections 4 and 5, and the non-perturbative projection between the N -graviton state and the black hole state, which is provided by the entropy factor e^N . This discussion about the perturbative insights into non-perturbative physics is continued in Section 7, where we also compare the gravitational case with the scalar ϕ^4 theory. Finally, Section 8 contains the outlook of the paper, including also a speculation about the planar limit of gauge theories with a large number of colors N_c and the limit of a large number N of gravitons, considered in this paper.

2. Recap of technical steps

As discussed above, the basic idea of this paper is to describe some key aspects of classicalization and black hole formation in the light of high-energy scattering amplitudes with a large number N of soft, elementary quanta in the final state. As already indicated, the phenomenon of classicalization implies a particular high-energy limit of the corresponding N -particle scattering amplitudes. Specifically we shall analyze the tree-level scattering in the kinematics of $2 \rightarrow (N - 2)$ particles, with N being arbitrarily large and with the high center of mass energy

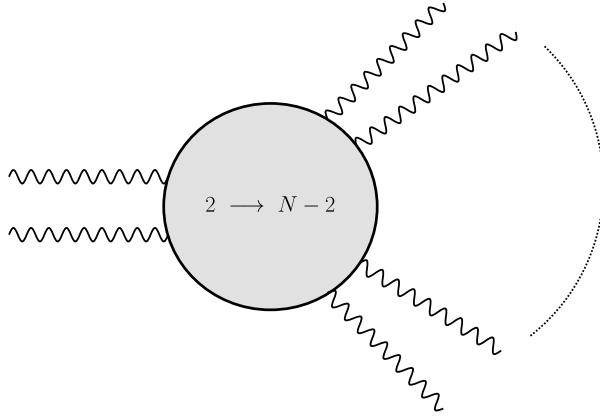


Fig. 4. Tree level scattering of 2 into $N - 2$ particles. The blob can be thought of as the sum over all Feynman diagrams at tree level.

\sqrt{s} uniformly distributed over the $N - 2$ particles in the final state (Fig. 4). We will call this particular kinematical limit the “classicalization” or eikonal Regge limit.

Since it is quite involved to compute the large N field theory amplitudes via standard Feynman diagrams we will employ novel methods in amplitude technology called on-shell methods which were developed over the recent years. See [15] and references therein for an overview of the vast progress. In addition, we make profit of deriving tree-level amplitudes by the scattering equations [9]. Concretely, we shall perform the following computations with the following main new results:

- Field theory gravity amplitudes in the classicalization regime: We first compute the $2 \rightarrow (N - 2)$ graviton amplitudes for arbitrary N in the eikonal Regge high-energy kinematics. In order to derive these amplitudes we use a version of the KLT relations for so-called maximally helicity violating (MHV) graviton amplitudes. These are amplitudes with two negative-helicity gravitons and the rest positive. The scattering equations allow us to fix the combinatorial factors of these amplitudes. From these amplitudes we extract key information about the underlying unitarization mechanism, based on the dominance of this kinematics, as well as on the perturbative suppression factors. These perturbative results provide a strong support both to the physics picture of unitarization by black hole formation as well as to the microscopic picture of black holes as bound states of gravitons (Fig. 5). This picture is completed once we superimpose these perturbative results with non-perturbative information derived from the many body physics of graviton condensates.
- Secondly we shall compute the high-energy open/closed string tree level scattering amplitudes for arbitrary (large) number of external legs. Furthermore, we will compare the string amplitudes with field theory amplitudes and discuss the classicalization regime in both cases. In particular, for fixed \sqrt{s} , the two agree for sufficiently large N . However, depending on the value of the string coupling, intermediate domains of N are possible when either perturbative stringy effects as Reggeization or non-perturbative field theory black hole regimes dominate and exclude certain regions. One generic observation is that the regions that are not unitary in perturbative treatment are cut-out by non-perturbative corpuscular black hole physics. On the basis of the concrete form of the string amplitudes we shall make some remarks about

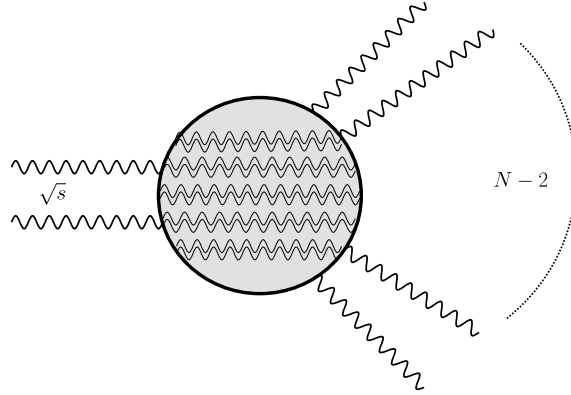


Fig. 5. Production of a black hole and decay into $N - 2$ soft quanta each with momenta $\sim \frac{\sqrt{s}}{N-2}$. The circle with the wiggly double lines depicts the Bose–Einstein condensate nature of the black hole.

some hidden color kinematics duality that at the threshold of black hole formation appears to be reminiscent of the well known gauge–gravity duality.

3. High-energy kinematical regimes

In this section we first review the various high-energy limits and their relevance. Generically, for N particle scattering amplitudes there are $\frac{1}{2}N(N - 3)$ kinematic invariants

$$s_{ij} \equiv s_{i,j} = (k_i + k_j)^2 = 2 k_i k_j . \quad (3.1)$$

However, the number of independent invariants depends on the number of space–time dimensions D under consideration. The number of independent Mandelstam variables will be reduced due to Gram determinant relations [16]. Depending on the number of dimensions D and particles N , the number of these constraints is given by

$$\#\text{constraints} = \frac{1}{2}(N - D)(N - D - 1) \quad (3.2)$$

reducing the number of independent invariants to

$$\#\{s_{ij}^{\text{indep}}\} = N(D - 1) - \frac{D(D + 1)}{2} . \quad (3.3)$$

In the sequel, however, we shall not be concerned with this issue and our results are independent of this number.

3.1. High-energy limits

For the high-energy limits of the four-point amplitude there are two regions of interest, which in this subsection we will review shortly. The kinematic invariants (3.1) for $N = 4$ are given by

$$s = (k_1 + k_2)^2 , \quad t = (k_1 + k_3)^2 , \quad u = (k_1 + k_4)^2 , \quad (3.4)$$

with $s + t + u = 0$. In the four-point scattering case we have the following relations

$$\begin{aligned}
s &= -E^2, \\
t &= E^2 \sin^2 \frac{\theta}{2}, \\
u &= E^2 \cos^2 \frac{\theta}{2},
\end{aligned} \tag{3.5}$$

with E the center-of-mass energy and θ being the angle between the external momenta k_1 and k_3 (center-of-mass scattering angle).

3.1.1. Regge limit

The Regge limit (also known as small fixed angle regime) is defined as

$$s \gg |t| \gg \Lambda \quad \text{with} \quad \left| \frac{s}{t} \right| \rightarrow \infty, \tag{3.6}$$

where Λ is some scale (usually that of QCD). In this regime scattering amplitudes of Yang–Mills and gravity field theory amplitudes exhibit a power-like behavior $\sim s^{\alpha(t)}$ with the exponent usually called the *Regge slope*. In the usual treatment, this slope is larger than unity meaning that the amplitude is not unitary at high energies. In order to unitarize the high-energy behavior of the amplitudes in e.g. Yang–Mills it was found that one has to take into account so-called multi-pomeron exchanges which are basically resummed all-loop information of certain ladder-type diagrams. In what follows we shall not touch this unitarization problem appearing in multi-Regge kinematics.

3.1.2. Hard scattering limit

The hard scattering limit (fixed finite angle regime) is the high-energy domain (ultra-high-energy limit) where all kinematic invariants become large while their ratios remain fixed. In the four-point scattering case this limit is defined by:

$$s, t \rightarrow \infty \quad \text{with} \quad \left| \frac{s}{t} \right| \sim \left| \frac{s}{u} \right| \sim \left| \frac{u}{t} \right| = \text{fixed}. \tag{3.7}$$

3.2. High-energy limit of four-point field-theory amplitudes

For the scattering $p_1 + p_2 \rightarrow p_3 + p_4$ of particles of different mass the differential cross section in the CM frame is given by a sum over all spins or helicities of scattering subamplitudes $A(1, 2, 3, 4)$

$$\frac{d\sigma}{d\Omega} \Big|_{CM} = \frac{1}{64\pi^2 E^2} \frac{|\vec{p}_3|}{|\vec{p}_1|} \sum_{\text{helicities}} |A(1, 2, 3, 4)|^2, \tag{3.8}$$

with E the CM energy. For $e^+e^- \rightarrow \mu^+\mu^-$ the sum over the matrix elements becomes $(p_{ij} = 2p_i p_j)$

$$\begin{aligned}
&\sum_{\text{helicities}} |A(1_e, 2_e, 3_\mu, 4_\mu)|^2 \\
&= \frac{2e^4}{(p_1 + p_2)^2} \left[p_{13} p_{24} + p_{14} p_{23} + 2m_\mu^2 p_{12} + 2m_e^2 p_{34} + 8 m_e^2 m_\mu^2 \right],
\end{aligned} \tag{3.9}$$

with the electron m_e and muon mass m_μ . In the ultra-high-energy limit $m_e, m_\mu = 0$, with $|\vec{p}_1| = |\vec{p}_3| = \frac{1}{2}E$ the cross section (3.8) becomes [17]

$$\left. \frac{d\sigma}{d\Omega} \right|_{CM} = \frac{e^4}{32\pi^2 E^2} \frac{t^2 + u^2}{s^2} = \frac{e^4}{64\pi^2 E^2} (1 + \cos^2 \theta) \quad (3.10)$$

in terms of the quantities (3.5). On the other hand, Rutherford scattering $e^- p^+ \rightarrow e^- p^+$ is obtained by using the corresponding t -channel matrix element

$$\begin{aligned} & \sum_{\text{helicities}} |A(1_e, 3_p, 2_e, 4_p)|^2 \\ &= \frac{2e^4}{(p_1 - p_3)^2} \left[p_{14} p_{23} + p_{12} p_{34} - 2m_p^2 p_{13} + 2m_e^2 p_{24} + 8m_e^2 m_p^2 \right], \end{aligned} \quad (3.11)$$

with the proton mass m_p . Using (3.11) and evaluating the corresponding differential cross section for elastic scattering (with $E \sim m_p$) gives the famous Rutherford scattering formula³

$$\left. \frac{d\sigma}{d\Omega} \right|_{CM} = \frac{e^4}{4\pi^2} \frac{m_e^2}{t^2} \quad (3.12)$$

in the non-relativistic limit (Born approximation) $m_e^2 + p^2 \sim m_e^2$ with $|\vec{p}_1| = |\vec{p}_3| = p$. On the other hand, in the ultra-high-energy limit $m_e^2 + p^2 \sim p^2$, i.e. $m_e \sim 0$ and $m_p \rightarrow \infty$ in terms of the quantities (3.5) we have:

$$\left. \frac{d\sigma}{d\Omega} \right|_{CM} = \frac{e^4}{64\pi^2} \frac{u}{t^2}. \quad (3.13)$$

Obviously, in (3.12) and (3.13) the propagator term $1/t^2$ dominates the high-energy behavior of the cross sections.

3.3. Eikonal constraints and high-energy limits

A special region of the space of kinematic invariants (3.1) describes the so-called eikonal constraints. In this limit two external momenta say k_1 and k_N are singled out and kinematic invariants (3.1) involving neither one of these two momenta nor two non-adjacent momenta are chosen to vanish. More precisely, the constraints on $\frac{1}{2}(N-3)(N-4)$ kinematic invariants (3.1) is

$$s_{ij} = 0, \quad i = 2, \dots, N-3, \quad i+2 \leq j \leq N-1, \quad (3.14)$$

while the remaining $2(N-3)$ invariants

$$\begin{aligned} s_{1j} &\neq 0, \quad j = 2, \dots, N, \quad s_{l,N} \neq 0, \quad l = 2, \dots, N-1 \\ s_{i,i+1} &\neq 0, \quad i = 2, \dots, N-2 \end{aligned} \quad (3.15)$$

are left free. E.g. we have:

$$\begin{aligned} N=5 : & s_{24} = 0, \\ N=6 : & s_{24} = 0, s_{25} = 0, s_{35} = 0, \\ N=7 : & s_{24} = 0, s_{25} = 0, s_{26} = 0, s_{35} = 0, s_{36} = 0, s_{46} = 0, \\ & \vdots \end{aligned} \quad (3.16)$$

³ Alternatively, in the non-relativistic limit the elastic cross section $\left. \frac{d\sigma}{d\Omega} \right|_{CM} = \frac{1}{64\pi^2 m_p^2} |A|^2$ can be approximated by the t -channel matrix element $A = (-2em_p) \frac{1}{t} (2em_e)$ also yielding (3.12) [18].

As we shall see in Section 5 in this eikonal limit the gauge and gravitational superstring amplitudes assume a form which is suited to study properties known from field-theory amplitudes in the large complex momentum limit [19]. The latter gives a relation between BCFW relations and the pomeron vertex in string theory [20], cf. also [21,22]. BCFW shifts in string amplitudes have also been studied in [23].

3.3.1. Eikonal hard scattering limit

For (3.14) in the hard scattering limit we consider the limit $s \rightarrow \infty$ for the non-vanishing invariants⁴ (3.15):

$$\begin{aligned} s_{1j} &\sim -s, \quad j = 2, \dots, N-1, \quad s_{l,N} \sim -s, \quad l = 2, \dots, N-1, \\ s_{i,i+1} &\sim s, \quad i = 2, \dots, N-2, \quad s_{1N} \sim s. \end{aligned} \quad (3.17)$$

3.3.2. Eikonal Regge limit

For (3.14) the non-vanishing invariants (3.15) can be parameterized as follows

$$s_{ij} = (k_i + k_j)^2 \sim \begin{cases} s, & i, j \in \{1, N\}, \\ -\epsilon s, & i \in \{1, N\}, j \notin \{1, N\}, \\ \epsilon^2 s, & i, j \notin \{1, N\}, \end{cases} \quad (3.18)$$

with some s and ϵ . The (adjacent) eikonal Regge limit is obtained for small ϵ and $s \rightarrow \infty$. This limit corresponds to a regime, where one subset of momenta (the adjacent momenta k_1 and k_N) is much greater than a given scale ϵ , while the other subset (all remaining momenta k_i , $i \neq 1, N$) is negligible compared to this scale, i.e. $k_i \sim \epsilon s^{1/2}$ and $k_1, k_N \sim s^{1/2}$.

3.4. Classicalization high-energy limit

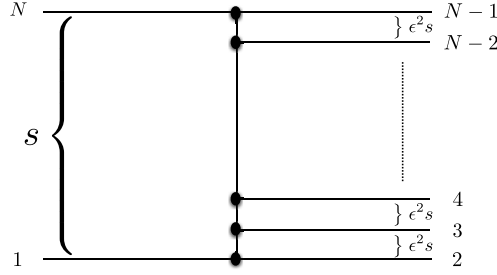
This is the limit in which we want to analyze the classicalization behavior of the scattering amplitudes. This behavior should manifest itself by preferring amplitudes with a greater number of external legs (and vice versa suppressing amplitudes with smaller number of external legs). Take particles 1 and N to be incoming with center of mass energy $s := s_{1N} = (k_1 + k_N)^2$ so that their momenta will be proportional to $\sqrt{s}/2$. Accordingly, we define the other $N-2$ particles to be outgoing with momenta proportional to $-\sqrt{s}/(N-2)$. This kinematical choice will lead to a particular scaling of momentum invariants given by

$$s_{ij} = (k_i + k_j)^2 \sim \begin{cases} s, & i, j \in \{1, N\}, \\ -\frac{s}{N-2}, & i \in \{1, N\}, j \notin \{1, N\}, \\ \frac{s}{(N-2)^2}, & i, j \notin \{1, N\}. \end{cases} \quad (3.19)$$

Note that (3.18) and (3.19) are closely related by identifying

$$\epsilon = \frac{1}{N-2}. \quad (3.20)$$

⁴ We can always find finite parameters to meet these conditions, e.g. for $N=5$ we may choose $s_{12} = s_{45} = -\frac{1}{2}s$, $s_{23} = s_{34} = \frac{3}{2}s$ and $s_{51} = s_{23} + s_{34} = 3s$.

Fig. 6. Kinematical configuration of the N -point amplitude.

The kinematical configurations (3.18) and (3.19) of the N -point amplitude are depicted in Fig. 6.

4. Field theory perspective

4.1. Large N field theory amplitudes in the high-energy classicalization limit

In this section the high-energy behavior of field theory scattering amplitudes of pure Yang–Mills theory and pure gravity will be investigated. It will be examined whether one can find hints for classicalization in the high-energy behavior of these amplitudes.

The starting point for the field theory computations in gravity are the Kawai–Lewellen–Tye (KLT) relations. These relations express an N -point graviton scattering amplitude at tree level $\mathcal{M}_{FT}(1, \dots, N)$ in terms of sums of products of Yang–Mills N -point tree amplitudes A_{YM} . This was first proven in string theory by exploiting the monodromy properties of the closed string world-sheet [10] and later cast into compact form in [24,25]. Along these lines the N -graviton scattering *field* theory tree amplitude becomes

$$\begin{aligned} \mathcal{M}_{FT}(1, \dots, N) = & (-1)^{N-3} \kappa^{N-2} \sum_{\sigma, \gamma \in S_{N-3}} A_{YM}(1, \sigma(2, \dots, N-2), N-1, N) \\ & \times S[\gamma(2, \dots, N-2) | \sigma(2, \dots, N-2)]_{N-1} \\ & \times A_{YM}(1, N-1, \gamma(2, \dots, N-2), N), \end{aligned} \quad (4.1)$$

where the sum runs over the permutations of the $N-3$ elements of the sets γ and σ . Above κ is the gravitational coupling constant with $\kappa^2 = 16\pi G_N$. The function $S[\dots | \dots]$ is called *momentum kernel* or *KLT kernel* and is defined via the Mandelstam variables (3.1) as

$$S[i_1, \dots, i_k | j_1, \dots, j_k]_P = \prod_{t=1}^k \left(s_{i_t, P} + \sum_{q>t}^k \theta(i_t, i_q) s_{i_t, i_q} \right). \quad (4.2)$$

Here, P is a lightlike reference momentum. The function θ is defined such that

$$\theta(i_a, i_b) = \begin{cases} 0, & \text{if } i_a \text{ sequentially comes before } i_b \text{ in } \{j_1, \dots, j_k\} \\ 1, & \text{otherwise.} \end{cases} \quad (4.3)$$

Note that while the gravity amplitude above is only manifestly S_{N-3} invariant, in fact it can be shown to be completely permutation symmetric.

