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PROVABLE WELLORDERINGS OF FORMAL THEORIES FOR TRANSFINITELY ITERATED INDUCTIVE DEFINITIONS

W. BUCHHOLZ and W. POHLERS

Introduction

By [12] we know that transfinite induction up to $\Theta\varepsilon_{\Omega_{N+1}}0$ is not provable in ID_N, the theory of N-times iterated inductive definitions. In this paper we will show that conversely transfinite induction up to any ordinal less than $\Theta\varepsilon_{\Omega_{N+1}}0$ is provable in ID_N^i, the intuitionistic version of ID_N, and extend this result to theories for transfinitely iterated inductive definitions.

In [14] Schütte proves the wellordering of his notational systems $\Sigma(N)$ using predicates $\mathfrak{W}_k(a) :\leftrightarrow (a \in M_k \wedge \{x \in M_k : x < a\}$ is wellordered) with $M_k := \{x \in \Sigma(N): \mathfrak{W}_0(K_0x) \wedge \cdots \wedge \mathfrak{W}_{k-1}(K_{k-1}x)\}$[1] and $0 \le k \le N$. Obviously the predicates $\mathfrak{W}_0, \ldots, \mathfrak{W}_{N-1}$ are definable in ID_N^i with the defining axioms:

$$(\mathfrak{W}_k 1) \qquad \mathrm{Prog}[M_k, \mathfrak{W}_k],$$

$$(\mathfrak{W}_k 2) \qquad \mathrm{Prog}[M_k, \mathfrak{F}] \rightarrow \forall x(\mathfrak{W}_k(x) \rightarrow \mathfrak{F}[x]),$$

where $\mathrm{Prog}[M_k, X]$ means that X is progressive with respect to M_k, i.e.

$$\mathrm{Prog}[M_k, X] :\leftrightarrow \forall x \in M_k(\forall y \in M_k(y < x \rightarrow X(y)) \rightarrow X(x)).$$

The crucial point in Schütte's wellordering proof is Lemma 19 [14, p. 130] which can be modified to

$$\text{(I)} \qquad \mathrm{TI}[M_{k+1}, a],\ Sb = k,\ \mathfrak{W}_k(b) \Rightarrow \mathfrak{W}_k((a, b)), \quad \text{for } 0 \le k \le N-1,$$

where $\mathrm{TI}[M_{k+1}, a]$ is the scheme of transfinite induction over M_{k+1} up to a[2]. Checking the proof of (I) it turns out that besides $(\mathfrak{W}_k 1)$ and $(\mathfrak{W}_k 2)$ $(0 \le k \le N-1)$ only finitary methods (including mathematical induction) are used. Since the proof uses "excluded middle" only for decidable formulas it is formalizable in ID_N^i. Following the proof of Lemma 17 in [14] one gets

$$\text{(II)} \qquad \mathrm{ID}_N^i \vdash \mathfrak{W}_0(1) \wedge \cdots \wedge \mathfrak{W}_{N-1}(\Omega_{N-1}) \quad \text{and}$$

$$\text{(III)} \qquad \mathrm{ID}_N^i \vdash \mathrm{TI}[M_N, \Omega_N].$$

From (III) one derives in the well-known way (due to Gentzen [5])

$$\text{(IV)} \qquad \mathrm{ID}_N^i \vdash \mathrm{TI}[M_N, c_n] \quad \text{for each } n \in \boldsymbol{N},$$

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[1] K_nx is a finite set of subterms of x. $\mathfrak{W}_n(K_nx)$ means $\forall y \in K_nx(\mathfrak{W}_n(y))$.

[2] For an exact definition see notational convention (5), page 3 of the present paper.

where $c_0 := \Omega_N$, $c_{n+1} := (1, c_n)$. By (I), (II), (IV) and the facts that $M_0 = \Sigma(N)$ and, $\mathfrak{B}_k(a)$ implies $\mathrm{TI}[M_k, a]$ one gets

(V) $$\mathrm{ID}^i_N \vdash \mathrm{TI}[\Sigma(N), \Omega[c_n, 0]] \quad \text{for each } n \in \boldsymbol{N},$$

where $\Omega[c_n, 0] := ((\cdots(c_n, \Omega_{N-1}), \ldots, \Omega_1), 1)$. Since $\sup_{n\in N} \Omega[c_n, 0] = \Omega[(1 \# 1, \Omega_N), 0]$ and the order type of $\{x \in \Sigma(N)\colon x < \Omega[(1 \# 1, \Omega_N), 0]\}$ is $\Theta\varepsilon_{\Omega_N+1}0$[3], *transfinite induction up to any ordinal less than* $\Theta\varepsilon_{\Omega_N+1}0$ *is provable in* ID^i_N, which we will abbreviate by $\mathrm{ID}^i_N \vdash \mathrm{TI}[<\Theta\varepsilon_{\Omega_N+1}0]$.

