AN INDEPENDENCE RESULT FOR $(\Pi^1_1$-CA) + BI

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Introduction

In Kirby and Paris [5] it was shown that a certain combinatorial statement (concerning finite trees) is independent of Peano Arithmetic. Here we present a not too complicated extension of this statement and prove its independence from the much stronger theory $(\Pi^1_1$-CA) + BI. This is done by refining the methods which we have developed in [2, Ch. IV, §1–§4].

Using the terminology of Kirby and Paris our result can be described as follows. A hydra is a finite labeled tree $A$ which has the following properties:

(i) the root of $A$ has label $+$,
(ii) any other node of $A$ is labeled by some ordinal $\nu \leq \omega$,
(iii) all nodes immediately above the root of $A$ have label $0$ (zero).

If Hercules chops off a head (i.e. top node) $\sigma$ of a given hydra $A$, the hydra will choose an arbitrary number $n \in \mathbb{N}$ and transform itself into a new hydra $A(\sigma, n)$ as follows. Let $\tau$ denote that node of $A$ which is immediately below $\sigma$, and let $A^-$ denote that part of $A$ which remains after $\sigma$ has been chopped off. The definition of $A(\sigma, n)$ depends on the label of $\sigma$:

Case 1: label $(\sigma) = 0$. If $\tau$ is the root of $A$, we set $A(\sigma, n):= A^-$. Otherwise $A(\sigma, n)$ results from $A^-$ by sprouting $n$ replicas of $A^-_\tau$ from the node immediately below $\tau$. Here $A^-_\tau$ denotes the subtree of $A^-$ determined by $\tau$.

\[
\begin{array}{c}
\text{Case 2: label } (\sigma) = u + 1. \text{ Let } \varepsilon \text{ be the first node below } \sigma \text{ with label } \nu \leq u. \text{ Let } B \text{ be that tree which results from the subtree } A_\varepsilon \text{ by changing the label of } \varepsilon \text{ to } u \text{ and the label of } \sigma \text{ to } 0. \text{ } A(\sigma, n) \text{ is obtained from } A \text{ by replacing } \sigma \text{ by } B. \text{ In this case } A(\sigma, n) \text{ does not depend on } n.
\end{array}
\]
Example \((u = 3, v = 1)\):

Case 3: label \((\sigma) = \omega\). \(A(\sigma, n)\) is obtained from \(A\) simply by changing the label of \(\sigma\): \(\omega\) is replaced by \(n + 1\).

Notation. If \(\sigma\) is the rightmost head of \(A\) (as in the pictures above) we write \(A(n)\) instead of \(A(\sigma, n)\). In the following we consider only the operation \(A \rightarrow A(n)\). By \(\oplus\) we denote the hydra which consists only of one node, namely its root.

The main results of the present paper are:

**Theorem I.** By always chopping off the rightmost head, Hercules is able to kill every hydra in a finite number of steps, i.e., for each hydra \(A\) and any sequence \((n_i)_{i \in \mathbb{N}}\) of natural numbers there exists \(k \in \mathbb{N}\) such that \(A(n_0)(n_1) \cdots (n_k) = \oplus\).

**Theorem II.** For every fixed hydra \(A\) the statement \(\forall (n_i)_{i \in \mathbb{N}} \exists k A(n_0)(n_1) \cdots (n_k) = \oplus\) is provable in \((\Pi^1_1\text{-CA}) + \text{BI}\).

**Theorem III.** Let

\[
A^n := \underbrace{\omega \cdots \omega}_{\text{n nodes with label } \omega} \uparrow \omega \uparrow 0
\]

Then the \(\Pi^0_2\)-sentence \(\forall n \exists k A^n(1)(2) \cdots (k) = \oplus\) is not provable in \((\Pi^1_1\text{-CA}) + \text{BI}\).
In Section 1 we prove Theorem I. In Section 2 we prove Theorem II. Section 3 contains some technical lemmata which will be used in Section 4 for the proof of Theorem III. In the appendix we characterize the proof-theoretic ordinals of the theories ID(v) (v ≤ ω) for v-times iterated inductive definitions by means of the term structure (T, [↑]).

1. Infinitary wellfounded trees and collapsing functions

In this section we introduce certain sets $T_v$ ($v ≤ ω$) of infinitary wellfounded trees together with a system of so-called collapsing functions $D_v : T_v → T_v$ ($v ≤ ω$). These functions are then used to associate with every hydra $A$ an element $∥A∥$ of $T_0$ in such a way that, for each $n ∈ N$, $∥A(n)∥$ is an immediate subtree of $∥A∥$. This yields Theorem I.

Definition of the tree classes $T_v$ ($v ≤ ω$)

Suppose that $T_u$ for $u < v$ is already defined. Then we define $T_v$ to be the least set which contains 0 (the empty set) and is closed under the following rule:

(T_v) If $α : I → T_v$ is a function with $I ∈ \{0, N\} ∪ \{T_u : u < v\}$, then $α ∈ T_v$.

According to the inductive definition of $T_v$ we have the following principle of transfinite induction over $T_v$:

$$\forall α ∈ T_v (\forall x ∈ \text{domain}(α) \: Ψ(α(x)) → Ψ(α)) → \forall α ∈ T_v \: Ψ(α).$$

Proposition. $u < v → T_u ⊆ T_v$.

Notations. $(α_x)_{x ∈ I} := \{(x, α_x) : x ∈ I\}$, i.e., $(α_x)_{x ∈ I}$ denotes the function $α$ with domain $I$ and $α(x) = α_x$ for all $x ∈ I$.

$$α^+ := (α_x)_{x ∈ \{0\}} := \{(0, α)\}$$

(the successor of $α$).

In the following $α$, $β$, $γ$ denote elements of $T_ω$.

Definition of $+ : T_ω × T_ω → T_ω$

We define $α + β$ by transfinite induction on $β$:

(i) $α + 0 := α$,

(ii) $α + (β_x)_{x ∈ I} := (α + β_x)_{x ∈ I}$. 

**Proposition.** (a) $\alpha + (\beta^+) = (\alpha + \beta)^+$. 
(b) $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$.  
(c) $\alpha, \beta \in \mathcal{F}_v \Rightarrow \alpha + \beta \in \mathcal{F}_v$.

**Definition.** $\alpha \cdot 0 := 0$, $\alpha \cdot (n + 1) := \alpha \cdot n + \alpha$.

**Definition of $D_v : \mathcal{F}_\omega \rightarrow \mathcal{F}_v$**

$D_v(\alpha)$ is defined by transfinite induction on $\alpha \in \mathcal{F}_\omega$ simultaneously for all $v \leq \omega$.

\begin{align*}
(D1) & \quad D_v(0) := \alpha^+, \quad D_{n+1}(0) := (x)_{x \in \mathcal{F}_n}, \quad D_\omega(0) := (D_{n+1}(0))_{n \in \mathbb{N}}. \\
(D2) & \quad D_v((\alpha_x)_{x \in I}) := \\
& \begin{cases} 
(D_v(\alpha_0) \cdot (n + 1))_{n \in \mathbb{N}}, & \text{if } I = \{0\}, \\
(D_v(\alpha_x))_{x \in I} & \text{if } I \in \{\mathbb{N}\} \cup \{\mathcal{F}_u : u < v\}, \\
(D_v(\alpha_z))_{n \in \mathbb{N}} \text{ with } z := D_u(\alpha_{0^+}), & \text{if } I = \mathcal{F}_u \text{ with } v \leq u < \omega.
\end{cases}
\end{align*}

**Remark.** If domain($\alpha$) $\in \{\mathcal{F}_u : v \leq u < \omega\}$, then $D_v(\alpha)$ is a constant function with domain $\mathbb{N}$.

**Definition of $\| A \|$**

For every finite labeled tree $A$ (with labels $\leq \omega$) we define $\| A \| \in \mathcal{F}_\omega$ by induction on the length (i.e. number of nodes) of $A$:

\begin{align*}
\| A \| := D_v(0), \\
\| A_0 : \cdots : A_k \| := D_v(\| A_0 \| + \cdots + \| A_k \|).
\end{align*}

If $A = \underbrace{A_0 : \cdots : A_k}$ is a hydra, we set $\| A \| := \| A_0 \| + \cdots + \| A_k \|$. For $\alpha \in \mathcal{F}_0$ with domain($\alpha$) $= \{0\}$ we set $\alpha(n) := \alpha(0)$.

**1.1. Theorem.** For every hydra $A \neq \emptyset$ and all $n \in \mathbb{N}$ the following holds: $\| A \| \in \mathcal{F}_0$ and $\| A(n) \| = \| A \| (n)$.

**Proof.** Easy exercise.

From 1.1 we obtain Theorem I by transfinite induction over $\mathcal{F}_0$.

**2. The term structure $(T, \cdot)$**

In this section we prove Theorem II. To this purpose we introduce the following set $T$ of terms, where $D_0, \ldots, D_\omega$ is a sequence of formal symbols.
Inductive definition of the set $T$

(T1) $0 \in T$.

(T2) If $a \in T$ and $v \leq \omega$, then $D_v a \in T$; we call $D_v a$ a principal term.

(T3) If $a_0, \ldots, a_k \in T$ are principal terms and $k \geq 1$, then $(a_0, \ldots, a_k) \in T$.

For each term $a \in T$ we define its value $\bar{a} \in \mathcal{T}_\omega$ by

$$\bar{0} := 0, \quad D_v \bar{a} := \mathcal{D}_v (\bar{a}), \quad (a_0, \ldots, a_k) := \bar{a}_0 + \cdots + \bar{a}_k.$$ 

This interpretation of terms as infinitary wellfounded trees will not be used in the proof of Theorem II. It serves only as a motivation for the following definitions of $a + b$, $T_v$, $\text{dom}(a)$ and $a[z]$.

The letters $a, b, c, z$ now always denote elements of $T$.

For principal terms $a_0, \ldots, a_k$ and $k \in \{-1, 0\}$ we set

$$(a_0, \ldots, a_k) := \begin{cases} 0, & \text{if } k = -1, \\ a_0, & k = 0. \end{cases}$$

Definition of $a + b$ and $a \cdot n \in T$

$$a + 0 := 0 + a := a,$$

$$(a_0, \ldots, a_k) + (b_0, \ldots, b_m) := (a_0, \ldots, a_k, b_0, \ldots, b_m) \quad (k, m \geq 0),$$

$$a \cdot 0 := 0, \quad a \cdot (n + 1) := a \cdot n + a.$$ 

Proposition. $(a + b) + c = a + (b + c)$.

Definition of $T_v$ for $v \leq \omega$

$$T_v := \{0\} \cup \{(D_{u_0} a_0, \ldots, D_{u_k} a_k) : k \geq 0, a_0, \ldots, a_k \in T, u_0, \ldots, u_k \leq v\}.$$ 

Remark. $T_0 \cong T_1 \cong \cdots \cong T_\omega = T$.

Abbreviation. $1 := D_0 0$.

Convention. We identify $\mathbb{N}$ with the subset $\{0, 1, 1 + 1, 1 + 1 + 1, \ldots\}$ of $T_0$.

Now we define, for every $a \in T$, a subset $\text{dom}(a)$ of $T$ and a function $z \mapsto a[z]$ from $\text{dom}(a)$ into $T$. This will be done in such a way that $\bar{z} \in \text{domain}(\bar{a})$ and $a[z] = \bar{a}(\bar{z})$, for all $z \in \text{dom}(a)$.

