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Generalized Galilean algebras and Newtonian gravity

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ABSTRACT

The non-relativistic versions of the generalized Poincaré algebras and generalized *AdS*-Lorentz algebras are obtained. These non-relativistic algebras are called, generalized Galilean algebras of type I and type II and denoted by $\mathcal{G}\mathfrak{B}_n$ and $\mathcal{G}\mathfrak{L}_n$ respectively. Using a generalized Inönü–Wigner contraction procedure we find that the generalized Galilean algebras of type I can be obtained from the generalized Galilean algebras type II. The *S*-expansion procedure allows us to find the $\mathcal{G}\mathfrak{B}_5$ algebra from the Newton Hooke algebra with central extension. The procedure developed in Ref. [1] allows us to show that the nonrelativistic limit of the five dimensional Einstein–Chern–Simons gravity is given by a modified version of the Poisson equation. The modification could be compatible with the effects of Dark Matter, which leads us to think that Dark Matter can be interpreted as a non-relativistic limit of Dark Energy.

1. Introduction

In Refs. [2,3] it was shown that the *S*-expansion procedure allows to construct Chern–Simons gravities in odd dimensions invariant under an algebra referred to as $\mathfrak{B}_m$ algebra and Born–Infeld gravities in even dimensions [4–7] invariant under a certain subalgebra of the $\mathfrak{B}_m$ algebra, leading to general relativity in a certain limit. The $\mathfrak{B}_m$ algebras, which could be also called 'generalized Poincaré algebras', were constructed from *AdS*-algebra and a particular semigroup denoted by $S_E^{(N)} = \{\lambda_\alpha\}_{\alpha=0}^{N+1}$, which is endowed with the multiplication rule $\lambda_\alpha\lambda_\beta = \lambda_{\alpha+\beta}$ if $\alpha+\beta \leq N+1$; $\lambda_\alpha\lambda_\beta = \lambda_{N+1}$ if $\alpha+\beta > N+1$.

In Ref. [8] it was shown that the so-called *AdS*-Lorentz algebra $\mathfrak{so}(D-1,1) \oplus \mathfrak{so}(D-1,2)$ algebra [9–11] in D dimensions can be obtained from *AdS*-algebra $\mathfrak{so}(D-1,2)$ by means of the *S*-expansion procedure with a semigroup which is known as $S_{\mathcal{M}}^{(2)}$. This *AdS*-Lorentz algebra is related to the so-called Maxwell algebra [12,13] via a contraction process [14].

Recently it was shown in Ref. [15] that the resonant *S*-expansion of the *AdS* Lie algebra leads to a generalization of the *AdS*-Lorentz algebra when $S_{\mathcal{M}}^{(N)} = \{\lambda_\alpha\}_{\alpha=0}^{N}$ is used as semigroup, which is endowed with the multiplication rule $\lambda_\alpha\lambda_\beta = \lambda_{\alpha+\beta}$ if $\alpha+\beta \leq N$; $\lambda_\alpha\lambda_\beta = \lambda_{\alpha+\beta-2[(N+1)/2]}$ if $\alpha+\beta > N$. These algebras are called generalized *AdS*-Lorentz algebras. In the same Ref. [15] it was found that a generalized Inönü–Wigner contraction of the generalized *AdS*-Lorentz algebras provides the so-called generalized Poincaré algebras, $\mathfrak{B}_m$.

On the other hand, in Ref. [1] it was shown how the Newton–Cartan formulation of Newtonian gravity can be obtained from gauging the Bargmann algebra, i.e. the centrally extended Galilean algebra.

This paper is organized as follows: In Section 2 it is shown that, using an analogous procedure to that used in Ref. [16] it is possible to obtain the non-relativistic versions of the generalized Poincaré algebras and generalized *AdS*-Lorentz algebras. The nonrelativistic algebras will be called, generalized Galilean algebras of type I and type II and denoted by $\mathcal{G}\mathfrak{B}_n$ and $\mathcal{G}\mathfrak{L}_n$ respectively. In Section 3 it is shown that the generalized Galilean algebras of type I can be obtained by a generalized Inönü–Wigner contraction of generalized Galilean algebras of type II. In this section it is also shown that the procedure of *S*-expansion allows us to find the $\mathcal{G}\mathfrak{B}_5$ algebra from the Newton Hooke algebra with central extension. In Section 4 it is shown that the non-relativistic limit of Einstein–Chern–Simons gravity is given by a modified version of the Poisson equation. In Section 5 it is found that, using an analogous procedure to that used in Ref. [1], it is possible to find a generalization of the Newtonian gravity. Finally our conclusions are presented in Section 6.

2. Generalized Galilean type I ($\mathcal{G}\mathfrak{B}_n$) and type II ($\mathcal{G}\mathfrak{L}_n$) algebras

The use of the procedure developed in Ref. [16], allows us to show that it is possible to obtain the non-relativistic ver-

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sions of the generalized Poincaré algebras and of the generalized *AdS*-Lorentz algebras. The nonrelativistic algebras will be called, generalized Galilean type I and type II algebras and denoted by $\mathcal{G}\mathfrak{B}_n$ and $\mathcal{G}\mathfrak{L}_n$ respectively. We consider the particular cases $n = 4, 5$.