Similar considerations apply to the wellordering proof of the system $\bar{\Theta}(\{g\})$ given in [2]. We will prove the following *results*:

(A) $$\mathrm{ID}^i_\nu \vdash \mathrm{TI}[<\bar{\Theta}\varepsilon_{\Omega_\nu+1}0] \qquad \text{for any countable } \nu \in \bar{\Theta}(\{g\})^4,$$

(B) $$\mathrm{ID}^i_{<\cdot} \vdash \mathrm{TI}[<\bar{\Theta}\varepsilon_{\Omega_{\Omega_1}+1}0]^5,$$

where ID^i_ν and $\mathrm{ID}^i_{<\cdot}$ are the intuitionistic versions of the theories ID_ν and $\mathrm{ID}_{<\cdot}$ defined in [4, pp. 307–308] (see also the last paragraph of page 3 of the present paper).

$\mathrm{ID}_{<\nu}$ is defined to be the theory $\bigcup_{\xi<\nu} \mathrm{ID}_\xi$. By $\overline{\mathrm{ID}}$ we mean the theory of autonomously iterated inductive definitions (i.e. if $\overline{\mathrm{ID}} \vdash \mathrm{TI}[\nu]$ then $\mathrm{ID}_\nu \subset \overline{\mathrm{ID}}$). For a theory Th we take $|\mathrm{Th}| := \sup\{\xi \in On\colon \mathrm{Th} \vdash \mathrm{TI}[\xi]\}$. There are the following ordinal theoretic relations:

(1) $\bar{\Theta}\varepsilon_{\Omega_{\Omega_1}+1}0 = \Theta\varepsilon_{\Omega_{\Omega_1}+1}0$ and $\bar{\Theta}\varepsilon_{\Omega_\nu+1}0 = \Theta\varepsilon_{\Omega_\nu+1}0$ for $\nu < \Theta\Omega_{\Omega_1}0$. (So the above derived results on transfinite induction in ID^i_N ($N < \omega$) are special cases of (A).)

(2) $\Theta\Omega_\nu 0 = \sup_{\xi<\nu} \Theta\varepsilon_{\Omega_\xi+1}0$ for limit $\nu \le \Theta\Omega_{\Omega_1}0$.

(3) $\Theta\Omega_\nu 0 = \nu$ for $\nu = \Theta\Omega_{\Omega_1}0$.

(4) $\Theta\Omega_{\Omega_1}0 = \sup_{n\in N} \nu_n$ with $\nu_0 := 1$, $\nu_{n+1} := \Theta\Omega_{\nu_n}0$.

By (A), (1)–(4) and [13] (cf. footnote 4) we get the equations:

(A1) $$|\mathrm{ID}_\nu| = |\mathrm{ID}^i_\nu| = \Theta\varepsilon_{\Omega_\nu+1}0 \quad \text{for } \nu < \Theta\Omega_{\Omega_1}0.$$

(A2) $$|\mathrm{ID}_{<\nu}| = |\mathrm{ID}^i_{<\nu}| = \Theta\Omega_\nu 0 \quad \text{for limit } \nu \le \Theta\Omega_{\Omega_1}0.$$

(A3)
$$|\mathrm{ID}^{(i)}_{<\Theta\Omega_{\Omega_1}0}| = |\overline{\mathrm{ID}}^{(i)}| = \Theta\Omega_{\Omega_1}0 \quad \text{and } \overline{\mathrm{ID}}^{(i)} \text{ has the same theorems as } \mathrm{ID}^{(i)}_{<\Theta\Omega_{\Omega_1}0}.$$

Preliminaries. In the sequel we assume an arithmetization of the notational system $\bar{\Theta}(\{g\})$, such that all relevant ordinal sets, functions and relations of [2] (as $\mathfrak{T}$, $\mathfrak{K}$, $K_u a$, S, $+$, $\bar{\Theta}$, $<$, etc.) become primitive recursive[6]. We will identify ordinal notations and their arithmetizations.

Though we presume some familiarity with [2], we will give a short *description of the system* $\bar{\Theta}(\{g\})$. $\bar{\Theta}(\{g\})$ is a *set* $\mathfrak{T}$ of ordinal notations ordered by a *relation* $<$. Each element of $\mathfrak{T}$ has the shape 0, $a + b$, $\bar{\Theta}ab$ or gab with

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[3] Cf. [3]. Note that the system $\Sigma(N)$ in [3] is a slight modification of that in [14]. In [3] the first element of $\Sigma(N)$ is 0 instead of 1.