Definition of $\text{dom}(a)$ and $a[z]$ for $a \in T$, $z \in \text{dom}(a)$

$$(\emptyset, 0) \quad \text{dom}(0) := \emptyset,$$

$$([ \ · ] 1) \quad \text{dom}(1) := \{0\} ; 1[0] := 0.$$
(\[ ]2) \text{dom}(D_{u+1}0) := T_u; (D_{u+1}0)[z] := z.

(\[ ]3) \text{dom}(D_00) := N; (D_00)[n] := D_{n+1}0.

(\[ ]4) \text{Let} a = D_v b \text{ with } b \neq 0:

(i) \text{dom}(b) = \{0\}; \text{dom}(a) = N, a[n] := (D_v b[0]) \cdot (n + 1).

(ii) \text{dom}(b) = T_u \text{ with } v \leq u < \omega; \text{dom}(a) := N, a[n] := D_v b[D_v b[1]].

(iii) \text{dom}(b) \in \{N\} \cup \{T_u : u < v\}; \text{dom}(a) := \text{dom}(b), a[z] := D_v b[z].

(\[ ]5) a = (a_0, \ldots, a_k) (k \geq 1); \text{dom}(a) := \text{dom}(a_k),

a[z] := (a_0, \ldots, a_{k-1}) + a_k[z].

\textbf{Definition.} \text{0}[n] := 0, a[n] := a[0] \text{ for } a \in T \text{ with } \text{dom}(a) = \{0\}.

\textbf{Proposition.} (a) \text{a} \neq 0 \Leftrightarrow \text{dom}(a) \neq 0.

(b) \text{dom}(a) = \{0\} \Leftrightarrow a = a[0] + 1.

(c) \text{0} \neq a \in T_v \Rightarrow \text{dom}(a) \in \{\{0\}, N\} \cup \{T_u : u < v\}, \text{ and } a[z] \in T_v \text{ for all } z \in \text{dom}(a).

Now we are going to compare terms and hydras. It will turn out that the term structure \((T_0, \cdot[\cdot])\) is isomorphic to the structure \((\mathcal{H}, \cdot[\cdot])\), where \(\mathcal{H}\) denotes the set of all hydras.

In fact \((\mathcal{H}, \cdot[\cdot])\) is nothing else than a geometric representation of \((T_0, \cdot[\cdot]).\) \((\mathcal{H}, \cdot[\cdot])\) has been defined just in such a way that it becomes isomorphic to \((T_0, \cdot[\cdot]).\)

\textbf{Definition of } |A|

If \(A = A_0 \cdots A_k (k \geq -1)\) is a hydra or any finite labeled tree with labels \(\leq \omega\) we define \(|A|\) to be that term \(a \in T\) which implicitly is given by the definition of \(|A|\) in Section 1, namely:

\(|A| :=\begin{cases}
D_\xi(|A_0|, \ldots, |A_k|), & \text{if } \xi \leq \omega, \\
(|A_0|, \ldots, |A_k|), & \text{if } \xi = +. 
\end{cases}\)

\textbf{2.1. Theorem.} (a) \text{The operation } A \mapsto |A| \text{ yields a } 1\text{–}1 \text{ correspondence between the set of all hydras and the set } T_0.

(b) \text{|A(n)| = |A|[n], for each hydra } A \text{ and all } n \in N.

\textbf{Proof.} (a) Obvious.

(b) Definition (for \(c, z \in T, c \neq 0\))

\[c[*/z] := \begin{cases}
z, & \text{if } c = D_v 0, \\
D_v b[*/z], & \text{if } c = D_v b \text{ with } b \neq 0, \\
(c_0, \ldots, c_{k-1}) + c_k[*/z], & \text{if } c = (c_0, \ldots, c_k), \quad k \geq 1.
\end{cases}\]

Now the reader can easily verify the following propositions and then also part (b) of the theorem.
Proposition 1. If \( z \) is a principal term, then \( c[\ast/z] \) results from \( c \) by replacing the rightmost subterm \( D_v0 \) of \( c \) by \( z \).

Proposition 2. If \( z \in T_u = \text{dom}(a) \), then \( a[z] = a[\ast/z] \).

Proposition 3. If \( \text{dom}(a) \in \{\{0\}, \mathbb{N}\} \), then one of the following cases holds:

(i) \( a = (a_0, \ldots, a_{k-1}, 1) \) and \( a[n] = (a_0, \ldots, a_{k-1}) \).
(ii) \( a = c[\ast/D_v(a_0, \ldots, a_{k-1}, 1)] \) and \( a[n] = c[\ast/D_v(a_0, \ldots, a_{k-1}) \cdot (n + 1)] \).
(iii) \( a = c[\ast/D_v0] \) and \( a[n] = c[\ast/D_v\{n\} + 10] \).
(iv) \( a = c[\ast/D_v b], \ \text{dom}(b) = T_u, \ v \leq u \) and \( a[n] = c[\ast/D_v b[D_v b[1]]] \).

Let \( W_0 \) denote the least subset of \( T_0 \) which contains \( 0 \) and is closed under the following rule:

\[
a \in T_0 \quad \text{and} \quad \forall n \in \mathbb{N}(a[n] \in W_0) \Rightarrow a \in W_0.
\]

Since every \( a \in T_0 \) corresponds to an infinitary wellfounded tree \( \bar{a} \in \mathcal{F}_0 \) with \( \bar{a}(n) = \overline{a[n]} \) (for all \( n \in \mathbb{N} \)), it follows that \( W_0 = T_0 \) and consequently \( \forall a \in T_0 \forall(n_i)_{i \in \mathbb{N}} \exists k \ a[n_0] \cdots [n_k] = 0 \).

Now we want to give a proof of "\( a \in W_0 \)" which, for every fixed term \( a \in T_0 \), can be formalized in \( \text{ID}_\omega \), the formal theory of \( \omega \)-times iterated inductive definitions. There we have to use methods which do not depend on the nonconstructive tree classes \( \mathcal{F}_v \). In fact, we will establish a more general result:

2.2. Theorem. Let \( 0 < v \leq \omega \). If \( a \in T_0 \) contains no symbol \( D_v \) with \( v < v \), then "\( a \in W_0 \)" is provable in \( \text{ID}_v \).

Since \( \text{ID}_\omega \) is contained in \( (\Pi_1^- \text{-CA}) + \text{BI} \) and since \( (\Pi_1^- \text{-CA}) + \text{BI} \) proves "\( a \in W_0 \Rightarrow \forall(n_i)_{i \in \mathbb{N}} \exists k \ a[n_0] \cdots [n_k] = 0 \)" , we obtain from 2.2:

2.3. Theorem. \( (\Pi_1^- \text{-CA}) + \text{BI} \Rightarrow \forall(n_i)_{i \in \mathbb{N}} \exists k \ a[n_0] \cdots [a_k] = 0, \) for each \( a \in T_0 \).

This theorem together with 2.1 yields Theorem II.

In the following let \( v \leq \omega \) be fixed. We use \( u, v \) to denote numbers \( \leq v \).

Iterated inductive definition of sets \( W_v (v < v) \)

(W1) \( 0 \in W_v \).
(W2) \( a \in T_v, \ \text{dom}(a) \in \{\{0\}, \mathbb{N}\}, \ \forall n \ (a[n] \in W_v) \Rightarrow a \in W_v \).
(W3) \( a \in T_v, \ \text{dom}(a) = T_u \) with \( u \leq v, \ \forall z \in W_u (a[z] \in W_v) \Rightarrow a \in W_v \).

Proposition. \( u \leq v < v \Rightarrow W_u \subseteq W_v \subseteq T_v \).

Abbreviations. Let \( X \) range over subsets of \( T \) which are definable in the language of \( \text{ID}_v \).
1. By $A_v(X, a)$ we denote the following statement:
\[ a = 0 \lor [\text{dom}(a) \in \{\{0\}, \mathbb{N}\} \land \forall n (a[n] \in X)] \]
\[ \lor \exists u < v [\text{dom}(a) = T_u \land \forall z \in W_u (a[z] \in X)]. \]

2. $A_v(X) := \{ x \in T : A_v(X, x) \}$.

3. $x < a) := \{ y \in T : a + y \in X \}$.

4. $X := \{ y \in T : \forall x (x \in X \lor D_y y \in X) \}.$

5. $W^* := \{ x \in T : \forall u < v (D_u x \in W_u) \}$.

By the definition of $W_v$, for all $v < v$ we have:

(A1) $A_v(W_v) = W_v,$
(A2) $A_u(X) \subseteq X \Rightarrow W_v \subseteq X.$

2.4. Lemma. (a) $A_v(X) \subseteq X$ and $a \in X \Rightarrow A_v(X^{(a)}) \subseteq X^{(a)} (v \leq v)$.
(b) $a, b \in W_v \Rightarrow a + b \in W_v (v < v)$.

Proof. (a) Suppose $A_v(X) \subseteq X, a \in X, A_v(X^{(a)}, b).$ We have to prove $a + b \in X$:
1. $b = 0$: Then $a + b = a \in X$.
2. $\exists b \in \{\{0\}, \mathbb{N}\}$ and $\forall n (b[n] \in X^{(a)})$: Then we have $\text{dom}(a + b) = \text{dom}(b)$ and $(a + b)[n] = a + b[n] \in X$, for all $n \in \mathbb{N}$. It follows that $a + b \in A_v(X) \subseteq X$.
3. $\exists b \in T_u$ with $u < v$: similar to 2.

(b) From (a) together with (A1), (A2) we obtain, for $v < v$, $a \in W_v \Rightarrow W_v \subseteq W_v^{(a)}$, i.e., $a \in W_v \Rightarrow (b \in W_v \Rightarrow a + b \in W_v)$.

2.5. Lemma. $A_v(X) \subseteq X \Rightarrow A_v(\tilde{X}) \subseteq \tilde{X}$

Proof. Assumptions: $A_v(X) \subseteq X, A_v(\tilde{X}, b), a \in X.$
We have to prove $a + D_v b \in X$. First we prove: (1) $\forall u < v (a + D_u + 1 0 \in X)$.
We have $\text{dom}(a + D_u + 1 0) = T_u$ and $(a + D_u + 1 0)[z] = a + z$. By 2.4 we obtain $A_v(X^{(a)}) \subseteq X^{(a)}$. Since $A_u(X^{(a)}) \subseteq A_v(X^{(a)})$, it follows by (A2) that $W_u \subseteq X^{(a)}$, i.e., $\forall z \in W_u (a + z \in X)$. Hence $A_v(X, a + D_u + 1 0)$ and therefore $a + D_u + 1 0 \in X$, since $A_v(X) \subseteq X$.

Proof of $a + D_v b \in X$:
1. $b = 0$ and $v = 0$: Then $a + D_v b = a + 1$; and $a + 1 \in X$ follows from $A_v(X) \subseteq X \land a \in X$.
2. $b = 0$ and $v = u + 1$: In this case we are done by (1).
3. $b = 0$ and $v = a$: Then $\text{dom}(a + D_v b) = \mathbb{N}$ and $(a + D_v b)[n] = a + D_n + 1 0$. By (1) we obtain $A_v(X, a + D_v b)$. Hence $a + D_v b \in X$.
4. $b = b_0 + 1$ with $b_0 \in \tilde{X}$: Then we have $\forall x \in X (x + D_v b_0 \in X)$. Using this and the assumption $a \in X$ we obtain $\forall n \in \mathbb{N} (a + (D_v b_0) \cdot (n + 1) \in X)$ by complete
induction. Since $\text{dom}(a + D_v b) = \mathbb{N}$ and $(a + D_v b)[n] = a + (D_v b_0) \cdot (n + 1)$ it follows that $a + D_v b \in A_v(X) \subseteq X$.