Consider now the non-relativistic versions of the Maxwell and $\mathfrak{B}_5$ algebras. Separating the spatial temporal components in the generators $\{P_a, J_{ab}, Z_a, Z_{ab}\}$, performing the rescaling $K_i \longrightarrow c^{-1} J_{i0}$, $P_i \longrightarrow R^{-1} P_i$, $H \longrightarrow cR^{-1} P_0 - c^2 M$, $Z_{i0} \longrightarrow c^{-1} Z_{i0}$, $Z_i \longrightarrow R^{-1} Z_i$, $Z_0 \longrightarrow cR^{-1} Z_0 - c^2 N$ and then taking the limit c, $R \to \infty$, we find that:

(i) the generators of the non-relativistic version of the Maxwell algebra, which we will denote by $\mathcal{G}\mathfrak{B}_4$, satisfy the following commutation relations

$$
\begin{aligned}
&[J_{ij}, J_{kl}] = \delta_{kj} J_{il} + \delta_{lj} J_{ki} - \delta_{ki} J_{jl} - \delta_{li} J_{kj},\\
&[J_{ij}, K_k] = \delta_{jk} K_i - \delta_{ik} K_j, \quad [K_i, P_j] = -\delta_{ij} M,\\
&[J_{ij}, P_k] = \delta_{jk} P_i - \delta_{ik} P_j, \quad [K_i, H] = -P_i,\\
&[J_{ij}, Z_{kl}] = \delta_{kj} Z_{il} + \delta_{lj} Z_{ki} - \delta_{ki} Z_{jl} - \delta_{li} Z_{kj},\\
&[J_{ij}, Z_{k0}] = \delta_{jk} Z_{i0} - \delta_{ik} Z_{j0}, \quad [P_i, H] = \nu^2 Z_{i0},\\
&[Z_{ij}, K_k] = \delta_{jk} Z_{i0} - \delta_{ik} Z_{j0},
\end{aligned}
\tag{1}
$$

and (ii) the generators of the non-relativistic version of the $\mathfrak{B}_5$ algebra [2,15] which we will denote by $\mathcal{G}\mathfrak{B}_5$, satisfy the commutation relations

$$
\begin{aligned}
&[J_{ij}, J_{kl}] = \eta_{kj} J_{il} + \eta_{lj} J_{ki} - \eta_{ki} J_{jl} - \eta_{li} J_{kj},\\
&[J_{ij}, K_k] = \eta_{jk} K_i - \eta_{ik} K_j, \quad [K_i, P_j] = -\delta_{ij} M,\\
&[J_{ij}, P_k] = \eta_{jk} P_i - \eta_{ik} P_j, \quad [K_i, H] = -P_i,\\
&[J_{ij}, Z_{kl}] = \eta_{kj} Z_{il} + \eta_{lj} Z_{ki} - \eta_{ki} Z_{jl} - \eta_{li} Z_{kj},\\
&[J_{ij}, Z_{k0}] = \eta_{jk} Z_{i0} - \eta_{ik} Z_{j0}, \quad [K_i, Z_j] = -\delta_{ij} N,\\
&[Z_{ij}, K_k] = \eta_{jk} Z_{i0} - \eta_{ik} Z_{j0}, \quad [K_i, Z_0] = -Z_i,\\
&[J_{ij}, Z_k] = \eta_{jk} Z_i - \eta_{ik} Z_j, \quad [Z_{i0}, P_j] = -\delta_{ij} N,\\
&[Z_{ij}, P_k] = \eta_{jk} Z_i - \eta_{ik} Z_j, \quad [Z_{i0}, H] = -Z_i,\\
&[P_i, H] = \nu^2 Z_{i0},
\end{aligned}
\tag{2}
$$

where $\nu = c/R$ is a finite constant with c the speed of light and R the universe radius. Following the same procedure used previously, it is possible to find non-relativistic versions of the generalized AdS-Lorentz algebras, which will be called generalized Galilean type II algebras and denoted and $\mathcal{G}\mathfrak{L}_n$. It is direct to show that using a Inönü–Wigner contraction procedure we can obtain the $\mathcal{G}\mathfrak{B}_n$ from $\mathcal{G}\mathfrak{L}_n$.

3. $\mathcal{G}\mathfrak{B}_n$ algebras

3.1. $\mathcal{G}\mathfrak{B}_4$ algebra from the $\mathcal{G}\mathfrak{L}_4$ algebra

From Ref. [15] we know that the generalized Poincaré algebras can be obtained from the generalized *AdS*-Lorentz algebras by means of a generalized Inönü–Wigner contraction. This property of these relativistic algebras is inherited by their corresponding nonrelativistic algebras. We consider, as an example, the contraction of the $\mathcal{G}\mathfrak{L}_4$ algebra. In fact, performing the following rescaling $P_i \longrightarrow \lambda P_i$, $H \longrightarrow \lambda H$, $M \longrightarrow \lambda M$, $Z_{i0} \longrightarrow \lambda^2 Z_{i0}$, $Z_{ij} \longrightarrow \lambda^2 Z_{ij}$ of the $\mathcal{G}\mathfrak{L}_4$ algebra provides in the limit $\lambda \longrightarrow 0$ the $\mathcal{G}\mathfrak{B}_4$ algebra (1). Similarly we can get the $\mathcal{G}\mathfrak{B}_5$ algebra from the $\mathcal{G}\mathfrak{L}_5$ algebra.

3.2. $\mathcal{G}\mathfrak{B}_5$ algebra from the Newton Hooke algebra

In Refs. [2,3] it was shown that the S-expansion procedure allows to obtain the 'generalized Poincaré algebras' $\mathfrak{B}_n$ from *AdS*-algebra. In this subsection it is shown that the procedure of S-expansion allows us to find the $\mathcal{G}\mathfrak{B}_5$ algebra from the Newton Hooke algebra with central extension. A representation of *AdS* algebra is given by the matrices

$$
\Gamma_{\mu\nu} = \frac{1}{4}\left[\gamma_\mu, \gamma_\nu\right],
$$

where the matrices γ_μ satisfy the Clifford algebra $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}$, with $\mu, \nu : 0, 1, \cdots, 5$ and where $\eta_{\mu\nu} = \mathrm{diag}\left(-c^2, 1, 1, 1, 1, -R^2\right)$. The identification $J_{ij} = \Gamma_{ij}$, $\Gamma_{i0} = cK_i$, $\Gamma_{i5} = RP_i$, $\Gamma_{05} = RP_0 = Rc^{-1}\left(H + c^2 M\right)$ and $\Gamma_* = 2M$ with $\Gamma_* = \gamma_0\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5$, it leads to the commutation relations of the Newton Hooke algebra with central extension Ref. [17].

