Anisotropic Jüttner (relativistic Boltzmann) distribution

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Abstract. A rigorous derivation of the Jüttner (covariant Boltzmann) distribution is provided for anisotropic pressure (or temperature) tensors. It was in similar form anticipated first by Gladd (1983). Its manifestly covariant version follows straightforwardly from its scalar property.

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The Jüttner distribution (Jüttner, 1911) is the relativistically generalized classical isotropic Maxwell–Boltzmann distribution, whether written in its dependence on relativistic particle energy $\epsilon_p = m\gamma(v)c^2$ or momentum $p$, with $\gamma = \sqrt{1 + p^2/m^2c^2}$. The phase-volume element $dx\,dp$ is covariant (a consequence of its scalar nature). Hence the Jüttner distribution is as well covariant but not manifestly covariant. In anisotropic relativistic gases or plasmas (for application to anisotropic plasmas cf. Yoon, 1989, 2007; Gladd, 1983) and in drifting plasmas (for recent examples cf. e.g. Swisdak, 2013; Lopez et al., 2014; Alves et al., 2015; Zenitani, 2015; DeVore et al., 2015) the form of the Jüttner distribution is usually assumed. Below we provide its simple analytical derivation and manifestly covariant version.

Thermally relativistic implies thermal speeds $v_t/v_c > 10^{-2}$. Avoiding creation, annihilation, and Compton interactions requires $T < mc^2$, with $T$ temperature in energy units, hence weakly relativistic thermal electrons of some 10 eV < $T < 0.5$ MeV, covering most hot classical plasmas.

Maxwell–Boltzmann distributions are solutions of the stationary one-particle Boltzmann equation with the argument of the ratio of the single particle to average thermal energies, viz. $\epsilon_p/T$. Properly normalized they give the probability at temperature $T$ for finding all particles of given momentum $p$ (or energy $\epsilon_p$) in the interval $dp$ (or $d\epsilon_p$) in momentum-space volume $dp$. With three-momentum vector $p = (p_\parallel, p_\perp)$ in index notation

$$\epsilon_p^2 = c^2p_\parallel^2 - m^2c^4$$

suggests introduction of temperature anisotropy guided by the diagonal anisotropy of pressure $P = N\{T_\parallel\delta^i_j + (T_\parallel - T_\perp)\delta^i_\perp\}$ (as for instance in magnetized plasma), with anisotropy in direction 3 (in plasma the direction of the magnetic field $b = B/B$, for instance). The inverse pressure/temperature tensor is $P^{-1} = (T_\parallel N)^{-1}\Theta$.

$\quad \Theta = \Theta^j_i = \delta^j_i + (A - 1)\delta^j_3$, with $A = T_\parallel/T_\perp$. (2)

Replacing $\delta^j_i$ in Eq. (1) with $\Theta^j_i$, valid in the four-velocity frame $U_j = (\epsilon_p/mc, 0)$, defining $\beta_\parallel = mc^2/T_\perp$, $\beta_\parallel = mc^2/T_\perp$, putting $p_\parallel = p\sin\theta$, $p_\perp = p\cos\theta$, and defining $p/mc \rightarrow p$ yields

$$\frac{\epsilon_p^2}{T_\perp^2} = \beta_\parallel^2 \left[1 + p^2 \left(\sin^2\theta + A\cos^2\theta\right)\right].$$

The square root of Eq. (3) enters the Boltzmann factor. Up to normalization $C$, the anisotropic Jüttner distribution function of the ideal gas becomes

$$F_j(p, A) = C \exp\left[-\beta_\perp \sqrt{1 + p_\perp^2 + A p_\parallel^2}\right].$$

With $A = 1$, $\beta_\perp = \beta_\parallel = \beta$ this is the ordinary Jüttner function. Expanding the root in the limit $c \rightarrow \infty$ reproduces the ordinary nonrelativistic anisotropic Maxwell–Boltzmann distribution. Extensions to drifting or non-ideal gases are straightforward.

This in principle trivial result was anticipated first without proof by Gladd (1983) in application to the whistler instabil-
Jüttner's anisotropic distribution is useful for analytical or numerical (as in Gladd, 1983) calculations. In particle-in-cell simulations the initial distribution is prescribed. In practice there is little need to choose it in the simulations to satisfy the Jüttner equilibrium requirement. Solving for all relativistic particle orbits in their self-consistent fields, the initial distribution readily adjusts itself to the physical distribution that evolves under the mutual interactions.

As a side product, this straightforward rigorous derivation indicates that in relativistic media the isotropic temperature $T$ and its inverse $\beta = 1/T$ should be understood as vectors (confirming Nakamura, 2009, who suggested it for different reasons). In presence of anisotropy they become tensors. Including particle spins requires a slightly different treatment.

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References


