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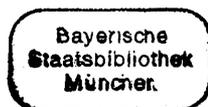
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## PROVABLE WELLORDERINGS OF FORMAL THEORIES FOR TRANSFINITELY ITERATED INDUCTIVE DEFINITIONS

W. BUCHHOLZ and W. POHLERS

### Introduction

By [12] we know that transfinite induction up to  $\Theta_{\varepsilon_{\Omega_{N+1}}}$  is not provable in  $ID_N$ , the theory of  $N$ -times iterated inductive definitions. In this paper we will show that conversely transfinite induction up to any ordinal less than  $\Theta_{\varepsilon_{\Omega_{N+1}}}$  is provable in  $ID_N^i$ , the intuitionistic version of  $ID_N$ , and extend this result to theories for transfinitely iterated inductive definitions.

In [14] Schütte proves the wellordering of his notational systems  $\Sigma(N)$  using predicates  $\mathfrak{B}_k(a) : \leftrightarrow (a \in M_k \wedge \{x \in M_k : x < a\} \text{ is wellordered})$  with  $M_k := \{x \in \Sigma(N) : \mathfrak{B}_0(K_0x) \wedge \dots \wedge \mathfrak{B}_{k-1}(K_{k-1}x)\}^1$  and  $0 \leq k \leq N$ . Obviously the predicates  $\mathfrak{B}_0, \dots, \mathfrak{B}_{N-1}$  are definable in  $ID_N^i$  with the defining axioms:

- ( $\mathfrak{B}_k 1$ )  $\text{Prog}[M_k, \mathfrak{B}_k]$ ,  
 ( $\mathfrak{B}_k 2$ )  $\text{Prog}[M_k, \mathfrak{F}] \rightarrow \forall x (\mathfrak{B}_k(x) \rightarrow \mathfrak{F}[x])$ ,

where  $\text{Prog}[M_k, X]$  means that  $X$  is progressive with respect to  $M_k$ , i.e.

$$\text{Prog}[M_k, X] : \leftrightarrow \forall x \in M_k (\forall y \in M_k (y < x \rightarrow X(y)) \rightarrow X(x)).$$

The crucial point in Schütte's wellordering proof is Lemma 19 [14, p. 130] which can be modified to

- (I)  $\text{TI}[M_{k+1}, a]$ ,  $Sb = k$ ,  $\mathfrak{B}_k(b) \Rightarrow \mathfrak{B}_k((a, b))$ , for  $0 \leq k \leq N-1$ ,

where  $\text{TI}[M_{k+1}, a]$  is the scheme of transfinite induction over  $M_{k+1}$  up to  $a^2$ . Checking the proof of (I) it turns out that besides ( $\mathfrak{B}_k 1$ ) and ( $\mathfrak{B}_k 2$ ) ( $0 \leq k \leq N-1$ ) only finitary methods (including mathematical induction) are used. Since the proof uses "excluded middle" only for decidable formulas it is formalizable in  $ID_N^i$ . Following the proof of Lemma 17 in [14] one gets

- (II)  $ID_N^i \vdash \mathfrak{B}_0(1) \wedge \dots \wedge \mathfrak{B}_{N-1}(\Omega_{N-1})$  and  
 (III)  $ID_N^i \vdash \text{TI}[M_N, \Omega_N]$ .

From (III) one derives in the well-known way (due to Gentzen [5])

- (IV)  $ID_N^i \vdash \text{TI}[M_N, c_n]$  for each  $n \in \mathbf{N}$ ,

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<sup>1</sup>  $K_n x$  is a finite set of subterms of  $x$ .  $\mathfrak{B}_n(K_n x)$  means  $\forall y \in K_n x (\mathfrak{B}_n(y))$ .

<sup>2</sup> For an exact definition see notational convention (5), page 3 of the present paper.

where  $c_0 := \Omega_N$ ,  $c_{n+1} := (1, c_n)$ . By (I), (II), (IV) and the facts that  $M_0 = \Sigma(N)$  and,  $\mathfrak{B}_k(a)$  implies  $\text{TI}[M_k, a]$  one gets

$$(V) \quad \text{ID}_N^i \vdash \text{TI}[\Sigma(N), \Omega[c_n, 0]] \quad \text{for each } n \in N,$$

where  $\Omega[c_n, 0] := ((\dots(c_n, \Omega_{N-1}), \dots, \Omega_1), 1)$ . Since  $\sup_{n \in N} \Omega[c_n, 0] = \Omega[(1 \neq 1, \Omega_N), 0]$  and the order type of  $\{x \in \Sigma(N) : x < \Omega[(1 \neq 1, \Omega_N), 0]\}$  is  $\Theta_{\varepsilon_{\Omega_{N+1}}0}^3$ , *transfinite induction up to any ordinal less than  $\Theta_{\varepsilon_{\Omega_{N+1}}0}$  is provable in  $\text{ID}_N^i$* , which we will abbreviate by  $\text{ID}_N^i \vdash \text{TI}[< \Theta_{\varepsilon_{\Omega_{N+1}}0}]$ .