4.1.1. Hard scattering limit

In the hard scattering limit, which was defined in Eq. (3.7) for four points, the behavior of field theory amplitudes is well known. Define $s = (k_1 + k_N)^2$ to be the center of mass energy. By counting mass dimensions one finds that an N -point Yang–Mills amplitude behaves as

$$A_{YM} \sim s^{\frac{4-N}{2}}, \quad (4.4)$$

i.e. it displays a power-fall off such that the amplitude decreases as the energy increases. In this way unitarity at tree level will not be violated. In contrast, an N -point gravity amplitude displays the following behavior

$$\mathcal{M}_{FT} \sim \kappa^{N-2} s. \quad (4.5)$$

In other words, the N -graviton amplitude grows monotonically as s increases. Alternatively, the high-energy behavior of the graviton amplitude can also be seen from the KLT formula (4.1) taking into account that each of the entries of the momentum kernel behave as

$$S[\sigma(2, \dots, N-2) | \gamma(2, \dots, N-2)]_P \sim s^{N-3} \quad (4.6)$$

in the hard scattering limit.

4.1.2. Classicalization of high-energy limit

Next, let us analyze whether we can find indications of classicalization in the high-energy limit of graviton scattering amplitudes utilizing (4.1). The classicalization of high-energy limit was already defined in Eq. (3.19).

Consider first Yang–Mills amplitudes in the high-energy limit as they are a building block of gravity amplitudes and so will be needed later. For simplicity, let us restrict ourselves to the case of MHV amplitudes, i.e. amplitudes with two particles of negative helicity and the other ones with positive helicity. In this case, the Yang–Mills amplitudes take a particularly simple form in four dimensions. In standard spinor helicity notation, it is given by [26]

$$A_{YM}(1^+, \dots, i^-, \dots, j^-, \dots, N^+) = \frac{\langle i j \rangle^4}{\langle 1 2 \rangle \langle 2 3 \rangle \dots \langle N-1 N \rangle \langle N 1 \rangle}. \quad (4.7)$$

The spinor helicity brackets are basically square roots of Mandelstam invariants and their precise relation is given by [27]

$$\langle i j \rangle = \sqrt{|s_{ij}|} e^{i\phi_{ij}} \quad (4.8)$$

with

$$\cos(\phi_{ij}) = \frac{k_i^1 k_j^+ - k_j^1 k_i^+}{\sqrt{|s_{ij}| k_i^+ k_j^+}}, \quad \sin(\phi_{ij}) = \frac{k_i^2 k_j^+ - k_j^2 k_i^+}{\sqrt{|s_{ij}| k_i^+ k_j^+}}, \quad \text{and} \quad k^\pm = k^0 \pm k^3. \quad (4.9)$$

In the classicalization kinematics region one can straight-forwardly find the scaling of the Yang–Mills amplitude by applying (4.8) and (3.19) to (4.7). Notice that one has to distinguish three cases: either both incoming particles have negative (positive) helicity or one particle might be positive and the other negative. One finds for these three cases (suppressing the absolute value)

$$A_{YM}(i^-, j^-) \sim s^{\frac{4-N}{2}} f(\phi) \times \begin{cases} (N-2)^{N-2}, & i, j \in \{1, N\}, \\ (N-2)^{N-4}, & i \in \{1, N\}, j \notin \{1, N\}, \\ (N-2)^{N-6}, & i, j \notin \{1, N\}, \end{cases} \quad (4.10)$$

with $f(\phi)$ shorthand for the phase-factors. In general $f(\phi)$ will be a very complicated function. The difference in scaling above can be easily understood: it originates in the numerator of the MHV expression (4.7). The high-energy scaling of $\langle i j \rangle^4$ depends on which particles are chosen to have negative helicity. The denominator, on the other hand, just encodes the pole structure of the amplitude which is independent of helicities. In other words, the denominator scales in the same way in all three cases. Note that the phase factors (4.9) do not scale with $N - 2$ since the scaling cancels.

Let us now turn to gravity amplitudes. To use the KLT relation (4.1) we need to know how the momentum kernel scales in this regime: its entries roughly scale as

$$S[\gamma(2, \dots, N-2), \sigma(2, \dots, N-2)]_{N-1} \sim \left(\frac{s}{(N-2)^2} \right)^{N-3} \quad (4.11)$$

as all particle labels involved in the momentum kernel belong to outgoing particles. Combining this with the scaling of the Yang–Mills amplitudes (4.10) via (4.1), leads to the following scaling for the gravity MHV amplitude

$$\mathcal{M}_{FT}(i^-, j^-) \sim \kappa^{N-2} \tilde{C}(N) s \times \begin{cases} (N-2)^2, & i, j \in \{1, N\}, \\ (N-2)^{-2}, & i \in \{1, N\}, j \notin \{1, N\}, \\ (N-2)^{-6}, & i, j \notin \{1, N\}, \end{cases} \quad (4.12)$$

where the function $\tilde{C}(N)$ is a complicated double sum over the phase factors arising when one rewrites the spinor brackets in terms of Mandelstam invariants and sums over the different permutations in the KLT sum. Unfortunately, it is very involved to evaluate this sum by using the techniques under consideration. However, fortunately as we shall see in Subsection 5.3.3, the factor can be computed in a straightforward way by making profit of the scattering equations [9]. It will be shown that (see Eq. (5.81))

$$\tilde{C}(N) = (N-1)! \quad (4.13)$$

Hence, in the classicalization regime the scaling of the gravity field theory amplitude at tree level should be given by

$$\mathcal{M}_{FT}(i^-, j^-) \sim \kappa^{N-2} (N-1)! s \times \begin{cases} (N-2)^2, & i, j \in \{1, N\}, \\ (N-2)^{-2}, & i \in \{1, N\}, j \notin \{1, N\}, \\ (N-2)^{-6}, & i, j \notin \{1, N\}. \end{cases} \quad (4.14)$$

Note that we could have chosen *any* two particles to be incoming in the analysis above. The only difference would be in the scaling of the momentum kernel: it would scale more complicated as a function of N since the kinematic invariants would not all scale homogeneously like $\frac{s}{N-2}$. Naively, the behavior of the gravity amplitude would then also be more complicated as a consequence. However, due to Bose symmetry the overall result is independent of the choice of incoming momenta and the seemingly more complex scaling will cancel in the sum over terms in (4.1) giving back Eq. (4.14).

The next step is to go from the on-shell scattering amplitude above to the physical (dimensionless) transition probability of two particles scattering into $N - 2$, i.e. to $|\langle 2|S|N-2 \rangle|^2$ (cf. Eq. (1.1)). In order to do so, one has to multiply \mathcal{M}_{FT} by the values of the outgoing and incoming momenta and take into account that the final states are identical. This amounts to dividing out by a factor of $(N-2)!$. Doing so one arrives at (indices suppressed)

$$|\langle 2|S|N-2\rangle|^2 = \frac{1}{(N-2)!} \left(\prod_{i=2}^{N-1} p_i p_1 p_N \mathcal{M}_{FT} \right)^2. \quad (4.15)$$

Finally, by identifying κ with L_P , taking $N \gg 1$, and remembering that $p_{out} \sim \frac{\sqrt{s}}{N}$ and $p_{in} \sim \sqrt{s}$ one arrives at Eq. (1.1)

$$|\langle 2|S|N\rangle|^2 \sim \left(\frac{L_P^2 s}{N^2} \right)^N N!. \quad (4.16)$$

Note that for large N we do not need to make the distinction between the three different cases anymore as they all scale the same. Moreover, the result (4.16) holds for both MHV and NMHV scattering since in Subsection 5.3.3 for the derivation of the factor (4.13) no specific helicity configuration is assumed.

As already advertised in the introduction this physical amplitude starts to unitarize for N given by $s = NM_P^2$. Moreover, the amplitude at this kinematical unitarity threshold is suppressed by e^{-N} anticipating at this perturbative level exactly the suppression factor that can be compensated by the entropy of a black hole with mass $M = \sqrt{s} = \sqrt{N}M_P$.

5. String theory perspective

The high-energy behavior (i.e. energies much larger than the string scale M_s) of perturbative string amplitudes is rather different than that of field-theory amplitudes. While the high-energy behavior in field-theory (3.12) furnishes a power fall-off behavior (in the kinematic invariant t) string theory exhibits an exponential fall-off. This opens the possibility to investigate the unitarity properties of the field theory amplitudes at high (or even trans-Planckian) energies within the framework of perturbative string amplitudes. The latter take into account effects from higher spin and black hole states, which may play a crucial role in the unitarization of the amplitude. In fact, higher spin states are vital for the consistency of weakly coupled gravity theories in the tree-level approximation with higher derivative corrections at intermediate energies (energies, at which the theory is still weakly coupled, but sensitive to higher derivative corrections) [28].

In string theory besides the string scale M_s , which is related to the string length L_s and string tension α' as

$$\alpha' \sim L_s^2 = M_s^{-2} \quad (5.1)$$

there is a string loop expansion parameter $g_s \sim e^\Phi$ (or closed string coupling constant $g_{closed} = g_s = \kappa_4/\alpha'^{1/2}$) controlling higher genus string effects with the dilaton field Φ . The latter determines the YM coupling $g_{YM} \sim e^{\Phi/2}$, which enters as the open string coupling $g_{open} = g_{YM}$. The relation between the Planck and string mass is given by $M_P = g_s^{-1} M_s$, i.e.

$$L_P = g_s L_s, \quad (5.2)$$

which in turn implies:

$$g_{closed} = g_{open}^2. \quad (5.3)$$

Hence, a small string loop expansion parameter g_s corresponds to:

$$g_s^{-2} \sim \frac{M_P^2}{M_s^2} = \frac{L_s^2}{L_P^2} \gg 1. \quad (5.4)$$

On the other hand, energies for which tree amplitudes are large correspond to $sL_p^2 \gg 1$. Hence, the legitimate kinematical regime to study high-energy string tree-level scattering is:

$$\alpha' s \gg \frac{L_s^2}{L_p^2} \gg 1. \quad (5.5)$$

The string theory scattering at fixed angles and large energy is determined by a classical solution, i.e. the high-energy scattering in string theory becomes semi-classical: the two-dimensional string world-sheet stretches at long distance and the classical solution minimizes its area. The world-sheet string integrals are dominated by saddle points.

The high-energy fixed-angle behavior of open string tree-level scattering was first investigated by Veneziano [29]. The high-energy behavior of four-point string scattering was then thoroughly analyzed by Gross, Mende and Manes [7], see also [6] for some complementary work. In this section for tree-level and a specific kinematical region (specified in Section 3) we shall generalize these results to an arbitrary number of external string states. Further interesting aspects of high-energy behavior in string theory have been discussed in [30].

Beyond the Born approximation (cf. e.g. Eq. (3.12)) for smaller impact parameter (larger t) the eikonal scattering regime is reached where ladder diagrams (and crossed ladder diagrams) in the s -channel become important⁵ [31]. In field theory the latter can be derived from or matched to available perturbative higher-loop supergravity computations [32]. On the other hand, in string theory it has already been argued in [7] that the higher genus high-energy behavior can also be approximated by one saddle point showing the universal exponential behavior (5.14). Hence, we believe, that our tree-level high-energy results may at least qualitatively also describe effects from higher genus string world-sheet topologies.

5.1. High-energy limit of four-point open and closed superstring amplitude

The color ordered open superstring four-point tree subamplitude reads

$$\mathcal{A}(1, 2, 3, 4) = g_{YM}^2 A_{YM}(1, 2, 3, 4) F_4, \quad (5.6)$$

with the SYM subamplitude $A_{YM}(1, 2, 3, 4)$ and the string form factor:

$$F_4 = \frac{\Gamma(1 + \alpha' s) \Gamma(1 + \alpha' u)}{\Gamma(1 + \alpha' s + \alpha' u)}. \quad (5.7)$$

The kinematic invariants are given in (3.4). In the hard scattering limit, i.e. for $\alpha' \rightarrow \infty$ the form factor (5.7) behaves as

$$F_4 \sim (2\pi\alpha')^{1/2} \left| \frac{su}{t} \right|^{1/2} \exp \left\{ \alpha' (s \ln |s| + t \ln t + u \ln u) \right\}, \quad (5.8)$$

to be contrasted with the corresponding field-theory expression (3.12) in the Born approximation. Eventually, with the fixed-angle parameterization (3.5) Eq. (5.8) can be cast into:

$$F_4 \sim (2\pi)^{1/2} \frac{E}{M_{\text{string}}} \cot \frac{\theta}{2} \exp \left\{ \frac{E^2}{M_{\text{string}}^2} \left(\sin^2 \frac{\theta}{2} \ln \sin^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2} \ln \cos^2 \frac{\theta}{2} \right) \right\}. \quad (5.9)$$

⁵ Some interesting connections between ladder diagrams and the picture of graviton Bose–Einstein condensates have recently been presented in [11].

The expression (5.8) follows by applying the Laplace method. The latter approximates the integral for a function f with a unique global maximum x_0 inside the integration region as:

$$\int_a^b g(x) \exp\{\alpha' f(x)\} dx \sim \sqrt{\frac{2\pi}{\alpha' |f''(x_0)|}} \left[g(x_0) + \mathcal{O}(\alpha'^{-1}) \right] \exp\{\alpha' f(x_0)\}. \quad (5.10)$$

Then, the result (5.8) follows from rewriting (5.7) as

$$F_4 = \alpha' s \int_0^1 dx x^{\alpha' s - 1} (1-x)^u = \alpha' s \int_0^1 dx x^{-1} \exp\{\alpha' s \ln x + \alpha' u \ln(1-x)\} \quad (5.11)$$

and applying Laplace's method (5.10) to approximate the latter. The stationary point $x_0 = -\frac{s}{t}$ (with $s < 0$) of the integrand of (5.11) follows from solving the equation:

$$\frac{s}{x} - \frac{u}{1-x} = 0. \quad (5.12)$$

With this information and the formula (5.10) we arrive at (5.8).

Furthermore, the closed-string four-point amplitude describing four graviton scattering is given by

$$\mathcal{M}(1, 2, 3, 4) = \kappa^2 |A_{YM}(1, 2, 3, 4)|^2 (\alpha' s)^2 \oint_{\mathcal{C}} d^2 z |z|^{\alpha' s/2-2} |1-z|^{\alpha' u/2} \quad (5.13)$$

with the SYM subamplitude $A_{YM}(1, 2, 3, 4)$ and gravitational coupling constant κ . Performing a saddle point approximation in the complex plane (w.r.t. polar coordinates) yields:

$$\begin{aligned} \mathcal{M}(1, 2, 3, 4) &\sim \kappa^2 |A_{YM}(1, 2, 3, 4)|^2 \\ &\times 4\pi\alpha' \left| \frac{su}{t} \right| \exp \left\{ \frac{\alpha'}{2} (s \ln |s| + t \ln t + u \ln u) \right\}. \end{aligned} \quad (5.14)$$

It is interesting to note, that Eq. (5.14) essentially is the square of the open string case (5.8) subject to a rescaling of the string tension α' as $\alpha' \rightarrow \alpha'/4$. Hence, qualitatively there is no difference between the high-energy behavior of the open and closed superstring tree-level amplitude. This fact becomes feasible by the single-valued projection [33,34], cf. the next subsection.

5.2. High-energy limit of N -point gauge and graviton amplitude

In this subsection for specific kinematical regions we shall derive the high-energy behavior of both open and closed superstring scattering amplitudes for an arbitrary number N of external string states.