From Ref. [18] we find that the non-vanishing components of the invariant tensor for $\mathfrak{so}(4, 2)$ are given by

$$
\begin{aligned}
&\langle \Gamma_* \{\Gamma_{\mu\nu}\Gamma_{\rho\sigma}\Gamma_{\delta\tau}\}\rangle = -8(Rc)^2 \epsilon_{\mu\nu\rho\sigma\delta\tau},\\
&\langle \Gamma_* \{\Gamma_{\mu\nu}\Gamma_{\rho\sigma}\Gamma_{\tau 5}\}\rangle = -8(Rc)^2 \epsilon_{\mu\nu\rho\sigma\tau 5}.
\end{aligned}
$$

Following an analogous procedure to that used in Ref. [16], we find that the only nonzero components of the invariant tensor for the 5-dimensional Newton Hooke algebra with central extension

$$
\begin{aligned}
&\langle J_{ij} J_{kl} M\rangle = -16\epsilon_{ijkl},\\
&\langle J_{ij} P_k K_l\rangle = -16\epsilon_{ijkl}.
\end{aligned}
$$

Following the definitions of Ref. [19] (see also [20]) let us consider the S-expansion of Newton Hooke algebra with central extension using as semigroup $S_E^{(3)} = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_3\}$ endowed with the multiplication rule $\lambda_\alpha\lambda_\beta = \lambda_{\alpha+\beta}$ if $\alpha + \beta \leq 4$; $\lambda_\alpha\lambda_\beta = \lambda_4$ if $\alpha + \beta > 4$. After extracting a resonant and reduced subalgebra, one finds the $\mathcal{G}\mathfrak{B}_5$ algebra, given by (2). The invariant tensors for $\mathcal{G}\mathfrak{B}_5$ can be obtained from Newton Hooke algebra with central extension. Using *VII*.2 from Ref. [19] we find

$$
\begin{aligned}
&\langle J_{ij} J_{kl} M\rangle = -\frac{4}{3}\alpha_1 l\nu\epsilon_{ijkl}, \quad \langle J_{ij} P_k K_l\rangle = -\frac{4}{3}\alpha_1 l\nu\epsilon_{ijkl},\\
&\langle J_{ij} Z_{kl} M\rangle = -\frac{4}{3}\alpha_3 l\nu\epsilon_{ijkl}, \quad \langle Z_{ij} P_k K_l\rangle = -\frac{4}{3}\alpha_3 l\nu\epsilon_{ijkl},\\
&\langle J_{ij} P_k Z_{l0}\rangle = -\frac{4}{3}\alpha_3 l\nu\epsilon_{ijkl}, \quad \langle J_{ij} J_{kl} N\rangle = -\frac{4}{3}\alpha_3 l\nu\epsilon_{ijkl},\\
&\langle J_{ij} Z_k K_l\rangle = -\frac{4}{3}\alpha_3 l\nu\epsilon_{ijkl},
\end{aligned}
\tag{3}
$$

where the constants α_1 and α_3 are dimensionless and the factors l, ν are introduced to display the dimension of $\langle\cdots\rangle$, and are parameters of dimension length and velocity respectively.

4. Non-relativistic limit of Einstein–Chern–Simons gravity

The five dimensional Chern–Simons lagrangian for the $\mathfrak{B}_5$ algebra is given by [2]

$$
\begin{aligned}
L_{\mathrm{ChS}}^{(5)} &= \alpha_1 l^2 \varepsilon_{abcde} R^{ab} R^{cd} e^e\\
&\quad + \alpha_3 \varepsilon_{abcde}\left(\frac{2}{3} R^{ab} e^c e^d e^e + 2l^2 k^{ab} R^{cd} T^e + l^2 R^{ab} R^{cd} h^e\right),
\end{aligned}
\tag{4}
$$

where α_1, α_3 are parameters of the theory, l is a coupling constant, $R^{ab} = d\omega^{ab} + \omega^a{}_c\omega^{cb}$ corresponds to the curvature 2-form in the first-order formalism. In Ref. [21] it was considered that in the presence of matter the lagrangian is given by

$$L = L^{(5)}_{\text{ChS}} + \kappa L_M,$$

where $L^{(5)}_{\text{ChS}}$ is the five-dimensional Chern–Simons lagrangian given by (4), $L_M = L_M(e^a, h^a, \omega^{ab})$ is the matter lagrangian and κ is a coupling constant related to the effective Newton's constant.

The Lagrangian (4) shows that standard, five-dimensional General Relativity emerges as the $l \to 0$ limit of a Chern–Simons theory for the generalized Poincaré algebra $\mathfrak{B}_5$. Here l is a length scale, a coupling constant that characterizes different regimes within the theory.

The variation of the lagrangian (4) w.r.t. the dynamical fields vielbein e^a, spin connection ω^{ab}, h^a and k^{ab}, leads to the following field equations [22]

$$\varepsilon_{abcde}(2\alpha_3 R^{ab}e^c e^d + \alpha_1 l^2 R^{ab}R^{cd}) = \kappa \frac{\delta L_M}{\delta e^e}, \tag{5}$$

$$\alpha_3 l^2 \varepsilon_{abcde} R^{ab}R^{cd} = \kappa \frac{\delta L_M}{\delta h^e}, \tag{6}$$

$$\alpha_3 l^2 \varepsilon_{abcde} R^{cd} Dh^e = 0, \tag{7}$$

where we have considered that the torsion vanishes $T^a = 0$ ($\frac{\delta L_M}{\delta \omega^{ab}} = 0$) and $k^{ab} = 0$, while the field h^a is associated, in the context of Einstein–Chern–Simons cosmology, with the dark energy, as shown in Refs. [21,22].