[4] Recently the second author [13] was able to show $\mathrm{ID}_\nu \nvdash \mathrm{TI}[\bar{\Theta}\varepsilon_{\Omega_\nu+1}0]$.

[5] Kino's wellordering proof for her ordinal diagrams $\mathrm{Od}(I)$ [8] is formalizable in $\mathrm{ID}^i_{<\cdot}$. Hence $\mathrm{ID}^i_{<\cdot} \vdash \mathrm{TI}[\|\mathrm{Od}(I), <_\infty\|]$. But as remarked in [2] $\|\mathrm{Od}(I), <_\infty\| \le \Theta\Omega_{\Omega_1}(\tau+1) < \Theta(\Omega_{\Omega_1}+1)0 < \Theta\varepsilon_{\Omega_{\Omega_1}+1}0$ for $\|I\| = 1 + \tau < \Theta(\Omega_{\Omega_1}+1)0$.

[6] For a subsystem of $\bar{\Theta}(\{g\})$ such an arithmetization will be carried out in [15].

$a, b \in \mathfrak{T}$. The symbols $+$ (ordinal sum), $\bar{\Theta}$ and g denote 2-place ordinal functions. So each term $a \in \mathfrak{T}$ canonically represents an ordinal $|a|$, for example $|\bar{\Theta}00| = 1$, $|\bar{\Theta}01| = \omega$, $|\bar{\Theta}10| = \varepsilon_0$. For the order relation $<$ we have $a < b \leftrightarrow |a| \in |b|$. Terms of the shape $\bar{\Theta}ab$ or gab are called main terms; they represent ordinals closed under $+$. $\mathfrak{K}$ is the set of all terms $gab \in \mathfrak{T}$; the elements of $\mathfrak{K}$ represent initial ordinals $> \omega$. There is a primitive recursive order isomorphism $a \mapsto \Omega_a$ from $\mathfrak{T}$ onto $\mathfrak{K}_0 := \{0\} \cup \mathfrak{K}$ with $\Omega_0 = 0$ and $|\Omega_a| = \Omega_{|a|}$ for $a \neq 0$. For each $a \in \mathfrak{T}$ there is exactly one $x \in \mathfrak{T}$ with $\Omega_x \leq a < \Omega_{x+1}$; we define $Sa := \Omega_x$ and call it the level (Stufe) of a. For all $a, b \in \mathfrak{T}$ we have $S\bar{\Theta}ab = Sb$, hence $\bar{\Theta}a0 < \Omega_1$. We define $M_0 := \{x \in \mathfrak{T}\colon Sx = 0\} = \{x \in \mathfrak{T}\colon x < \Omega_1\}$. M_0 represents a segment of the countable ordinals, i.e.

(1) $\{\xi \in On\colon \xi < |a|\} = \{|x|\colon x \in M_0 \wedge x < a\}$ for $a \in M_0$.

For $a \in \mathfrak{T}$ and $u \in \mathfrak{K}_0$, $K_u a$ is a finite set of main terms with levels $\leq u$. The sets $K_u a$ have the following properties:

(2) If $Sa \leq u$, then $K_u a$ is the set of components of a [7].

(3) $K_u b \subset K_u(a + b) \subset K_u a \cup K_u b$.

(4) $w \leq v \wedge c \in K_v a \rightarrow K_w c \subset K_w a$.

(5) $v < \Omega_a \rightarrow K_v \Omega_a \subset K_v a \subset K_v \Omega_a \cup \{1\}$.

We fix the following notational conventions:

(1) a, b, c, x, y, z denote elements of $\mathfrak{T}$.

(2) u, v, w denote elements of $\mathfrak{K}_0$.

(3) $\mathfrak{N}, \mathfrak{X}$ serve as syntactical variables for sets $\{x\colon \mathfrak{F}[x]\} \subset \mathfrak{T}$, where $\mathfrak{F}[x]$ is a formula of the theory considered.

(4) $\mathfrak{X} \cap a := \{x \in \mathfrak{X}\colon x < a\}$, $\mathfrak{X} \cap u^+ := \{x \in \mathfrak{X}\colon Sx \leq u\}$.

(5) $\mathrm{Prog}[\mathfrak{N}, \mathfrak{X}]$ abbreviates the formula $\forall x \in \mathfrak{N}(\mathfrak{N} \cap x \subset \mathfrak{X} \rightarrow x \in \mathfrak{X})$. $\mathrm{TI}[\mathfrak{N}, \mathfrak{X}, a]$ abbreviates the fromula $a \in \mathfrak{N} \wedge (\mathrm{Prog}[\mathfrak{N}, \mathfrak{X}] \rightarrow \mathfrak{N} \cap a \subset \mathfrak{X})$, and $\mathrm{TI}[\mathfrak{N}, a]$ denotes the scheme $\{\mathrm{TI}[\mathfrak{N}, \mathfrak{X}, a]\}_{\mathfrak{X}}$ expressing the principle of transfinite induction over $\mathfrak{N}$ up to a.