5. $\text{dom}(b) = \mathbb{N}$ and $\forall n \in \mathbb{N}$: Then we have $\text{dom}(a + D_v b) = \mathbb{N}$ and $(a + D_v b)[n] = a + D_v b[n] \in X$, for all $n \in \mathbb{N}$. Hence $a + D_v b \in A_v(X)$.

6. $\text{dom}(b) = T_u, u < v$ and $\forall z \in W_u(b[z]) \in \bar{X}$: similar to 5.

2.6. Lemma. $A_v(W^*) \subseteq W^*$.

Proof. Suppose $b \in A_v(W^*)$ and $v < v$. We have to show $D_v b \in W_v$.

1. $b = 0$ and $v = 0$: From $0 \in W_v$ we get $D_0 0 = 1 \in W_v$ by (W2).

2. $b = 0$ and $v = u + 1$: Then $\text{dom}(D_v b) = T_u, (D_v b)[z] = z$ and $W_u \subseteq W_v$.

3. $b = b_0 + 1$ and $b_0 \in W^*$: Then we have $\text{dom}(b[v]) = \mathbb{N}$, $(b[v])[n] = (b[v][0]) \cdot (n + 1)$ and $D_v b_0 \in W_v$. Using 2.4(b) we obtain $\forall n \in \mathbb{N}$ by induction on $n$. Hence $D_v b \in W_v$.

4. $\text{dom}(b) = T_u, u < v$ and $b[z] \in W^*$ for all $z \in W_u$:

4.1. $u < v$: Then we have $\text{dom}(D_v b) = T_u$ and $(D_v b)[z] = D_v b[z] \in W_v$ for all $z \in W_u$, i.e., $D_v b \in W_v$.

4.2. $v \leq u < v$: Then we have $\text{dom}(D_v b) = \mathbb{N}$ and $(D_v b)[n] = D_v b[z] \in W^*$ with $z := D_u b[1]$. Obviously $1 \in W_u$ and therefore $b[1] \in W^*$. It follows that $z \in W_u$. From this we obtain $b[z] \in W^*$ and then $W_v b[z] \in W_v$, i.e., $\forall n \in \mathbb{N}$ by induction hypothesis.

Hence $D_v b \in W_v$.

5. $\text{dom}(b) = \mathbb{N}$ and $b[n] \in W^*$ for all $n \in \mathbb{N}$: similar to 4.1.

2.7. Lemma. If $a \in T$ contains no symbol $D_v$ with $v > v$, then $A_v(X) \subseteq X \rightarrow a \in X$.

Proof. By induction on the length of $a$: suppose $A_v(X) \subseteq X$.

1. $a = 0$: In this case $a \in A_v(X) \subseteq X$.

2. $a = (a_0, \ldots, a_k)(k \geq 1)$: Let $c := (a_0, \ldots, a_k-1)$. Then we have:

(1) $c \in X \rightarrow A_v(X^{(c)}) \subseteq X^{(c)}$ (by 2.4(a)).

(2) $c \in X$ (by induction hypothesis).

(3) $A_v(X^{(c)}) \subseteq X^{(c)} \rightarrow a_k \in X^{(c)}$ (by induction hypothesis).

From this we get $a = c + a_k \in X$.

3. $a = D_v b$: From $A_v(X) \subseteq X$ we get $0 \in X$ and $A_v(\bar{X}) \subseteq \bar{X}$ by 2.5. By I.H. (induction hypothesis) we have $A_v(\bar{X}) \subseteq \bar{X} \rightarrow b \in \bar{X}$. By definition of $\bar{X}$ we have $b \in \bar{X} \rightarrow (0 \in X \rightarrow D_v b \in X)$. Hence $D_v b \in X$.

4. $a = D_v b$ with $v < v$: By I.H. we have $A_v(W^*) \subseteq W^* \rightarrow b \in W^*$. Using 2.6 we obtain $b \in W^*$. Hence $a = D_v b \in W_v$. From $A_v(X) \subseteq X$ we get $A_v(X) \subseteq X$ and then $W_v \subseteq X$.

2.8. Lemma. If $a \in T_0$ contains no symbol $D_v$ with $v > v$, then $a \in W_0$. 

Proof. Let \( a \neq 0 \). Then \( a = D_0a_0 + \cdots + D_0a_k \) with \( a_0, \ldots, a_k \in T \), and by Lemmata 2.6, 2.7 we have \( a_0, \ldots, a_k \in W^* \). Hence \( D_0a_0, \ldots, D_0a_k \in W_0 \). From this we obtain \( a \in W_0 \) by 2.4(b).

By formalizing in \( \text{ID}_\nu \) the definition of \( W_\nu \) \((\nu < \nu)\) and the proofs of 2.4–2.8 we obtain Theorem 2.2.

3. The relations \( \ll_k \) and the functions \( H_\alpha : \mathbb{N} \rightarrow \mathbb{N} \)

In Section 4 we will use terms \( a \in T \) instead of ordinals to measure the lengths of infinitary derivations. In this context we need certain relations \( \ll_k \) on \( T \) which we introduce now. We also introduce a hierarchy \( (H_\alpha)_{a \in T_0} \) of number-theoretic functions which is closely related to the so called Hardy hierarchy. The relation \( \ll_0 \) restricted to \( T_0 \) is just the step-down relation of Schmidt [6]; cf. also Ketonen and Solovay [4] where similar relations are studied.

As before the letters \( a, b, c, d, e, z \) will always denote elements of \( T \). As mentioned in Section 2 every \( a \in T \) can be considered as a notation for a wellfounded tree \( \bar{a} \in \mathcal{F}_w \) in such a way that \( \bar{z} \in \text{domain}(\bar{a}) \) and \( \bar{a}(\bar{z}) = \bar{a}[z] \) holds for all \( z \in \text{dom}(a) \). Consequently we have the following principle of transfinite induction over \( T \):

\[
\forall a \in T \ [\forall z \in \text{dom}(a) \ \Psi(a[z]) \rightarrow \Psi(a)] \rightarrow \forall a \in T \ \Psi(a).
\]

Definition of \( c \ll_k a \) by transfinite induction on \( a \in T \)

\[
c \ll_k a \iff a \neq 0 \quad \text{and} \quad \forall z \in d_k(a) (c \ll_k a[z])
\]

where

\[
d_k(a) := \begin{cases} 
\{k\}, & \text{if} \ \text{dom}(a) \in \{0, \mathbb{N}\} \\
\{D_e e : 0 \neq e \in T\}, & \text{if} \ \text{dom}(a) = T_u 
\end{cases}
\]

and

\[
c \ll_k a \iff c \ll_k a \quad \text{or} \quad c = a.
\]

3.1. Lemma. (a) \( c \ll_k a \) and \( a \ll_k b \Rightarrow c \ll_k b \).

(b) \( c \ll_k b \Rightarrow a + c \ll_k a + b \).

(c) \( b \neq 0 \Rightarrow a \ll_k a + b \).

Proof by transfinite induction on \( b \).

3.2. Lemma. (a) \( n \leq k + 1 \Rightarrow (D_\nu a) \cdot n \ll_k D_\nu (a + 1) \).

(b) \( c \ll_k a \Rightarrow D_\nu c \ll_k D_\nu a \).

Proof. (a) By 3.1(c) we have \( (D_\nu a) \cdot n \ll_k (D_\nu a) \cdot (k + 1) = D_\nu (a + 1)[k] \). Hence \( (D_\nu a) \cdot n \ll_k D_\nu (a + 1) \), since \( d_k(D_\nu (a + 1)) = \{k\} \).
(b) Transfinite induction on $a$: Suppose $a \neq 0$ and $\forall z \in d_k(a)(c \leq_k a[z])$.

1. $a = a_0 + 1$: By I.H. and 3.2(a) we have $D_v c \leq_k D_v a_0 \leq_k D_v a$.

2. $\text{dom}(a) \in \{N\} \cup \{T_v: v < v\}$: Then $d_k(D_v a) = d_k(a)$ and $\forall z \in d_k(a)$ ($(D_v a)[z] = D_v a[z]$). By I.H. we have $\forall z \in d_k(a)(D_v c \leq_k D_v a[z])$. Hence $D_v c \leq_k D_v a$.

3. $\text{dom}(a) = T_u$ with $v \leq u$: Then $d_k(D_v a) = \{k\}$ and $(D_v a)[k] = D_v a[z]$ with $z := D_u a[1] \in d_k(a)$. By I.H. we have $D_v c \leq_k D_v a[z]$. Hence $D_v c \leq_k D_v a$.

3.3. Lemma. $\text{dom}(a) = \mathbb{N} \Rightarrow a[n] \leq_k a[n + 1]$.

Proof. By induction on the length of $a$:

1. $a = D_\omega 0$: Then we have $a[n + 1] = D_{n+2} 0$ and therefore $d_k(a[n + 1]) = \{D_{n+1} e: 0 \neq e \in T\}$, $a[n + 1][z] = z$. Using 3.1(c) and 3.2(b) we obtain $\forall z \in d_k(a[n + 1])(D_{n+1} 0 \leq_k z)$. Hence $a[n] \leq_k a[n + 1]$.  

2. For $a = b + c$ or $a = D_v b$ with $\text{dom}(b) = \mathbb{N}$ the assertion follows immediately from I.H. and 3.1(b), 3.2(b).

3. For $a = D_v b$ with $\text{dom}(b) \in \{T_v: v \leq u\}$ we have $a[n] = a[n + 1]$.

4. For $a = D_v (b_0 + 1)$ we have $a[n] = (D_v b_0)(n + 1) \leq_k (D_v b_0)(n + 2) = a[n + 1]$ by 3.1(c).

3.4. Lemma. (a) $a \leq_k b$ and $k \leq_1 m \Rightarrow a \leq_1 b$.

(b) $\text{dom}(a) = \mathbb{N}$ and $n \leq k \Rightarrow a[n] \leq_1 a$.

Proof. (a) Transfinite induction on $b$: Suppose $b \neq 0$ and $\forall z \in d_k(b)(a \leq_k b[z])$. For $\text{dom}(b) = \{0\}$ or $\text{dom}(b) = T_u$ the assertion follows immediately from I.H. Otherwise the I.H. and 3.3 yield $a \leq_1 b[k] \leq_1 b[m]$. Hence $a \leq_1 b$.

(b) By 3.3 we get $a[n] \leq_k a[k]$. Hence $a[n] \leq_k a$.

3.5. Lemma. (a) $a \neq 0 \Rightarrow 1 \leq_0 a$.

(b) $D_v a + 1 \leq_1 D_v (a + 1)$.

(c) $D_v 1 \leq_0 D_v a + 1$ and $D_v 1 \leq_0 D_v 0$.

(d) $a \neq 0$ or $v \neq 0 \Rightarrow k + 1 \leq_k D_v a$ and for $k \neq 0$, $D_v a + k + 1 \leq_k D_v (a + 1)$.

Proof. (a) For $a \notin \{0, 1\}$ we have $\forall z \in d_0(a)(a[z] \neq 0)$. From this the assertion follows by transfinite induction on $a$.

(b) We have $D_v a + 1 \leq_0 D_v a + D_v a = D_v (a + 1)[1]$.

(c) By 3.5(a) and 3.2(b) we have $D_v 1 \leq_0 z = (D_v a + 1)[z]$ for all $z \in d_0(D_v a + 1)$. Hence $D_v 1 \leq_0 D_v a + 1$. Especially $D_v 1 \leq_0 D_v 0 = (D_v a)[0]$ and thus $D_v 1 \leq_0 D_v 0$.