A first glance at the high-energy behavior of the N -point open superstring amplitude can be gained by considering the Selberg integral [35]

$$\begin{aligned} S_n(\alpha, \beta, \gamma) &= \left(\prod_{i=1}^n \int_0^1 dt_i \right) \prod_{l=1}^n t_l^{\alpha-1} (1-t_l)^{\beta-1} \prod_{1 \leq i < j \leq n} |t_i - t_j|^{2\gamma} \\ &= n! \prod_{l=0}^{n-1} \frac{\Gamma(\alpha + l\gamma) \Gamma(\beta + l\gamma) \Gamma(\gamma + l\gamma)}{\Gamma(\alpha + \beta + (n+l-1)\gamma) \Gamma(\gamma)}, \end{aligned} \quad (5.15)$$

with complex parameters α, β, γ such that $\Re\alpha, \Re\beta > 0$ and $\Re\gamma > -\text{Min}\{\frac{1}{n}, \frac{\Re\alpha}{n-1}, \frac{\Re\beta}{n-1}\}$. The Selberg integral (5.15) may be thought as the straightforward multi-dimensional generalization of (5.11) to be suited to describe the N -point case. In fact, for $n = N - 3$ and the parameterization

$$\begin{aligned}\alpha &= \alpha' s_{1i} \quad , \quad \beta = \alpha' s_{i,N-1}, \quad i = 2, \dots, N-2, \\ 2\gamma &= \alpha' s_{ij} \quad , \quad 2 \leq i < j \leq N-2\end{aligned}\quad (5.16)$$

Eq. (5.15) describes a generic world-sheet disk integral involving N open strings. The integral (5.15) sums up $n!$ iterated real integrals with identical contributions. The latter corresponds to $(N-3)!$ (independent) color ordered subamplitudes. With Stirling's formula [36]

$$\begin{aligned}\Gamma(z) &= (2\pi)^{1/2} \exp\left\{\left(z - \frac{1}{2}\right) \ln z - z\right\} \\ &\times \left(1 + \frac{1}{12z} + \frac{1}{288z^2} - \dots\right) \quad , \quad \text{for } z \rightarrow \infty \text{ in } |\arg(z)| < \pi\end{aligned}\quad (5.17)$$

we may easily determine the $\alpha' \rightarrow \infty$ limit of (5.15). E.g. for $N = 5$, i.e. $n = 2$ we find the behavior:

$$\begin{aligned}S_2 &\sim (2\pi) 2^{\frac{1}{2}+2\gamma} \left(\frac{(\alpha + \beta + \gamma)(\alpha + \beta + 2\gamma)}{\alpha\beta(\alpha + \gamma)(\beta + \gamma)}\right)^{1/2} \\ &\times \exp\left\{\frac{\alpha \ln \alpha + \beta \ln \beta + \gamma \ln \gamma + (\alpha + \gamma) \ln(\alpha + \gamma) + (\beta + \gamma) \ln(\beta + \gamma)}{(\alpha + \beta + \gamma) \ln(\alpha + \beta + \gamma) + (\alpha + \beta + 2\gamma) \ln(\alpha + \beta + 2\gamma)}\right\} \\ &+ \mathcal{O}(\alpha'^{-1}) \quad .\end{aligned}\quad (5.18)$$

5.2.1. High-energy limit of N -point gauge amplitude

The open superstring N -gluon tree-level amplitude \mathfrak{A}_N describing the scattering of N gluons decomposes into a sum

$$\mathfrak{A}_N = \sum_{\Pi \in S_N/\mathbf{Z}_2} \text{Tr}(T^{a_{\Pi(1)}} \dots T^{a_{\Pi(N)}}) \mathcal{A}(\Pi(1), \dots, \Pi(N)) \quad (5.19)$$

over color ordered subamplitudes $\mathcal{A}(\Pi(1), \dots, \Pi(N))$ supplemented by a group trace in the adjoint representation. The sum runs over all permutations S_N of labels $i = 1, \dots, N$ modulo cyclic permutations \mathbf{Z}_2 , which preserve the group trace. The $(N-3)!$ independent open N -point superstring subamplitudes can be cast into the compact form [37]

$$\mathcal{A}(1, \pi(2), \dots, N-2, N-1, N) = g_{YM}^{N-2} \sum_{\sigma \in S_{N-3}} F_{\pi\sigma} A_{YM}(\sigma) \quad , \quad \pi \in S_{N-3} \quad , \quad (5.20)$$

with the $(N-3)!$ (independent) SYM subamplitudes $A_{YM}(\sigma) := A_{YM}(1, \sigma(2), \dots, N-2, N-1, N)$, $\sigma \in S_{N-3}$ and the $(N-3)! \times (N-3)!$ matrix F , whose entries $F_{\pi\sigma}$ can be expressed as [38]

$$F_{\pi\sigma} = (-\alpha')^{N-3} \sum_{\rho \in S_{N-3}} Z_{\pi}(\rho) S[\rho|\sigma] \quad , \quad (5.21)$$

with the world-sheet disk integrals specified in Eq. (5.59) and some variant of the KLT kernel (4.2)

$$S[\rho|\sigma] := S[\rho(2, \dots, N-2) | \sigma(2, \dots, N-2)] = \prod_{j=2}^{N-2} \left(s_{1,j_\rho} + \sum_{k=2}^{j-1} \theta(j_\rho, k_\rho) s_{j_\rho, k_\rho} \right), \quad (5.22)$$

with $j_\rho = \rho(j)$ and $\theta(j_\rho, k_\rho) = 1$ if the ordering of the legs j_ρ, k_ρ is the same in both orderings $\rho(2, \dots, N-2)$ and $\sigma(2, \dots, N-2)$, and zero otherwise. The matrix entries $F_{\pi\sigma}$ given in Eq. (5.21) represent generalized Euler integrals integrating to multiple hypergeometric functions [37]:

$$F_{\pi\sigma} = (-\alpha')^{N-3} \int_{D(\pi)} \left(\prod_{j=2}^{N-2} dz_j \right) \left(\prod_{i < l} |z_{il}|^{\alpha' s_{il}} \right) \left\{ \prod_{k=2}^{N-2} \sum_{m=1}^{k-1} \frac{s_{mk}}{z_{mk}} \right\}. \quad (5.23)$$

Above, the permutations $\sigma \in S_{N-3}$ act on all indices $\{2, \dots, N-2\}$ within the curly bracket. Due to conformal invariance on the world-sheet we have fixed three of the N world-sheet positions as

$$z_1 = 0, \quad z_{N-1} = 1, \quad z_N = \infty, \quad (5.24)$$

and the remaining $N-3$ positions z_i are integrated along the boundary of the disk subject to the ordering $D(\pi) = \{z_j \in \mathbf{R} \mid z_1 < z_{\pi(2)} < \dots < z_{\pi(N-2)} < z_{N-1} < z_N\}$. Furthermore, we have $z_{ij} \equiv z_{i,j} = z_i - z_j$. Integration by parts admits to simplify the integrand in (5.23). As a result the length of the sum over m becomes shorter for $k > \lfloor N/2 \rfloor$

$$F_{\pi\sigma} = (-\alpha')^{N-3} \int_{D(\pi)} \left(\prod_{j=2}^{N-2} dz_j \right) \left(\prod_{i < l} |z_{il}|^{s_{il}} \right) \times \left\{ \left(\prod_{k=2}^{\lfloor N/2 \rfloor} \sum_{m=1}^{k-1} \frac{s_{mk}}{z_{mk}} \right) \left(\prod_{k=\lfloor N/2 \rfloor + 1}^{N-2} \sum_{n=k+1}^{N-1} \frac{s_{kn}}{z_{kn}} \right) \right\}, \quad (5.25)$$

with $\lfloor x \rfloor$ the integer part of x .

In the sequel, in (5.20), without loss of generality let us concentrate on the canonical color ordering $\pi = 1$ describing the string subamplitude $\mathcal{A}(1, \dots, N)$ and work out the latter in the eikonal limit (3.14) and (3.15). By applying partial integrations w.r.t. to the world-sheet coordinates it can be evidenced, that for the case $\sigma \neq id$ all functions $F_{1\sigma}$ have one of the invariants (3.14) as prefactor. Note, that in a gluon subamplitude with canonical color ordering $\pi = 1$ the constraints (3.14) do not cause any singularities as the latter would correspond to unphysical poles. So we can safely take the limit (3.14), i.e. $F_{1\sigma} = 0$, $\sigma \neq 1$ and only $F_N := F_{11}$ is non-vanishing in the eikonal limit. As a consequence in the eikonal limit the full open superstring subamplitude (5.20) reduces to one term

$$\mathcal{A}(1, \dots, N) = g_{YM}^{N-2} F_N A_{YM}(1, \dots, N), \quad (5.26)$$

with the string form factor F_N given by:

$$F_N = (-\alpha')^{N-3} \int_{z_i < z_{i+1}} \left(\prod_{j=2}^{N-2} dz_j \right) \left(\prod_{i < l} |z_{il}|^{\alpha' s_{il}} \right) \frac{s_{12}}{z_{12}} \left(\prod_{l=1}^{N-4} \frac{s_{N-l-1, N-l}}{z_{N-l-1, N-l}} \right). \quad (5.27)$$

In the eikonal limit the kinematical factor $A_{YM}(1, \dots, N)$ of the superstring amplitude (5.26) is identical in form to that of the corresponding field theory amplitude. Hence, in the four-dimensional MHV case⁶ in Eq. (5.26) the SYM amplitude factor is given by the Parke–Taylor amplitude (4.7).

For the choice of vertex operator positions (5.24) and the parameterization

$$z_l = \prod_{i=1}^{N-l-1} x_i \quad , \quad l = 2, \dots, N-2 \quad (5.30)$$

the string form factor (5.27) becomes:

$$\begin{aligned} F_N = & \alpha' s_{12} \left(\prod_{i=1}^{N-3} \int_0^1 dx_i \right) x_{N-3}^{-1+\alpha' s_{12}} (1 - x_{N-3})^{\alpha' s_{23}} \\ & \times \prod_{l=1}^{N-4} x_l^{-\alpha' s_{N-l-1, N-l} + \alpha' \sum_{j=2}^{N-l-1} s_{1j} + s_{j, j+1}} (1 - x_l)^{-1+\alpha' s_{N-1-l, N-l}} \alpha' s_{N-l-1, N-l} . \end{aligned} \quad (5.31)$$

As a result of taking the eikonal limit, the $N-3$ world-sheet integrations become independent Euler integrals, which integrate to Beta functions

$$F_N = \frac{\Gamma(1 + \alpha' s_{12}) \Gamma(1 + \alpha' s_{23})}{\Gamma(1 + \alpha' s_{12} + \alpha' s_{23})} \prod_{l=1}^{N-4} \frac{\Gamma(1 + \alpha' x_l) \Gamma(1 + \alpha' y_l)}{\Gamma(1 + \alpha' x_l + \alpha' y_l)} , \quad (5.32)$$

with

⁶ According to [39] the rational function in the world-sheet positions has a simple representation in terms of tree diagrams. In the MHV case (with gluon 1 and N of negative helicity) the N -point open superstring subamplitude (5.20) can be expressed as [39,40]

$$\begin{aligned} \mathcal{A}(1, \dots, N) = & g_{YM}^{N-2} \frac{\langle 1N \rangle^4}{\langle 1, (N-1) \rangle \langle (N-1), N \rangle \langle N1 \rangle} \\ & \times \int \left(\prod_{j=2}^{N-2} dz_j \right) \left(\prod_{i < l} |z_{il}|^{\alpha' s_{il}} \right) \\ & \times \sum_{\sigma \in S_{N-2}} \frac{1}{z_{12}} \prod_{k=2}^{N-2} \frac{\langle N | N-1 + \dots + (k+1) | k \rangle}{\langle kN \rangle} \frac{\alpha'}{z_{k(k+1)}} , \end{aligned} \quad (5.28)$$

where the permutations σ act on the set $\{2, \dots, N-2\}$. In the above form (5.28) one easily observes the effect of taking the eikonal limit (3.14). Any of the $(N-3)$ brackets $\langle N | N-1 + \dots + (k+1) | k \rangle$ would vanish for $\sigma \neq 1$. Hence the eikonal limit of (5.29) gives

$$\begin{aligned} \mathcal{A}(1, \dots, N) = & g_{YM}^{N-2} \frac{\langle 1N \rangle^4}{\langle 1, (N-1) \rangle \langle (N-1), N \rangle \langle N1 \rangle} \\ & \times \int \left(\prod_{j=2}^{N-2} dz_j \right) \left(\prod_{i < l} |z_{il}|^{s_{il}} \right) \frac{1}{z_{12}} \prod_{k=2}^{N-2} \frac{\langle N | (k+1) | k \rangle}{\langle kN \rangle} \frac{\alpha'}{z_{k(k+1)}} , \end{aligned} \quad (5.29)$$

which can be shown to agree with (5.26) in the MHV case.

$$\begin{aligned}
 x_l &= s_{N-1-l, N-l} = (k_{N-1-l} + k_{N-l})^2, \\
 y_l &= \sum_{j=2}^{N-l-1} s_{1j} + \sum_{j=2}^{N-l-2} s_{j, j+1} = \begin{cases} (k_{N-l} + \dots + k_N)^2, & l < \lfloor \frac{N}{2} \rfloor, \\ (k_1 + k_2 + \dots + k_{N-1-l})^2, & l \geq \lfloor \frac{N}{2} \rfloor, \end{cases}
 \end{aligned} \quad (5.33)$$

for $l = 1, \dots, N-4$. E.g. we have

$$\begin{aligned}
 N=5 : x_1 &= s_{34}, \quad y_1 = s_{45} \\
 N=6 : x_1 &= s_{45}, \quad x_2 = s_{34}, \quad y_1 = s_{56}, \quad y_2 = s_{123}, \\
 &\vdots
 \end{aligned} \quad (5.34)$$

with $s_{ijl} = \alpha' (k_i + k_j + k_l)^2$. Note, that the first factor of (5.32) simply represents the four-point result (5.7). In [41] for even N a similar expression than (5.32) has been considered in describing a very restricted and constrained subset of the full kinematics of the SYM factor of (5.26).

Let us now compute the hard scattering high-energy limit $\alpha' \rightarrow \infty$ of the result (5.32). In the sequel we apply the asymptotic formula [36]

$$\Gamma(az + b) \sim (2\pi)^{1/2} \exp \left\{ \left(az + b - \frac{1}{2} \right) \ln(az) - az \right\}, \quad |\arg(z)| < \pi, \quad a > 0, \quad (5.35)$$

to find the following approximation:

$$\begin{aligned}
 \frac{\Gamma(1 + \alpha'x) \Gamma(1 + \alpha'y)}{\Gamma(1 + \alpha'x + \alpha'y)} &\sim (2\pi\alpha')^{1/2} \left(\frac{xy}{x+y} \right)^{1/2} \\
 &\times \exp \left\{ \alpha' [x \ln x + y \ln y - (x+y) \ln(x+y)] \right\}.
 \end{aligned} \quad (5.36)$$

With (5.36) we now can extract the high-energy limit of the function (5.32)

$$\begin{aligned}
 F_N &\sim (2\pi\alpha')^{\frac{N-3}{2}} \left(\frac{s_{12} s_{23}}{s_{2N}} \right)^{1/2} \exp \left\{ \alpha' (s_{12} \ln s_{12} + s_{23} \ln s_{23} + s_{2N} \ln s_{2N}) \right\} \\
 &\times \prod_{l=1}^{N-4} \left(\frac{x_l y_l}{z_l} \right)^{1/2} \exp \left\{ \alpha' (x_l \ln x_l + y_l \ln y_l + z_l \ln z_l) \right\},
 \end{aligned} \quad (5.37)$$

with:

$$\begin{aligned}
 z_l &= -x_l - y_l = - \sum_{j=2}^{N-l-1} s_{1j} + s_{j, j+1} = \sum_{j=2}^{N-l-1} s_{jN} + \sum_{j=2}^{N-l-2} s_{j, j+1} \\
 &= \begin{cases} (k_1 + k_{N-l} + \dots + k_{N-1})^2, & l < \lfloor \frac{N}{2} \rfloor, \\ (k_2 + \dots + k_{N-1-l} + k_N)^2, & l \geq \lfloor \frac{N}{2} \rfloor, \end{cases}
 \end{aligned} \quad (5.38)$$

for $l = 1, \dots, N-4$. Note, that in deriving (5.37) we have not used the scattering equations [9]. We have extracted the limit $\alpha' \rightarrow \infty$ directly from the explicit expression (5.32). As a consequence the final result (5.26) is given by a single term.

There are two different situations to be discussed. The latter correspond to the two regimes (1.7) and (1.8), respectively.

Case (i) $\frac{\sqrt{s}}{N} > M_s$

For finite N this case is met for small string mass $M_s \rightarrow 0$ (i.e. $\alpha' \rightarrow \infty$) or large momenta $s \rightarrow \infty$. Then (5.37) can be used to approximate the string form factor (5.32). With (3.20) (i.e. finite ϵ) for this region all invariants of the eikonal parameterization (3.18) are of the same order

and can be approximated by (3.17). For the parameterization (3.17), i.e. $|s_{ij}| \sim s$ we roughly have

$$\begin{aligned} x_l &\sim s, \quad y_l \sim -s, \\ z_l &\sim -s, \quad l = 1, \dots, N-4, \end{aligned} \quad (5.39)$$

and the high-energy behavior $s \rightarrow \infty$ of the N -gluon form factor (5.37) behaves as:

$$F_N \sim (\alpha' s)^{\frac{1}{2}(N-3)} e^{-(N-3) \alpha' s \ln(\alpha' s)}. \quad (5.40)$$

Together with the YM behavior $A_{YM} \sim s^{-\frac{1}{2}(N-4)}$ (given in (4.4)) we obtain the following high-energy behavior $s \rightarrow \infty$ of the open superstring N -point amplitude (5.26) in the eikonal constraints (3.14) and (3.15):

$$\mathcal{A}_N \sim g_{YM}^{N-2} \alpha'^{\frac{1}{2}(N-3)} s^{1/2} e^{-(N-3) \alpha' s \ln(\alpha' s)}. \quad (5.41)$$

Case (ii) $\frac{\sqrt{s}}{N} < M_s$

Finally, for small $\epsilon \rightarrow 0$ (corresponding to $N \rightarrow \infty$) the eikonal parameterization (3.18) describes the eikonal Regge regime. In this regime some of the quantities (5.33) vanish

$$x_l \sim 0, \quad (5.42)$$

and the string form factor (5.32) becomes trivial:

$$F_N = 1. \quad (5.43)$$

Hence, in the eikonal Regge regime, the open superstring amplitude (5.26) becomes identical in form to the field-theory amplitude:

$$\mathcal{A}(1, \dots, N) = g_{YM}^{N-2} A_{YM}(1, \dots, N). \quad (5.44)$$

This fact has been conjectured for the MHV case in [21]. For the latter we recover the SYM result (4.10):

$$\mathcal{A}_N \sim g_{YM}^{N-2} \left(\frac{s}{(N-2)^2} \right)^{\frac{1}{2}(4-N)} f(\phi) \times \begin{cases} (N-2)^2, & i, j \in \{1, N\}, \\ 1, & i \in \{1, N\}, j \notin \{1, N\}, \\ (N-2)^{-2}, & i, j \notin \{1, N\}. \end{cases} \quad (5.45)$$

The eikonal Regge regime corresponds to a limit in which the positions z_1, z_N of string vertex operators $V(z_1), V(z_N)$ are close to each other and generate a pomeron vertex operator [20].

Note, that the two results (5.41) and (5.45) represent two different high-energy limits: while in (5.41) for finite N the large s (or large α') limit is taken, in (5.45) for infinite N ($\epsilon = 0$) the large s limit is considered.

5.2.2. High-energy limit of N -point graviton amplitude

Let us now move on to the closed string N -point amplitude describing the scattering of N gravitons. A useful way of expressing the latter has recently been presented in the works [33,34] in the form:

$$\mathcal{M}(1, \dots, N) = (-1)^{N-3} \kappa^{N-2} A_{YM}^t S_0 \text{sv}(\mathcal{A}). \quad (5.46)$$

Above \mathcal{A} is a $(N-3)!$ -dimensional vector encompassing the independent open string subamplitudes (5.20). Similarly, A_{YM} denotes an $(N-3)!$ -dimensional vector of independent SYM

subamplitudes. The map sv denotes the single-valued map, which roughly speaking projects an open string amplitude such that it describes a closed string amplitude, cf. Ref. [34] for more details. Furthermore, S_0 is a $(N-3)! \times (N-3)!$ matrix given by $S_0 = SK$, with S being the momentum kernel (4.2) and K accounting for the basis change of SYM subamplitudes $A_{YM}(1, \rho(2, \dots, N-2), N, N-1) = K_\rho^\sigma A_{YM}(1, \sigma(2, \dots, N-2), N-1, N)$.