Transfinite inductions provable in ID^i_ν *and* $\mathrm{ID}^i_{<*}$. Our main tool in proving transfinite inductions will be the concept of the *accessible part* $W[\mathfrak{N}]$ *of a set* $\mathfrak{N}$, usually defined by $W[\mathfrak{N}] = \{x \in \mathfrak{N}\colon \mathfrak{N} \cap x$ is wellordered$\}$, which is a second-order definition. This definition however can be replaced by an inductive definition, which is expressible in a first-order language by the infinite list of axioms:

(i) $\mathrm{Prog}[\mathfrak{N}, W[\mathfrak{N}]]$ and

(ii) $\mathrm{Prog}[\mathfrak{N}, \mathfrak{X}] \rightarrow W[\mathfrak{N}] \subset \mathfrak{X}$ for each $\mathfrak{X}$.

The theories ID^i_ν (*with* $\nu \in M_0$) *and* $\mathrm{ID}^i_{<*}$ are formal theories for iterations of such inductive definitions. They are first-order extensions of Heyting's arithmetic, where ID^i_ν allows iteration of monotone inductive definitions along the segment $M_0 \cap \nu$, while $\mathrm{ID}^i_{<*}$ allows iteration along the accessible part $W_0 := W[M_0]$ of M_0. Besides the axioms for iteration of inductive definitions (cf. [4, p. 307, (i), (ii)]) there are the axioms:

$$(\mathrm{TI}_\nu) \qquad \mathrm{Prog}[M_0, \mathfrak{X}] \rightarrow M_0 \cap \nu \subset \mathfrak{X} \qquad \text{for each } \mathfrak{X}, \text{ in } \mathrm{ID}^i_\nu,$$

[7] For each $a \neq 0$ there are uniquely determined main terms $a_1 \geq \cdots \geq a_n$ $(n \geq 1)$ such that $a = a_1 + \cdots + a_n$. We call $a_1, \ldots, a_n$ the components of a. 0 is defined to have no components.

asserting the wellordering of M_0,

$(W_0 1)$ $\quad\quad \mathrm{Prog}[M_0, W_0]$ and

$(W_0 2)$ $\quad\quad \mathrm{Prog}[M_0, \mathcal{X}] \to W_0 \subset \mathcal{X}$ for each $\mathcal{X}$, in $\mathrm{ID}^i_{<\cdot}$,

defining the accessible part W_0 of M_0.

In order to treat ID^i_ν and $\mathrm{ID}^i_{<\cdot}$ simultaneously as far as possible, we refer to both as ID^i and define A to be the set $\{\Omega_x : x < \nu\}$ in the case of ID^i_ν and the set $\{\Omega_x : x \in W_0\}$ in the case of $\mathrm{ID}^i_{<\cdot}$. Then A is a segment of $\mathfrak{K}_0 \cap \Omega_{\Omega_1}$ with $0 \in A$.

In the sequel u, v are reserved to denote elements of A !

Define $\mathfrak{A}[X, Y, x, y]$ to be the formula $\mathfrak{F}[x] \wedge \forall x_0 < x(\mathfrak{F}[x_0] \to x_0 \in X)$, where $\mathfrak{F}[x]$ stands for $Sx \leq \Omega_y \wedge \forall z_1 < y(\{\langle z_0, z_1\rangle : z_0 \in K_{\Omega_{z_1}} x\} \subset Y)$. Then $\mathfrak{A}[X, Y, x, y]$ is an arithmetic formula such that each occurrence of X is positive. To $\mathfrak{A}$ corresponds a set constant $P^{\mathfrak{A}}$ (cf. [4, p. 307]). We define

$$W_{\Omega_y} := \{x : \langle x, y\rangle \in P^{\mathfrak{A}}\} \quad \text{and}$$

$$M_u := \{x : Sx \leq u \wedge \forall v < u(K_v x \subset W_v)\}.$$

Then the axioms (i) and (ii) of [4, p. 307] become

$(W1)$ $\quad\quad \forall u \in A(\mathrm{Prog}[M_u, W_u])$ and

$(W2)$ $\quad\quad \forall u \in A(\mathrm{Prog}[M_u, \mathcal{X}] \to W_u \subset \mathcal{X})$ for each $\mathcal{X}$.