(d) We have $k + 1 = (D_v a)[k]$ and therefore $k + 1 \leq_k D_v 0$. By (c) it follows that $k + 1 \leq_k D_v a$ for all $v \neq 0$. If $a \neq 0$, then we have $k + 1 \leq_k D_v 1 \leq_0 D_v a$ by (a) and 3.2(b). Using $k + 1 \leq_k D_v a$ we get $D_v a + k + 1 \leq_k (D_v a) \cdot 2 \leq_1 D_v (a + 1)$.  

Definition of $H_a: \mathbb{N} \rightarrow \mathbb{N}$ for $a \in T_0$

$H_0(n) := n,$
$H_a(n) := H_a[n](n + 1), \text{ if } a \neq 0.$

3.6. Lemma. Let $a, b, c \in T_0$.

(a) $H_a(n) = \min\{k > n: a[n][n + 1] \cdots [k - 1] = 0\}$, if $a \neq 0$.
(b) $H_{a+b} = H_a \circ H_b$
(c) $H_a(n) < H_a(n + 1)$.
(d) $c \ll_k a \Rightarrow H_c(n) < H_a(n)$, for all $n \geq k$.

Proof. (a) Let $m := \min\{k > n: a[n][n + 1] \cdots [k - 1] = 0\}$. Then we have

$H_a(n) = H_a[n](n + 1) = \cdots = H_{a[n] \cdots [m-1]}(m) = H_0(m) = m.$

(b) Let $b \neq 0$ and $m := H_b(n)$. Then $(a + b)[n] \cdots [m - 1] = a + (b[n] \cdots [m - 1]) = a + 0 = a$ and thus $H_{a+b}(n) = H_a(m) = H_a(H_b(n))$.
(c) and (d) are proved simultaneously by transfinite induction on $a$: Let $a \neq 0$.
(c) By 3.3 we have $a[n] \leq_0 a[n + 1]$, and therefore by I.H.

$H_a(n) = H_a[n](n + 1) \leq H_a[n+1](n + 1) < H_a[n+1](n + 2) = H_a(n + 1)$.
(d) Suppose $c \ll_k a[k]$ and $n \geq k$: By 3.3 we get $c \ll_k a[n]$ and then by I.H.

$H_c(n) \leq H_a[n](n) < H_a[n](n + 1) = H_a(n)$.

Definition.

$D_0^a := D_a, \quad D_{v}^{m+1}a := D_vD_v^ma, \quad c_v^n := D_0D_v^m0.$

7. Lemma. (a) $(D_v^m a) \cdot n \ll_k D_v^m(a + 1)$, for $n \leq k + 1$.
(b) $(D_v^m 0) \cdot n \ll_k D_v^{m+1}0$, for $n \leq k + 1$.

Proof. (a) From 3.1(c) and 3.2(b) we obtain $D_v^m a \ll_0 D_v^m(a + 1)$. For $k \neq 0$ we proceed by induction on $m$:

1. $m = 0$: $(D_v^m a) \cdot n = (D_a a) \cdot n \ll_k D_v(a + 1) = D_v^m(a + 1)$ by 3.2.
2. $m \neq 0$: Using 3.2(a), 3.5(a) and the I.H. we obtain

$(D_v^m a) \cdot n = D_v(D_v^{m-1} a) \cdot n \ll_k D_v(D_v^{m-1} a + 1)$

and

$D_v^{m-1} a + 1 \leq_0 (D_v^{m-1} a) \cdot 2 \ll_1 D_v^{m-1}(a + 1)$.
From this the assertion follows by 3.2(b).
(b) $(D_v^m 0) \cdot n \ll_k D_v^{m+1}0 \leq_0 D_vD_0 0 = D_v^{m+1} 0$ by 3.7(a), 3.5(a), 3.2(b).

3.8. Lemma. (a) $m \geq 1$ and $n \geq 1 \Rightarrow H_{c_v}(4n + 6) < H_{c_v+1}(n)$.
(b) $n \geq m + 1 \Rightarrow H_{c_v}(n) < H_{c_v}(1)$.
Proof. (a) Let \( a := D_0^m 0 \). Obviously \( H(n) = i + n \) and therefore \( H_{D_1}(n) = H_{i + 1}(n + 1) = 2n + 2 \). By 3.6(b) we obtain \( H_{D_{m+1}}(4n + 6) = H_{D_{m+1}D_1}(n) \). By 3.5(d) we have \( 2 \ll a \) (since \( m \neq 0 \)) and thus
\[
D_0 a + (D_0 1) \cdot 2 \ll D_0 a + D_0 2 \ll D_0 a \ll D_0 (a + 1)
\]
and \( a + 1 \ll a + a = (D_0^m 0) \cdot 2 \ll D_0^m + 1 0 \). From this together with 3.2(b) we get
\[
D_0 a + (D_0 1) \cdot 2 \ll D_0 D_0^{m+1} 0 = c_v^{m+1}.
\]
Hence \( H_{D_{m+1}}(4n + 6) < H_{c_v^{m+1}}(n) \) for \( n \geq 1 \).

(b) By 3.7(b) and 3.2(b) we have \( c_v^n < c_v^{n+1} \). Hence \( n \ll H_{c_v^n}(0) \) and \( n + 1 \ll H_{c_v^n}(1) \) by 3.6(c, d). For \( n \geq m + 1 \) we have
\[
c_v^{n-1} + c_v^{n-1} = (D_0 D_0^{n-1} 0) \cdot 2 \ll D_0 (D_0^{n-1} 0 + 1)
\]
and thus
\[
H_{c_v^n}(n) \ll H_{c_v^{n-1}}(n) \ll H_{c_v^{n-1}}(H_{c_v^{n-1}}(1)) = H_{c_v^{n-1} + c_v^{n-1}}(1) < H_{c_v^n}(1).
\]

4. The infinitary system \( \text{ID}_\omega^\infty \)

In this section we prove the following theorem:

4.0. Theorem. If a \( \Pi_0^0 \)-sentence \( \forall x \exists y \varphi(x, y) \) \((\varphi \in \Sigma_1^0)\) is provable in \( \text{ID}_\nu \) \((\nu \ll \omega)\), then there exists \( p \in \mathbb{N} \) such that \( \forall n \geq p \exists k < H_{D_0 D_0^0}(1) \varphi(n, k) \).

Corollary. \( \text{ID}_\nu \not\vdash \forall n \exists k (D_0 D_0^0)[1][2] \cdot \cdot \cdot [k] = 0 \).

Proof. Suppose \( \text{ID}_\nu \vdash \forall n \exists k (D_0 D_0^0)[1] \cdot \cdot \cdot [k] = 0 \). Then also \( \text{ID}_\nu \vdash \forall n \exists k (D_0 D_0^0)[1] \cdot \cdot \cdot [k-1] = 0 \) and therefore by 4.0 there exists \( p \in \mathbb{N} \) such that \( \forall n \geq p \exists k < H_{D_0 D_0^0}(1) (D_0 D_0^0)[1] \cdot \cdot \cdot [k-1] = 0 \). Hence \( \min \{ k \in \mathbb{N} : (D_0 D_0^0)[1] \cdot \cdot \cdot [k-1] = 0 \} < H_{D_0 D_0^0}(1) \), which is a contradiction to 3.6(a).

From this corollary together with 2.1 and the fact that \( \text{ID}_\omega \) proves the same arithmetic sentences as \( (\Pi_1^1-\text{CA}) + \text{BI} \) we obtain Theorem III, i.e.,
\[
(\Pi_1^1-\text{CA}) + \text{BI} \not\vdash \forall n \exists k A^*(1) \cdot \cdot \cdot (k) = \ominus.
\]

Theorem 4.0 is obtained by embedding \( \text{ID}_\nu \) into an infinitary proof system \( \text{ID}_\omega^\infty \) which allows cut elimination.

Preliminaries. Let \( L \) denote the first-order language consisting of the following symbols:

(i) the logical constants \( \neg, \land, \lor, \forall, \exists \),

(ii) number variables (indicated by \( x, y \)),

(iii) a constant 0 (zero) and a unary function symbol \( ' \) (successor),

(iv) constants for primitive recursive predicates (among them the symbol \( < \) for the arithmetic ‘less’ relation).
By $s, t, t_0, \ldots$ we denote arbitrary $L$-terms. The constant terms $0, 0', 0'', \ldots$ are called numerals; we identify numerals and natural numbers and denote them by $i, j, k, m, n, u, v, w$. A formula of the shape $Rt_1 \cdots t_n$ or $\neg Rt_1 \cdots t_n$, where $R$ is a $n$-ary predicate symbol of $L$, is called an arithmetic prime formula (abbreviated by a.p.f.).

Let $X$ be a unary and $Y$ a binary predicate variable. A positive operator form is a formula $\forall (X, Y, y, x)$ of $L(X, Y)$ in which only $X, Y, y, x$ occur free and all occurrences of $X$ are positive. The language $L_{ID}$ is obtained from $L$ by adding a binary predicate constant $P_0$ and a 3-ary predicate constant $P_3$ for each positive operator form $\forall$.

Abbreviations

\[
\begin{align*}
t \in P_0 & \ := P_0t := P_0st, \\
& t \notin P_0 := \neg (t \in P_0), \\
P_{<} st_0 t_1 & := P_{<} st_0 t_1, \\
\forall (X, x) & := \forall (X, P_0, s, x).
\end{align*}
\]

The formal theory $ID_\omega$ is an extension of Peano Arithmetic, formulated in the language $L_{ID}$, by the following axioms:

\begin{enumerate}
\item $(P_0.1)$ $\forall y \forall x \left( \exists_y (P_0(x, y) \rightarrow x \in P_0y) \right)$. \\
\item $(P_0.2)$ $\forall y \left( \forall x \left( \exists_y (F(x) \rightarrow F(x)) \right) \rightarrow \forall x (x \in P_0y \rightarrow F(x)) \right)$, \\
for every $L_{ID}$-formula $F(x)$.
\end{enumerate}

\begin{enumerate}
\item $(P_0)$ $\forall y \forall x_0 \forall x_1 \left( P_0 x_0 x_1 \leftrightarrow x_0 < y \wedge x_1 \in P_0 x_0 \right)$.
\end{enumerate}

The infinitary system $ID_\omega$ will be formulated in the language $L_{ID}(N)$ which arises from $L_{ID}$ by adding a new unary predicate symbol $N$. This is a technical tool which shall help us to keep control over the numerals $n$ occurring in $\exists$-inferences $A(n) \vdash \exists x A(x)$ of $ID_\omega$-derivations. Following Tait [8] we assume all formulas to be in negation normal form, i.e., the formulas are built up from atomic and negated atomic formulas by means of $\wedge, \vee, \forall \exists$ If $A$ is a complex formula we consider $\neg A$ as a notation for the corresponding negation normal form.

Definition of the length $|A|$ of a $L_{ID}(N)$-formula $A$

1. $|Nt| := |\neg Nt| := 0$.
2. $|A| := 1$, if $A$ is an a.p.f. or a formula $\neg P_{<} st_0 t_1$.
3. $|P_{<} st_0 t_1| := |\neg P_{<} st_0 t_1| := 2$.
4. $|A \land B| := |A \lor B| := \max\{|A|, |B|\} + 1$.
5. $|\forall x A| := |\exists x A| := |A| + 1$.

Proposition. $|\neg A| = |A|$, for each $L_{ID}(N)$-formula $A$.

As before we use the letters $u, v$ to denote numbers $\leq \omega$. 