In the case where the equations (5)–(7) satisfy the cosmological principle and the ordinary matter is negligible compared to the dark energy, we find that the equations (5)–(7) take the form [22]

$$6\left(\frac{\dot{a}^2 + k}{a^2}\right) = \kappa_5 \alpha \rho^{(h)}, \tag{8}$$

$$3\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}^2 + k}{a^2}\right)\right] = -\kappa_5 \alpha p^{(h)}, \tag{9}$$

$$\frac{3l^2}{\kappa_5}\left(\frac{\dot{a}^2 + k}{a^2}\right)^2 = \rho^{(h)}, \tag{10}$$

$$\frac{3l^2}{\kappa_5}\frac{\ddot{a}}{a}\left(\frac{\dot{a}^2 + k}{a^2}\right) = -p^{(h)}, \tag{11}$$

$$\left(\frac{\dot{a}^2 + k}{a^2}\right)\left[(h - h(0))\frac{\dot{a}}{a} + \dot{h}\right] = 0. \tag{12}$$

The field equations (8)–(12) were completely resolved for the age of Dark Energy in Ref. [22], where was find that the field h^a has a similar behavior of a cosmological constant.

In fact, in Section 3 of Ref. [22] has been found solutions that describes accelerated expansion for the three possible cosmological models of the universe. Namely, spherical expansion ($k = 1$), flat expansion ($k = 0$) and hyperbolic expansion ($k = -1$). This means that the Einstein–Chern–Simons field equations have as one of their solutions an universe in accelerated expansion. This result allows us to conjecture that this solutions are compatible with the era of Dark Energy and that the energy-momentum tensor for the field h^a corresponds to a kind of positive cosmological constant.

Introducing (6) in (5) we find

$$*\varepsilon_{abcde}(2\alpha_3 R^{ab}e^c e^d) = \left(\beta_1 \Upsilon_e - \frac{\alpha_1}{\alpha_3}\beta_2 \Upsilon^{(h)}_e\right), \tag{13}$$

where

$$\beta_1 \Upsilon_a = \kappa * \left(\frac{\delta L_M}{\delta e^a}\right), \quad \beta_2 \Upsilon^{(h)}_a = \kappa * \left(\frac{\delta L_M}{\delta h^a}\right),$$

and $*$ is the Hodge star operator.

In the limit of weak gravitational field one assumes that the world metric tensor $g_{\mu\nu}$ is not very much different from the Minkowski metric $\eta_{\mu\nu} = \text{diag}(-1, 1, \ldots, 1)$. In fact, it can be then written in the form $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ represents the small corrections to the flat space–time metric $\eta_{\mu\nu}$ due to the presence of a weak gravitational field. In this approximation $|h_{\mu\nu}| << 1$, so that terms of order higher than the first in $h_{\mu\nu}$ can be neglected in the field equations. So that,

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu = \eta_{ab}e^a e^b \\ &= -(1 - h_{00})dt^2 + (1 + h_{11})(dx^1)^2 + (1 + h_{11})(dx^2)^2 \\ &\quad + (1 + h_{11})(dx^3)^2 + (1 + h_{11})(dx^4)^2. \end{aligned}$$

Introducing an orthonormal basis

$$e^0 \approx \left(1 + \frac{h^0{}_0}{2}\right)dt, \; e^1 \approx \left(1 + \frac{h^1{}_1}{2}\right)dx^1, \; e^2 \approx \left(1 + \frac{h^2{}_2}{2}\right)dx^2,$$

$$e^3 \approx \left(1 + \frac{h^3{}_3}{2}\right)dx^3, \; e^4 \approx \left(1 + \frac{h^4{}_4}{2}\right)dx^4,$$

and using the first and second structural equations $T^a = de^a + \omega^a_b e^b = 0$, $R^a_b = d\omega^a_b + \omega^a_c \omega^c_b$ we have

$$R_{00} = -\frac{\nabla^2 h_{00}}{2},$$

$$R_{11} = \frac{\partial_1{}^2 h_{00}}{2} - \frac{\partial_2{}^2 h_{11}}{2} - \frac{\partial_3{}^2 h_{11}}{2} - \frac{\partial_4{}^2 h_{11}}{2},$$

$$R_{22} = \frac{\partial_2{}^2 h_{00}}{2} - \frac{\partial_1{}^2 h_{22}}{2} - \frac{\partial_3{}^2 h_{22}}{2} - \frac{\partial_4{}^2 h_{22}}{2},$$

$$R_{33} = \frac{\partial_3{}^2 h_{00}}{2} - \frac{\partial_1{}^2 h_{33}}{2} - \frac{\partial_2{}^2 h_{33}}{2} + \frac{\partial_4{}^2 h_{33}}{2},$$

$$R_{44} = \frac{\partial_4{}^2 h_{00}}{2} - \frac{\partial_1{}^2 h_{44}}{2} - \frac{\partial_2{}^2 h_{44}}{2} - \frac{\partial_3{}^2 h_{44}}{2}.$$

From (13) we can see

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{1}{8\alpha_3}\left(\beta_1 \Upsilon_{\mu\nu} - \frac{\alpha_1}{\alpha_3}\beta_2 \Upsilon^{(h)}_{\mu\nu}\right). \tag{14}$$