Similar considerations apply to the wellordering proof of the system  $\bar{\Theta}(\{g\})$  given in [2]. We will prove the following *results*:

- (A)  $\text{ID}_\nu^i \vdash \text{TI}[< \bar{\Theta}_{\varepsilon_{\Omega_{\nu+1}}0}]$  for any countable  $\nu \in \bar{\Theta}(\{g\})^4$ ,
- (B)  $\text{ID}_{<\cdot}^i \vdash \text{TI}[< \bar{\Theta}_{\varepsilon_{\Omega_{\nu+1}}0}]^5$ ,

where  $\text{ID}_\nu^i$  and  $\text{ID}_{<\cdot}^i$  are the intuitionistic versions of the theories  $\text{ID}_\nu$  and  $\text{ID}_{<\cdot}$  defined in [4, pp. 307–308] (see also the last paragraph of page 3 of the present paper).

$\text{ID}_{<\cdot}$  is defined to be the theory  $\bigcup_{\xi < \cdot} \text{ID}_\xi$ . By  $\overline{\text{ID}}$  we mean the theory of autonomously iterated inductive definitions (i.e. if  $\overline{\text{ID}} \vdash \text{TI}[\nu]$  then  $\text{ID}_\nu \subset \overline{\text{ID}}$ ). For a theory  $\text{Th}$  we take  $|\text{Th}| := \sup\{\xi \in \text{On} : \text{Th} \vdash \text{TI}[\xi]\}$ . There are the following ordinal theoretic relations:

- (1)  $\bar{\Theta}_{\varepsilon_{\Omega_{\nu+1}}0} = \Theta_{\varepsilon_{\Omega_{\nu+1}}0}$  and  $\bar{\Theta}_{\varepsilon_{\Omega_{\nu+1}}0} = \Theta_{\varepsilon_{\Omega_{\nu+1}}0}$  for  $\nu < \Theta_{\Omega_{\nu+1}}0$ . (So the above derived results on transfinite induction in  $\text{ID}_\nu^i$  ( $N < \omega$ ) are special cases of (A).)
  - (2)  $\Theta_{\Omega_\nu}0 = \sup_{\xi < \nu} \Theta_{\varepsilon_{\Omega_\xi}0}$  for limit  $\nu \leq \Theta_{\Omega_{\nu+1}}0$ .
  - (3)  $\Theta_{\Omega_\nu}0 = \nu$  for  $\nu = \Theta_{\Omega_{\nu+1}}0$ .
  - (4)  $\Theta_{\Omega_{\nu+1}}0 = \sup_{n \in N} \nu_n$  with  $\nu_0 := 1$ ,  $\nu_{n+1} := \Theta_{\Omega_{\nu_n}}0$ .
- By (A), (1)–(4) and [13] (cf. footnote 4) we get the equations:

- (A1)  $|\text{ID}_\nu| = |\text{ID}_\nu^i| = \Theta_{\varepsilon_{\Omega_{\nu+1}}0}$  for  $\nu < \Theta_{\Omega_{\nu+1}}0$ .
- (A2)  $|\text{ID}_{<\cdot}| = |\text{ID}_{<\cdot}^i| = \Theta_{\Omega_\nu}0$  for limit  $\nu \leq \Theta_{\Omega_{\nu+1}}0$ .
- (A3)  $|\text{ID}_{<\Theta_{\Omega_{\nu+1}}0}^{(i)}| = |\overline{\text{ID}}^{(i)}| = \Theta_{\Omega_{\nu+1}}0$  and  $\overline{\text{ID}}^{(i)}$  has the same theorems as  $\text{ID}_{<\Theta_{\Omega_{\nu+1}}0}^{(i)}$ .

**Preliminaries.** In the sequel we assume an arithmetization of the notational system  $\bar{\Theta}(\{g\})$ , such that all relevant ordinal sets, functions and relations of [2] (as  $\mathfrak{L}$ ,  $\mathfrak{R}$ ,  $K_\mu a$ ,  $S$ ,  $+$ ,  $\bar{\Theta}$ ,  $<$ , etc.) become primitive recursive<sup>6</sup>. We will identify ordinal notations and their arithmetizations.

Though we presume some familiarity with [2], we will give a short *description of the system  $\bar{\Theta}(\{g\})$* .  $\bar{\Theta}(\{g\})$  is a set  $\mathfrak{X}$  of ordinal notations ordered by a relation  $<$ . Each element of  $\mathfrak{X}$  has the shape  $0$ ,  $a + b$ ,  $\bar{\Theta}ab$  or  $gab$  with

<sup>3</sup> Cf. [3]. Note that the system  $\Sigma(N)$  in [3] is a slight modification of that in [14]. In [3] the first element of  $\Sigma(N)$  is 0 instead of 1.

<sup>4</sup> Recently the second author [13] was able to show  $\text{ID}_\nu \not\vdash \text{TI}[\bar{\Theta}_{\varepsilon_{\Omega_{\nu+1}}0}]$ .