An independence result

Inductive definition of formula sets $\text{Pos}_v (v < \omega)$

1. All $L(N)$-formulas belong to $\text{Pos}_v$.
2. All formulas $P^\text{iv}_u t$, $(\neg)P^\text{iv}_{<u} t_1$ with $u \leq v$ belong to $\text{Pos}_v$.
3. All formulas $\neg P^\text{iv}_u t$ with $u < v$ belong to $\text{Pos}_v$.
4. If $A$ and $B$ belong to $\text{Pos}_v$, then the formulas $A \land B$, $A \lor B$, $\forall x A$, $\exists x A$ also belong to $\text{Pos}_v$.

Remark. If $P^\text{iv}_u t \in \text{Pos}_v$, then also $\forall u (P^\text{iv}_u t) \in \text{Pos}_v$.

Notations

- In the following $A$, $B$, $C$ always denote closed $L_{ID}(N)$-formulas.
- $\Gamma$, $\Delta$, $\Delta$ denote finite sets of closed $L_{ID}(N)$-formulas; we write, e.g., $\Gamma$, $\Delta$, $A$ for $\Gamma \cup \Delta \cup \{A\}$.
- $A^N$ denotes the result of restricting all quantifiers in $A$ to $N$.
- $t \in N : = Nt$, $t \notin N : = \neg Nt$.
- As before we use the letters $a$, $b$, $c$, $d$, $z$ to denote elements of $T$.

Definition

$$c \ll_{\Gamma} a \iff c \ll_k a, \text{ where } k := \max(\{2\} \cup \{3n : \neg Nn \in \Gamma\}).$$

4.1. Proposition. (a) $c \ll_{\Gamma} a$ and $\Gamma \subseteq \Delta \Rightarrow c \ll_{\Delta} a$ (cf. 3.4(a)).

(b) $c \ll_{\Gamma \cup (0 \in N)} a \Rightarrow c \ll_{\Gamma} a$.

Basic inference rules

$$(\land) \quad A_0, A_1 \vdash A_0 \land A_1.$$
$$(\lor) \quad A \vdash A \lor B; \quad B \vdash A \lor B.$$

$$(\forall^n) \quad (A(n))_{n \in N} \vdash \forall x A(x).$$
$$(\exists) \quad A(n) \vdash \exists x A(x).$$
$$(N) \quad n \in N \vdash n \in N.$$  

$$(P^\text{iv}_u) \quad P^\text{iv}_{j} n \vdash P^\text{iv}_{<u} j n, \quad \text{if } j < u < \omega.$$  

$$(\neg P^\text{iv}_u) \quad \neg P^\text{iv}_{j} n \vdash \neg P^\text{iv}_{<u} j n, \quad \text{if } j < u < \omega.$$  

Every instance $(A_i)_{i \in I} \vdash A$ of these rules is called a basic inference. If $(A_i)_{i \in I} \vdash A$ is a basic inference with $A \in \text{Pos}_v$, then $A_i \in \text{Pos}_v$ for all $i \in I$. This property will be used in the proof of 4.6.

The system $ID_{\omega}^\omega$ consists of the language $L_{ID}(N)$ and a certain derivability relation $\vdash_{\omega}^\omega \Gamma$ ("$\Gamma$ is derivable with order $a \in T$ and cutdegree $m \in \mathbb{N}"$) which we introduce below by an iterated inductive definition similar to that of the tree classes $T_v$ in Section 1. The main feature in the definition of $\vdash_{\omega}^\omega \Gamma$ is the $\Omega_{u+1}$-rule.
which we have developed in Buchholz [1], [2]. We try to give a short explanation of this inference rule. To this purpose let us consider \("\vdash_{i} A\) as a notion of realizability similar to modified realizability. Now suppose that \(\vdash_{i} \Gamma\) is already defined for all \(z \in T_{u}\). Then, according to the fact that

\[
f^{\sigma} \circ r \quad \text{mr} A \rightarrow B \quad \text{iff} \quad \forall g^{\sigma} (g^{\sigma} \text{ mr} A \Rightarrow f^{\sigma} \circ r(g^{\sigma}) \text{ mr} B),
\]

it seems reasonable to define:

\[
a \text{ realizes } (P^{\mathfrak{u}}_{u} \rightarrow B) \quad \iff \quad \{ \text{dom}(a) = T_{u} \quad \text{and} \quad \forall z \in T_{u} (z \text{ realizes } P^{\mathfrak{u}}_{u} \Rightarrow a[z] \text{ realizes } B) \}.
\]

This motivates the following inference rule:

\[
(\Omega_{u+1})' \quad \text{dom}(a) = T_{u} \quad \text{and} \quad \forall z \in T_{u} (\vdash_{i} P^{\mathfrak{u}}_{u} \Rightarrow a[z] B) \Rightarrow \vdash_{m}^{a} P^{\mathfrak{u}}_{u} \rightarrow B.
\]

The next step is a straightforward modification of this rule:

\[
(\Omega_{u+1})'' \quad \text{dom}(a) = T_{u} \quad \text{and} \quad \forall z \in T_{u} \forall A \in \text{Pos}_{u} (\vdash_{i} A \lor P^{\mathfrak{u}}_{u} \Rightarrow a[z] A \lor B) \Rightarrow \vdash_{m}^{a} P^{\mathfrak{u}}_{u} \rightarrow B.
\]

For technical reasons we combine every application of \((\Omega_{u+1})''\) with a cut \(B \lor P^{\mathfrak{u}}_{u} \Rightarrow P^{\mathfrak{u}}_{u} \rightarrow B\). This gives the final version of the \(\Omega_{u+1}\)-rule.

**Inductive definition of \(\vdash_{m}^{a} \Gamma\) (\(a \in T, m \in \mathbb{N}\))**

(Ax1) \(\vdash_{m}^{a} \Gamma, A\), if \(A\) is a true a.p.f. or \(A = 0 \in N\) or \(A = \neg P^{\mathfrak{u}}_{u} \in n\) with \(u \leq j\).

(Ax2) \(\vdash_{m}^{a} \Gamma, \neg A\), if \(A = n \in N\) or \(A = P^{\mathfrak{u}}_{u}\).

(Bas) If \((A_{i})_{i \in I} \vdash_{i} A\) is a basic inference with \(A \in \Gamma\) and \(\forall i \in I (\vdash_{m}^{a} \Gamma, A_{i})\), then \(\vdash_{m}^{a} \Gamma\).

(Pu) \(\vdash_{m}^{a} \Gamma, n \in N \land \forall N^{N}(P^{\mathfrak{u}}_{u}, n)\) and \(P^{\mathfrak{u}}_{u} \in \Gamma \Rightarrow \vdash_{m}^{a+1} \Gamma\).

(Cut) \(\vdash_{m}^{a} \Gamma, \neg C\) and \(\vdash_{m}^{a} \Gamma, C\) and \(|C| < m \Rightarrow \vdash_{m}^{a+1} \Gamma\).

(\Omega_{u+1}) \(\text{dom}(a) = T_{u}\) and \(\vdash_{m}^{a} \Gamma, P^{\mathfrak{u}}_{u} \text{ and} \forall z \in T_{u} \forall A \in \text{Pos}_{u} (\vdash_{i} A, P^{\mathfrak{u}}_{u} \Rightarrow a[z] A \lor B) \Rightarrow \vdash_{m}^{a} \Gamma\).

(\ll) \(\vdash_{m}^{b} \Gamma \text{ and } b \ll_{F} a \Rightarrow \vdash_{m}^{a} \Gamma\).

**4.2. Lemma.** (a) \(\vdash_{m}^{a} \Gamma \text{ and } m \leq k, \Gamma \subseteq \Delta \Rightarrow \vdash_{k}^{a} \Delta\).

(b) \(\vdash_{m}^{a} \Gamma \Rightarrow \vdash_{m}^{k+a} \Gamma\).

(c) \(\vdash_{m}^{a} \Gamma, 0 \notin N \Rightarrow \vdash_{m}^{a} \Gamma\).

**Proof.** By transfinite induction on \(a\) using 3.1(b) and 4.1 and the fact that \((c + a)[z] = c + a[z]\) for all \(z \in \text{dom}(a)\).
4.3. **Lemma** (Inversion). Let \((A_i)_{i \in I} \vdash A\) be a basic inference \((\land), (\lor), (P^u), (P^u n), (\neg P^u n)\). Then \(\vdash_m \Gamma, A\) implies \(\forall i \in I (\vdash_m^\alpha \Gamma, A_i)\).

**Proof.** By transfinite induction on \(a\).

4.4. **Lemma** (Reduction). Suppose \(\vdash_m \Gamma_0, \neg C\) and \(|C| \leq m\), where \(C\) is a formula of the shape \(A \lor B\) or \(\exists x A(x)\) or \(P^u n\) or \(P^u <u n\) or \(\neg P^u n\) or a false a.p.f. Then \(\vdash_m^a \Gamma_0, \Gamma\).

**Proof.** By transfinite induction on \(b\):

(Ax1) If \(\vdash_m^a \Gamma, C\) holds by (Ax1), then also \(\vdash_m^{a+b} \Gamma\) by (Ax1).

(Ax2) If \(\vdash_m^a \Gamma, C\) holds by (Ax2), then either \(\vdash_m^{a+b} \Gamma\) by (Ax2) or \(\neg C \in \Gamma\). In the latter case \(\vdash_m^{a+b} \Gamma_0, \Gamma\) follows from \(\vdash_m^a \Gamma_0, \neg C\).

(Bas) Suppose \(b = b_0 + 1\), \(A \in \Gamma \cup \{C\}\) and \(\forall i \in I (\vdash_m^{b_0} \Gamma, C, A_i)\) where \((A_i)_{i \in I} \vdash A\) is a basic inference \((\land)\). Then by I.H. we have (1) \(\forall i \in I (\vdash_m^{a+b_0} \Gamma_0, \Gamma, A_i)\).

Case 1: \(A \in \Gamma\). Then the assertion follows immediately from (1).

Case 2: \(A \equiv C\). Then, according to the assumption we have made on \(C\), \((\land)\) is an inference \((\lor), (\exists), (P^u n)\) with \(I = \{0\}\). By 4.3, 4.2(a) and \((\land)\) from \(\vdash_m^a \Gamma_0, \neg C\) we get (2) \(\vdash_m^{a+b_0} \Gamma_0, \Gamma, \neg A_0\). From (1), (2) and \(|A_0| < |C| \leq m\) we obtain \(\vdash_m^{a+b} \Gamma_0, \Gamma\) by a cut with cutformula \(A_0\).

\((\land)\) Suppose \(\vdash_m^{b_0} \Gamma, C\) with \(b_0 \ll_{\Gamma, C} b\). Since \(C\) is not a formula \(n \notin N\), it follows that \(a + b_0 \ll_{\Gamma, C} a + b\). By I.H. we have \(\vdash_m^{a+b_0} \Gamma_0, \Gamma\). Hence \(\vdash_m^{a+b} \Gamma_0, \Gamma\) by \((\land)\).

In all other cases the assertion follows immediately from I.H.

4.5. **Theorem** (Cutelimination). \(\vdash_m^{a+1} \Gamma\) and \(a \in T_\rho\), \(\rho \leq \omega\), \(m > 0 \Rightarrow \vdash_m^{D_\rho} \Gamma\).