$(W1)$ and $(W2)$ assert that *W_u is the accessible part of M_u*. Clearly for $u = 0$, M_u coincides with the previously defined set $M_0 = \mathfrak{T} \cap \Omega_1$, and in the case of $\mathrm{ID}^i_{<\cdot}$ the set W_0 defined by $(W1)$, $(W2)$ coincides with the set W_0 defined by $(W_0 1)$, $(W_0 2)$. As immediate consequences of $(W1)$, $(W2)$ the following formulas are provable in ID^i:

(6) $\forall x \in W_u(x \in M_u \wedge M_u \cap x = W_u \cap x)$, i.e. W_u is a segment of M_u.

(7) $a \in W_u \to \mathrm{TI}[M_u, \mathcal{X}, a]$.

By (3) and the definition of M_u we get

(8) $a, b \in M_u \to a + b \in M_u$ and $a + b \in M_u \to b \in M_u$.

The following lemmata 1–3 are straightforward modifications of corresponding lemmata in [9], [10], [11] and [14].

Lemma 1. (a) *$a, b \in W_u \to a + b \in W_u$ and*

(b) *$Sa \leq u \wedge K_u a \subset W_u \to a \in W_u$ are provable in* ID^i.

Proof. By (6) and (8). $a, b \in W_u \wedge \forall x \in M_u \cap b(a + x \in W_u) \to a + b \in M_u \wedge M_u \cap (a + b) \subset W_u$. Hence by $(W1)$, $a \in W_u \to \mathrm{Prog}[M_u, \{x : x + a \in W_u\}]$ and thence by $(W2)$, $a, b \in W_u \to a + b \in W_u$. Part (b) is an immediate consequence of (a) and (2).

Lemma 2. (a) *$a \in W_u \to K_v a \subset W_v$ and*

(b) *$v < u \to W_v = W_u \cap v^+$ are provable in* ID^i.

Proof. Suppose $a \in W_u \wedge v = \Omega_x$. Using (TI_ν) or $(W_0 2)$ resp. we prove $K_v a \subset W_v$ by transfinite induction on x. For $v < u$ we have $K_v a \subset W_v$ by $a \in W_u \subset M_u$. From $u \leq v$ we get $M_v \cap u^+ \subset M_u$, hence by $(W1)$, $\mathrm{Prog}[M_u, \{x : x \in M_v \to x \in W_v\}]$ and thence by $(W2)$, $W_u \subset \{x : x \in M_v \to x \in W_v\}$, i.e. $W_u \cap M_v \subset W_v$. By the induction hypothesis we have $K_w a \subset W_w$ for all

$w < v$ and hence $a \in W_u \cap M_v \subset W_v$. By (2), (4) and (6) we then get $K_v a \subset M_v \cap (a+1) \subset W_v$. Part (b) follows from (a) by Lemma 1(b).

LEMMA 3. *$u \in W_u$ is provable in* ID^i.

PROOF. We have $\{x: \Omega_x \in A\} \subset W_0$, which is trivial for $\mathrm{ID}^i_{<\nu}$ and proved by (TI_ν) and $(W1)$ in ID^i_ν. So for $u = \Omega_x$ we get $x \in W_0$ and thence by (5) and Lemma 2(a), $K_v u \subset K_v x \subset W_v$ for all $v < u$, which implies $u \in M_u$. Now suppose $a \in M_u \cap u$, then $Sa \in A \cap u$, $K_{Sa} a \subset W_u$ and by the lemmata 1(b), 2(b), $a \in W_u$. Hence $M_u \cap u \subset W_u$ and by $(W1)$, $u \in W_u$.

DEFINITION. $Q := \{x: \exists u(x \in W_u)\} = \bigcup_{u \in A} W_u$; $M := \{x: \forall v(K_v x \subset W_v)\}$.

Consequences. 1. Obviously $Sa \in A$ for all $a \in Q$. By Lemma 3 we then have $\forall x(x \in Q \rightarrow Sx \in Q)$ and $Q \cap \mathfrak{K}_0 = A$. Hence by Lemma 2(b) for all $u \in Q$

(9) $Q \cap u^+ = W_u$

and $M_u = \{x: Sx \leq u \wedge \forall w(w \in Q \cap u \rightarrow K_w x \subset Q)\}$. That means the set Q is "ausgezeichnet" in the sense of [2, p. 18] with $M^Q_u \cap u^+ = M_u$ and $W^Q_u = W[M^Q_u \cap u^+] = W_u$.