One important observation is the fact, that in the eikonal limit (3.14) only the first element

$$\sigma_N := (SK)_{11} \quad (5.47)$$

of the matrix product SK is non-vanishing. E.g. we have:

$$\begin{aligned} \sigma_4 &= \frac{su}{t} \quad , \quad N=4 \quad , \\ \sigma_5 &= \frac{s_{12} s_{23} s_{34} s_{45}}{(s_{12} + s_{23})(s_{34} + s_{45})} \quad , \quad N=5 \quad , \\ \sigma_6 &= \frac{s_{12} s_{23} s_{34} s_{45} s_{56} s_{123}}{(s_{12} + s_{23})(s_{45} + s_{56})(s_{34} + s_{123})} \quad , \quad N=6 \quad . \\ &\vdots \end{aligned} \quad (5.48)$$

The general expression for σ_N can be given as

$$\sigma_N = \left(\frac{s_{12} s_{23}}{s_{12} + s_{23}} \right) \prod_{l=1}^{N-4} \left(\frac{x_l y_l}{x_l + y_l} \right) = (-1)^{N-3} \left(\frac{s_{12} s_{23}}{s_{2N}} \right) \prod_{l=1}^{N-4} \left(\frac{x_l y_l}{z_l} \right) \quad , \quad (5.49)$$

with x_l, y_l given in (5.33) and z_l displayed in (5.38). The function σ_N is a rational function in kinematic invariants of degree $N-3$, i.e. $\sigma_N \sim s^{N-3}$. Hence, in the eikonal limit the N -graviton amplitude (5.46) becomes

$$\mathcal{M}(1, \dots, N) = \kappa^{N-2} |A_{YM}(1, \dots, N)|^2 M_N \quad , \quad (5.50)$$

with the form factor

$$M_N = (-1)^{N-3} \sigma_N \text{sv}(F_N) \quad , \quad (5.51)$$

and the function F_N given in (5.32). To extract the high-energy limit $\alpha' \rightarrow \infty$ of the latter we use [34]

$$\text{sv} \left(\frac{\Gamma(1+x) \Gamma(1+y)}{\Gamma(1+x+y)} \right) = - \frac{\Gamma(x) \Gamma(y) \Gamma(-x-y)}{\Gamma(-x) \Gamma(-y) \Gamma(x+y)} \quad , \quad (5.52)$$

and

$$\begin{aligned} \frac{\Gamma(\alpha'x) \Gamma(\alpha'y) \Gamma(-\alpha'x - \alpha'y)}{\Gamma(-\alpha'x) \Gamma(-\alpha'y) \Gamma(\alpha'x + \alpha'y)} &\sim \exp \left\{ \alpha' \left[2x \ln x + 2y \ln y - 2(x+y) \ln(x+y) \right] \right\} \quad , \\ x, y &\rightarrow \infty \end{aligned} \quad (5.53)$$

to arrive at

$$\begin{aligned} M_N &\sim (4\pi\alpha')^{N-3} \left(\frac{s_{12} s_{23}}{s_{2N}} \right) \exp \left\{ \frac{\alpha'}{2} (s_{12} \ln s_{12} + s_{23} \ln s_{23} + s_{2N} \ln s_{2N}) \right\} \\ &\times \prod_{l=1}^{N-4} \left(\frac{x_l y_l}{z_l} \right) \exp \left\{ \frac{\alpha'}{2} (x_l \ln x_l + y_l \ln y_l + z_l \ln z_l) \right\} \quad , \end{aligned} \quad (5.54)$$

with x_l, y_l introduced in (5.33) and z_l defined in (5.38).

Again, there are two different cases to be discussed. The latter corresponds to the two regimes (1.7) and (1.8), respectively.

Case (i) $\frac{\sqrt{s}}{N} > M_s$

For finite N this case is met for small string mass $M_s \rightarrow 0$ (i.e. $\alpha' \rightarrow \infty$) or large momenta $s \rightarrow \infty$. Then, (5.54) can be used to approximate the string form factor (5.51). With (3.20) (i.e. finite ϵ) for this region all invariants of the eikonal parameterization (3.18) are of the same order and can be approximated by (3.17). For the parameterization (3.17), i.e. $|s_{ij}| \sim s$ and (5.39) the high-energy behavior $s \rightarrow \infty$ of the N -graviton form factor (5.54) behaves as:

$$M_N \sim (\alpha' s)^{N-3} e^{-\frac{\alpha'}{2}(N-3)s \ln(\alpha' s)}. \quad (5.55)$$

Together with the YM behavior $A_{YM} \sim s^{-\frac{1}{2}(N-4)}$ (given in (4.4)) we obtain the following high-energy behavior $s \rightarrow \infty$ of the closed superstring N -point amplitude (5.26) in the eikonal constraints (3.14) and (3.15):

$$\mathcal{M}_N \sim \kappa^{N-2} \alpha'^{N-3} s e^{-\frac{\alpha'}{2}(N-3)s \ln(\alpha' s)}. \quad (5.56)$$

It is interesting to note, that (5.55) essentially is the square of the open string case (5.40) subject to a rescaling of the string tension α' as $\alpha' \rightarrow \alpha'/4$. This fact becomes feasible by the single-valued projection [34]. The limit discussed above corresponds to the stringy region (1.7). It is important to note that for large s (or small string scale M_s) the high-energy limit of the N -point graviton string amplitude is exponentially suppressed in contrast to the corresponding field theory amplitude.

Case (ii) $\frac{\sqrt{s}}{N} < M_s$

Finally, in the eikonal Regge regime $\epsilon \rightarrow 0$ (corresponding to $N \rightarrow \infty$) some of the quantities (5.33) vanish (5.42). In this limit the factor (5.49) scales as $\sigma_N \sim \left(\frac{s}{(N-2)^2}\right)^{N-3}$. As a consequence the whole string form factor (5.51) becomes

$$M_N = \left(\frac{s}{(N-2)^2}\right)^{N-3}, \quad (5.57)$$

and the gravitational string amplitude (5.50) becomes identical to the corresponding field-theory amplitude (4.1):

$$\mathcal{M}(1, \dots, N) = \mathcal{M}_{FT}(1, \dots, N). \quad (5.58)$$

Note, that this limit corresponds to the region (1.8). Hence, in the eikonal Regge regime the closed superstring amplitude becomes the field-theory graviton amplitude. For the MHV case this fact has also been conjectured in [21]. In the MHV case we recover the explicit field-theory expression (4.14).

5.3. Scattering equations and classicalization high-energy limit

In this subsection we shall show that, in a parameterization of the classicalization limit, the scattering equations [9] can be solved exactly allowing us to obtain a closed expression for the high-energy limit of the open and closed superstring tree-level scattering amplitudes for a generic number N of external legs. In addition, we obtain compact expressions for the field-theory N -gluon and N -graviton amplitudes in the classicalization limit.

5.3.1. Saddle point approximation and scattering equations

The generic expression for an open string N -point form factor is given by the real iterated disk integral (cf. Eq. (5.21))

$$\begin{aligned} Z_\pi(\rho) &:= Z_\pi(1, \rho(2), \dots, \rho(N-2), N, N-1): \\ &= V_{\text{CKG}}^{-1} \int_{D(\pi)} \left(\prod_{l=1}^N dz_l \right) \\ &\quad \times \frac{\prod_{i < j}^N |z_{ij}|^{\alpha' s_{ij}}}{z_{1\rho(2)} z_{\rho(2), \rho(3)} \cdots z_{\rho(N-3), \rho(N-2)} z_{\rho(N-2), N} z_{N, N-1} z_{N-1, 1}} \end{aligned} \quad (5.59)$$

specified by some ordering of N points as $D(\pi) = \{z_j \in \mathbf{R} \mid z_1 < z_{\pi(2)} < \dots < z_{\pi(N-2)} < z_{N-1} < z_N\}$ (cf. comment below Eq. (5.24)) and the permutations $\rho, \pi \in S_{N-3}$. Furthermore, the

Koba–Nielsen factor $\prod_{i < j}^N |z_i - z_j|^{\alpha' s_{ij}}$ with the kinematic invariants (3.1) enters in the integrand.

In (5.59) the factor V_{CKG} accounts for the volume of the conformal Killing group of the disk after choosing the conformal gauge. It will be canceled by fixing three vertex positions z_i, z_j, z_k , i.e. $V_{\text{CKG}} = \frac{dz_i dz_j dz_k}{z_{ij} z_{jk} z_{ki}}$. The factor $z_{ij} z_{jk} z_{ki}$ can be identified as the standard reparameterization ghost correlator.

For fixed-angle scattering, the high-energy limit $\alpha' \rightarrow \infty$ of the disk integral (5.59) can be determined by performing a saddle-point approximation [42]. Rewriting the Koba–Nielsen factor of the integrand of (5.59) as

$$\prod_{i < j}^N |z_{ij}|^{\alpha' s_{ij}} = \exp \left\{ \frac{\alpha'}{2} \sum_{i \neq j} s_{ij} \ln |z_{ij}| \right\}$$

yields the saddle point equations

$$\sum_{j \neq i}^N \frac{s_{ij}}{z_i - z_j} = 0 \quad , \quad i = 1, \dots, N, \quad (5.60)$$

whose $(N-3)!$ solutions determine the locations

$$\{z_1^{(l)}, \dots, z_N^{(l)}\} \in \mathbf{C} \quad , \quad l = 1, \dots, (N-3)! \quad (5.61)$$

of the saddle points. Note, that the stationary points (5.61) do not have to lie⁷ within the real integration region $D(\pi)$, but may also be complex. By Cauchy's theorem the saddle point approximation then implies the continuous deformation of the integral along $D(\pi)$ (without leaving the domain of analyticity of the integrand) to a new (admissible) contour C_π (saddle contour) having the same endpoints as $D(\pi)$ and passing through the stationary points (5.61) in the direc-

tion of the steepest descent of $\Re \left(\sum_{i \neq j} s_{ij} \ln |z_{ij}| \right)$ [42]. Then, the maximum of the integrand is assumed at the isolated points and the full contribution to the asymptotic expansion of the original integral (5.59) is obtained by adding the amounts (of the integrals over small arcs containing

⁷ Their actual positions depend on the choice of kinematic invariants (3.1).

these points) from all relevant saddle points (5.61). Eventually, the saddle points (5.61) enter the disk integral (5.59) as

$$Z_\pi(\rho) = \left(\frac{2\pi}{\alpha'}\right)^{\frac{N-3}{2}} V_{\text{CKG}}^{-1} \int_{\tilde{C}_\pi} \left(\prod_{l=1}^N dz_l\right) (\det' \Phi)^{1/2} \prod_{a=1}^N \delta\left(\sum_{b \neq a}^N \frac{s_{ab}}{z_a - z_b}\right) \\ \times \frac{\prod_{i < j}^N |z_i - z_j|^{\alpha' s_{ij}}}{z_{1\rho(2)} z_{\rho(2), \rho(3)} \cdots z_{\rho(N-3), \rho(N-2)} z_{\rho(N-2), N} z_{N, N-1} z_{N-1, 1}} + \mathcal{O}(\alpha'^{-1}), \quad (5.62)$$

with the Jacobian:

$$\Phi_{ab} = \frac{1}{2} \frac{\partial^2}{\partial z_a \partial z_b} \sum_{i \neq j} s_{ij} \ln |z_{ij}| = \begin{cases} \frac{s_{ab}}{z_{ab}^2} & a \neq b, \\ -\sum_{c \neq a} \frac{s_{ac}}{z_{ac}^2} & a = b. \end{cases} \quad (5.63)$$

Of the latter a specific minor $|\Phi|_{pqr}^{ijk}$, arising after deleting three rows p, q, r and three columns i, j, k of the matrix Φ , enters the determinant $\det' \Phi$ as:

$$\det' \Phi = \frac{|\Phi|_{pqr}^{ijk}}{(z_{ij} z_{jk} z_{ki}) (z_{pq} z_{qr} z_{pr})}. \quad (5.64)$$

Furthermore, in (5.62) there is the product of delta-functions

$$\prod_{a=1}^N \delta\left(\sum_{b \neq a}^N \frac{s_{ab}}{z_a - z_b}\right) = z_{ij} z_{jk} z_{ki} \prod_{a \neq i, j, k} \delta\left(\sum_{b \neq a}^N \frac{s_{ab}}{z_a - z_b}\right), \quad (5.65)$$

which is independent on the choice i, j, k and hence permutation invariant. Eventually, (5.62) can be written as

$$Z_\pi(\rho) = \left(\frac{2\pi}{\alpha'}\right)^{\frac{N-3}{2}} \sum_{l=1}^{(N-3)!} [\det' \Phi(z^{(l)})]^{-1/2} \\ \times \frac{\prod_{i < j}^N |z_i^{(l)} - z_j^{(l)}|^{\alpha' s_{ij}}}{z_{1\rho(2)}^{(l)} z_{\rho(2), \rho(3)}^{(l)} \cdots z_{\rho(N-3), \rho(N-2)}^{(l)} z_{\rho(N-2), N}^{(l)} z_{N, N-1}^{(l)} z_{N-1, 1}^{(l)}} + \mathcal{O}(\alpha'^{-1}). \quad (5.66)$$

The world-sheet string integral (5.62) is dominated by the contributions of saddle points (5.61) yielding the sum (5.66). Although the latter may be complex their total contributions to the sum (5.66) must sum up to a real value.

In [7] the open string saddle points are obtained from saddle points of the closed string scattering by some reflection principle. On the other hand, by the single-valued projection [33,34] the high-energy limit of closed world-sheet sphere integrals can be obtained from the analog limit of open string integrals (5.66).

The set of equations (5.60) also appears in the context of describing Yang–Mills theory by twistor string theory [43] or recently as so-called scattering equations relating the space of kinematic invariants (3.1) and locations of N punctures on the complex sphere [9]. Hence, as already

pointed out in [43,9] there seems to be a striking relation between Yang–Mills theory and string theory at high energies communicated by the equations (5.60).

Clearly, for $N = 4$ Eq. (5.60) boils down to (5.12). In the general case there are $N - 3$ (independent) non-linear equations (5.60) to be solved and their solutions (5.61) are difficult to find. Yet for $N = 5$ explicit expressions for (5.66) can still be evaluated in general and for $N = 6$ the explicit solution can be written in $D = 4$ in terms of spinor helicity variables [44].

In the high-energy limit in (5.21) each integral $Z_\pi(\rho)$ gives rise to a sum (5.66) over $(N - 3)!$ saddle points (5.61). A similar sum over the $(N - 3)!$ solutions (5.61) of the scattering equations (5.60) can be used to specify the SYM factors $A_{YM}(\sigma)$ in (5.20) as [9]

$$A_{YM}(1, \dots, N) = \frac{\int \left(\prod_{l=1}^N d\sigma_l \right)}{\text{Vol SL}(2, \mathbf{C})} \prod_{a=1}^N \delta \left(\sum_{b \neq a}^N \frac{s_{ab}}{\sigma_{ab}} \right) \frac{E_N(\{k, \xi, \sigma\})}{\sigma_{12} \dots \sigma_{N1}}, \quad (5.67)$$

with N inhomogeneous coordinates $\sigma_l \in \mathbf{CP}^1$ ($\sigma_{ab} = \sigma_a - \sigma_b$) and $E_N(\{k, \xi, \sigma\})$ given by some Pfaffian encoding the external gluon kinematics with momenta k_i and gluon polarizations ξ_j . As consequence the high-energy limit of the open superstring amplitude (5.20) becomes a double sum over solutions (5.61) of the scattering equations (5.60)

$$\begin{aligned} \mathcal{A}(1, \dots, N) &= g_{YM}^{N-2} (2\pi\alpha')^{\frac{N-3}{2}} \\ &\times \sum_{a,b=1}^{(N-3)!} \frac{\left(\prod_{i < j}^N |z_{ij}^{(a)}|^{\alpha' s_{ij}} \right)}{\det' \Phi(z^{(a)})^{1/2}} \frac{E_N(\{k, \xi, \sigma^{(b)}\})}{\det' \Phi(\sigma^{(b)})} \det' \Psi(\{z^{(a)}\}, \{\sigma^{(b)}\}) \\ &+ \mathcal{O}(\alpha'^{-1}), \end{aligned} \quad (5.68)$$

with the generalized Hodges' determinant $\det' \Psi$ encoding the KLT kernel (5.22) and specified in [45,46]. Eventually, by applying the KLT orthogonality property [45,9]

$$\det' \Phi(\sigma^{(a)})^{-1/2} \det' \Phi(\sigma^{(b)})^{-1/2} \det' \Psi(\{\sigma^{(a)}\}, \{\sigma^{(b)}\}) = \delta^{ab} \quad (5.69)$$

of two solutions a, b of the scattering equation (5.60), one can cast the high-energy limit of (5.20) into a single sum over $(N - 3)!$ solutions (5.61):

$$\mathcal{A}(1, \dots, N) = g_{YM}^{N-2} (2\pi\alpha')^{\frac{N-3}{2}} \sum_{a=1}^{(N-3)!} \frac{\left(\prod_{i < j}^N |z_{ij}^{(a)}|^{\alpha' s_{ij}} \right)}{\det' \Phi(z^{(a)})^{1/2}} E_N(\{k, \xi, z^{(a)}\}) + \mathcal{O}(\alpha'^{-1}). \quad (5.70)$$

In $D = 4$ the sum (5.70) decomposes into k R -charge sectors describing N^{k-2} MHV amplitudes (with k negative-helicity states) labeled by $k = 2, \dots, N - 2$, with each sector having $\binom{N-3}{k-2}$ solutions and $\sum_{k=2}^{N-2} \binom{N-3}{k-2} = (N - 3)!$ [47]. The latter describes the RSVW residua in super-twistor space [48,43].