**Proof.** By transfinite induction on \(a\):

1. If \(\vdash_m^{a+1} \Gamma\) holds by (Ax1) or (Ax2), then the assertion is trivial.

2. Suppose \(a = a_0 + 1\), \(A \in \Gamma\) and \(\forall i \in I (\vdash_m^{a_0} \Gamma, A_i)\), where \((A_i)_{i \in I} \vdash A\) is a basic inference \((\land)\). Then by I.H. we have \(\forall i \in I (\vdash_m^{D_\rho a_0} \Gamma, A_i)\). By \((\land)\) we obtain \(\vdash_m^{D_\rho a_0 + 1} \Gamma\) and then \(\vdash_m^{D_\rho a} \Gamma\) by \((\land)\) and 3.5(a).

3. Suppose \(a = a_0 + 3\), \(P^u n \in \Gamma\) and \(\vdash_m^{a_0 + 1} \Gamma, B\) with \(B = n \in N \wedge \forall u (P^u n)\). Then by I.H. and \((\land)\) we have \(\vdash_m^{D_\rho a_0 + 2} \Gamma, B\). By \((P^u n)\) we get \(\vdash_m^{D_\rho(a_0 + 2) + 3} \Gamma\) and then \(\vdash_m^{D_\rho a} \Gamma\) by \((\land)\) and 3.5(d).

4. Suppose \(\text{dom}(a) = T_u\), \(\vdash_m^{a[l]} \Gamma, P^u n \in \Gamma\) and \(\vdash_m^{a[z]} \Delta, \Gamma\) for all \(z \in T_u\), \(\Delta \subseteq \text{Pos}_u\) with \(\vdash_1 \Delta, P^u n\). Since \(a \in T_\rho\), we have \(u < \rho\) and thus \(\text{dom}(D_\rho a) = T_u\) and \(D_\rho a[a] = D_\rho a[a]\). By I.H. we have \(\vdash_m^{D_\rho a[l]} \Gamma, P^u n\) and \(\vdash_m^{D_\rho a[z]} \Delta, \Gamma\) for all \(z \in T_u\), \(\Delta \subseteq \text{Pos}_u\) with \(\vdash_1 \Delta, P^u n\). From this we obtain \(\vdash_m^{D_\rho a} \Gamma\) by an application of \((\Omega_{a+1})\).

5. Suppose \(\vdash_m^{a_0 + 1} \Gamma\) and \(a_0 \ll_{\Gamma, a}\). Then by I.H. and 3.2(b) we have \(\vdash_m^{D_\rho a_0} \Gamma\) and \(D_\rho a_0 \ll_{\Gamma, a}\). Hence \(\vdash_m^{D_\rho a} \Gamma\).

6. Suppose \(a = a_0 + 1\), \(\vdash_m^{a_0 + 1} \Gamma, \neg C\), \(\vdash_m^{a_0 + 1} \Gamma, C\) and \(|C| < m + 1\). Then by I.H. we have \(\vdash_m^{D_\rho a} \Gamma, \neg C\) and \(\vdash_m^{D_\rho a} \Gamma, C\).
6.1. $|C| < m$: In this case we obtain $\vdash_{m,a_0+1} \Gamma$ by a cut with cut formula $C$. The assertion follows by $(\ll)$ and 3.5(b).

6.2. $|C| = m$: Since $m > 0$, we may assume that $C$ fulfills the condition of 4.4. Then by 4.4 we obtain $\vdash_{m,a_0+D,a_0} \Gamma$, and from this $\vdash_{m,a} \Gamma$ by $(\ll)$ and 3.2(a).

The following theorem shows that if $\Gamma \subseteq \text{Pos}_u$ is derivable with cutdegree 1, then one can eliminate all $\Omega_{u+1}$-inferences with $u \geq v$ from the derivation of $\Gamma$.

4.6 Theorem (Collapsing). $\vdash_{1} \Gamma$ and $\Gamma \subseteq \text{Pos}_u \Rightarrow \vdash_{1}^{D,a} \Gamma$.

Proof. By transfinite induction on $a$:

1. Suppose $\text{dom}(a) = T_u$, $\vdash_{1}^{q[1]} \Gamma$, $P_u^{x}\Delta$ and $\vdash_{1}^{q[z]} \Delta$, $\Gamma$ for all $z \in T_u$, $\Delta \subseteq \text{Pos}_u$ with $\vdash_{1}^{z} \Delta$, $P_u^{x}\Delta$.

Case 1: $u < v$. Then by I.H. we have $\vdash_{1}^{D,a[1]} \Gamma$, $P_u^{x}\Delta$ and $\vdash_{1}^{D,a[z]} \Delta$, $\Gamma$ for all $z \in T_u$, $\Delta \subseteq \text{Pos}_u$ with $\vdash_{1}^{z} \Delta$, $P_u^{x}\Delta$. Moreover, $\text{dom}(D,a) = T_u$ and $(D,a)[z] = D_u[z]$. The assertion follows by $(\Omega_{u+1})$.

Case 2: $u = v$. Then $\Gamma \cup \{P_u^{x}\Delta\} \subseteq \text{Pos}_u$ and therefore by I.H. $\vdash_{1}^{D,a[1]} \Gamma$, $P_u^{x}\Delta$. Since $z := D_u[a] \in T_u$, we get $\vdash_{1}^{q[z]} \Gamma$. Now we apply the I.H. again and obtain $\vdash_{1}^{D,a[z]} \Gamma$. But $D_u[a] = (D_u[a])[0] \ll_r D_u[a]$, and therefore $\vdash_{1}^{D,a} \Gamma$.

2. In all other cases the assertion follows immediately from the I.H. by 3.5(b, d), 3.4(a), $(\ll)$.

Definition

$L(N)_+ := \{A : A$ is a sentence of $L(N)$ in which $N$ occurs only positively}$.

For $\Gamma = \{A_1, \ldots, A_n\} \subseteq L(N)_+$ we define:

$\vdash \Gamma(k) := \{A_1 \vee \cdots \vee A_n$ is true in the standard model when $N$ is interpreted as $\{i \in \mathbb{N} : 3i < k\}$.

4.7 Lemma.

$\vdash_{1}^{q} i_1 \notin N, \ldots, i_m \notin N, \Gamma \text{ and } \Gamma \subseteq L(N)_+, n \geq \max\{2, 3i_1, \ldots, 3i_m\} \Rightarrow \vdash \Gamma(H_{D_o}(n))$.

Proof. By transfinite induction on $a$: Let

$\Gamma_0 := \{i_1 \notin N, \ldots, i_m \notin N\}, \quad k := \max\{2, 3i_1, \ldots, 3i_m\} \leq n$.

1. If $\vdash_{1}^{q} \Gamma_0$, $\Gamma$ holds by (Ax1), then the assertion is trivial.

2. If $\vdash_{1}^{q} \Gamma_0$, $\Gamma$ holds by (Ax2), then the assertion follows from $n < H_{D_o}(n)$.

3. If $\vdash_{1}^{q} \Gamma_0$, $\Gamma$ is the conclusion of a basic inference $\neq(N)$, then the assertion follows immediately from the I.H. and the relation $H_{D_o}(n) < H_{D_o(b+1)}(n)$.

4. Suppose $a = b + 1$, $N(j + 1) \in \Gamma$, $\vdash_{1}^{q} \Gamma_0$, $\Gamma, N_j$. By I.H. we obtain $\vdash \Gamma \cup$
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\( \{N_j\}(H_{D_0b}(n)) \). By 3.1(c), 3.2(a), 3.6(d) we have

\( H_{D_0b}(n) < H_{(D_0b)^2}(n) < H_{(D_0b)^3}(n) < H_{D_0a}(n) \) and therefore \( H_{D_0}(n) + 3 \leq H_{D_0a}(n) \). Hence \( \varepsilon(G(H_{D_0a}(n))) \).

5. Suppose \( \varepsilon \Gamma_0, \Gamma \) with \( b < k \ D_0a \) and therefore \( H_{D_0b}(n) < H_{D_0a}(n) \), since \( n \geq k \). Now the assertion follows immediately from the I.H.

6. Suppose \( a = b + 1, \varepsilon \Gamma_0, \Gamma, i_0 \in N \) and \( \varepsilon i_{i_0} \notin N, \Gamma_0, \Gamma \). Let \( \tilde{n} := H_{D_0b}(n) \). Then we have

\[ n < \tilde{n} < H_{D_0b}(\tilde{n}) = H_{(D_0b)^2}(n) < H_{D_0a}(n) \]

6.1. \( \tilde{n} < 3i_0 \): From \( \varepsilon \Gamma_0, \Gamma, i_0 \in N \) we obtain by the I.H. \( \varepsilon \Gamma \cup \{i_0 \in N\}(\tilde{n}) \) and then \( \varepsilon \Gamma(\tilde{n}) \), since \( 3i_0 \not< \tilde{n} \). Using \( \tilde{n} < H_{D_0a}(n) \) we get the assertion.

6.2. \( 3i_0 \leq \tilde{n} \): From \( \varepsilon \Gamma_0, \Gamma, i_0 \in N \) and \( \max\{k, 3i_0\} \leq \tilde{n} \) we obtain by the I.H. \( \varepsilon \Gamma(H_{D_0a}(n)) \) and thus \( \varepsilon \Gamma(H_{D_0a}(n)) \).

7. Suppose \( \text{dom}(a) = T_u, \varepsilon \Gamma_0, \Gamma, P_{a}^{\{z\}}, \varepsilon \Gamma_0, \Gamma \) for all \( z \in T_u, \Delta \subseteq \text{Pos}_{a} \) with \( \varepsilon \Gamma_0, \Gamma, P_{a}^{\{z\}} \). By 4.6 we obtain \( \varepsilon \Gamma_0, \Gamma, P_{a}^{\{z\}} \) with \( z := D_oa[1] \in T_u \). From this we get \( \varepsilon \Gamma_0, \Gamma, \Gamma \). Now we apply the I.H. and obtain \( \varepsilon \Gamma(H_{D_0a}(n)) \).

Hence \( \varepsilon \Gamma(H_{D_0a}(n)) \), since \( D_oa[z] = (D_oa)[0] \).

4.8. Theorem. If \( \varepsilon P_{a}^{\omega_\varepsilon} \forall x \in N \exists y \in N \varphi^N(x, y) \), where \( \varphi \leq \omega, m \neq 0 \) and \( \varphi(x, y) \) a \( \Sigma_1^0 \)-formula of the language \( L \), then there exists \( p \in \mathbb{N} \) such that \( \forall n \geq p \ \exists k < H_{D_0D_0^\omega_\varepsilon}(1) \varphi(n, k) \).

Proof. Let \( a := D_o\omega_\varepsilon \). From the premise we obtain \( \varepsilon n \in N, \exists y \in N \varphi^N(n, y) \) for all \( n \in \mathbb{N} \). Then by 4.7 we get \( \varepsilon \exists y \in N \varphi^N(n, y) (H_{D_0a}(\tilde{n})) \) for all \( n \in \mathbb{N} \) and all \( \tilde{n} \geq \max\{2, 3n\} \). Hence \( \forall n \exists k < H_{D_0a}(3n + 2) \varphi(n, k) \). By 3.8 we have \( H_{D_0a}(3n + 2) < H_{D_0a^\omega_\varepsilon}(1) \) for all \( n \geq m + 2 \).

In the remaining part of this section we show that \( ID_{\omega_\varepsilon}^{\omega_\varepsilon} \omega_\varepsilon \) can be embedded into \( ID_{\omega_\varepsilon}^{\omega_\varepsilon} \) and finally we prove Theorem 4.0. Let \( \nu \leq \omega \) be fixed.

Abbreviations

\[ k := D^k_{\omega_\varepsilon + 2}, \]

\[ a \rightarrow_n b \quad \iff \quad \exists a_0, \ldots, a_n(a_0 = a \land a_n = b \land \forall i < n (a_i + 1 \leq a_{i+1})) \]

4.9. Lemma. (a) \( k \ll k + 1 \), (b) \( k \rightarrow_3 k + 1 \).

Proof. (a) follows from 3.7(b).