In the limit of weak gravitational field one assumes that the leading term in the energy–momentum tensors are $\Upsilon_{00} = \rho$ and $\Upsilon^{(h)}_{00} = \rho^{(h)}$ so that

$$R_{00} = \frac{1}{12\alpha_3}\left(\beta_1 \rho - \frac{\alpha_1}{\alpha_3}\beta_2 \rho^{(h)}\right). \tag{15}$$

On the another hand the motion of a particle described by the geodesic equation

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{ds}\frac{dx^\rho}{ds} = 0, \tag{16}$$

where $x^\mu = \{x^0, x^i\} = \{t, x^i\}$. In the nonrelativistic limit eq. (16) becomes

$$\frac{d^2x^\mu}{dt^2} = -\Gamma^\mu_{00}\left(\frac{dx^0}{dt}\right)^2 = -\Gamma^\mu_{00}, \tag{17}$$

and in the limit of weak gravitational field we can put $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $|h_{\mu\nu}| << 1$, and we can neglect terms of order h^2 and higher. From the definition of the Christoffel symbols we have $\Gamma^i_{00} = -\frac{1}{2}\delta^i{}_j \partial_j h_{00}$, where we have assumed that the field is static, i.e., $\partial_0 g_{\mu\nu} = 0$. So that the geodetic equation is given by

$$\frac{d^2x^i}{dt^2} = \frac{1}{2}\delta^i{}_j \partial_j h_{00}, \tag{18}$$

which coincides with Newton equation of motion $\frac{d^2x^i}{dt^2} = -\partial_i\phi$ provided that $h_{00} = -2\phi$, therefore $\Gamma^i_{00} = \delta^{ij}\partial_j\phi$. This means that the only non-zero component of the Riemann tensor corresponding to connection Γ^i_{00} is given by $R^i_{0j0} = \delta^{ik}\partial_k\partial_j\phi$, so from (15) we can conclude that the nonrelativistic limit of the five dimensional Einstein–Chern–Simons gravity is a modified version of the Poisson equation given by

$$\nabla^2\phi = \frac{2}{3}\left(k_1\rho - \alpha k_2\rho^{(h)}\right), \tag{19}$$

where $k_1 = \frac{\beta_1}{8\alpha_3}$, $k_2 = \frac{\beta_2}{24\alpha_3}$ and $\alpha = \frac{3\alpha_1}{\alpha_3}$ [21]. If in (19) we choose $\alpha = 0$ or $k_2 = 0$ we obtain the Poisson equation in five dimensions provided that $k_1 = 8\pi G$.

5. Newton–Chern–Simons gravity

In Ref. [1] it was shown how the Newton–Cartan formulation of Newtonian gravity can be obtained from gauging the Bargmann algebra. In Refs. [2] it was shown that the gauging of $\mathfrak{B}_5$ leads to a five-dimensional Chern–Simons gravity which empties into general relativity in a certain limit. On the other hand, we have seen that the non-relativistic version of the $\mathfrak{B}_5$ algebra is given by the $\mathcal{G}\mathfrak{B}_5$ algebra and that the procedure of S-expansion allows us to find the $\mathcal{G}\mathfrak{B}_5$ algebra from the Newton Hooke algebra with central extension. In this Section we show that, using an analogous procedure to that used in Ref. [1], it is possible to find a generalization of the Newtonian gravity.

5.1. Gauging the $\mathcal{G}\mathfrak{B}_5$ algebra

We start with a one-form gauge connection A valued in the $\mathcal{G}\mathfrak{B}_5$ algebra is given by

$$\begin{aligned} A = {} & \frac{v}{l}\tau H + \frac{1}{l}e^iP_i + \frac{v}{l}h^0Z_0 + \frac{1}{l}h^iZ_i + \frac{1}{vl}mM + \frac{1}{vl}nN \\ & + \frac{1}{v}\omega^iK_i + \frac{1}{v}k^iZ_{i0} + \frac{1}{2}\omega^{ij}J_{ij} + \frac{1}{2}k^{ij}Z_{ij}, \end{aligned} \tag{20}$$

where l and v are parameters of dimension length and velocity respectively. The corresponding two-form curvature is given by

$$\begin{aligned} F = {} & \frac{v}{l}R(H)H + \frac{1}{l}R^i(P_i)P_i + \frac{v}{l}R(Z_0)Z_0 + \frac{1}{l}R^i(Z_i)Z_i \\ & + \frac{1}{vl}R(M)M + \frac{1}{vl}R(N)N + \frac{1}{v}R^i(K_i)K_i + \frac{1}{v}R^i(Z_{i0})Z_{i0} \\ & + \frac{1}{2}R^{ij}\left(J_{ij}\right)J_{ij} + \frac{1}{2}R^{ij}\left(Z_{ij}\right)Z_{ij}, \end{aligned} \tag{21}$$

where

$$\begin{aligned} & R(H) = d\tau,\ R^i(P_i) = T^i - \omega^i\tau, \\ & R(Z_0) = dh^0,\ R(M) = dm - \omega^ie_i, \\ & R^i(Z_i) = Dh^i - \omega^ih^0 - k^i\tau + k^i{}_je^j, \\ & R(N) = dn - \omega^ih_i - k^ie_i, \\ & R^i(Z_{i0}) = Dk^i + e^i\tau + k^i{}_j\omega^j,\ R^i(K_i) = D\omega^i, \\ & R^{ij}(J_{ij}) = R^{ij},\quad R^{ij}(Z_{ij}) = Dk^{ij}, \end{aligned} \tag{22}$$

with $T^i = de^i + \omega^{ij}e_j$, $R^{ij} = d\omega^{ij} + \omega^i{}_k\omega^{kj}$.