<sup>5</sup> Kino's wellordering proof for her ordinal diagrams  $\text{Od}(I)$  [8] is formalizable in  $\text{ID}_{<\cdot}^i$ . Hence  $\text{ID}_{<\cdot}^i \vdash \text{TI}[\|\text{Od}(I), <_{\infty}\|]$ . But as remarked in [2]  $\|\text{Od}(I), <_{\infty}\| \leq \Theta_{\Omega_{\nu+1}}(\tau + 1) < \Theta_{\Omega_{\nu+1}}(1) < \Theta_{\varepsilon_{\Omega_{\nu+1}}0}$  for  $\|I\| = 1 + \tau < \Theta_{\Omega_{\nu+1}}(1)$ .

<sup>6</sup> For a subsystem of  $\bar{\Theta}(\{g\})$  such an arithmetization will be carried out in [15].

$a, b \in \mathfrak{T}$ . The symbols  $+$  (ordinal sum),  $\bar{\Theta}$  and  $g$  denote 2-place ordinal functions. So each term  $a \in \mathfrak{T}$  canonically represents an ordinal  $|a|$ , for example  $|\bar{\Theta}00| = 1$ ,  $|\bar{\Theta}01| = \omega$ ,  $|\bar{\Theta}10| = \varepsilon_0$ . For the order relation  $<$  we have  $a < b \leftrightarrow |a| \in |b|$ . Terms of the shape  $\bar{\Theta}ab$  or  $gab$  are called main terms; they represent ordinals closed under  $+$ .  $\mathfrak{R}$  is the set of all terms  $gab \in \mathfrak{T}$ ; the elements of  $\mathfrak{R}$  represent initial ordinals  $> \omega$ . There is a primitive recursive order isomorphism  $a \mapsto \Omega_a$  from  $\mathfrak{T}$  onto  $\mathfrak{R}_0 := \{0\} \cup \mathfrak{R}$  with  $\Omega_0 = 0$  and  $|\Omega_a| = \Omega_{|a|}$  for  $a \neq 0$ . For each  $a \in \mathfrak{T}$  there is exactly one  $x \in \mathfrak{T}$  with  $\Omega_x \leq a < \Omega_{x+1}$ ; we define  $Sa := \Omega_x$  and call it the level (Stufe) of  $a$ . For all  $a, b \in \mathfrak{T}$  we have  $S\bar{\Theta}ab = Sb$ , hence  $\bar{\Theta}a0 < \Omega_1$ . We define  $M_0 := \{x \in \mathfrak{T} : Sx = 0\} = \{x \in \mathfrak{T} : x < \Omega_1\}$ .  $M_0$  represents a segment of the countable ordinals, i.e.

(1)  $\{\xi \in On : \xi < |a|\} = \{x : x \in M_0 \wedge x < a\}$  for  $a \in M_0$ .

For  $a \in \mathfrak{T}$  and  $u \in \mathfrak{R}_0$ ,  $K_u a$  is a finite set of main terms with levels  $\leq u$ . The sets  $K_u a$  have the following properties:

(2) If  $Sa \leq u$ , then  $K_u a$  is the set of components of  $a^7$ .

(3)  $K_u b \subset K_u(a + b) \subset K_u a \cup K_u b$ .

(4)  $w \leq v \wedge c \in K_w a \rightarrow K_w c \subset K_v a$ .

(5)  $v < \Omega_a \rightarrow K_v \Omega_a \subset K_v a \subset K_v \Omega_a \cup \{1\}$ .

We fix the following notational conventions:

(1)  $a, b, c, x, y, z$  denote elements of  $\mathfrak{T}$ .

(2)  $u, v, w$  denote elements of  $\mathfrak{R}_0$ .

(3)  $\mathfrak{R}, \mathfrak{X}$  serve as syntactical variables for sets  $\{x : \mathfrak{F}[x]\} \subset \mathfrak{T}$ , where  $\mathfrak{F}[x]$  is a formula of the theory considered.

(4)  $\mathfrak{X} \cap a := \{x \in \mathfrak{X} : x < a\}$ ,  $\mathfrak{X} \cap u^+ := \{x \in \mathfrak{X} : Sx \leq u\}$ .

(5)  $\text{Prog}[\mathfrak{R}, \mathfrak{X}]$  abbreviates the formula  $\forall x \in \mathfrak{R} (\mathfrak{R} \cap x \subset \mathfrak{X} \rightarrow x \in \mathfrak{X})$ .  $\text{TI}[\mathfrak{R}, \mathfrak{X}, a]$  abbreviates the formula  $a \in \mathfrak{R} \wedge (\text{Prog}[\mathfrak{R}, \mathfrak{X}] \rightarrow \mathfrak{R} \cap a \subset \mathfrak{X})$ , and  $\text{TI}[\mathfrak{R}, a]$  denotes the scheme  $\{\text{TI}[\mathfrak{R}, \mathfrak{X}, a]\}_x$  expressing the principle of transfinite induction over  $\mathfrak{R}$  up to  $a$ .