(b) By 3.5(d) and 3.7(b) we have \( 3 \ll_2 k \) and \( k \cdot 3 \ll_2 k + 1 \). Hence \( k + 3 \ll k \cdot 2, k \cdot 2 + 3 \ll k \cdot 3 \ll k + 1 \) and consequently \( k \rightarrow_3 k \cdot 2 \rightarrow_3 k + 1 \).

4.10. Lemma. \( \varepsilon 0 \neg A, A \) where \( k := |A| \)
Proof. By induction on $|A|$:  
1. If $A$ is atomic, then $\vdash_0^k \neg A$, $A$ by (Ax1) or (Ax2).
2. $A = A_0 \land A_1$: Then $k = m + 1$ with $m := \max\{|A_0|, |A_1|\}$. By I.H., 4.9(a) and $(\ll)$ we get $\vdash_0^m \neg A_i, A_i$ for $i = 0, 1$, and then $\vdash_0^{m+1} \neg A_0 \lor \neg A_1, A_0 \land A_1$ by $(\lor), (\land), 4.9(b)$.
3. $A = \forall x B(x)$: This case is treated as 2.

4.11. Lemma. $\vdash_0^{k+Dh_1} \neg F(0), \neg \forall x \in N (F(x) \rightarrow F(x'))$, $n \notin N$, $F(n)$, where $k := |F|$.

Proof. Let $G := \forall x \in N (F(x) \rightarrow F(x'))$. By induction on $n$ we show:

(1) $\vdash_0^{k+3n} \neg F(0), \neg G, F(n)$.

From (1) we obtain $\vdash_0^{k+Dh_1} \neg F(0), \neg G, F(n), n \notin N$, since $\bar{k} + 3n \ll_3 \bar{k} + D_0$.

Proof of (1). For $n = 0$ the assertion holds by 4.10.

Induction step: Suppose $\vdash_0^{k+3n} \neg F(0), \neg G, F(n)$. By 4.10 we have $\vdash_0^{k+3n} \neg F(n'), F(n')$. Hence $\vdash_0^{k+3n+1} \neg F(0), \neg G, F(n) \land \neg F(n'), F(n')$. By (Ax1) and $n$ applications of (N) we get $\vdash_0^{k+3n+1} n \in N$, and then by $(\land) \vdash_0^{k+3n+2} \neg F(0), \neg G, n \in N \land (F(n) \land \neg F(n'))$, $F(n')$. Now we apply $(\exists)$ and obtain $\vdash_0^{k+3n+1} \neg F(0), \neg G, F(n')$, since $\neg G \equiv \exists x (x \in N \land (F(x) \land \neg F(x')))$. The following lemma will be used to show that the induction scheme $\forall x \in N (\forall x' (F, x) \rightarrow F(x')) \rightarrow \forall x \in N (P_{x} \rightarrow F(x))$ is derivable in $ID_\omega$.


$$
\begin{align*}
&\text{Let } a \in T_u, \Delta \subseteq \text{Pos}_u, \varphi \vdash_\Delta \Delta, P_{u}^n \\
&k = |F|, G = \forall x \in N (\forall x' (F, x) \rightarrow F(x)) \\
&\Rightarrow \vdash_1^{k+a} \Delta, \neg G, F(n).
\end{align*}
$$

Proof. Informal description: Let $\Pi$ be a derivation of $\Delta, P_{u}^n$. In $\Pi$ we replace every occurrence of $P_{u}^n$, which is linked to the endformula $P_{u}^n$, by $F(\cdot)$. Let $\Pi'$ denote the result of this transformation. $\Pi'$ may contain certain inferences of the kind $j \in N \land \forall x'(F, j) \vdash F(j)$, and therefore $\Pi'$ may fail to be an $ID_\omega$-derivation. From $\Pi'$ we obtain an $ID_\omega$-derivation of $\Delta, \neg G, F(n)$ as follows: First we adjoin $\neg G$ to each $\Gamma$ in $\Pi'$, and then we replace every inference $\neg G, \Gamma, j \in N \land \forall x'(F, j) \vdash \neg G, \Gamma, F(j)$ by the following inferences:

$$
\begin{align*}
&\neg G, \Gamma, j \in N \land \forall x'(F, j) \vdash \neg F(j), F(j) \\
&\neg G, \Gamma, j \in N \land \forall x'(F, j) \land \neg F(j), F(j) \quad (\lor) \\
&\neg G, \Gamma, F(j) \quad (\exists).
\end{align*}
$$
In order to get a rigorous proof of the lemma we have to prove a more general proposition.

**Definition.** For $A \in \text{Pos}_u$ let $A^*$ denote the result of replacing all occurrences of $P_u^\mathcal{W}$ in $A$ by $F(\cdot)$. \{${A_1, \ldots, A_m}$\}*: = \{${A_1^*, \ldots, A_m^*}$\}.

**Proposition.** $I_0 \cup \Gamma \subseteq \text{Pos}_u$, $a \in T_u$, $k = |F|$, $\vdash_1 I_0, \Gamma \Rightarrow \vdash_1 I_0^+ a, \neg G, \Gamma^*$.

**Proof.** By transfinite induction on $a$:

1. If $\vdash_1 I_0, \neg \neg G$ holds by (Ax1) or (Ax2), then also $\vdash_1 I_0^+ a, \neg G, \Gamma^*$ by (Ax1), (Ax2), since $\neg P_u^\mathcal{W}$ does not occur in $I_0 \cup \Gamma$.

2. Suppose that $a = a_0 + 1$ and $(A_i)_{i \in I} \vdash A$ is a basic inference with $A \in I_0 \cup \Gamma$ and $\forall i \in I (\vdash_1 I_0, \Gamma, A_i)$. Then $\forall i \in I (A_i \in \text{Pos}_u)$ and therefore we can apply the I.H. to $I_0, \Gamma, A_i$.

2.1. $A \in I_0$: By I.H. we get $\forall i \in I (\vdash_1 I_0^+ a_0, \neg G, \Gamma^*, A_i)$ and from this $\vdash_1 I_0^+ a, \neg G, \Gamma^*$ by the respective basic inference.

2.2. $A \in \Gamma$: Then $A^* \in \Gamma^*$ and $(A_i^*)_{i \in I} \vdash A^*$ is a basic inference. By I.H. we have $\forall i \in I (\vdash_1 I_0^+ a_0, \neg G, \Gamma^*, A_i^*)$. Hence $\vdash_1 I_0^+ a, \neg G, \Gamma^*$.

3. Suppose that $\text{dom}(a) = T_w$, $\vdash_1 \Delta, I_0, \Gamma, P^\mathcal{W}$ and $\vdash_1 I_0^+ a, \Delta, I_0, \Gamma$ for all $z \in T_w$, $\Delta \subseteq \text{Pos}_w$ with $\vdash_1 \Delta, P^\mathcal{W}$.

Since $a \in T_u$, we have $w < u$ and therefore by I.H.

$\vdash_1 I_0^+ a_0, \neg G, \Gamma^*, P^\mathcal{W}$ and $\vdash_1 I_0^+ a, \Delta, I_0, \Gamma^*$ for all $z \in T_w$, $\Delta \subseteq \text{Pos}_w$ with $\vdash_1 \Delta, P^\mathcal{W}$.

Now by an application of $(Q_{w+1})$ we get the assertion.

4. Suppose $a = a_0 + 3$, $P^\mathcal{W}$ and $\vdash_1 I_0^+ a_0, \Gamma, j \in N \setminus \text{Pos}_u(P^\mathcal{W}, j)$. Then $F(j) \in \Gamma^*$ and therefore $\vdash_1 I_0^* \neg F(j)$ by 4.10. By I.H. and 4.3 we have $\vdash_1 I_0^+ a_0, \Gamma^*, \neg G, j \in N$ and $\vdash_1 I_0^+ a, \Gamma^*, \neg G, \text{Pos}_u(F, j)$. Now we obtain $\vdash_1 I_0^+ a_0, \Gamma^*, \neg G, j \in N \setminus (\text{Pos}_u(F, j) \setminus \neg F(j))$ and then by $(P \setminus 1) \vdash_1 I_0^+ a_0, \Gamma^*, \neg G$.

5. In all other cases the assertion follows immediately from I.H.

4.13. **Lemma.** $\vdash_1 I_0^+ a_0, \forall x \in N (\forall_u(F, x) \rightarrow F(x)), \neg P^\mathcal{W} n, F(n), \text{with } k := |F|$.

**Proof.** Let $b := k + D + 0$ and $G := \forall x \in N (\forall_u(F, x) \rightarrow F(x))$. Then $\text{dom}(b) = T_u$ and $b[z] = k + z$. Therefore by 4.12 we have $\vdash_1 I_0^+ a_0, \neg G, \neg P^\mathcal{W} n, F(n)$ for all $z \in T_u$, $\Delta \subseteq \text{Pos}_u$ with $\vdash_1 \Delta, P^\mathcal{W}$. By (Ax2) we also have $\vdash_1 I_0^+ a_0, \neg G, \neg P^\mathcal{W} n, F(n), P^\mathcal{W} n$. Now we apply the $\Omega_{w+1}$-rule and obtain $\vdash_1 I_0^+ a, \neg G, \neg P^\mathcal{W} n, F(n)$.

**Remark.** The theory $\text{ID}_v$ with $v < \omega$ is the same as $\text{ID}_\omega$ except that the axioms $(P^\mathcal{W})_v$ are replaced by

$(P^\mathcal{W})_v \forall x (\forall_u(F, x) \rightarrow F(x)) \rightarrow \forall x (P^\mathcal{W} x \rightarrow F(x))$,

for each $L_{\text{ID}}$-formula $F(x)$ and each $u < v$.

4.14. **Theorem.** If the sentence $A$ is provable in $\text{ID}_v (v \leq \omega)$, then there exists $k \in \mathbb{N}$ such that $\vdash_1 I_0^l A^N$. 

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Proposition 1. For every mathematical axiom $A(v_1, \ldots, v_m)$ of ID, there exists $k \in \mathbb{N}$ such that $\vdash_k A(i_1, \ldots, i_m)^N$ for all $i_1, \ldots, i_m \in \mathbb{N}$. ($v_1, v_2, \ldots$ denote variables of the language $L$.)

Proof. We assume $m = 1$.

1. $A(v) = B(0, v) \land \forall x (B(x, v) \rightarrow B(x', v)) \rightarrow \forall x B(x, v)$.

Let $F(x) := B(x, v) \land (\forall x \in N (F(x) \rightarrow F(x')))$. Then $A(\Delta N) := \forall v (\forall x \in N (F(N + F(x)))$ and $k := |F(x)|$. By 4.11, 3.5(c), 4.9(a) we have $\vdash_k F(0)$, $\neg G$, $n \notin N$, $F(n)$ for all $n \in \mathbb{N}$. Since $k \cdot 2 \rightarrow_9 k + 2$, we obtain $\vdash_k A(\Delta N)$.

2. For any other axiom of PA the assertion is trivial.

3. $A(v) = \forall x (\forall x (F(x', v) \rightarrow B(x', v))) \rightarrow \forall x (F(x', v) \rightarrow B(x', v)))$, $u < v < \omega$.