Since the gauge connection A transforms as

$$\delta A = d\Lambda + [A, \Lambda],$$

where

$$\begin{aligned} \Lambda = {} & \frac{v}{l}\zeta^0H + \frac{1}{l}\zeta^iP_i + \frac{v}{l}\rho^0Z_0 + \frac{1}{l}\rho^iZ_i + \frac{1}{vl}\sigma M + \frac{1}{vl}\gamma N \\ & + \frac{1}{v}\lambda^iK_i + \frac{1}{v}\chi^iZ_{i0} + \frac{1}{2}\lambda^{ij}J_{ij} + \frac{1}{2}\chi^{ij}Z_{ij}, \end{aligned}$$

we find, using the $\mathcal{G}\mathfrak{B}_5$ algebra, that the variations of the gauge fields are given by

$$\begin{aligned} & \delta\tau = d\zeta^0,\ \delta e^i = D\zeta^i - \omega^i\zeta^0 - \lambda^{ij}e_j + \tau\lambda^i, \\ & \delta h^i = D\rho^i - \omega^i\rho^0 - \lambda^{ij}h_j + h^0\lambda^i + k^{ij}\zeta_j - k^i\zeta^0 - \chi^{ij}e_j + \tau\chi^i, \\ & \delta m = d\sigma - \omega^i\zeta_i + e^i\lambda_i,\ \delta\omega^i = D\lambda^i - \lambda^{ij}\omega_j, \\ & \delta n = d\gamma - k^i\zeta_i + h^i\lambda_i - \omega^i\rho_i + e^i\chi_i,\ \delta h^0 = d\rho^0, \\ & \delta k^i = D\chi^i - \lambda^{ij}k_j - \chi^{ij}\omega_j + k^{ij}\lambda_j + e^i\zeta^0 - \zeta^i\tau,\ \delta\omega^{ij} = D\lambda^{ij}, \\ & \delta k^{ij} = D\chi^{ij} + k^i{}_k\lambda^{kj} + k^j{}_k\lambda^{ik}, \end{aligned} \tag{23}$$

where the derivative D is covariant with respect to the J-transformations.

Following Ref. [1] we impose now several curvature constraints. These constraints convert the P and H transformations into general coordinate transformations in space and time. We write the parameter of the general coordinate transformations ξ^λ as

$$\xi^\lambda = e^\lambda_i\zeta^i + \tau^\lambda\zeta^0.$$

Here we have used the inverse spatial vielbein $e^\lambda{}_i$ and the inverse temporal vielbein τ^λ defined by [1]

$$\begin{aligned} & e^i{}_\mu e^\mu{}_j = \delta^i{}_j,\quad \tau^\mu\tau_\mu = 1,\quad \tau^\mu e^i{}_\mu = 0 \\ & \tau_\mu e^\mu{}_i = 0,\quad e^i{}_\mu e^\nu{}_i = \delta^\nu{}_\mu - \tau_\mu\tau^\nu. \end{aligned} \tag{24}$$

From (23) we can see that only the gauge fields $e_\mu{}^i$, τ_μ, m_μ, $h_\mu{}^i$, h^0_μ and n_μ transform under the P and H transformations. These are the fields which should remain independents, while the remaining fields will be dependent upon the aforementioned fields. This can be achieved with the following constraints

$$\begin{aligned} & R(H) = d\tau = 0,\ R^i(P_i) = T^i - \omega^i\tau = 0, \\ & R(M) = dm - \omega^ie_i = 0,\ R(Z_0) = dh^0 = 0, \end{aligned} \tag{25}$$

$$\begin{aligned} & R^i(Z_i) = Dh^i - \omega^ih^0 - k^i\tau + k^i_je^j = 0, \\ & R(N) = dn - \omega^ih_i - k^ie_i = 0. \end{aligned} \tag{26}$$

An analogous procedure to that used in Ref. [1] allows us to obtain the $k_\mu{}^{ij}$ and $k_\mu{}^i$ fields. In fact, using the constraints (26) we find

$$\begin{aligned} \omega_\mu{}^{ij} = {} & (\partial_{[\mu}e_{\nu]})^ie^{\nu j} - (\partial_{[\mu}e_{\nu]})^je^{\nu i} + e_{\mu k}(\partial_{[\nu}e_{\rho]})^ke^{\nu i}e^{\rho j} \\ & - \tau_\mu e^{\nu[i}\omega_\nu{}^{j]}, \end{aligned} \tag{27}$$

$$\begin{aligned} \omega_\mu{}^i = {} & e^{\nu i}\partial_{[\mu}m_{\nu]} + e^{\nu i}\tau^\rho e_{\mu j}\partial_{[\nu}e_{\rho]}{}^j + \tau_\mu\tau^\nu e^{\rho i}\partial_{[\nu}m_{\rho]} \\ & + \tau^\nu\partial_{[\mu}e_{\nu]}{}^i, \end{aligned} \tag{28}$$

$$\begin{aligned} k_\mu{}^{ij} = {} & (D_{[\mu}h_{\nu]})^ie^{\nu j} - (D_{[\mu}h_{\nu]})^je^{\nu i} + e_{\mu k}(D_{[\nu}h_{\rho]})^ke^{\nu i}e^{\rho j} \\ & - \omega^i_{[\mu}h^0_{\nu]}e^{\nu j} + \omega^j_{[\mu}h^0_{\nu]}e^{\nu i} - e_{\mu k}\omega^k_{[\nu}h^0_{\nu]}e^{\nu i}e^{\rho j} - \tau_\mu e^{\nu[i}k_\nu{}^{j]}, \end{aligned} \tag{29}$$

$$\begin{aligned} k_\mu{}^i = {} & e^{\nu i}\partial_{[\mu}n_{\nu]} - e^{\nu i}\omega^k{}_{[\mu}h_{\nu]k} + e^{\nu i}\tau^\rho e_{\mu j}D_{[\nu}h_{\rho]}{}^j \\ & - e^{\nu i}\tau^\rho e_{\mu j}\omega^j{}_{[\nu}h^0{}_{\rho]} + \tau_\mu\tau^\nu e^{\rho i}\partial_{[\nu}n_{\rho]} - \tau_\mu\tau^\nu e^{\rho i}\omega^k{}_{[\nu}h_{\rho k]} \\ & + \tau^\nu D_{[\mu}h_{\nu]}{}^i - \tau^\nu\omega^i{}_{[\mu}h^0{}_{\nu]}. \end{aligned} \tag{30}$$