*Transfinite inductions provable in  $\text{ID}_\nu^i$  and  $\text{ID}_\nu^{i-}$ .* Our main tool in proving transfinite inductions will be the concept of the *accessible part*  $W[\mathfrak{R}]$  of a set  $\mathfrak{R}$ , usually defined by  $W[\mathfrak{R}] = \{x \in \mathfrak{R} : \mathfrak{R} \cap x \text{ is wellordered}\}$ , which is a second-order definition. This definition however can be replaced by an inductive definition, which is expressible in a first-order language by the infinite list of axioms:

(i)  $\text{Prog}[\mathfrak{R}, W[\mathfrak{R}]]$  and

(ii)  $\text{Prog}[\mathfrak{R}, \mathfrak{X}] \rightarrow W[\mathfrak{R}] \subset \mathfrak{X}$  for each  $\mathfrak{X}$ .

The theories  $\text{ID}_\nu^i$  (with  $\nu \in M_0$ ) and  $\text{ID}_\nu^{i-}$  are formal theories for iterations of such inductive definitions. They are first-order extensions of Heyting's arithmetic, where  $\text{ID}_\nu^i$  allows iteration of monotone inductive definitions along the segment  $M_0 \cap \nu$ , while  $\text{ID}_\nu^{i-}$  allows iteration along the accessible part  $W_0 := W[M_0]$  of  $M_0$ . Besides the axioms for iteration of inductive definitions (cf. [4, p. 307, (i), (ii)]) there are the axioms:

( $\text{TI}_\nu$ )  $\text{Prog}[M_0, \mathfrak{X}] \rightarrow M_0 \cap \nu \subset \mathfrak{X}$  for each  $\mathfrak{X}$ , in  $\text{ID}_\nu^i$ ,

<sup>7</sup>For each  $a \neq 0$  there are uniquely determined main terms  $a_1 \geq \dots \geq a_n$  ( $n \geq 1$ ) such that  $a = a_1 + \dots + a_n$ . We call  $a_1, \dots, a_n$  the components of  $a$ . 0 is defined to have no components.

asserting the wellordering of  $M_0$ ,

$$(W_01) \quad \text{Prog}[M_0, W_0] \quad \text{and}$$

$$(W_02) \quad \text{Prog}[M_0, \mathfrak{X}] \rightarrow W_0 \subset \mathfrak{X} \quad \text{for each } \mathfrak{X}, \text{ in } \text{ID}^{i_<},$$

defining the accessible part  $W_0$  of  $M_0$ .

In order to treat  $\text{ID}^i$  and  $\text{ID}^{i_<}$  simultaneously as far as possible, we refer to both as  $\text{ID}^i$  and define  $A$  to be the set  $\{\Omega_x: x < \nu\}$  in the case of  $\text{ID}^i$  and the set  $\{\Omega_x: x \in W_0\}$  in the case of  $\text{ID}^{i_<}$ . Then  $A$  is a segment of  $\mathfrak{R}_0 \cap \Omega_\Omega$ , with  $0 \in A$ .

*In the sequel  $u, v$  are reserved to denote elements of  $A$ !*

Define  $\mathfrak{U}[X, Y, x, y]$  to be the formula  $\mathfrak{F}[x] \wedge \forall x_0 < x (\mathfrak{F}[x_0] \rightarrow x_0 \in X)$ , where  $\mathfrak{F}[x]$  stands for  $Sx \leq \Omega_y \wedge \forall z_1 < y (\{\{z_0, z_1\}: z_0 \in K_{\Omega_z, x}\} \subset Y)$ . Then  $\mathfrak{U}[X, Y, x, y]$  is an arithmetic formula such that each occurrence of  $X$  is positive. To  $\mathfrak{U}$  corresponds a set constant  $P^{\mathfrak{U}}$  (cf. [4, p. 307]). We define

$$W_{\Omega_y} := \{x: \langle x, y \rangle \in P^{\mathfrak{U}}\} \quad \text{and}$$

$$M_u := \{x: Sx \leq u \wedge \forall v < u (K_v x \subset W_v)\}.$$

Then the axioms (i) and (ii) of [4, p. 307] become

$$(W1) \quad \forall u \in A (\text{Prog}[M_u, W_u]) \quad \text{and}$$

$$(W2) \quad \forall u \in A (\text{Prog}[M_u, \mathfrak{X}] \rightarrow W_u \subset \mathfrak{X}) \quad \text{for each } \mathfrak{X}.$$

(W1) and (W2) assert that  $W_u$  is the accessible part of  $M_u$ . Clearly for  $u=0$ ,  $M_u$  coincides with the previously defined set  $M_0 = \mathfrak{X} \cap \Omega_1$ , and in the case of  $\text{ID}^{i_<}$  the set  $W_0$  defined by (W1), (W2) coincides with the set  $W_0$  defined by (W01), (W02). As immediate consequences of (W1), (W2) the following formulas are provable in  $\text{ID}^i$ :

$$(6) \quad \forall x \in W_u (x \in M_u \wedge M_u \cap x = W_u \cap x), \text{ i.e. } W_u \text{ is a segment of } M_u.$$

$$(7) \quad a \in W_u \rightarrow \text{TI}[M_u, \mathfrak{X}, a].$$

By (3) and the definition of  $M_u$  we get

$$(8) \quad a, b \in M_u \rightarrow a + b \in M_u \quad \text{and} \quad a + b \in M_u \rightarrow b \in M_u.$$

The following lemmata 1–3 are straightforward modifications of corresponding lemmata in [9], [10], [11] and [14].