Next, for the high-energy limit of the closed superstring N -graviton amplitude we start from the expression [9]

$$\begin{aligned}
\mathcal{M}(1, \dots, N) &= \kappa^{N-2} V_{\text{CKG}}^{-1} \left(\prod_{j=1, z_j \in \mathbb{C}}^N \int d^2 z_j \right) \left(\prod_{i < j}^N |z_{ij}|^{\frac{\alpha'}{2} s_{ij}} \right) \\
&\times \sum_{a,b=1}^{(N-3)!} \frac{E_N(\{k, \xi, \sigma^{(a)}\}) E_N(\{k, \tilde{\xi}, \tilde{\sigma}^{(b)}\})}{\det' \Phi(\sigma^{(a)}) \det' \Phi(\tilde{\sigma}^{(b)})} \\
&\times \det' \Psi(\{z\}, \{\sigma^{(a)}\}) \det' \Psi(\{\tilde{z}\}, \{\tilde{\sigma}^{(b)}\}). \tag{5.71}
\end{aligned}$$

Note, that the saddle-point method described above, relies on Cauchy's theorem for the integration of analytic functions to deform the path of integration $D(\pi)$ in the complex plane onto a path of steepest descent. The integration over a domain in the multi-dimensional complex plane requires some sort of multi-dimensional generalization of the Laplace method (5.10) [49]. For the one-dimensional complex plane ($N = 4$ case) in Eq. (5.14) we have accomplished this by using polar coordinates. The saddle points are given by the same equations (5.60) with solutions (5.61). After using the KLT orthogonality (5.69) we have [9]:

$$\begin{aligned}
\mathcal{M}(1, \dots, N) &= \kappa^{N-2} (4\pi\alpha')^{N-3} \sum_{a=1}^{(N-3)!} \frac{\left(\prod_{i < j}^N |z_{ij}^{(a)}|^{\frac{\alpha'}{2} s_{ij}} \right)}{\det' \Phi(z^{(a)})^{1/2} \det' \Phi(\bar{z}^{(a)})^{1/2}} E_N(\{k, \xi, z^{(a)}\})^2 \\
&+ \mathcal{O}(\alpha'^{-1}). \tag{5.72}
\end{aligned}$$

In the previous subsection we have discussed the eikonal constraints (3.14). For the latter the scattering equations (5.60) separate. More precisely, for the region $z_1 < \dots < z_N$, after gauge fixing three positions as (5.24) and introducing the parameterization (5.30) the scattering equations (5.60) boil down to the $N - 3$ equations

$$\begin{aligned}
&\frac{s_{12}}{x_{N-3}} - \frac{s_{23}}{1 - x_{N-3}} = 0, \\
&\frac{-s_{N-l-1, N-l} + \sum_{j=2}^{N-l-1} s_{1j} + s_{j, j+1}}{x_l} - \frac{s_{N-l-1, N-l}}{1 - x_l} = 0, \quad l = 2, \dots, N-2, \tag{5.73}
\end{aligned}$$

each depending on only one of the remaining $N - 3$ positions. As a consequence in the limit (3.14) the high-energy behavior $\alpha' \rightarrow \infty$ of the string form factor (5.59) is given by a single term in agreement with the results in the previous subsection.

5.3.2. Solutions of scattering equations in the classicalization high-energy limit

Interestingly, for special subspaces of kinematics (3.1), the scattering equations (5.60) become Stieltjes sums for zeros of special functions [50] and can be solved analytically. In this subsection we shall see, that a parameterization of the classicalization high-energy limit (3.19) allows for solutions (5.61) of the scattering equations (5.60) given by the zeros of a Jacobi polynomial.

In units of $\frac{s}{(N-2)^2}$ the classicalization high-energy limit (3.19) can qualitatively be described by the following parameterization

$$\begin{aligned}
s_{1,N} &= \frac{1}{2} (N-3) (N - \mathfrak{a} - \mathfrak{b}), \\
s_{N-1,N} &= -\frac{1}{2} (N-3) (2 - \mathfrak{b}), \quad s_{1,N-1} = -\frac{1}{2} (N-3) (2 - \mathfrak{a}),
\end{aligned}$$

$$\begin{aligned}
s_{1,i} &= -\frac{1}{2} (N-2-b), \quad s_{i,N} = -\frac{1}{2} (N-2-a), \\
s_{N-1,i} &= \frac{1}{2} (4-a-b), \quad s_{ij} = 1, \quad i, j \in \{2, \dots, N-2\},
\end{aligned} \tag{5.74}$$

with finite a, b (e.g. $-1 < a, b < 0$). With the identification

$$a = \alpha + N - 1, \quad b = \beta + N - 1 \tag{5.75}$$

the parameterization (5.74) can then be adjusted to:

$$\begin{aligned}
s_{1,N} &= \frac{1}{2} (3-N) (\alpha + \beta + N - 2), \\
s_{N-1,N} &= \frac{1}{2} (N-3) (N-3+\beta), \quad s_{1,N-1} = \frac{1}{2} (N-3) (N-3+\alpha), \\
s_{1,i} &= \frac{1}{2} (1+\beta), \quad s_{i,N} = \frac{1}{2} (1+\alpha), \\
s_{N-1,i} &= \frac{1}{2} (6-2N-\alpha-\beta), \quad s_{ij} = 1, \quad i, j \in \{2, \dots, N-2\}.
\end{aligned} \tag{5.76}$$

For this special two parameter family of kinematics (5.76) (described by α, β) the scattering equations (5.60) allow for solutions (5.61), which can be related to the $N-3$ zeros x_a , $a = 1, \dots, N-3$ of the Jacobi polynomial $P_{N-3}^{(\alpha, \beta)}(x)$ [51]. Actually, this solution is degenerate by $(N-3)!$, i.e. each solution $z_i^{(l)} = x_{\pi_l(i-1)}$, $i = 2, \dots, N-2$ is specified by a permutation $\pi_l \in S_{N-3}$, $l = 1, \dots, (N-3)!$ of the $N-3$ zeros x_a . For this solution the SYM amplitude (5.67) and the graviton amplitude have been worked out in compact form⁸ [51]

$$\begin{aligned}
A_{YM}(1, \dots, N) &= \sum_{l=1}^{(N-3)!} \frac{1}{\sigma_{12}^{(l)} \dots \sigma_{N1}^{(l)}} \frac{E_N(\{k, \xi, \sigma^{(l)}\})}{\det' \Phi(\sigma^{(l)})} \\
&= 2^{4-\frac{N}{2}} (N-3)!! \frac{\Gamma\left(\frac{N-1+\alpha}{2}\right) \Gamma\left(1+\frac{\beta}{2}\right) \Gamma\left(\frac{N-1+\alpha+\beta}{2}\right)}{\Gamma\left(\frac{1+\alpha}{2}\right) \Gamma\left(\frac{N-2+\beta}{2}\right) \Gamma\left(\frac{2N-5+\alpha+\beta}{2}\right)} H_N(\alpha, \beta),
\end{aligned} \tag{5.77a}$$

$$\begin{aligned}
\mathcal{M}_{FT}(1, \dots, N) &= \kappa^{N-2} \sum_{l=1}^{(N-3)!} \frac{E_N(\{k, \xi, \sigma^{(l)}\})^2}{\det' \Phi(\sigma^{(l)})} \\
&= -\kappa^{N-2} 2^{8-N} [(N-3)!!]^2 \frac{\Gamma\left(\frac{N-1+\alpha}{2}\right) \Gamma\left(1+\frac{\beta}{2}\right) \Gamma\left(\frac{N-1+\alpha+\beta}{2}\right)}{\Gamma\left(\frac{1+\alpha}{2}\right) \Gamma\left(\frac{N-2+\beta}{2}\right) \Gamma\left(\frac{2N-5+\alpha+\beta}{2}\right)} \\
&\quad \times \frac{\Gamma\left(1+\frac{\alpha}{2}\right) \Gamma\left(\frac{N-1+\beta}{2}\right) \Gamma\left(\frac{2N-4+\alpha+\beta}{2}\right)}{\Gamma\left(\frac{1+\beta}{2}\right) \Gamma\left(\frac{N-2+\alpha}{2}\right) \Gamma\left(\frac{N-2+\alpha+\beta}{2}\right)} H_N(\alpha, \beta)^2,
\end{aligned} \tag{5.77b}$$

respectively. Above, H_N is the helicity dependent part depending on the external kinematics of momenta k_i and polarizations ξ_j to be specified below.

⁸ Note, that we have corrected the gauge amplitude by a factor of $\frac{1}{(N-3)!}$, which is missing on the r.h.s. of Eq. (11) in [51].

For $\alpha, \beta > -1$ the n -th order Jacobi polynomial $P_n^{(\alpha, \beta)}(x)$ has n distinct (real) roots in the interval $(-1, 1)$. The conditions $\alpha, \beta > -1$ are to be imposed for the orthogonality of the Jacobi polynomials [52]. However, we may relax these constraints. Therefore, in (5.77a) and (5.77b) we may consider α and β as two distinct arbitrary real parameters:

$$\alpha, \beta \in \mathbf{R}. \quad (5.78)$$

In this case $P_n^{(\alpha, \beta)}(x)$ denote generalized Jacobi polynomials [52]. Note, that the zeros of the latter, and therefore the solutions of (5.61), may be complex and the comments below Eq. (5.61) apply. We have verified, that the results [51] can be derived for generic parameters $\alpha, \beta \in \mathbf{R}$ as long as no singularity occurs. So the amplitudes (5.77a) and (5.77b) are valid for generic parameters α and β (5.78). Hence, we may simply rewrite (5.77a) and (5.77b) in terms of the parametrization (5.74)

$$A_{YM}(1, \dots, N) = 2^{4-\frac{N}{2}} \left(\frac{s}{(N-2)^2} \right)^{\frac{4-N}{2}} (N-3)!! \times \frac{\Gamma\left(\frac{a}{2}\right) \Gamma\left(\frac{3}{2} + \frac{b-N}{2}\right) \Gamma\left(\frac{1-N+a+b}{2}\right)}{\Gamma\left(1 + \frac{a-N}{2}\right) \Gamma\left(\frac{b-1}{2}\right) \Gamma\left(\frac{a+b-3}{2}\right)} H_N(a, b), \quad (5.79a)$$

$$\mathcal{M}_{FT}(1, \dots, N) = -\kappa^{N-2} 2^{8-N} \frac{s}{(N-2)^2} [(N-3)!!]^2 \times \frac{\Gamma\left(\frac{a}{2}\right) \Gamma\left(\frac{3}{2} + \frac{b-N}{2}\right) \Gamma\left(\frac{1-N+a+b}{2}\right)}{\Gamma\left(1 + \frac{a-N}{2}\right) \Gamma\left(\frac{b-1}{2}\right) \Gamma\left(\frac{a+b-3}{2}\right)} \times \frac{\Gamma\left(\frac{3}{2} + \frac{a-N}{2}\right) \Gamma\left(\frac{b}{2}\right) \Gamma\left(\frac{a+b-2}{2}\right)}{\Gamma\left(1 + \frac{b-N}{2}\right) \Gamma\left(\frac{a-1}{2}\right) \Gamma\left(\frac{a+b-N}{2}\right)} H_N(a, b)^2, \quad (5.79b)$$

respectively. Above, we have reinstated the s -dependence by inspecting (4.10) and (4.14). This s -behavior may also be easily extracted from considering the behavior of the determinants entering in (5.79a) and (5.79b). For the $(N-3) \times (N-3)$ reduced matrix (5.63) we have $\det' \Phi \sim \left(\frac{s}{(N-2)^2} \right)^{N-3}$, while the determinant of the relevant $(N-2) \times (N-2)$ submatrix of Ψ scales as $E_N^2 = \det' \Psi \sim \left(\frac{s}{(N-2)^2} \right)^{N-2}$. Furthermore, we have the kinematical factor [51]

$$H_N(a, b) = \frac{c_2^{\frac{N}{2}-3}}{a+b-4} \left(\frac{2(N-3)(N-4)c_1 c_{N-1} c_N}{(2-N+a)(2-N+b)} - c_2 c_{N-1} \xi_{1,N} \right) + c_2^{\frac{N}{2}-2} \left(\frac{c_1 \xi_{N-1,N}}{2-N+b} + \frac{c_N \xi_{1,N-1}}{2-N+a} \right), \quad (5.80)$$

with $\xi_{a,b} \equiv \xi_{ab} = \xi_a \xi_b$, $c_1 = \xi_{1,i}$, $c_{N-1} = \xi_{i,N-1}$, $c_N = \xi_{i,N}$ and $c_2 := \xi_{i,j}$, $i, j \in \{2, \dots, N-2\}$. This choice of polarization vectors (with arbitrary parameters c_1, c_2, c_{N-1} and c_N) guarantees the on-shell condition $\xi_a k_a = 0$, $a = 1, \dots, N$ and momentum conservation.

5.3.3. Fixing combinatorics from scattering equations

From (5.79b) let us now extract the large N behavior (classicalization limit) of the graviton amplitude for some $-1 < a, b < 0$. First, the kinematical factor H_N behaves as $c^{N/2}$, with some finite constant c . The ratio of Gamma-functions (depending on N) can be approximated by (5.35) as:

$$\frac{\Gamma\left(\frac{3}{2} + \frac{a-N}{2}\right) \Gamma\left(\frac{3}{2} + \frac{b-N}{2}\right) \Gamma\left(\frac{1-N+a+b}{2}\right)}{\Gamma\left(1 + \frac{a-N}{2}\right) \Gamma\left(1 + \frac{b-N}{2}\right) \Gamma\left(\frac{a+b-N}{2}\right)} \sim -\left(\frac{N}{2}\right)^{3/2}.$$

Hence, in total with $[(N-3)!!]^2 \sim \sqrt{\frac{2}{\pi}} \frac{(N-2)!}{\sqrt{(N-2)}}$ we have

$$\begin{aligned} \mathcal{M}_{FT}(1, \dots, N) &\sim \kappa^{N-2} 2^{8-N} c^N \frac{s}{(N-2)^2} \left(\frac{N}{2}\right)^{3/2} [(N-3)!!]^2 \\ &\sim \kappa^{N-2} \frac{s}{(N-2)^2} (N-1)!, \end{aligned} \quad (5.81)$$

in lines with the behavior (4.14) for the field theory graviton amplitudes. It is interesting to note, that for $s \sim N$ the number on the r.h.s. of (5.81) approximately coincides with the dimension $(N-3)!$ of the period matrix of the moduli space $\mathcal{M}_{0,N}$ of curves of genus zero with N labeled points, which in turn is the set of Riemann spheres with N marked points modulo isomorphisms of Riemann surfaces sending marked points to marked points, i.e. $\mathcal{M}_{0,N} \simeq \{(z_1, \dots, z_N) \in \mathbf{P}^1(\mathbf{C}) \mid z_i \neq z_j\} / \text{PSL}(2, \mathbf{C})$.

5.3.4. High-energy classicalization limit of string amplitudes from scattering equations

Stieltjes has already discovered a relation between the zeros of classical polynomials and the electrostatic equilibrium interpretation of the saddle point approximation, which is closely connected with the calculation of the discriminant of these polynomials. In fact, in a moment we shall see that in the classicalization parameterization (5.76) the discriminant of generalized Jacobi polynomials is related to the Koba–Nielsen factor. Here, we shall compute the high-energy open superstring N -gluon amplitude (5.70) and the high-energy closed superstring N -graviton amplitude (5.72) in the classicalization parameterization (5.76). Therefore, we shall evaluate (5.70) and (5.72) at the solutions of the scattering equations (5.60), which are described by the $N-3$ zeros x_a , $a = 1, \dots, N-3$, of the generalized Jacobi polynomial $P_{N-3}^{(\alpha, \beta)}(x)$.

To proceed we first need to work out some properties of the zeros x_a of generalized Jacobi polynomials. With

$$l = \frac{1}{(N-3)!} \frac{\partial^{N-3}}{\partial x^{N-3}} P_{N-3}^{(\alpha, \beta)}(x) = \frac{2^{3-N}}{(N-3)!} \frac{\Gamma(2N-5+\alpha+\beta)}{\Gamma(N-2+\alpha+\beta)}, \quad (5.82)$$

being the coefficient of the highest term x^{N-3} of the Jacobi polynomial $P_{N-3}^{(\alpha, \beta)}(x)$ the discriminant of the latter is given by [52]:

$$\begin{aligned} \Delta_{N-3} &:= l^{2N-8} \prod_{1 \leq a < b \leq N-3} (x_a - x_b)^2 \\ &= 2^{-(N-3)(N-4)} \prod_{v=1}^{N-3} v^{v-2N+8} (\alpha+v)^{v-1} (\beta+v)^{v-1} \\ &\quad \times (\alpha+\beta+N-3+v)^{N-3-v}. \end{aligned} \quad (5.83)$$

Furthermore, we derive the following identities:

$$\begin{aligned}
 \prod_{a=1}^{N-3} (1-x_a) &= (N-3)! \frac{P_{N-3}^{(\alpha,\beta)}(1)}{P_{N-3}^{(\alpha,\beta)(N-3)}(x)} = 2^{N-3} \frac{\Gamma(N-2+\alpha)}{\Gamma(1+\alpha)} \frac{\Gamma(N-2+\alpha+\beta)}{\Gamma(2N-5+\alpha+\beta)} \\
 &= 2^{N-3} \prod_{\nu=1}^{N-3} (\alpha+\nu) (\alpha+\beta+N-3+\nu)^{-1}, \\
 \prod_{a=1}^{N-3} (1+x_a) &= (-1)^{N+1} (N-3)! \frac{P_{N-3}^{(\alpha,\beta)}(-1)}{P_{N-3}^{(\alpha,\beta)(N-3)}(x)} \\
 &= 2^{N-3} \frac{\Gamma(N-2+\beta)}{\Gamma(1+\beta)} \frac{\Gamma(N-2+\alpha+\beta)}{\Gamma(2N-5+\alpha+\beta)} \\
 &= 2^{N-3} \prod_{\nu=1}^{N-3} (\beta+\nu) (\alpha+\beta+N-3+\nu)^{-1}. \tag{5.84}
 \end{aligned}$$