5.2. Newton–Chern–Simons lagrangian

A Chern–Simons lagrangian form $L_{\text{ChS}}(A,0) \equiv Q_{2n+1}(A,0)$ is a differential form defined for a connection, whose exterior derivative yields a Chern class. Although the Chern classes are gauge invariant, the Chern–Simons forms are not; under gauge transformations they change by a closed form. A transgression form $Q_{2n+1}(A_1,A_2)$ on the other hand, is an invariant differential form whose exterior derivative is the difference of two Chern classes. It generalizes the Chern–Simons form with the additional advantage that it is gauge invariant.

To obtain the lagrangian for 5-dimensional Chern–Simons gravity we use subspaces separation method introduced in Ref. [23] and write L_{ChS} in terms of a transgression form, a Chern–Simons form and a total exact form

$$Q_5(A_1,0) = Q_5(A_1,A_2) + Q_5(A_2,0) + dQ_4(A_1,A_2,0),$$

where,

$$Q_5(A_1,A_2) = 3\int_0^1 dt \left\langle \theta F_t^2 \right\rangle, \tag{31}$$

with $\theta = A_1 - A_2$, $A_t = A_2 + t\theta$, $A_1 = A$, $A_2 = \omega = \frac{1}{2}\omega^{ij} J_{ij}$, $F_t = dA_t + A_t A_t$ and

$$Q_5(A_2,0) = 3\int_0^1 dt \left\langle A_2 F_t^2 \right\rangle, \tag{32}$$

where now $A_t = tA_2 = t\omega$.

So that if we don't consider boundary terms the Chern–Simons lagrangian is given by:

$$\begin{aligned} L_{\text{ChS},\mathcal{GB}_5} = {} & \alpha_1 \varepsilon_{ijkl} \left(-2R^{ij}T^k\omega^l - \frac{4}{3}R^{ij}\omega^k\omega^l\tau + 2R^{ij}D\omega^k e^l \right. \\ & \left. - R^{ij}R^{kl}m \right) + \alpha_3 \varepsilon_{ijkl} \left(\frac{4}{3}\nu^2 R^{ij}e^k e^l\tau - 2R^{ij}Dh^k\omega^l \right. \\ & - \frac{4}{3}R^{ij}k^k\omega^l\tau - \frac{4}{3}R^{ij}\omega^k\omega^l\hat{\tau} + 2R^{ij}D\omega^k h^l \\ & - \frac{4}{3}Dk^{ij}T^k\omega^l - Dk^{ij}\omega^k\omega^l\tau - R^{ij}k^{kl}dm \\ & - \frac{2}{3}R^{ij}k^{kl}e^m\omega_m - \frac{2}{3}R^{ij}\omega_m^k k^{ml}m - \frac{4}{3}k^{ij}T^k D\omega^l \\ & - k^{ij}D\omega^k\omega^l\tau - 2R^{ij}T^k k^l - \frac{4}{3}R^{ij}\omega^k k^l\tau + \frac{2}{3}R^{ij}k^{km}\omega_m e^l \\ & \left. + \frac{2}{3}\omega_m^i k^{jm}D\omega^k e^l - R^{ij}R^{kl}n - 2R^{ij}\omega^{km}k_m e^l \right). \end{aligned} \tag{33}$$

The lagrangian variation of (33) leads to the following equations of motion:

$$\varepsilon_{ijkl}\left(-\frac{4}{3}\alpha_1 R^{ij}\omega^k\omega^l + \frac{4}{3}\nu^2\alpha_3 R^{ij}e^k e^l \right) = \kappa\frac{\delta L_M}{\delta\tau}, \tag{34}$$

$$\frac{4}{3}\alpha_3\varepsilon_{ijkl0}R^{ij}\omega^k\omega^l = -\kappa\frac{\delta L_M}{\delta\hat{\tau}}, \tag{35}$$

$$4\varepsilon_{ijkl}\left(\alpha_1 R^{ij}D\omega^k - \frac{2}{3}\nu^2\alpha_3 R^{ij}e^k\tau \right) = \kappa\frac{\delta L_M}{\delta e^l}, \tag{36}$$

$$2\alpha_3\varepsilon_{ijkl}R^{ij}D\omega^k = \kappa\frac{\delta L_M}{\delta h^l}, \tag{37}$$

$$\alpha_1\varepsilon_{ijkl}R^{ij}R^{kl} = -\kappa\frac{\delta L_M}{\delta m}, \tag{38}$$

$$\alpha_3\varepsilon_{ijkl}R^{ij}R^{kl} = -\kappa\frac{\delta L_M}{\delta n}, \tag{39}$$

$$\begin{aligned} & 4\varepsilon_{ijkl}\left(\frac{2\alpha_1}{3}R^{ij}\omega^k\tau - \alpha_1 R^{ij}T^k + \frac{2\alpha_3}{3}R^{ij}\omega^k\hat{\tau} - \alpha_3 R^{ij}Dh^k \right) \\ & = \kappa\frac{\delta L_M}{\delta\omega^l}, \end{aligned} \tag{40}$$