LEMMA 1. (a)  $a, b \in W_u \rightarrow a + b \in W_u$  and

(b)  $Sa \leq u \wedge K_u a \subset W_u \rightarrow a \in W_u$  are provable in  $\text{ID}^i$ .

PROOF. By (6) and (8).  $a, b \in W_u \wedge \forall x \in M_u \cap b (a + x \in W_u) \rightarrow a + b \in M_u \wedge M_u \cap (a + b) \subset W_u$ . Hence by (W1),  $a \in W_u \rightarrow \text{Prog}[M_u, \{x: x + a \in W_u\}]$  and thence by (W2),  $a, b \in W_u \rightarrow a + b \in W_u$ . Part (b) is an immediate consequence of (a) and (2).

LEMMA 2. (a)  $a \in W_u \rightarrow K_v a \subset W_v$  and

(b)  $v < u \rightarrow W_v = W_u \cap v^+$  are provable in  $\text{ID}^i$ .

PROOF. Suppose  $a \in W_u \wedge v = \Omega_x$ . Using  $(\text{TI}_v)$  or (W02) resp. we prove  $K_v a \subset W_v$  by transfinite induction on  $x$ . For  $v < u$  we have  $K_v a \subset W_v$  by  $a \in W_u \subset M_u$ . From  $u \leq v$  we get  $M_v \cap u^+ \subset M_u$ , hence by (W1),  $\text{Prog}[M_u, \{x: x \in M_v \rightarrow x \in W_v\}]$  and thence by (W2),  $W_u \subset \{x: x \in M_v \rightarrow x \in W_v\}$ , i.e.  $W_u \cap M_v \subset W_v$ . By the induction hypothesis we have  $K_w a \subset W_w$  for all

$w < v$  and hence  $a \in W_u \cap M_v \subset W_v$ . By (2), (4) and (6) we then get  $K_v a \subset M_v \cap (a + 1) \subset W_v$ . Part (b) follows from (a) by Lemma 1(b).

LEMMA 3.  $u \in W_u$  is provable in  $ID^i$ .

PROOF. We have  $\{x : \Omega_x \in A\} \subset W_0$ , which is trivial for  $ID^i_{<}$  and proved by  $(TI_\nu)$  and  $(W1)$  in  $ID^i_\nu$ . So for  $u = \Omega_x$  we get  $x \in W_0$  and thence by (5) and Lemma 2(a),  $K_v u \subset K_v x \subset W_v$  for all  $v < u$ , which implies  $u \in M_u$ . Now suppose  $a \in M_u \cap u$ , then  $Sa \in A \cap u$ ,  $K_{Sa} a \subset W_u$  and by the lemmata 1(b), 2(b),  $a \in W_u$ . Hence  $M_u \cap u \subset W_u$  and by  $(W1)$ ,  $u \in W_u$ .

DEFINITION.  $Q := \{x : \exists u(x \in W_u)\} = \bigcup_{u \in A} W_u$ ;  $M := \{x : \forall v(K_v x \subset W_v)\}$ .

Consequences. 1. Obviously  $Sa \in A$  for all  $a \in Q$ . By Lemma 3 we then have  $\forall x(x \in Q \rightarrow Sx \in Q)$  and  $Q \cap \mathfrak{R}_0 = A$ . Hence by Lemma 2(b) for all  $u \in Q$

$$(9) \quad Q \cap u^+ = W_u$$

and  $M_u = \{x : Sx \leq u \wedge \forall w(w \in Q \cap u \rightarrow K_w x \subset Q)\}$ . That means the set  $Q$  is "ausgezeichnet" in the sense of [2, p. 18] with  $M_u^Q \cap u^+ = M_u$  and  $W_u^Q = W[M_u^Q \cap u^+] = W_u$ .

2. By (6) and (9)  $\text{Prog}[Q, \mathfrak{X}] \rightarrow \text{Prog}[M_u, \{x : x \in W_u \rightarrow x \in \mathfrak{X}\}]$  and thence by  $(W2)$

$$(10) \quad \text{Prog}[Q, \mathfrak{X}] \rightarrow Q \subset \mathfrak{X} \text{ for each } \mathfrak{X},$$

which is the first-order formulation of the fact that  $Q$  is wellordered.