2. By (6) and (9) $\mathrm{Prog}[Q, \mathcal{X}] \rightarrow \mathrm{Prog}[M_u, \{x: x \in W_u \rightarrow x \in \mathcal{X}\}]$ and thence by $(W2)$

(10) $\mathrm{Prog}[Q, \mathcal{X}] \rightarrow Q \subset \mathcal{X}$ for each $\mathcal{X}$,

which is the first-order formulation of the fact that *Q is wellordered.*

3. Since Q is "ausgezeichnet" (provable in ID^i) we may follow the proof of Theorem 15(b) in [2, p. 19] and get the formula

$$a \in M \wedge \forall x \in M \cap a(Q \subset R_x) \rightarrow \mathrm{Prog}[Q, R_a]^8,$$

where $R_a := \{y: \bar{\Theta} ay \in \mathfrak{T} \rightarrow \bar{\Theta} ay \in Q\}$. Here besides the premise "Q ausgezeichnet" only methods formalizable in Heyting's arithmetic are used. By (10) it follows

(11) $\mathrm{Prog}[M, \{x: \forall y \in Q(\bar{\Theta} xy \in \mathfrak{T} \rightarrow \bar{\Theta} xy \in Q)\}]$.

4. From outside we know that $\bar{\Theta}(\{g\}) = (\mathfrak{T}, <)$ is wellordered and hence $W_u = M_u = \{x: Sx \leq u\}$ and $M = \mathfrak{T}$ which implies $Q = M \cap \Omega_\sigma$, σ defined by:

DEFINITION.

$$\sigma := \begin{cases} \nu, & \text{in the case of } \mathrm{ID}^i_\nu, \\ \Omega_1, & \text{in the case of } \mathrm{ID}^i_{<\nu}. \end{cases}$$

Of course $W_u = M_u = \{x: Sx \leq u\}$ is not provable in ID^i, but the weaker assertion $Q = M \cap \Omega_\sigma$ is provable as the following theorem shows.

THEOREM 1. *$\Omega_\sigma \in M$ and $Q = M \cap \Omega_\sigma$ are provable in* ID^i.

PROOF. By Lemma 2(a) we have $Q \subset M \cap \Omega_\sigma$. If $a \in M$ and $Sa \in A$ we get $K_{Sa}\, a \subset W_{Sa}$ and by Lemma 1(b), $a \in W_{Sa} \subset Q$. So we just have to prove $a \in M \cap \Omega_\sigma \rightarrow Sa \in A$ and $\Omega_\sigma \in M$. The proofs differ for ID^i_ν, $\mathrm{ID}^i_{<\nu}$.

1. ID^i_ν. Then $A = \{w: w < \Omega_\sigma\}$ and trivially $a \in M \cap \Omega_\sigma \rightarrow Sa \in A$ holds. By (TI_ν) and $(W1)$ we get $\sigma = \nu \in W_0$. Hence by Lemma 2(a) and (5) $\forall v(K_v \Omega_\sigma \subset K_v \sigma \subset W_v)$ which means $\Omega_\sigma \in M$.

[8] In [2] this formula is proved with Q in place of M, but an analysis of the proof shows that it is enough to have the premise $a \in M \wedge \forall x \in M \cap a(Q \subset R_x)$.

2. $\mathrm{ID}^i_{<\cdot}$. Suppose $a \in M \cap \Omega_\sigma$. Then $K_0 a \subset W_0$ and $Sa = \Omega_x$ for some $x < \Omega_1 = \sigma$. By (5) $K_0 x \subset K_0 \Omega_x \cup \{1\}$. Further $K_0 Sa \subset K_0 a$ and $1 \in W_0$. We therefore get by Lemma 1(b) $x \in W_0$ and thence $Sa \in A$. For $0 < u$, $K_0 \Omega_{\Omega_1} = \emptyset$ and $K_u \Omega_{\Omega_1} = \{\Omega_1\}$. Obviously $\Omega_1 \in A$ and hence $\Omega_1 \in W_{\Omega_1}$ by Lemma 3. By Lemma 2(b) it follows $\forall u(K_u \Omega_{\Omega_1} \subset W_u)$, which means $\Omega_\sigma = \Omega_{\Omega_1} \in M$.

Now by (10) and Theorem 1 we get

(12) $\mathrm{TI}[M, \Omega_\sigma]$.

Hence by (11) $\forall y \in Q(\bar{\Theta}\Omega_\sigma y \in \mathfrak{T} \to \bar{\Theta}\Omega_\sigma y \in Q)$. Since $0 \in Q \wedge \bar{\Theta}\Omega_\sigma 0 \in \mathfrak{T} \cap \Omega_1$ we obtain $\bar{\Theta}\Omega_\sigma 0 \in Q \cap \Omega_1$. Hence by (9) and (7) $\mathrm{TI}[M_0, \bar{\Theta}\Omega_\sigma 0]$. This means we are able to *collapse* the wellordering $M \cap \Omega_\sigma$ to the provable ordinal $\bar{\Theta}\Omega_\sigma 0$[9]. This is a special case of the following theorem, which is proved by the above considerations with c in place of Ω_σ.