Let $F(x) := B(x, v) \land (\forall x \in N (F(x) \rightarrow F(x')))$. Then $A(\Delta N) := \forall y (\forall x \in N (F(N + F(x)))$ and $k := |F(x)|$. By 4.13 $\vdash_{k+2} A(\Delta N)$, $\forall x \in N (F(N + F(x)))$, for all $n \in \mathbb{N}$. Since $k + D_u + 0 = (k + D_0) u < u$, we obtain by (v), $(\forall \omega)$, $(\forall \omega)$, $(\forall \omega)$. From this we get $\vdash_{k+2} A(\Delta N)$, since $k + D_0 + 0 < k + 2 \rightarrow_9 k + 2$.

4. $A(v) = \forall y (\forall x \in N (F(N + F(x))) \rightarrow \forall x (F(x', v) \rightarrow B(x', v)))$, $v = \omega$.

Let $F(x) := B(x, y, v) \land (\forall x \in N (F(N + F(x)))$. Then $A(v) = \forall y (\forall x \in N (F(N + F(x)))$ and $k := |F(x)|$. By 4.13 $\vdash_{k+2} A(\Delta N)$, $\forall x \in N (F(N + F(x)))$, for all $n \in \mathbb{N}$. Since $k + D_u + 0 = (k + D_0) u < u$, we obtain by (v), $(\forall \omega)$, $(\forall \omega)$, $(\forall \omega)$. From this we get $\vdash_{k+2} A(\Delta N)$, since $k + D_0 + 0 < k + 2 \rightarrow_9 k + 2$.

5. $A(v) = \forall y (\forall x \in N (F(N + F(x))) \rightarrow \forall x (F(x', v) \rightarrow B(x', v)))$, $v = \omega$.

Let $F(x) := B(x, y, v) \land (\forall x \in N (F(N + F(x)))$. Then $A(v) = \forall y (\forall x \in N (F(N + F(x)))$ and $k := |F(x)|$. By 4.13 $\vdash_{k+2} A(\Delta N)$, $\forall x \in N (F(N + F(x)))$, for all $n \in \mathbb{N}$. Since $k + D_u + 0 = (k + D_0) u < u$, we obtain by (v), $(\forall \omega)$, $(\forall \omega)$, $(\forall \omega)$. From this we get $\vdash_{k+2} A(\Delta N)$, since $k + D_0 + 0 < k + 2 \rightarrow_9 k + 2$.

6. $A(v) = \forall y (\forall x \in N (F(N + F(x))) \rightarrow \forall x (F(x', v) \rightarrow B(x', v)))$, $v = \omega$.

Let $F(x) := B(x, y, v) \land (\forall x \in N (F(N + F(x)))$. Then $A(v) = \forall y (\forall x \in N (F(N + F(x)))$ and $k := |F(x)|$. By 4.13 $\vdash_{k+2} A(\Delta N)$, $\forall x \in N (F(N + F(x)))$, for all $n \in \mathbb{N}$. Since $k + D_u + 0 = (k + D_0) u < u$, we obtain by (v), $(\forall \omega)$, $(\forall \omega)$, $(\forall \omega)$. From this we get $\vdash_{k+2} A(\Delta N)$, since $k + D_0 + 0 < k + 2 \rightarrow_9 k + 2$.

Proposition 2. By PL1 we denote Tait's calculus for first-order predicate logic in the language $L_{ID}$ (cf. [8]). If $\Gamma(v_1, \ldots, v_m)$ is derivable in PL1, then there exists $k \in \mathbb{N}$ such that $\vdash_k A(i_1, \ldots, i_m)^N$ for all $i_1, \ldots, i_m \in \mathbb{N}$.

Proof. By induction on the derivation of $\Gamma$: Let $m = 1$.

1. $\Gamma = \Gamma_1 \cup \{\neg A, A\}$: cf. 4.10.

2. If $\Gamma$ is the conclusion of a ($\wedge$)- or ($\vee$)-inference, then the assertion follows immediately from the I.H.

3. $\Gamma(v) = \Gamma_0(v)$, $\forall x A(v, x)$ and PL1 $\vdash \Gamma(v)$, $A(v, x)$ with $x \neq v$: By I.H. there exists $k$ such that $\vdash_k A(i_1, \ldots, i_m)^N$ for all $i, n \in \mathbb{N}$. Then by (v) and ($\forall \omega$) we get $\vdash_k A(i_1, \ldots, i_m)^N$.

4. $\Gamma(v) = \Gamma_0(v)$, $\exists x A(v, x)$ and PL1 $\vdash \Gamma(v)$, $A(v, i)$:

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4.2. \( t \equiv v^{i_0} \): By I.H. there exists \( k \gg k_0 \) such that \( \vdash_k^i i \notin N \), \( \Gamma(i)^N \), \( A(i, i_0) \subseteq N \) for all \( i \in \mathbb{N} \). Since \( k \gg k_0 \), we have \( \vdash_k^{i+1} i \notin N \), \( i_0 \in N \) for all \( i \in \mathbb{N} \). Hence \( \vdash_k^{i+1} i \notin N \), \( \Gamma(i)^N \). Now we apply (\( \exists \)) and get \( \vdash_k^{i+1} i \notin N \), \( \Gamma(i)^N \).

**Proof of 4.14.** Suppose \( \text{ID}_v \vdash A \) (A closed). Then \( \text{PL}_1 \vdash \neg(A_1 \land \cdots \land A_n) \), \( A \) where every \( A_i \) is the universal closure of an axiom of \( \text{ID}_v \). By Propositions 1 and 2 there exists \( m \) such that \( \vdash_0^{\alpha} (A_1 \land \cdots \land A_n)^N \) and \( \vdash_0^{\alpha} \neg(A_1 \land \cdots \land A_n)^N, A^N \). By a cut with cut formula \( (A_1 \land \cdots \land A_n)^N \) we obtain now \( \vdash_k^k A^N \) with \( k :=\max\{|(A_1 \land \cdots \land A_n)^N|, m\} + 1 \).

**Conclusion.** By combining the Theorems 4.14, 4.5, 4.8 we obtain Theorem 4.0 which was stated at the beginning of this section.

**Appendix: The proof-theoretic ordinal of \( \text{ID}_v \)**

**Definitions.** 1. By transfinite induction on \( a \) we define an ordinal \( \text{rk}(a) \) for every \( a \in T_0 \):

\[
\text{rk}(a) := \sup(\text{rk}(a[n]) + 1 : n \in \text{dom}(a)).
\]

2. By transfinite induction on \( \alpha \in \text{On} \) we define the sets \( I^\alpha_a \) and \( I^\leq_\alpha a \) for every positive operator form \( a \):

\[
I^\alpha_a := \{ n \in \mathbb{N} : \mathbb{A}_0(I^\leq_\alpha a, n) \text{ is true in the standard model} \},
\]

\[
I^\leq_\alpha a := \bigcup_{a < \alpha} I^\alpha_a.
\]

3. For \( n \in \bigcup_{a \in \text{On}} I^\alpha_a \) we set \( |n|_\alpha := \min\{\alpha : n \in I^\alpha_a\} \).

4. \( |\text{ID}_v| := \sup\{|n|_\alpha : \text{ID}_v \vdash P^{\alpha}_0 n\} \). \( |\text{ID}_v| \) is called the proof-theoretic ordinal of \( \text{ID}_v \).

We will prove the following result:

\[
|\text{ID}_v| = \sup\{\text{rk}(D_0 D_{\mathbb{B}}^0) : k \in \mathbb{N}\} \quad (v \ll \omega).
\]

**Definition.** Let \( \Gamma = \{A_1, \ldots, A_n\} \subseteq \text{Pos}_0 \):

\[
\vdash^\alpha \Gamma \iff \begin{cases} A_1 \lor \cdots \lor A_n \text{ is true in the standard model when} \\ P^{\alpha}_0, P^\leq_\alpha, N \text{ are interpreted by } I^\leq_\alpha, \emptyset, \mathbb{N} \text{ resp.} \end{cases}
\]

**A.1. Lemma.** \( \vdash_{\mathbb{B}} \Gamma, \Gamma \subseteq \text{Pos}_0, a \in T_0, \text{rk}(a) \ll \alpha \Rightarrow \vdash^\alpha \Gamma \).

**Proof.** By transfinite induction on \( a \):

1. If \( \vdash^\alpha \Gamma \) holds by (Ax1), then \( \vdash^\alpha \Gamma \) for every \( \alpha \).

2. Suppose that \( \vdash^\alpha \Gamma \) holds by (Ax2). Then, since \( \Gamma \subseteq \text{Pos}_0 \), we have \( \Gamma = \Gamma_0 \), \( n \notin N \), \( n \in N \) and thus \( \vdash^\alpha \Gamma \) for every \( \alpha \).
3. If $\vdash_4 \Gamma$ is the conclusion of a basic inference ($\mathcal{I}$), then ($\mathcal{I}$) is an inference $(\wedge), (\vee), (\forall')$, $(\exists)$ or $(N)$, and the assertion follows immediately from the I.H.

4. Suppose $\vdash_4 \Gamma, \ n \in N \wedge \mathcal{V}_\alpha^\mathcal{N}(P_0^\mathcal{N}, n)$ with $a = b + 1$ and $\Gamma = \Delta, P_0^\mathcal{N} n$. Then $\beta := \text{rk}(b) < \alpha$. By I.H. we get "$\vdash^{\beta} \Delta$ or $n \in I_\alpha^\mathcal{N}$ or $\mathcal{V}_\alpha(I_\alpha^\mathcal{N}, n)$" and from this "$\vdash^{\alpha} \Delta$ or $n \in I_\alpha^\mathcal{N}$", i.e., $\vdash^{\alpha} \Gamma$.

5. If $\vdash_4 \Gamma$ is the conclusion of a cut, then the cut formula is of the kind $n \in N$, and the assertion follows immediately from the I.H.

6. If $\vdash_4 \Gamma$ with $b \ll a$, then $\text{rk}(b) < \text{rk}(a)$ and thus $\vdash_4 \Gamma$ by I.H.

From $a \in T_0$ it follows that $\vdash_4 \Gamma$ cannot be the conclusion of an application of the $\Omega_{u+1}$-rule.

A.2. Lemma. $|\text{ID}_v| \leq \sup\{\text{rk}(D_0 D_0^v 0): k \in \mathbb{N}\}$.

Proof. Suppose $\text{ID}_v \vdash P_0^\mathcal{N} n$. Then by 4.14, 4.5, 4.6 we obtain $\vdash_4 P_0^\mathcal{N} 0 P_0^\mathcal{N} n$, for some $k \in \mathbb{N}$. By A.1 this yields $n \in I_\alpha^{\mathcal{N}}$ with $\alpha := \text{rk}(D_0 D_0^v 0)$. Hence $|n|_\mathcal{N} < \text{rk}(D_0 D_0^v 0)$.

A.3. Lemma. $\sup\{\text{rk}(D_0 D_0^v 0): k \in \mathbb{N}\} \leq |\text{ID}_v|.$

Proof. Here we make use of Theorem 2.2 which claims that "$a \in W_0$" is provable in $\text{ID}_v$, for every $a \in T_0$ which contains no symbol $D_0$ with $v > v$. From this we get, for all $k \in \mathbb{N}$,

(1) $\text{ID}_v \vdash P_0^\mathcal{N}[D_0 D_0^v 0]^k$

where $a \rightarrow [a]$ is any reasonable Gödel numbering of the terms in $T$, and $\mathcal{N}$ is a positive operator form which on the basis of this Gödel numbering formalizes the inductive definition of the sets $W_v$ ($v < v$) in Section 2. Then we also have

(2) $|[a]|_\mathcal{N} = \text{rk}(a), \quad \text{for all } a \in T_0.$

The assertion follows immediately from (1) and (2).

References