With these preliminaries for the kinematic invariants (5.76), the Koba–Nielsen factor of (5.70) can be worked out for any solution (5.61). The latter is specified by some permutation $\pi_l \in S_{N-3}$, $l = 1, \dots, (N-3)!$ acting on the $N-3$ zeros x_a of the generalized Jacobi polynomials $P_{N-3}^{(\alpha,\beta)}(x)$ as $\{z_i^{(l)} = x_{\pi_l(i-1)} \mid i = 2, \dots, N-2\}$. Together with the three $SL(2, \mathbb{C})$ fixed positions $z_1^{(l)} = -1$, $z_{N-1}^{(l)} = \infty$ and $z_N^{(l)} = 1$ we obtain

$$\begin{aligned}
 \prod_{i < j} |z_{ij}^{(l)}|^{\alpha' s_{ij}} &= 2^{\alpha' s_{1N}} \prod_{a=2}^{N-2} |z_1^{(l)} - z_a^{(l)}|^{\alpha' s_{1a}} |z_N^{(l)} - z_a^{(l)}|^{\alpha' s_{aN}} \prod_{2 \leq a < b \leq N-2} |z_a^{(l)} - z_b^{(l)}|^{\alpha' s_{ab}} \\
 &= 2^{\alpha' s_{1N}} \prod_{a=1}^{N-3} |1+x_a|^{\alpha' s_{1m}} |1-x_a|^{\alpha' s_{mN}} \prod_{1 \leq a < b \leq N-3} |x_a - x_b|^{\alpha' s_{mn}}, \\
 &= \prod_{\nu=1}^{N-3} \left(\frac{\nu^\nu (\alpha+\nu)^{\alpha+\nu} (\beta+\nu)^{\beta+\nu}}{(\alpha+\beta+N-3+\nu)^{\alpha+\beta+N-3+\nu}} \right)^{\alpha'/2}, \tag{5.85}
 \end{aligned}$$

with any $m, n \in \{2, \dots, N-2\}$. Note, that the above expression is independent on the permutation π_l under consideration, i.e. for the parameterization (5.76) each solution (5.61) of the scattering equation yields the same Koba–Nielsen factor. In addition, in the sum (5.70), the quotient $\frac{E_N(\{k, \xi, z^{(a)}\})}{\det' \Phi(z^{(a)})^{1/2}}$ is independent on the particular solution a . As a consequence we can rewrite this sum as

$$\begin{aligned}
 &\sum_{a=1}^{(N-3)!} \frac{\left(\prod_{i < j}^N |z_{ij}^{(a)}|^{\alpha' s_{ij}} \right)}{\det' \Phi(z^{(a)})^{1/2}} E_N(\{k, \xi, z^{(a)}\}) \\
 &= (N-3)! \left(|\Phi|_{1,N-1,N}^{1,N-1,N} \right)^{1/2} \left(\prod_{a=1}^{N-3} (1+x_a) \right) \left(\prod_{i < j} |z_{ij}^{(l)}|^{\alpha' s_{ij}} \right) \\
 &\quad \times \sum_{a=1}^{(N-3)!} \frac{1}{\sigma_{12}^{(a)} \dots \sigma_{N1}^{(a)}} \frac{E_N(\{k, \xi, \sigma^{(a)}\})}{\det' \Phi(\sigma^{(a)})}, \tag{5.86}
 \end{aligned}$$

with l denoting any solution. In (5.86) the last factor yields the SYM amplitude (5.77a). On the other hand, based on the results in [51] we have:

$$|\Phi|_{1,N-1,N}^{1,N-1,N} = [(N-3)!]^2 \frac{l^3}{P_{N-3}^{(\alpha,\beta)}(1) P_{N-3}^{(\alpha,\beta)}(-1)} = (-1)^{N+1} 2^{9-3N} (N-3)! \\ \times \frac{\Gamma(1+\alpha)}{\Gamma(N-2+\alpha)} \frac{\Gamma(1+\beta)}{\Gamma(N-2+\beta)} \left(\frac{\Gamma(2N-5+\alpha+\beta)}{\Gamma(N-2+\alpha+\beta)} \right)^3. \quad (5.87)$$

With (5.84) this gives:

$$(|\Phi|_{1,N-1,N}^{1,N-1,N})^{1/2} \left(\prod_{a=1}^{N-3} (1+x_a) \right) \\ = \sqrt{(-1)^{N+1} 2^{3-N} (N-3)!} \\ \times \left\{ \frac{\Gamma(1+\alpha)}{\Gamma(N-2+\alpha)} \frac{\Gamma(N-2+\beta)}{\Gamma(1+\beta)} \frac{\Gamma(2N-5+\alpha+\beta)}{\Gamma(N-2+\alpha+\beta)} \right\}^{1/2}.$$

After putting all expressions together we arrive at the final result of (5.70)

$$\mathcal{A}(1, \dots, N) = g_{YM}^{N-2} (2\pi\alpha')^{\frac{N-3}{2}} (N-3)! \prod_{\nu=1}^{N-3} \left(-\frac{\nu(\beta+\nu)(\alpha+\beta+N-3+\nu)}{2(\alpha+\nu)} \right)^{1/2} \\ \times \prod_{\nu=1}^{N-3} \left(\frac{\nu^\nu (\alpha+\nu)^{\alpha+\nu} (\beta+\nu)^{\beta+\nu}}{(\alpha+\beta+N-3+\nu)^{\alpha+\beta+N-3+\nu}} \right)^{\alpha'/2} A_{YM}(1, \dots, N) + \mathcal{O}(\alpha'^{-1}), \quad (5.88)$$

with the field-theory gluon amplitude given in (5.77a). Note, that with the parameterization (5.76) for $N=4$ the result (5.88) boils down to (5.6) with (5.8). Furthermore, the analytic structure of the result (5.88) is very reminiscent of the functional dependence appearing in (5.18).

Next, let us compute the closed superstring N -graviton amplitude in the high-energy classicalization parameterization (5.74). We start from the expression (5.72). For our solutions (5.61) the determinants $\det' \Phi(z^{(a)})$ and $\det' \Phi(\bar{z}^{(a)})$ are real quantities (5.87). The same is true for the Shapiro–Virasoro factor. As a consequence the latter can be expressed as a square root of the Koba–Nielsen factor (5.85) and the sum in (5.72) can be written as

$$\sum_{a=1}^{(N-3)!} \frac{\left(\prod_{i<j}^N |z_{ij}^{(a)}|^{\frac{\alpha'}{2} s_{ij}} \right)}{\det' \Phi(z^{(a)})^{1/2} \det' \Phi(\bar{z}^{(a)})^{1/2}} E_N(\{k, \xi, z^{(a)}\})^2 \\ = \left(\prod_{i<j}^N |z_{ij}^{(l)}|^{\frac{\alpha'}{2} s_{ij}} \right) \sum_{a=1}^{(N-3)!} \frac{E_N(\{k, \xi, z^{(a)}\})^2}{\det' \Phi(z^{(a)})},$$

with l denoting any solution and the last factor being the field-theory graviton amplitude (5.77b). Eventually after putting all expressions together we obtain

$$\mathcal{M}(1, \dots, N) = (4\pi\alpha')^{N-3} \prod_{\nu=1}^{N-3} \left(\frac{\nu^\nu (\alpha+\nu)^{\alpha+\nu} (\beta+\nu)^{\beta+\nu}}{(\alpha+\beta+N-3+\nu)^{\alpha+\beta+N-3+\nu}} \right)^{\alpha'/4} \\ \times \mathcal{M}_{FT}(1, \dots, N) + \mathcal{O}(\alpha'^{-1}), \quad (5.89)$$

with the field-theory graviton amplitude given in (5.77b). Again, with the parametrization (5.76) for $N = 4$ the result (5.89) yields (5.14).

The high-energy limits (5.88) and (5.89) correspond to the Case (i) discussed in the previous subsection, i.e. $M_s \rightarrow 0$ (and $\alpha' \rightarrow \infty$) for finite N and large momenta $s \rightarrow \infty$. If the parametrization (5.76) is taken in units of s , i.e. $|s_{ij}| \sim s$ we can easily reinstate the s -dependence in (5.88) and (5.89) and find agreement with the results (5.41) and (5.56), respectively.

6. Black hole dominance

6.1. Black hole dominance and a cross-check by semi-classical estimates

An useful cross-check of large- N scaling of amplitudes is provided by applying them to the production of generic classical states composed of much softer gravitons than a black hole of the same mass. It is obvious that such states are in a very weak λ domain and thus the semi-classical estimates are expected to be applicable. We shall then match the perturbative quantum and non-perturbative semi-classical estimates.

Such a matching serves us for a double purpose. First, it enables us to obtain an independent input about the scaling of large- N amplitude. It also shows how the suppression of production of non-black hole classical configurations can be understood from N -particle perturbative amplitudes. This understanding gives a valuable information, as it uncovers the corpuscular quantum nature behind the exponential suppression of the production of classical configurations, described by soft coherent states, in high energy two-particle collision processes.

As an example, let us estimate the production rate of a *classical* gravitational wave in the above-discussed graviton–graviton scattering. For simplicity, we shall take the wave to be monochromatic, of characteristic wavelength L and the amplitude A_{cl} . For such a monochromatic wave, the classical energy per wavelength-cubed is $E = A_{cl}^2 L$. In order to be both in a weak gravity regime as well as in the domain of semi-classical approximation, we shall demand that the Schwarzschild radius corresponding to this energy is much shorter than the wavelength, $R = EG_N \ll L$. Or equivalently,

$$A_{cl}^2 G_N \ll 1. \quad (6.1)$$

The leading behavior of the transition probability to such a classical wave can be reliably estimated in the semi-classical approximation, and is given by,

$$P_{2 \rightarrow \text{Wave}} = e^{-\frac{A_{cl}^2 L^2}{\hbar}} \times (\text{coupling-dependent factor}), \quad (6.2)$$

where the quantity in the exponent is the Euclidean action, $S_E = A_{cl}^2 L^2$.

In order to make contact between the perturbative matrix element (1.3) and the semi-classical one (6.2), we have to translate the monochromatic wave in the quantum language. In this language, the wave is a coherent state $|N\rangle_{coh}$ of gravitons of momenta $p = \hbar/L$ and the average occupation number $N = \frac{A_{cl}^2 L^2}{\hbar}$,

$$|N\rangle_{coh} \equiv e^{-\frac{N}{2}} \sum_n \frac{N^{\frac{n}{2}}}{\sqrt{n!}} |n\rangle, \quad (6.3)$$

where $|n\rangle$ are n -graviton Fock states of momenta $p = \hbar/L$. Notice, that the condition (6.1) is simply $\lambda \ll 1$, signaling that we are in a weak-coupling regime in which gravitons can be treated as free and thus the perturbative amplitudes must be fully applicable.

By choosing L and A_{cl} appropriately, we can make the parameter N of the coherent state arbitrarily-large for an arbitrary choice of E . In this way, we can create an *arbitrarily-classical* wave of arbitrarily low or high energy. In particular, E can be chosen to be ultra-Planckian or well below the Planck scale, without affecting the validity of the classical approximation for the final monochromatic wave. This fact suggests that for the estimate of the transition probability we should be able to reliably use both semi-classical as well as perturbative quantum amplitudes, and the two must match to the leading order.

The rest of the analysis is straightforward. We need to estimate the perturbative S -matrix element $|\langle 2|S|N \rangle_{coh}|^2_{pert}$ using (1.3) and match it with (6.2). Notice, that since the Fock states that enter in the coherent state (6.3) correspond to different occupation numbers of the same fixed momentum (or wavelength) gravitons, for each choice of this wavelength only one Fock state from this sum matches the center of mass energy of the initial 2-graviton states. This is the state $|n\rangle$ with $n = N = \sqrt{s}/p$. Correspondingly, only the transition to this particular state is possible. That is, $\langle 2|S|n \rangle = \delta_{n,N} \langle 2|S|N \rangle$, where $\langle 2|S|N \rangle$ is given by (1.3). We thus obtain,

$$|\langle 2|S|N \rangle_{coh}|^2 = |e^{-\frac{N}{2}} \sum_n \frac{N^{\frac{n}{2}}}{\sqrt{n!}} \langle 2|S|n \rangle|^2 = e^{-N} \lambda^N. \quad (6.4)$$

Matching this expression with (6.2) reproduces the exponential suppression of the classical state.

The factor λ^N reveals an extra suppression, due to weak coupling. This is expected, since the transition must be absent in a free theory. This extra suppression is absent for the case of black hole production, since $\lambda = 1$, which is one of the reasons of black hole dominance. The other, as explained, is the enhancement by an e^N factor due to multiplicity of states at the quantum-critical point.

6.2. Possible subtleties of the perturbative description

We would like to stress the possible subtleties of the perturbative framework we are working in and its validity for black hole physics. *A priori*, it is not obvious that signatures of black hole formation in two-particle scattering can be captured by perturbative amplitudes. In particular, by tree-level amplitudes that are suppressed by the powers of some weak coupling, such as the gravitational or string coupling. It could happen that no single class of Feynman diagrams describing such weak coupling expansion can be pin-pointed as a source of black hole formation in two-particle scattering. The answer instead could require either a full re-summation of infinite number of diagrams, or even inclusion of contributions of yet unknown non-perturbative processes.

So what makes us think that black hole formation can be captured perturbatively?

First, an encouragement comes from the fact that our results allow to create a link between the production of black holes and other classical objects, composed out of softer gravitons than a would-be black hole at a given \sqrt{s} . In other words, we identify a kinematical regime in which the questions of reliability of black hole production description is linked, with the reliability of the description of production of other classical objects, whose quantum composition can be identified beyond any reasonable doubt.

However, we are going beyond this link by postulating that there exists a part of the information that can be extracted from a class of perturbative diagrams within a properly identified kinematical regime. These are the $2 \rightarrow N$ transition processes.

What we are suggesting is that in the process of black hole formation, which in general is expected to be a highly non-perturbative phenomenon, there exists a well-defined division be-

tween the contribution that can be interpreted in the language of perturbative diagrams and the fully non-perturbative one. What is important is that the knowledge of the latter contribution is crucial for identifying the former one. In other words, without having the non-perturbative input that black holes represent a N -graviton bound-state it would be impossible to look for the perturbative counterpart of the process in the form of $2 \rightarrow N$ scattering.

Thus, we are postulating that it is meaningful to represent, schematically, the black hole formation probability as the sum over probabilities,

$$\sum_j |\langle 2|S|N \rangle_{\text{pert}}|^2 |\langle N|BH \rangle_j|^2, \quad (6.5)$$

with each member of the sum representing a product of perturbative and non-perturbative matrix elements. Here the sum over j runs over non-perturbative black hole states $|BH\rangle_j$, with their multiplicity scaling as e^N . Of course, one can say that such a scaling is expected from the black hole entropy counting, and one does not need any microscopic theory for postulating it. This is certainly true, but solely knowledge of the multiplicity of unknown hypothetical micro-states is useless for understanding the mechanism of black hole production.

The new ingredient is contained in the identification of the projection $\langle N|BH \rangle_j$ of these states on a N -graviton state. It is this identification what enables to conclude that black hole formation process includes a perturbative part in the form of the perturbative amplitude of N -graviton production. Of course, drawing such a connection is impossible without a microscopic theory and this is where the black hole corpuscular portrait enters in our analysis. Since in this picture black hole represents an N soft graviton bound-state at the critical point, it naturally suggest a significant projection on an out-state of N free gravitons of wavelengths equal to the ones of the black hole constituents.

The subtle point here is not in accepting such an overlap between the N -graviton state and a black hole state, but rather in the perturbative part of the probability, which assumes that we can reliably estimate the N -graviton production in perturbation theory. Viability of the latter assumption has nothing to do with a particular microscopic theory of a black hole and, as shown above, is generic for perturbative computation of the production rate of arbitrary N -particle states in two-particle collision, including the ones not even remotely related to black holes. This separation of the issues is crucial for understanding the framework we are working in.

In order to explain why this latter assumption is so subtle, let us consider the two-particle scattering at ultra-Planckian center of mass energy from a fully non-perturbative corpuscular point of view. In fact, we can very quickly realize that the initial state can be represented as a genuine two-particle state only at *infinite* separation. At finite separation, L , the center of mass energy sources a Newtonian gravitational field $\phi(\vec{x})$, which in the corpuscular language itself represents a coherent state of longitudinal gravitons in which the gravitons of wavelength L have average occupation number $N = EL_p^2$ [3,4,53,54]. Schematically, we can write this in the following form,

$$|Newton\rangle = \sum_{n_k=0, \dots, n_k=\infty} \prod_k e^{-\frac{N_k}{2}} \frac{N_k^{\frac{n_k}{2}}}{\sqrt{n_k!}} |n_k=0, \dots, n_k=\infty\rangle, \quad (6.6)$$

where $|n_k=0, \dots, n_k=\infty\rangle$ are the Fock states with definite occupation numbers of longitudinal gravitons of wavenumber k and the summation is taken over all possible distributions of n_k -s. The function N_k represents the data that determine the average occupation number of gravitons

of wavenumber k in the given coherent state. The function $N_{|k|}$ is exponentially decaying for $|k| \gg 1/L$. The dominant contribution to gravitational self-energy,

$$E_{grav} = \frac{\hbar}{L_P^2} \int d^3\vec{x} \vec{\nabla} \phi \vec{\nabla} \phi \sim \frac{\hbar}{L} \frac{s}{M_P^2} \quad (6.7)$$

is coming from the modes of momenta $k \sim 1/L$, with their number being, $N = \frac{s}{M_P^2}$. Notice that this number coincides with the number of black hole constituent gravitons. The only difference is that the gravitons that are present in the initial state have extremely long wavelengths and their collective coupling λ is negligible. Correspondingly, neither they contribute significantly to the energy, nor are they capable of forming a bound-state. Nevertheless, the message is that an ultra-Planckian initial state for any finite value of L is secretly a multi-particle state that on top of the two source particles contains N additional gravitons. As the system evolves in time, decreasing the separation between the initial two source particles, L , the multi-particle nature of the initial state becomes more and more apparent. The peak of the dominant graviton distribution in the coherent state evolves towards the higher momenta. The non-perturbative N -particle physics becomes fully important for \hbar/L of order $\frac{\sqrt{s}}{N}$. At this stage λ becomes order one signaling that the constituent gravitons are driven into the quantum critical point at which they form the bound-state and Bogoliubov modes become gapless.

What we are suggesting in our current analysis is that the above fully non-perturbative evolution can be substituted by a perturbative creation of N -graviton state and its projection on a black hole state using the non-perturbative input from the microscopic theory.