THEOREM 2 (COLLAPSING PROPERTY). *If* $\mathrm{TI}[M, c]$ *is provable in* ID^i *and* $\bar{\Theta}c0 \in \mathfrak{T}$, *then* $\mathrm{TI}[M_0, \bar{\Theta}c0]$ *is provable in* ID^i.

Starting from (12) we now prove $\mathrm{TI}[M, c]$ for each $c \in M \cap \bar{\Theta}1\Omega_\sigma$ using Gentzen's [5] method for proving $\mathrm{TI}[<\varepsilon_0]$ in number theory.

DEFINITION. $\bar{\mathfrak{X}} := \{x : \bar{\Theta}1\Omega_\sigma \leq x \vee \forall y(M \cap y \subset \mathfrak{X} \to M \cap (y + \bar{\Theta}0x) \subset \mathfrak{X})\}$.

LEMMA 4. $\forall y(M \cap y \subset \mathfrak{X} \to M \cap (y + \Omega_\sigma) \subset \mathfrak{X}) \to \mathrm{Prog}[M, \bar{\mathfrak{X}}]$ *is provable in* ID^i.

PROOF. We have to prove $M \cap (b + \bar{\Theta}0a) \subset \mathfrak{X}$ under the assumptions (1) $\forall y(M \cap y \subset \mathfrak{X} \to M \cap (y + \Omega_\sigma) \subset \mathfrak{X})$, (2) $M \cap a \subset \bar{\mathfrak{X}}$, (3) $a < \bar{\Theta}1\Omega_\sigma$, (4) $M \cap b \subset \mathfrak{X}$.

By (1) and (4) we get $M \cap (b + \Omega_\sigma \cdot n) \subset \mathfrak{X}$ for all $n \in \mathbf{N}$ using mathematical induction. Hence $M \cap (b + \bar{\Theta}0\Omega_\sigma) \subset \mathfrak{X}$ because of $\sup_{n \in \mathbf{N}} \Omega_\sigma \cdot n = \bar{\Theta}0\Omega_\sigma$. Suppose $z \in M \cap (b + \bar{\Theta}0a)$. We may assume $b + \bar{\Theta}0\Omega_\sigma \leq z$. By (3) $z < b + \bar{\Theta}0a < b + \bar{\Theta}1\Omega_\sigma$. Hence $z = b + \bar{\Theta}0a_1 \cdot n + z_1$ with $1 \leq n \in \mathbf{N}$, $\Omega_\sigma \leq a_1 < a$, $z_1 < \bar{\Theta}0a_1$. By $\forall v(v < \Omega_\sigma = Sa_1)$ and the definition of K_v—it is the case that $K_v a_1 = K_v \bar{\Theta}0a_1 \subset K_v z$. So we get $a_1 \in M \cap a$ since $z \in M$ is assumed. By (2), (3), (4) we get $M \cap (b + \bar{\Theta}0a_1 \cdot (n+1)) \subset \mathfrak{X}$ using mathematical induction. Hence $z \in \mathfrak{X}$.

DEFINITION. $c_0 := \Omega_\sigma$, $c_{n+1} := \bar{\Theta}0c_n$.

One easily proves $c_n \in M$, $c_n < \bar{\Theta}1\Omega_\sigma = \sup_{k \in \mathbf{N}} c_k$ and $\bar{\Theta}c_n 0 \in \mathfrak{T}$.

THEOREM 3. $\mathrm{TI}[M, c_n]$ *is provable in* ID^i *for each* $n \in \mathbf{N}$.

PROOF. We prove the theorem by 'metainduction' on n. By (3) it follows that $a + b \in M \to b \in M$. Hence $\mathrm{Prog}[M, \mathfrak{X}] \wedge M \cap a \subset \mathfrak{X} \to \mathrm{Prog}[M, \{x : a + x \in M \to a + x \in \mathfrak{X}\}]$ and thence by (12)

$$(*) \qquad \mathrm{Prog}[M, \mathfrak{X}] \to \forall y(M \cap y \subset \mathfrak{X} \to M \cap (y + \Omega_\sigma) \subset \mathfrak{X}).$$

By (*) and $c_0 = \Omega_\sigma \in M$ we have $\mathrm{TI}[M, c_0]$. For $n > 0$ we have the induction hypothesis $\mathrm{TI}[M, c_{n-1}]$. Hence $\mathrm{Prog}[M, \bar{\mathfrak{X}}] \to c_{n-1} \in \bar{\mathfrak{X}}$ which implies $\mathrm{Prog}[M, \bar{\mathfrak{X}}] \to M \cap c_n \subset \mathfrak{X}$. By (*) and Lemma 4 we get $\mathrm{Prog}[M, \mathfrak{X}] \to \mathrm{Prog}[M, \bar{\mathfrak{X}}]$ and therefore $\mathrm{Prog}[M, \mathfrak{X}] \to M \cap c_n \subset \mathfrak{X}$. Hence $\mathrm{TI}[M, c_n]$.