$$\begin{aligned} & \varepsilon_{ijkl}\left(-2\alpha_1 R^{km}e_m\omega^l - 4\alpha_1 T^k D\omega^l - \frac{4\alpha_1}{3}\omega^k\omega^l d\tau - \frac{8\alpha_1}{3}D\omega^k\omega^l\tau \right. \\ & + 2\alpha_1 R^{km}\omega_m e^l - 2\alpha_1 R^{kl}dm + \frac{8}{3}\nu^2\alpha_3 T^k e^l\tau + \frac{4\alpha_3}{3}e^k e^l d\tau \\ & - 2\alpha_3 R^{km}h_m\omega^l - 4\alpha_3 Dh^k D\omega^l - \frac{4\alpha_3}{3}\omega^k\omega^l d\hat{\tau} - \frac{8\alpha_3}{3}D\omega^k\omega^l\hat{\tau} \\ & \left. + 2\alpha_3 R^{km}\omega_m h^l - \alpha_3 R^{kl}dn \right) = \kappa\frac{\delta L_M}{\delta\omega^{ij}}, \end{aligned} \tag{41}$$

where, we have considered, in analogy with the Section 4, $k^{ij} = k^i = 0$. The first four equations corresponding to the non-relativistic version of the Einstein equations. The equations (38) and (39) are second order curvatures, then in the limit of weak gravitational field $\frac{\delta L_M}{\delta m} = \frac{\delta L_M}{\delta n} = 0$. The equations (40) and (41) corresponding to the non-relativistic version of the torsion equation.

In analogy with the Section 4, the first two lead us to

$$*\varepsilon_{ijkl}R^{ij}e^k e^l = \frac{3}{4\nu^2}\left(\frac{\beta_1}{\alpha_3}\Upsilon_0 - \frac{\alpha_1\beta_2}{\alpha_3{}^2}\Upsilon_0^{(h)} \right) = \Omega_0, \tag{42}$$

where we found that $4R_{00} = \Omega_{00}$ with $R_{00} = \nabla^2\phi$. Finally from (42) we obtain

$$\nabla^2\phi = \frac{3}{2\nu^2}(k_1\rho - \alpha k_2\rho^{(h)}), \tag{43}$$

where the constants $k_1 = \frac{\beta_1}{8\alpha_3} = 8\pi G$, $k_2 = \frac{\beta_2}{24\alpha_3}$ and $\alpha = \frac{3\alpha_1}{\alpha_3}$. This result coincides with the equation (19) if $\nu = \frac{3}{2}$. This result shows that the non-relativistic limit of Einstein–Chern–Simons gravity, invariant under the $\mathfrak{B}_5$ algebra coincides with Newton–Chern–Simons gravity invariant under the algebra $\mathcal{G}\mathfrak{B}_5$.

6. Comments

In the present work we have shown that: (**i**) it is possible to obtain the non-relativistic versions of both generalized Poincaré algebras and generalized *AdS*-Lorentz algebras. These non-relativistic algebras are called generalized Galilean type I and type II algebras and denoted by $\mathcal{G}\mathfrak{B}_n$ and $\mathcal{G}\mathfrak{L}_n$ respectively. (**ii**) The procedure of *S*-expansion allows us to find the $\mathcal{G}\mathfrak{B}_5$ algebra from the Newton–Hooke algebra with central extension. (**iii**) Using an analogous procedure to that used in Ref. [1], it is possible to find the non-relativistic limit of the five dimensional Einstein–Chern–Simons gravity which lead us to a modified version of the Poisson equation.

It is interesting to note that the $\mathfrak{B}_5$ algebra is a generalization of the Poincaré algebra which includes the extra generators Z_{ab} and Z_a. This algebra leads to a Chern–Simons lagrangian which coincides with the Einstein–Hilbert lagrangian in a certain limit, even if the new gauge field vanishes and therefore leads to newtonian gravity in the non-relativistic limit. The generators Z_{i0}, Z_{ij}, are the space–time components of the $Z_{ab} = (Z_{i0}, Z_{ij})$ relativistic generators, whose gauge field $k^{ab} = (k^i, k^{ij})$ we fix to $k^{ab} = 0$ in the field equations.

On the other hand the gauge field $h^a = (h^0, h^i)$ associated to the generators Z_a generates modifications in the Einstein equations which can be interpretated, in the cosmological context, as an effect due to the dark energy [21,22]. This modification leads, in the non-relativistic limit, to a modification in the Poisson equation shown in (43), which could be compatible with the Dark Matter.

This would allow us conjecture that Dark Matter could be interpreted as the non-relativistic limit of Dark Energy.

The modified form of Poisson equation (43) suggests a possible connection with the so-called MOND approach to gravity interactions. In fact the first complete theory of MOND was constructed by Milgrom and Bekenstein in Ref. [24]. This theory is based on the lagrangian

$$\mathcal{L} = -\frac{a_0^2}{8\pi G} f\left(\frac{\left|\vec{\nabla}\phi\right|}{a_0^2}\right) - \rho\phi, \tag{44}$$

where ϕ is the gravitational potential (meaning that for a test particle $\vec{a} = -\vec{\nabla}\phi$), and ρ denotes the matter mass density. The corresponding equation for ϕ is given by

$$\vec{\nabla}\cdot\left[\mu\left(\frac{\left|\vec{\nabla}\phi\right|}{a_0}\vec{\nabla}\phi\right)\right] = 4\pi G\rho, \tag{45}$$

where $\mu(\sqrt{y}) = df(y)/dy$, which can be written as

$$\mu\nabla^2\phi = 4\pi G\rho - \vec{\nabla}\mu\cdot\vec{\nabla}\phi. \tag{46}$$

Comparing this last equation with equation (43), we can see that in some particular cases the MOND approach to gravity could coincide with the modified Poisson equation (43).

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