The fact that we are able to cross-check the result by normalizing the amplitude to the creation of a generic N -particle state, indicates that the failure of the above program would imply a problem in the description of the production of the N -graviton state in perturbation theory, rather than in the projection of such state into a black hole quantum state. It is interesting that at the level of the studied kinematic regimes the perturbative treatment comes up with the adequate physical results.

7. Lessons from gravitational multi-particle amplitudes

7.1. Peculiarities of multi-particle amplitudes in gravity

One of the outcomes of our analysis is to reveal a special property of multi-particle gravitational amplitudes in contrast to similar amplitudes in *non-derivatively* interacting bosonic theories, such as for example, in a self-interacting scalar theory $\alpha_\phi \phi^4$, with a non-derivative coupling α_ϕ .

It has been known for some time [55] that multi-scalar production amplitudes in such theories exhibit (at least at the threshold of producing N on-shell massive scalars of mass m_ϕ out of some initial few-particle state, the simplest being a single virtual boson of energy $\sqrt{s} = Nm_\phi$) a factorial growth,

$$\mathcal{A}_{1 \rightarrow N} \sim \alpha_\phi^{N/2} N!, \quad (7.1)$$

and a corresponding growth of the cross-section,

$$\sigma_{1 \rightarrow N} \sim \frac{1}{N!} |\mathcal{A}_{1 \rightarrow N}|^2 \sim \alpha_\phi^N N!, \quad (7.2)$$

where we have omitted the phase-space and other irrelevant factors. In non-derivatively coupled theory the tree-level coupling α_ϕ is momentum-independent and the factorial growth violates unitarity at sufficiently large N . Perturbation theory breaks down for $N \gg \alpha_\phi^{-1}$. The physical implications of this phenomenon is not fully understood. It may signal inapplicability of the perturbative treatment or even an inconsistency of the theory. Since, this question is not the focus of our paper we shall not discuss it further, but rather confront the growth of the scalar amplitude with the analogous factorial behavior in gravity and stress the important differences.

Notice that Eq. (7.1) is very similar to (1.3) with the difference that α of gravity is replaced by the scalar self-coupling α_ϕ . However, the momentum dependence of the gravitational coupling, $\alpha = L_p^2 s / N^2$, makes a dramatic difference. In particular, for large N it overpowers the factorial growth of diagrams. The resulting amplitude in gravity is exponentially-suppressed as opposed to the factorially-exploding counterpart in non-derivative ϕ^4 theory.

Notice, that the perturbative tree-level amplitudes in gravity and in non-derivative scalar theory have problems in the opposite domains of N , with the dramatic difference that in the problematic domain gravity amplitudes are cured by black holes, whereas in ϕ^4 theory no obvious helper is visible.

On the other hand, in ϕ^4 theory scattering for $N \ll \alpha_\phi^{-1}$ is unitary, whereas for gravity tree-level unitarity is violated for small N and large s . However, as discussed above, in gravity this very domain is excluded by the black hole quantum portrait, due to collective effects of graviton Bose-gas. Thus, the black hole physics prevents us from entering there.

In contrast, the domain $N \gg \alpha^{-1}$ in gravity is perturbatively-safe, since in this domain $\alpha^N \sim \lambda^N e^{-N} / N!$, whereas the analogous domain in ϕ^4 violates unitarity. In particular, as we have seen, in gravity this large- N behavior takes care of the exponential suppression in the production of classical configurations composed of gravitons softer than the Schwarzschild radius of a \sqrt{s} mass black hole.

The property of suppression of multi-particle amplitudes in $N \gg \alpha^{-1}$ domain is expected to be shared by other derivatively-coupled theories, which are also considered as candidates for classicalization. For example, in a theory $(\partial_\mu \phi \partial_\mu \phi)^2$ the effective quartic coupling scales as the fourth power of momentum and the multi-particle production must be suppressed in the domain $N \gg \alpha^{-1}$.

7.2. Perturbative insights into non-perturbative black hole production

The former discussion on the factorial growth of the cross section for scalar theories of type ϕ^4 sheds light on how the perturbative amplitudes can foresee the non-perturbative existence of black holes. The simplest way to identify non-perturbative physics within perturbation theory is to look for the limits of applicability of perturbation theory. As previously discussed a key aspect of the approach to quantum gravity based on classicalization lies in replacing ultra-Planckian $2 \rightarrow 2$ strongly coupled processes, violating unitarity already at tree level, by $2 \rightarrow N$ weakly coupled processes where the total center of mass energy \sqrt{s} is equi-distributed into the N soft outgoing gravitons. Irrespectively how large is \sqrt{s} the corresponding process at tree level is, for large enough N , well defined perturbatively. Indeed, all vertices involved in the process can be made, tuning N , arbitrarily small. However, there is a prize that we need to pay when we proceed in this way, namely the growth of the number of tree Feynman diagrams contributing to the $2 \rightarrow N$ process. This growth is at the origin of the factorials discussed in the previous subsection. The interplay between the effective coupling constant and the growth of the number of diagrams sets the regime where weakly coupled perturbative analysis at tree level is reliable.

To fix ideas let us consider a generic bosonic theory where amplitudes $2 \rightarrow N$ for arbitrary large value of N are not forbidden by any form of the selection rule. To characterize the theory we need to know the number of vertices involved in the process, the number $C(N)$ of tree level diagrams and the value of the effective coupling g entering into each vertex. If we assume N to be large enough and we consider a three point vertex the number of vertices will be order N and the number of trees will grow with N as (up to exponents unimportant for this discussion),⁹

$$C(N) \sim N!. \quad (7.3)$$

consequently the cross section will behave as

$$\sigma_{2 \rightarrow N} \sim N! \alpha^N \quad (7.4)$$

for $\alpha \equiv g^2$. Note that for a ϕ^4 theory we get (7.2). The leading dependence of $\sigma_{2 \rightarrow N}$ on the center of mass momentum is implicitly contained in the effective coupling $\alpha \equiv g^2$. The effective coupling is defining the interaction in the underlying Lagrangian. This interaction term can define a relevant or an irrelevant operator depending on the spin of the bosonic field involved in the process. The ϕ^4 case corresponds to the marginal case. If it is a relevant operator (as it will be with a $g\phi^3$ type of theory) then the effective coupling α will depend on the corresponding momentum transfer $\sqrt{t(N)}$ – which for the classicalization kinematics (where \sqrt{s} is equi-distributed) is of order $\frac{\sqrt{s}}{N}$ – as $\frac{1}{t(N)}$. However, if the interaction vertex defines an irrelevant operator, as it is the case for the three-point vertex of gravitons, α goes as $t(N)$. In this case we obtain

$$\sigma_{2 \rightarrow N} \sim N! \left(\frac{s L_P^2}{N^2} \right)^N \quad (7.5)$$

that is precisely what we have reached for these amplitudes both in the KLT approach (supplemented by the results from scattering equations) as well as in the string approach.

Once we have fixed the effective coupling and its dependence both on s and N we can set the limits of perturbation theory. The perturbative approach to multi-particle scattering is reliable only if

$$\frac{\sigma_{2 \rightarrow N+1}}{\sigma_{2 \rightarrow N}} \lesssim 1, \quad (7.6)$$

which leads to

$$\alpha \lesssim \frac{1}{N}. \quad (7.7)$$

Although the amplitude in absolute terms may not violate unitarity, the turning point indicates that some non-perturbative information must be included for the corresponding value of N . Thus, for gravity the bound (7.6) implies.

$$N \gtrsim s L_P^2. \quad (7.8)$$

This is a very interesting result since this bound is telling us that N should be larger or equal to the corresponding black hole entropy (equivalently number of constituents) of a black hole with

⁹ To be more precise if we use Cayley's formula we should expect $C(N) \sim N!/2^N$ where the factor 2^N depends on the specific assumption that the vertex is a three point vertex. Incidentally, note that the factor 2^{-N} is consistent with the similar factor appearing in the string result (5.81) presented at the end of Section 5.

mass equal to the center of mass energy. It is instructive to see how the limits of applicability of perturbation theory are teaching us about the underlying physics. For large value of s and N much smaller than s in Planck units, the amplitude is obviously violating unitarity. At this point you can wonder if increasing the value of N for the same value of s will improve the situation. What you observe is that while you are in the regime with N much smaller than s , increasing N is not making the situation better but worst. In other words in this regime the ratio (7.6) is bigger than one. This perturbative situation changes only when you reach a critical value of N where the ratio (7.6) reaches one and starts to decrease. The regime where the ratio (7.6) is bigger than one is precisely the regime corresponding, in the black hole portrait, to the strong coupling regime with λ larger than one. Thus, whenever we violate the above perturbative bound we enter into a regime that requires, in order to be analyzed, non-perturbative input. Nicely enough this regime precisely agrees with the region $\lambda \geq 1$, i.e., with the region that is cut out using the non-perturbative corpuscular information of the black hole portrait. Moreover, the turning point happens precisely when N is equal to the black hole entropy. This makes explicit the way perturbation theory anticipates not only the non-perturbative black hole formation but also, as already stressed many times, its corpuscular constituency. In other words the perturbative analysis, both in field theory as well as in string theory, sets the limit of applicability of perturbation theory in the classicalization kinematics precisely at the point where the system of outgoing gravitons reaches the dynamical condition defining the critical point of the black hole portrait. Furthermore, perturbation theory encodes information about the black hole existence, despite the fact that for corresponding value of N the amplitude is still unitary in the absolute sense.

Finally let us stress the difference with the case where the three point interaction vertex is a relevant operator. In this case the former bound becomes $s \geq N^3$. This means that we don't have problems for arbitrarily large s and small N but instead for large N and small s . This is a key difference with the case of irrelevant operators i.e. with the case of gravity.

8. Outlook: classicalization and black holes in the light of graviton amplitudes

In previous sections we have collected some results regarding tree level N -graviton amplitudes in the eikonal-Regge kinematical regime. In this summary section we shall complement the discussion, already initiated in the introduction, on the physical meaning of these findings.

In the field theory context we have focused our attention on two key issues. First of all, we have analyzed how for ultra-Planckian values of \sqrt{s} the amplitude is smoothed-out once we increase the number N of outgoing gravitons. This kinematic mechanism of unitarization – which is at the core of the idea of classicalization – would be nevertheless completely useless if the contribution of this kinematics to the total scattering rate were very much suppressed. So our second task has been to extract from the concrete expressions of the amplitudes this suppression factor. By using the scattering equations in the classicalization limit this has been accomplished for graviton amplitudes (4.16) in Eq. (5.81). For large N the graviton scattering matrix element in this kinematic regime depends on s and N as $\sim (\frac{sL_p^2}{N^2})^N N!$. From this expression we observe that the amplitude starts to be smoothed-out for $N = sL_p^2$. We can interpret this value of N as the unitarity threshold for the given value of s . In other words, a slower growth of N in the double-scaling limit ($s, N \rightarrow \infty$) violates unitarity. Indeed, parameterizing the scaling as $N^{1+\gamma} = (sL_p^2)$, the matrix element in large N scales as $\sim N^{\gamma N} e^{-N}$, which for $\gamma > 0$ blows up for sufficiently large N . However, notice that the final states obtained in unitarity-violating scaling are precisely the

ones excluded by non-perturbative corpuscular physics, since they correspond to the over-critical region of the graviton bound-state, with $\lambda > 1$, since $\lambda = sL_p^2/N$.

The key lesson we learn from the expression of the amplitude is how much this concrete kinematical configuration contributes to the total amplitude. Indeed, for this threshold value the suppression factor is $\sim e^{-N}$. The amplitudes with faster growing values of N with s are more suppressed, while the slower-growing ones, that would naively violate unitarity, are excluded by the non-perturbative many-body physics of soft gravitons.

As we have discussed, the physics interpretation of the previous result is quite transparent. The value of N at which the amplitude starts to smooth-out is precisely what would be the Bekenstein–Hawking entropy of a black hole of mass equal to the center of mass energy, i.e., \sqrt{s} . Moreover, the suppression factor is precisely what would be the multiplicity of states of such a black hole according to the corpuscular quantum portrait. The crucial information we extract from here is how the amplitude reveals the microscopic structure of the black hole as being composed of the N soft outgoing gravitons. This is precisely, as already stressed in the introduction, what we expect from the N -portrait of black holes as composite systems of soft gravitons. Moreover, the kinematical conditions of the outgoing gravitons for $N = sL_p^2$ are the ones determining the critical point of the graviton Bose–Einstein condensate. It is this criticality what accounts for the entropy needed to compensate the exponential suppression factor in the form of a large multiplicity of gapless Bogoliubov modes.

As already argued the regime with N larger than the threshold value determined by the perturbative amplitude corresponds to $\lambda < 1$ and although is not violating unitarity is very much suppressed. From the microscopic point of view we understand the large suppression of this multi-particle kinematics as due to the fact that the system defined by the outgoing gravitons is far from the critical point with a well-defined finite gap for the Bogoliubov modes. The regime with small number of outgoing gravitons violates unitarity and corresponds from the microscopic point of view to the strong collective-coupling regime $\lambda > 1$. This is the regime that the microscopic non-perturbative dynamics is cutting out and this is the key of the unitarization mechanism through the black hole formation.

But what have we learned from the string theory amplitudes in this kinematical regime? In this eikonal-Regge kinematics it is easy to identify when purely stringy effects become relevant. Indeed, the effective center of mass energy $\sqrt{s_{i,i+1}}$ between two consecutive final state gravitons goes like $\sqrt{s_{i,i+1}} \sim \frac{\sqrt{s}}{N}$. Thus, this partial contribution to the total amplitude becomes sensitive to string effects if

$$\frac{\sqrt{s}}{N} \geq M_s, \quad (8.1)$$

for M_s being the string mass scale. In such a case each of the $N - 3$ vertical graviton propagators (see Fig. 7) should be effectively Reggeized. In other words, in this multi-Regge kinematics and in the regime (8.1), we should effectively dress each propagator with the Regge factor

$$s_{i,i+1}^{(\alpha' s_{i,i+1})}, \quad (8.2)$$

leading to an overall contribution of the order of $e^{-(N-3)\frac{s}{N^2} \ln(\frac{s}{N^2})}$ with s measured in string units. This estimate is to be compared with the result (5.56) from the string theory computation.

Consequently, the field theory computation is reliable if $\frac{\sqrt{s}}{N} \leq M_s$. In this case the factor coming from the Reggeization of the exchanged gravitons becomes one. In Section 5 we have considered both situations in the double-scaling limit of both s and N large. In this double-scaling

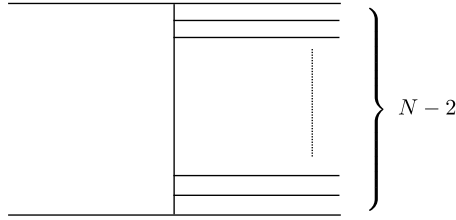


Fig. 7. N -graviton scattering with $N - 3$ graviton propagators.

limit stringy Reggeization effects become relevant for $\frac{\sqrt{s}}{N}$ larger than one (case (i)) while they are suppressed for $\frac{\sqrt{s}}{N}$ smaller than one (case (ii)) (in both cases written in string units). Notice that the concrete value of L_s setting the regime where string effects are relevant only enters, in this kinematics, in the form of the formerly-described Reggeization of the exchanged gravitons.

In order to compare the field theoretic and the string theoretic pictures we need the relation between the two relevant mass scales, namely L_P and L_s given in Eq. (5.2). With this relation we can, as described already in the introduction, consider different regimes. In the regime where $g_s^2 N < 1$ stringy effects due to the Reggeization of the exchanged gravitons starts to be relevant before the created soft gravitons organize themselves into a field theoretic self-sustained condensate i.e. in the weak coupling regime $\lambda < 1$. For $g_s^2 N > 1$ instead the string effects are relevant only in the regime where the outgoing gravitons would be strongly coupled and therefore we could wonder if these Regge effects tame the field theoretical violation of unitarity.

However, the interesting value at which we want to focus our attention is $g_s^2 N = 1$, i.e., when the threshold of string effects exactly matches the field-theoretic critical point of black hole formation. For this special point we have,

$$g_s = \frac{1}{\sqrt{N}}. \quad (8.3)$$

What is the meaning of this relation? The answer is simply that this value corresponds to the well known string–black hole correspondence. The previous discussion sheds however a new light on this correspondence as determining the point where – for given kinematics – the threshold of string effects coincides with the critical point of the graviton Bose–Einstein condensate. Or, as already stressed in the introduction, this is the situation when at the would-be critical point the string coupling between the constituent quanta becomes equally important as the gravitational coupling [56] (see also [57,58]).

Finally, we would like to put forward a slightly more speculative observation. Until this point although we have been working within the general frame ($closed = open \times open$) or equivalently ($gravity = YM^2$) we have not used in any explicit way the information about the color of the YM gauge sector. What do we get if we naively use it? As it is customary, we have to use $g_s = g_{open}^2$, cf. Eq. (5.3). Moreover, if we think of the open string, as originally pointed out by 't Hooft, as the planar limit of the gauge theory, we should identify $g_{open}^2 = \frac{1}{N_c}$ for N_c being the rank of the gauge group. If we naively combine these two ingredients we arrive to the formal “color-kinematics” relation,

$$N = N_c^2. \quad (8.4)$$

Of course in this formal relation N refers to the number of created soft gravitons and thus it must be interpreted with a bit of care. Note that we arrive to this formal relation only when we

put ourselves at the threshold of black hole formation. In these conditions the former relation between N and N_c becomes very reminiscent of the gauge/gravity duality relations. Indeed what this relation does is to identify the black hole entropy N with what would be the c -function of the gauge theory. Incidentally, an information that we have never used in our computation of graviton amplitudes. Pushing a bit forward the analogy, it seems to indicate a deep connection, taking place at the black hole threshold formation, between the hidden Chan–Paton factors dressing the *open* string we have used in the computation of the gravitational amplitudes and the gauge holographic dual. Obviously, this observation should be taken with a grain of salt but we feel it certainly deserves a further study.