THEOREM 4. *In* ID^i, $\mathrm{TI}[M_0, a]$ *is provable for each* $a < \bar{\Theta}(\bar{\Theta}1\Omega_\sigma)0$.

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[9] Remember that by (1) $M_0 \cap \bar{\Theta}\Omega_\sigma 0$ represents the ordinal $\bar{\Theta}\Omega_\sigma 0$.

PROOF. For $a < \bar{\Theta}(\bar{\Theta}1\Omega_\sigma)0$ there is an $n \in N$ with $a < \bar{\Theta}c_n 0$. By Theorem 3 we have $\mathrm{TI}[M, c_n]$, which can be collapsed to $\mathrm{TI}[M_0, \bar{\Theta}c_n 0]$ by Theorem 2. So $\mathrm{TI}[M_0, a]$ holds.

Since $M_0 \cap \bar{\Theta}(\bar{\Theta}1\Omega_\sigma)0$ represents the segment $\{\xi \in On: \xi < \bar{\Theta}\varepsilon_{\Omega_\sigma+1}0\}$, the results (A) and (B) stated in the introduction follow from Theorem 4.

FINAL REMARKS. The above proof of transfinite induction admits the following generalization. Let Th be a theory containing Heyting's arithmetic and axioms for iterations of inductive definitions along a provably wellordered subset A of $\mathfrak{K}_0$. Then the sets $W_u := W[M_u]$, $M_u := \{x: Sx \leq u \wedge \forall v \in A \cap u(K_v x \subset W_v)\}$ $(u \in A)$, $Q := \{x: \exists u \in A(x \in W_u)\}$ and $M := \{x: \forall u \in A(K_u x \subset W_u)\}$ are definable in Th, and if the formula $\forall u \in A(u \in W_u \wedge \forall x \in W_u(Sx \in A))$ is provable in Th, one gets:

I. *Q is wellordered*, i.e.

$$\mathrm{Th} \vdash \mathrm{Prog}[Q, \mathfrak{X}] \to Q \subset \mathfrak{X}.$$

II. *Collapsing property.*

$$\mathrm{Th} \vdash \mathrm{TI}[M, c] \Rightarrow \mathrm{Th} \vdash \mathrm{TI}[M_0, \bar{\Theta}c0] \qquad (\text{for } \bar{\Theta}c0 \in \mathfrak{T}).$$

III. *Extension to the next ε-number.*

$$\mathrm{Th} \vdash \mathrm{TI}[M, \Omega_a] \Rightarrow \mathrm{Th} \vdash \mathrm{TI}[M, c] \qquad \text{for each } c \in M \cap \bar{\Theta}1\Omega_a.$$

From I, II, III it follows:

IV.

$$\mathrm{Th} \vdash \Omega_a \in M \wedge M \cap \Omega_a \subset Q \Rightarrow \mathrm{Th} \vdash \mathrm{TI}[< \bar{\Theta}(\bar{\Theta}1\Omega_a)0] \qquad \text{for } \bar{\Theta}(\bar{\Theta}1\Omega_a)0 \in \mathfrak{T}.$$

As an example we regard the following definition by transfinite recursion on $\nu \in \mathfrak{T} \cap \Omega_1$:

$\lambda(0) := 0,$ $\qquad A_0 := \emptyset.$

$\lambda(\nu+1) := \Omega_{\lambda(\nu)+1},$ $\qquad A_{\nu+1} := A_\nu \cup \{\Omega_x: \lambda(\nu) \leq x \in W[M^\nu]\}$, with $M^\nu := \{x: x < \lambda(\nu+1) \wedge \forall u \in A_\nu(K_u x \subset W_u)\}$ and W_u defined as above by iteration of inductive definitions along A_ν.

$\lambda(\nu) := \sup_{\xi<\nu}\lambda(\xi)$[10], $\qquad A\nu := \bigcup_{\xi<\nu} A_\xi$ for limit ordinals ν.

Let ID_ν^* be the theory, which allows to define A_ν and to iterate inductive definitions along A_ν (ID_1^* for example is $\mathrm{ID}_{<\cdot}$). Then by the above considerations we get:

$$\mathrm{ID}_\nu^* \vdash \mathrm{TI}[< \bar{\Theta}(\bar{\Theta}1\Omega_{\lambda(\nu)})0].$$

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