3. Since  $Q$  is "ausgezeichnet" (provable in  $ID^i$ ) we may follow the proof of Theorem 15(b) in [2, p. 19] and get the formula

$$a \in M \wedge \forall x \in M \cap a(Q \subset R_x) \rightarrow \text{Prog}[Q, R_a]^8,$$

where  $R_a := \{y : \bar{\Theta}ay \in \mathfrak{X} \rightarrow \bar{\Theta}xy \in Q\}$ . Here besides the premise " $Q$  ausgezeichnet" only methods formalizable in Heyting's arithmetic are used. By (10) it follows

$$(11) \quad \text{Prog}[M, \{x : \forall y \in Q(\bar{\Theta}xy \in \mathfrak{X} \rightarrow \bar{\Theta}xy \in Q)\}].$$

4. From outside we know that  $\bar{\Theta}(\{g\}) = (\mathfrak{X}, <)$  is wellordered and hence  $W_u = M_u = \{x : Sx \leq u\}$  and  $M = \mathfrak{X}$  which implies  $Q = M \cap \Omega_\sigma$ ,  $\sigma$  defined by:

DEFINITION.

$$\sigma := \begin{cases} \nu, & \text{in the case of } ID^i_\nu, \\ \Omega_1, & \text{in the case of } ID^i_{<}. \end{cases}$$

Of course  $W_u = M_u = \{x : Sx \leq u\}$  is not provable in  $ID^i$ , but the weaker assertion  $Q = M \cap \Omega_\sigma$  is provable as the following theorem shows.

THEOREM 1.  $\Omega_\sigma \in M$  and  $Q = M \cap \Omega_\sigma$  are provable in  $ID^i$ .

PROOF. By Lemma 2(a) we have  $Q \subset M \cap \Omega_\sigma$ . If  $a \in M$  and  $Sa \in A$  we get  $K_{Sa} a \subset W_{Sa}$  and by Lemma 1(b),  $a \in W_{Sa} \subset Q$ . So we just have to prove  $a \in M \cap \Omega_\sigma \rightarrow Sa \in A$  and  $\Omega_\sigma \in M$ . The proofs differ for  $ID^i_\nu$ ,  $ID^i_{<}$ .

1.  $ID^i_\nu$ . Then  $A = \{w : w < \Omega_\sigma\}$  and trivially  $a \in M \cap \Omega_\sigma \rightarrow Sa \in A$  holds. By  $(TI_\nu)$  and  $(W1)$  we get  $\sigma = \nu \in W_0$ . Hence by Lemma 2(a) and (5)  $\forall v(K_v \Omega_\sigma \subset K_v \sigma \subset W_v)$  which means  $\Omega_\sigma \in M$ .

<sup>8</sup>In [2] this formula is proved with  $Q$  in place of  $M$ , but an analysis of the proof shows that it is enough to have the premise  $a \in M \wedge \forall x \in M \cap a(Q \subset R_x)$ .

2.  $ID^i$ . Suppose  $a \in M \cap \Omega_\sigma$ . Then  $K_0 a \subset W_0$  and  $Sa = \Omega_x$  for some  $x < \Omega_1 = \sigma$ . By (5)  $K_0 x \subset K_0 \Omega_x \cup \{1\}$ . Further  $K_0 Sa \subset K_0 a$  and  $1 \in W_0$ . We therefore get by Lemma 1(b)  $x \in W_0$  and thence  $Sa \in A$ . For  $0 < u$ ,  $K_0 \Omega_{\Omega_1} = \emptyset$  and  $K_u \Omega_{\Omega_1} = \{\Omega_1\}$ . Obviously  $\Omega_1 \in A$  and hence  $\Omega_1 \in W_{\Omega_1}$ , by Lemma 3. By Lemma 2(b) it follows  $\forall u (K_u \Omega_{\Omega_1} \subset W_u)$ , which means  $\Omega_\sigma = \Omega_{\Omega_1} \in M$ .

Now by (10) and Theorem 1 we get

(12)  $TI[M, \Omega_\sigma]$ .

Hence by (11)  $\forall y \in Q (\bar{\Theta} \Omega_\sigma y \in \mathcal{X} \rightarrow \bar{\Theta} \Omega_\sigma y \in Q)$ . Since  $0 \in Q \wedge \bar{\Theta} \Omega_\sigma 0 \in \mathcal{X} \cap \Omega_1$ , we obtain  $\bar{\Theta} \Omega_\sigma 0 \in Q \cap \Omega_1$ . Hence by (9) and (7)  $TI[M_0, \bar{\Theta} \Omega_\sigma 0]$ . This means we are able to *collapse* the wellordering  $M \cap \Omega_\sigma$  to the provable ordinal  $\bar{\Theta} \Omega_\sigma 0$ . This is a special case of the following theorem, which is proved by the above considerations with  $c$  in place of  $\Omega_\sigma$ .