The former “color kinematics” relation could be anticipated from a different point of view directly working with the gauge theory (or equivalently open string) amplitudes by simply imposing a good planar limit for higher loop amplitudes built using as generalized vertex the gauge theory amplitudes A_N . For instance with the gauge theory amplitudes A_N interpreted as effective vertex with N external lines we can define a $2 \rightarrow 2$ scattering amplitude with $N - 3$ internal loops. This loop gauge amplitude scales with the number of colors as N_c^{N-3} and therefore in order to get a good large N_c planar limit we need to impose (for N large enough)

$$g_{YM}^{2N} s^{-N} N^{2N} \sim N_c^{-N}. \quad (8.5)$$

Using now $g_{YM}^2 \sim \frac{1}{N_c}$ the former condition becomes $\sqrt{s} = N$ (in open string units) which translated into Planckian language with $g_s = \frac{1}{\sqrt{N}}$ leads to the black hole threshold relation $\sqrt{s} = \sqrt{N} M_P$. In other words what we observe is that at the level of the gauge amplitudes the condition of having a good planar limit in ’t Hooft sense (for loop amplitudes) underlies the mechanism of unitarization by black hole formation for the corresponding tree level gravitational amplitudes.

In conclusion, in this paper we have discussed a particular high-energy limit of Yang–Mills and gravity scattering amplitudes, both from the field theory as well as from the string perspective. From the technical side, we have derived new closed expressions for the tree-level string scattering amplitudes at high energies, which are valid for an arbitrarily large number of external particles. Moreover, we have considered a particular high-energy limit, which corresponds to the case of classicalization via black hole production, where black holes are bound states of a large number of very soft gravitons. As discussed, this correspondence finds additional support by the existence of a potentially new kind of large N gauge–gravity correspondence with N the number of external particles in eikonal Regge kinematics. As pointed out it would be interesting to relate this large N duality to the standard large N duality arising in the context of holography and the AdS/CFT correspondence.

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References

- [1] G. Dvali, C. Gomez, Self-completeness of Einstein gravity, arXiv:1005.3497 [hep-th].
- [2] G. Dvali, G.F. Giudice, C. Gomez, A. Kehagias, UV-completion by classicalization, J. High Energy Phys. 1108 (2011) 108, arXiv:1010.1415 [hep-ph].
- [3] G. Dvali, C. Gomez, A. Kehagias, Classicalization of gravitons and goldstones, J. High Energy Phys. 1111 (2011) 070, arXiv:1103.5963 [hep-th].
- [4] G. Dvali, C. Gomez, Black hole's quantum N -portrait, Fortschr. Phys. 61 (2013) 742, arXiv:1112.3359 [hep-th];
G. Dvali, C. Gomez, Quantum compositeness of gravity: black holes, AdS and inflation, J. Cosmol. Astropart. Phys. 1401 (2014) 023, arXiv:1312.4795.
- [5] G. 't Hooft, Graviton dominance in ultrahigh-energy scattering, Phys. Lett. B 198 (1987) 61–63.
- [6] D. Amati, M. Ciafaloni, G. Veneziano, Superstring collisions at Planckian energies, Phys. Lett. B 197 (1987) 81;
D. Amati, M. Ciafaloni, G. Veneziano, Classical and quantum gravity effects from Planckian energy superstring collisions, Int. J. Mod. Phys. A 3 (1988) 1615;
D. Amati, M. Ciafaloni, G. Veneziano, Can space–time be probed below the string size? Phys. Lett. B 216 (1989) 41;
D. Amati, M. Ciafaloni, G. Veneziano, Higher order gravitational deflection and soft bremsstrahlung in Planckian energy superstring collisions, Nucl. Phys. B 347 (1990) 550;
D. Amati, M. Ciafaloni, G. Veneziano, Effective action and all order gravitational eikonal at Planckian energies, Nucl. Phys. B 403 (1993) 707.
- [7] D.J. Gross, P.F. Mende, The high-energy behavior of string scattering amplitudes, Phys. Lett. B 197 (1987) 129;
D.J. Gross, P.F. Mende, String theory beyond the Planck scale, Nucl. Phys. B 303 (1988) 407;
D.J. Gross, J.L. Manes, The high-energy behavior of open string scattering, Nucl. Phys. B 326 (1989) 73.
- [8] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, G.R. Dvali, New dimensions at a millimeter to a Fermi and superstrings at a TeV, Phys. Lett. B 436 (1998) 257, arXiv:hep-ph/9804398;
T. Banks, W. Fischler, A model for high-energy scattering in quantum gravity, arXiv:hep-th/9906038;
G.R. Dvali, G. Gabadadze, M. Kolanovic, F. Nitti, Scales of gravity, Phys. Rev. D 65 (2002) 024031, arXiv:hep-th/0106058;
S.B. Giddings, S.D. Thomas, High-energy colliders as black hole factories: the end of short distance physics, Phys. Rev. D 65 (2002) 056010, arXiv:hep-ph/0106219;
S. Dimopoulos, G.L. Landsberg, Black holes at the LHC, Phys. Rev. Lett. 87 (2001) 161602, arXiv:hep-ph/0106295.
- [9] F. Cachazo, S. He, E.Y. Yuan, Scattering of massless particles: scalars, gluons and gravitons, J. High Energy Phys. 1407 (2014) 033, arXiv:1309.0885 [hep-th];
F. Cachazo, S. He, E.Y. Yuan, Scattering of massless particles in arbitrary dimensions, Phys. Rev. Lett. 113 (2014) 171601, arXiv:1307.2199 [hep-th];
F. Cachazo, S. He, E.Y. Yuan, Scattering equations and Kawai–Lewellen–Tye orthogonality, Phys. Rev. D 90 (2014) 065001, arXiv:1306.6575 [hep-th].
- [10] H. Kawai, D.C. Lewellen, S.H.H. Tye, A relation between tree amplitudes of closed and open strings, Nucl. Phys. B 269 (1986) 1.
- [11] F. Kuhnel, B. Sundborg, High-energy gravitational scattering and Bose–Einstein condensates of gravitons, arXiv:1406.4147 [hep-th].
- [12] G. Dvali, C. Gomez, Black holes as critical point of quantum phase transition, Eur. Phys. J. C 74 (2014) 2752, arXiv:1207.4059 [hep-th];
D. Flassig, A. Pritzel, N. Wintergerst, Black holes and quantumness on macroscopic scales, Phys. Rev. D 87 (2013) 084007, arXiv:1212.3344;
G. Dvali, D. Flassig, C. Gomez, A. Pritzel, N. Wintergerst, Scrambling in the black hole portrait, Phys. Rev. D 88 (12) (2013) 124041, arXiv:1307.3458 [hep-th].
- [13] G. Dvali, C. Gomez, Black hole's $1/N$ hair, Phys. Lett. B 719 (2013) 419, arXiv:1203.6575 [hep-th];
G. Dvali, C. Gomez, D. Lüst, Black hole quantum mechanics in the presence of species, Fortschr. Phys. 61 (2013) 768, arXiv:1206.2365 [hep-th];
G. Veneziano, Quantum hair and the string–black hole correspondence, Class. Quantum Gravity 30 (2013) 092001, arXiv:1212.2606 [hep-th];
P. Binetruy, Vacuum energy, holography and a quantum portrait of the visible Universe, arXiv:1208.4645 [gr-qc];
F. Berkhahn, S. Muller, F. Niedermann, R. Schneider, Microscopic picture of non-relativistic classicalons, J. Cosmol. Astropart. Phys. 1308 (2013) 028, arXiv:1302.6581 [hep-th];

- R. Casadio, Charged shells and elementary particles, *Int. J. Mod. Phys. A* 28 (2013) 1350088, arXiv:1303.1274 [gr-qc];
- R. Casadio, A. Orlandi, Quantum harmonic black holes, *J. High Energy Phys.* 1308 (2013) 025, arXiv:1302.7138 [hep-th];
- R. Casadio, A. Giugno, O. Micu, A. Orlandi, Black holes as self-sustained quantum states, and Hawking radiation, arXiv:1405.4192 [hep-th];
- W. Mück, Counting photons in static electric and magnetic fields, *Eur. Phys. J. C* 73 (2013) 2679, arXiv:1310.6909 [hep-th];
- R. Casadio, O. Micu, F. Scardigli, Quantum hoop conjecture: black hole formation by particle collisions, *Phys. Lett. B* 732 (2014) 105, arXiv:1311.5698 [hep-th];
- F. Kuhnel, Bose–Einstein condensates with derivative and long-range interactions as set-ups for analog black holes, arXiv:1312.2977 [gr-qc];
- F. Kuhnel, B. Sundborg, Modified Bose–Einstein condensate black holes in d dimensions, arXiv:1401.6067 [hep-th];
- M. Chen, Y.C. Huang, Gas model of gravitons with light speed, arXiv:1402.5767 [gr-qc];
- S. Hofmann, T. Rug, A quantum bound state description of black holes, arXiv:1403.3224 [hep-th];
- F. Kuhnel, B. Sundborg, Decay of graviton condensates and their generalizations in arbitrary dimensions, arXiv:1405.2083 [hep-th];
- C.J. Hogan, Directional entanglement of quantum fields with quantum geometry, arXiv:1312.7798 [gr-qc];
- R. Casadio, A. Giugno, O. Micu, A. Orlandi, Black holes as self-sustained quantum states, and Hawking radiation, arXiv:1405.4192 [hep-th];
- R. Casadio, O. Micu, P. Nicolini, Minimum length effects in black hole physics, arXiv:1405.1692 [hep-th];
- L. Gruending, S. Hofmann, S. Müller, T. Rug, Probing the constituent structure of black holes, arXiv:1407.1051 [hep-th].
- [14] S.B. Giddings, Possible observational windows for quantum effects from black holes, arXiv:1406.7001 [hep-th].
- [15] H. Elvang, Y.-t. Huang, Scattering amplitudes, arXiv:1308.1697 [hep-th].
- [16] V.E. Asriekov, Choice of invariant variables for the many-point functions, *J. Exptl. Theoret. Phys. (USSR)* 42, 565–576.
- [17] L.J. Dixon, A brief introduction to modern amplitude methods, arXiv:1310.5353 [hep-ph].
- [18] M.D. Schwartz, *Quantum Field Theory and the Standard Model*, Cambridge University Press, 2014.
- [19] N. Arkani-Hamed, J. Kaplan, On tree amplitudes in gauge theory and gravity, *J. High Energy Phys.* 0804 (2008) 076, arXiv:0801.2385 [hep-th].
- [20] R.C. Brower, J. Polchinski, M.J. Strassler, C.-I. Tan, The Pomeron and gauge/string duality, *J. High Energy Phys.* 0712 (2007) 005, arXiv:hep-th/0603115.
- [21] C. Cheung, D. O’Connell, B. Wecht, BCFW recursion relations and string theory, *J. High Energy Phys.* 1009 (2010) 052, arXiv:1002.4674 [hep-th].
- [22] R.H. Boels, D. Marmiroli, N.A. Obers, On-shell recursion in string theory, *J. High Energy Phys.* 1010 (2010) 034, arXiv:1002.5029 [hep-th].
- [23] R. Boels, K.J. Larsen, N.A. Obers, M. Vonk, MHV, CSW and BCFW: field theory structures in string theory amplitudes, *J. High Energy Phys.* 0811 (2008) 015, arXiv:0808.2598 [hep-th].
- [24] Z. Bern, L.J. Dixon, M. Perelstein, J.S. Rozowsky, Multileg one loop gravity amplitudes from gauge theory, *Nucl. Phys. B* 546 (1999) 423, arXiv:hep-th/9811140.
- [25] N.E.J. Bjerrum-Bohr, P.H. Damgaard, T. Sondergaard, P. Vanhove, The momentum kernel of gauge and gravity theories, *J. High Energy Phys.* 1101 (2011) 001, arXiv:1010.3933 [hep-th].
- [26] S.J. Parke, T.R. Taylor, An amplitude for n gluon scattering, *Phys. Rev. Lett.* 56 (1986) 2459.
- [27] L.J. Dixon, Calculating scattering amplitudes efficiently, in: *QCD and Beyond, Proceedings of TASI 1995*, World Scientific, 1996, arXiv:hep-ph/9601359.
- [28] X.O. Camanho, J.D. Edelstein, J. Maldacena, A. Zhiboedov, Causality constraints on corrections to the graviton three-point coupling, arXiv:1407.5597 [hep-th].
- [29] G. Veneziano, Construction of a crossing – symmetric, Regge behaved amplitude for linearly rising trajectories, *Nuovo Cimento A* 57 (1968) 190.
- [30] G. D’Appollonio, P. Di Vecchia, R. Russo, G. Veneziano, High-energy string–brane scattering: Leading eikonal and beyond, *J. High Energy Phys.* 1011 (2010) 100, arXiv:1008.4773 [hep-th];
- G. D’Appollonio, P. Di Vecchia, R. Russo, G. Veneziano, The leading eikonal operator in string–brane scattering at high energy, arXiv:1310.4478 [hep-th].
- [31] S.B. Giddings, The gravitational S-matrix: Erice lectures, arXiv:1105.2036 [hep-th].

- [32] S.B. Giddings, M. Schmidt-Sommerfeld, J.R. Andersen, High energy scattering in gravity and supergravity, *Phys. Rev. D* 82 (2010) 104022, arXiv:1005.5408 [hep-th].
- [33] S. Stieberger, Closed superstring amplitudes, single-valued multiple zeta values and the Deligne associator, *J. Phys. A* 47 (2014) 155401, arXiv:1310.3259 [hep-th].
- [34] S. Stieberger, T.R. Taylor, Closed string amplitudes as single-valued open string amplitudes, *Nucl. Phys. B* 881 (2014) 269, arXiv:1401.1218 [hep-th].
- [35] A. Selberg, Remarks on a multiple integral, *Norsk Mat. Tidsskr.* 26 (1944) 71–78.
- [36] M. Abramowitz, I.A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover Publications, New York, 1972.
- [37] C.R. Mafra, O. Schlotterer, S. Stieberger, Complete N -point superstring disk amplitude I. Pure spinor computation, *Nucl. Phys. B* 873 (2013) 419, arXiv:1106.2645 [hep-th];
C.R. Mafra, O. Schlotterer, S. Stieberger, Complete N -point superstring disk amplitude II. Amplitude and hypergeometric function structure, *Nucl. Phys. B* 873 (2013) 461, arXiv:1106.2646 [hep-th].
- [38] J. Broedel, O. Schlotterer, S. Stieberger, Polylogarithms, multiple zeta values and superstring amplitudes, *Fortschr. Phys.* 61 (2013) 812, arXiv:1304.7267 [hep-th].
- [39] S. Stieberger, T.R. Taylor, Maximally helicity violating disk amplitudes, twistors and transcendental integrals, *Phys. Lett. B* 716 (2012) 236, arXiv:1204.3848 [hep-th].
- [40] S. Stieberger, T.R. Taylor, Superstring amplitudes as a Mellin transform of supergravity, *Nucl. Phys. B* 873 (2013) 65, arXiv:1303.1532 [hep-th].
- [41] M.R. Garousi, Disk level S -matrix elements at eikonal Regge limit, *Phys. Lett. B* 696 (2011) 291, arXiv:1010.4950 [hep-th].
- [42] M.V. Fedoryuk, *Asymptotic Expansion: Integrals and Series*, Nauka, Moscow, 1987;
N.G. de Bruijn, *Asymptotic Methods in Analysis*, Dover Publications, 2010.
- [43] E. Witten, Parity invariance for strings in twistor space, *Adv. Theor. Math. Phys.* 8 (2004) 779, arXiv:hep-th/0403199.
- [44] S. Weinzierl, On the solutions of the scattering equations, *J. High Energy Phys.* 1404 (2014) 092, arXiv:1402.2516 [hep-th];
L. Dolan, P. Goddard, The polynomial form of the scattering equations, *J. High Energy Phys.* 1407 (2014) 029, arXiv:1402.7374 [hep-th].
- [45] F. Cachazo, Y. Geyer, A ‘Twistor String’ inspired formula for tree-level scattering amplitudes in $N = 8$ SUGRA, arXiv:1206.6511 [hep-th].
- [46] S. Stieberger, T.R. Taylor, Superstring/supergravity Mellin correspondence in Grassmannian formulation, *Phys. Lett. B* 725 (2013) 180, arXiv:1306.1844 [hep-th].
- [47] F. Cachazo, S. He, E.Y. Yuan, Scattering in three dimensions from rational maps, *J. High Energy Phys.* 1310 (2013) 141, arXiv:1306.2962 [hep-th].
- [48] R. Roiban, M. Spradlin, A. Volovich, On the tree level S matrix of Yang–Mills theory, *Phys. Rev. D* 70 (2004) 026009, arXiv:hep-th/0403190.
- [49] R. Wong, *Asymptotic Approximations of Integrals*, Academic Press, 2001.
- [50] T.J. Stieltjes, Sur les racines de l’équation $X_n = 0$, *Acta Math.* 9 (1886) 385–400 (Oeuvres Complètes II, 73–88).
- [51] C. Kalousios, Massless scattering at special kinematics as Jacobi polynomials, *J. Phys. A* 47 (2014) 215402, arXiv:1312.7743 [hep-th].
- [52] G. Szegoe, *Orthogonal Polynomials*, American Mathematical Society Colloquium Publications, vol. 23, American Mathematical Society, 1939.
- [53] W. Mück, On the number of soft quanta in classical field configurations, *Can. J. Phys.* 92 (9) (2014) 973–975, arXiv:1306.6245 [hep-th];
W. Mück, G. Pozzo, Quantum portrait of a black hole with Pöschl–Teller potential, *J. High Energy Phys.* 1405 (2014) 128, arXiv:1403.1422 [hep-th].
- [54] See the reference by Casadio, Giugno, Micu and Orlandi in [13].
- [55] H. Goldberg, Breakdown of perturbation theory at tree level in theories with scalars, *Phys. Lett. B* 246 (1990) 445;
J.M. Cornwall, On the high-energy behavior of weakly coupled gauge theories, *Phys. Lett. B* 243 (1990) 271.
- [56] G.T. Horowitz, J. Polchinski, A correspondence principle for black holes and strings, *Phys. Rev. D* 55 (1997) 6189, arXiv:hep-th/9612146.
- [57] G. Dvali, D. Lüst, Evaporation of microscopic black holes in string theory and the bound on species, *Fortschr. Phys.* 58 (2010) 505, arXiv:0912.3167 [hep-th].
- [58] G. Dvali, C. Gomez, Species and strings, arXiv:1004.3744 [hep-th].