**THEOREM 2 (COLLAPSING PROPERTY).** *If  $TI[M, c]$  is provable in  $ID^i$  and  $\bar{\Theta} c 0 \in \mathcal{X}$ , then  $TI[M_0, \bar{\Theta} c 0]$  is provable in  $ID^i$ .*

Starting from (12) we now prove  $TI[M, c]$  for each  $c \in M \cap \bar{\Theta} 1 \Omega_\sigma$  using Gentzen's [5] method for proving  $TI[< \varepsilon_0]$  in number theory.

**DEFINITION.**  $\bar{\mathcal{X}} := \{x : \bar{\Theta} 1 \Omega_\sigma \leq x \vee \forall y (M \cap y \subset \bar{\mathcal{X}} \rightarrow M \cap (y + \bar{\Theta} 0 x) \subset \bar{\mathcal{X}})\}$ .

**LEMMA 4.**  $\forall y (M \cap y \subset \bar{\mathcal{X}} \rightarrow M \cap (y + \Omega_\sigma) \subset \bar{\mathcal{X}}) \rightarrow \text{Prog}[M, \bar{\mathcal{X}}]$  is provable in  $ID^i$ .

**PROOF.** We have to prove  $M \cap (b + \bar{\Theta} 0 a) \subset \bar{\mathcal{X}}$  under the assumptions (1)  $\forall y (M \cap y \subset \bar{\mathcal{X}} \rightarrow M \cap (y + \Omega_\sigma) \subset \bar{\mathcal{X}})$ , (2)  $M \cap a \subset \bar{\mathcal{X}}$ , (3)  $a < \bar{\Theta} 1 \Omega_\sigma$ , (4)  $M \cap b \subset \bar{\mathcal{X}}$ .

By (1) and (4) we get  $M \cap (b + \Omega_\sigma \cdot n) \subset \bar{\mathcal{X}}$  for all  $n \in \mathbf{N}$  using mathematical induction. Hence  $M \cap (b + \bar{\Theta} 0 \Omega_\sigma) \subset \bar{\mathcal{X}}$  because of  $\sup_{n \in \mathbf{N}} \Omega_\sigma \cdot n = \bar{\Theta} 0 \Omega_\sigma$ . Suppose  $z \in M \cap (b + \bar{\Theta} 0 a)$ . We may assume  $b + \bar{\Theta} 0 \Omega_\sigma \leq z$ . By (3)  $z < b + \bar{\Theta} 0 a < b + \bar{\Theta} 1 \Omega_\sigma$ . Hence  $z = b + \bar{\Theta} 0 a_1 \cdot n + z_1$  with  $1 \leq n \in \mathbf{N}$ ,  $\Omega_\sigma \leq a_1 < a$ ,  $z_1 < \bar{\Theta} 0 a_1$ . By  $\forall v (v < \Omega_\sigma = Sa_1)$  and the definition of  $K_v$ —it is the case that  $K_v a_1 = K_v \bar{\Theta} 0 a_1 \subset K_v z$ . So we get  $a_1 \in M \cap a$  since  $z \in M$  is assumed. By (2), (3), (4) we get  $M \cap (b + \bar{\Theta} 0 a_1 \cdot (n+1)) \subset \bar{\mathcal{X}}$  using mathematical induction. Hence  $z \in \bar{\mathcal{X}}$ .

**DEFINITION.**  $c_0 := \Omega_\sigma$ ,  $c_{n+1} := \bar{\Theta} 0 c_n$ .

One easily proves  $c_n \in M$ ,  $c_n < \bar{\Theta} 1 \Omega_\sigma = \sup_{k \in \mathbf{N}} c_k$  and  $\bar{\Theta} c_n 0 \in \mathcal{X}$ .

**THEOREM 3.**  $TI[M, c_n]$  is provable in  $ID^i$  for each  $n \in \mathbf{N}$ .

**PROOF.** We prove the theorem by 'metainduction' on  $n$ . By (3) it follows that  $a + b \in M \rightarrow b \in M$ . Hence  $\text{Prog}[M, \bar{\mathcal{X}}] \wedge M \cap a \subset \bar{\mathcal{X}} \rightarrow \text{Prog}[M, \{x : a + x \in M \rightarrow a + x \in \bar{\mathcal{X}}\}]$  and thence by (12)

(\*)  $\text{Prog}[M, \bar{\mathcal{X}}] \rightarrow \forall y (M \cap y \subset \bar{\mathcal{X}} \rightarrow M \cap (y + \Omega_\sigma) \subset \bar{\mathcal{X}})$ .

By (\*) and  $c_0 = \Omega_\sigma \in M$  we have  $TI[M, c_0]$ . For  $n > 0$  we have the induction hypothesis  $TI[M, c_{n-1}]$ . Hence  $\text{Prog}[M, \bar{\mathcal{X}}] \rightarrow c_{n-1} \in \bar{\mathcal{X}}$  which implies  $\text{Prog}[M, \bar{\mathcal{X}}] \rightarrow M \cap c_n \subset \bar{\mathcal{X}}$ . By (\*) and Lemma 4 we get  $\text{Prog}[M, \bar{\mathcal{X}}] \rightarrow \text{Prog}[M, \bar{\mathcal{X}}]$  and therefore  $\text{Prog}[M, \bar{\mathcal{X}}] \rightarrow M \cap c_n \subset \bar{\mathcal{X}}$ . Hence  $TI[M, c_n]$ .

**THEOREM 4.** In  $ID^i$ ,  $TI[M_0, a]$  is provable for each  $a < \bar{\Theta}(\bar{\Theta} 1 \Omega_\sigma) 0$ .

<sup>9</sup>Remember that by (1)  $M_0 \cap \bar{\Theta} \Omega_\sigma 0$  represents the ordinal  $\bar{\Theta} \Omega_\sigma 0$ .

PROOF. For  $a < \bar{\Theta}(\bar{\Theta}1\Omega_\sigma)0$  there is an  $n \in \mathbf{N}$  with  $a < \bar{\Theta}c_n 0$ . By Theorem 3 we have  $\text{TI}[M, c_n]$ , which can be collapsed to  $\text{TI}[M_0, \bar{\Theta}c_n 0]$  by Theorem 2. So  $\text{TI}[M_0, a]$  holds.

Since  $M_0 \cap \bar{\Theta}(\bar{\Theta}1\Omega_\sigma)0$  represents the segment  $\{\xi \in \text{On} : \xi < \bar{\Theta}\varepsilon_{\Omega_\sigma+1}0\}$ , the results (A) and (B) stated in the introduction follow from Theorem 4.

FINAL REMARKS. The above proof of transfinite induction admits the following generalization. Let  $\text{Th}$  be a theory containing Heyting's arithmetic and axioms for iterations of inductive definitions along a provably wellordered subset  $A$  of  $\mathfrak{R}_0$ . Then the sets  $W_u := W[M_u]$ ,  $M_u := \{x : Sx \leq u \wedge \forall v \in A \cap u(K_v x \subset W_v)\}$  ( $u \in A$ ),  $Q := \{x : \exists u \in A(x \in W_u)\}$  and  $M := \{x : \forall u \in A(K_u x \subset W_u)\}$  are definable in  $\text{Th}$ , and if the formula  $\forall u \in A(u \in W_u \wedge \forall x \in W_u(Sx \in A))$  is provable in  $\text{Th}$ , one gets:

I.  $Q$  is wellordered, i.e.

$$\text{Th} \vdash \text{Prog}[Q, \mathfrak{X}] \rightarrow Q \subset \mathfrak{X}.$$

II. *Collapsing property.*

$$\text{Th} \vdash \text{TI}[M, c] \Rightarrow \text{Th} \vdash \text{TI}[M_0, \bar{\Theta}c 0] \quad (\text{for } \bar{\Theta}c 0 \in \mathfrak{I}).$$

III. *Extension to the next  $\varepsilon$ -number.*

$$\text{Th} \vdash \text{TI}[M, \Omega_a] \Rightarrow \text{Th} \vdash \text{TI}[M, c] \quad \text{for each } c \in M \cap \bar{\Theta}1\Omega_a.$$

From I, II, III it follows:

IV.

$$\text{Th} \vdash \Omega_a \in M \wedge M \cap \Omega_a \subset Q \Rightarrow \text{Th} \vdash \text{TI}[\langle \bar{\Theta}(\bar{\Theta}1\Omega_a)0]$$

$$\text{for } \bar{\Theta}(\bar{\Theta}1\Omega_a)0 \in \mathfrak{I}.$$

As an example we regard the following definition by transfinite recursion on  $\nu \in \mathfrak{I} \cap \Omega_1$ :

$$\lambda(0) := 0,$$

$$A_0 := \emptyset.$$

$$\lambda(\nu + 1) := \Omega_{\lambda(\nu)+1},$$

$$A_{\nu+1} := A_\nu \cup \{\Omega_x : \lambda(\nu) \leq x \in W[M^\nu]\},$$

with  $M^\nu := \{x : x < \lambda(\nu + 1) \wedge \forall u \in A_\nu(K_u x \subset W_u)\}$   
and  $W_u$  defined as above by iteration of  
inductive definitions along  $A_\nu$ .

$$\lambda(\nu) := \sup_{\xi < \nu} \lambda(\xi)^{10},$$

$$A_\nu := \bigcup_{\xi < \nu} A_\xi \text{ for limit ordinals } \nu.$$

Let  $\text{ID}_\nu^*$  be the theory, which allows to define  $A_\nu$  and to iterate inductive definitions along  $A_\nu$  ( $\text{ID}_\nu^*$  for example is  $\text{ID}_{<}$ ). Then by the above considerations we get:

$$\text{ID}_\nu^* \vdash \text{TI}[\langle \bar{\Theta}(\bar{\Theta}1\Omega_{\lambda(\nu)})0].$$

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<sup>10</sup>For  $\nu = \omega(1 + a)$  it is  $\Omega_{\lambda(\nu)} = \lambda(\nu) = g1a$ .

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