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Notation systems for infinitary derivations

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Dedicated to Kurt Schütte on the occasion of his 80th birthday

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It is one of Kurt Schütte's great merits to have established cut-elimination on infinitary derivations as a powerful and elegant tool for proof-theoretic investigations. Compared to the Gentzen-Takeuti approach where ordinals are assigned to finite derivations in a rather cryptic way, the use of infinitary derivations together with the canonical assignment of ordinals as lengths of derivations provides a very perspicious and conceptually clear-cut method which has proved successful even with respect to the strongest systems analyzed till now. But on the other side something is lost when passing from finite to (unrestricted) infinite derivations, in so far as along these lines one only obtains information on the provable Π^1_1-sentences of a formal theory, while Gentzen's method – if successfully applied – yields stronger results, e.g. bounds for provable Π^0_2-sentences (provably recursive functions) or the unprovability of primitive recursive wellfoundedness *PRWO*. Of course, as pointed out by Kreisel [7] such stronger results can be recaptured by arithmetizing the cut-elimination procedure for (primitive) recursively represented infinite derivations via the (Primitive) Recursion Theorem (cf. Schwichtenberg [15], Girard [5]). But this requires a lot of cumbersome and boring coding machinery which on the other side is not completely trivial, and it seems to me that all presentations of this subject in the existing literature are more or less unsatisfactory.

Our purpose here is to provide a technically smooth method for the finitary treatment of infinite derivations in ω-arithmetic Z^∞, where we don't need numerical codes but instead are working with natural *notations* for infinite derivations. These notations are finite terms generated from finite derivations (considered as constants) by certain function symbols $I_{k,A}$, R_C, E corresponding to the operations $\mathcal{I}_{k,A}: Z^\infty \to Z^\infty, \mathcal{R}_C: Z^\infty \times Z^\infty \to Z^\infty, \mathcal{E}: Z^\infty \to Z^\infty$ which make up the cut-elimination procedure for Z^∞ developed by Schütte [12] and Minc [10]. (Minc' contribution was to modify Schütte's cut-elimination procedure by incorporating the so-called repetition-rule, which is crucial for the subsequent work.)

In order to demonstrate the working of our method we will prove two wellknown results of classical proof theory for the system $Z+TI_{\prec\upharpoonright}$ (i.e. Peano-Arithmetic together with the scheme of transfinite induction along any proper segment of some prim. rec. wellordering $\prec$). These results are

(I) If $Z+TI_{\prec\upharpoonright}\vdash\forall x\exists y R(x,y)$ ($R\in\Sigma^0_1$) then there are prim. rec. functions g, $o:\mathbb{N}^2\to\mathbb{N}$, such that $\min\{m:R(n,m)\}=g(n,\min\{k:o(n,k+1)\not\prec o(n,k)\})$, for all $n\in\mathbb{N}$.

(II) $PRA\vdash PRWO(\prec)\to\Pi^0_2$-Reflection $(Z+TI_{\prec\upharpoonright})$.

Of course in the proof of (I) and (II) we cannot completely dispense with coding. But we only need the comparatively trivial coding of syntactic objects (such as formulas, sequents, finite derivations, etc.) and even this plays a rather marginal rôle, while the central part of our proof is coding-free.

Content

Section 1 contains besides some preliminary definitions and abbreviations a precise definition of the set $Z_\prec$ of all $(Z+TI_{\prec\upharpoonright})$-derivations. In Sect. 2 we introduce the set Z^∞ of derivations of ω-arithmetic and define the Schütte-Minc operator $\mathscr{E}:Z^\infty\to Z^\infty$ by transfinite recursion on wellfounded trees. The effect of $\mathscr{E}$ is to lower the cutrank $deg(\varphi)$ of each $\varphi\in Z^\infty$ (with $0<deg(\varphi)<\omega$) at least by 1. In [10] $\mathscr{E}$ is denoted $\mathscr{R}_1$. In fact the material of this section is not necessary for the proof of (I), (II) above; it only serves as a semantical basis for the syntactic definitions of Sect. 3. In Sect. 3 we define the notation system $Z^*_\prec$ which contains notations for all $\varphi\in Z^\infty$ arising from finite derivations $d\in Z_\prec$ via embedding $^\infty:Z_\prec\to Z^\infty$ and subsequent cut-elimination in Z^∞. The main point is that (by a simple recursion on the built up of terms) from every notation $h\in Z^*_\prec$ one can compute the endsequent of $\mathfrak{v}(h)$ (the Z^∞-derivation denoted by h) and notations $\mathfrak{s}_0 h, \mathfrak{s}_1 h, \ldots$ for the immediate subderivations of $\mathfrak{v}(h)$. In Sect. 4 we prove (I) and (II) using the work of Sect. 3. In Sect. 5 we generalize the approach of Sect. 3 and introduce the notion of an arbitrary notation system for ω-derivations. This is then used in Sect. 6 to give an alternative description of Minc's continuous cut-elimination operator $\mathscr{E}'$ for arbitrary (not necessarily wellfounded) proof-figures of ω-arithmetic. In fact $\mathscr{E}'$ is an extension of $\mathscr{E}$.

Remarks

1. We want to emphasize that the present paper has profited considerably by previous presentations of the subject by Girard [5], Minc [10], Schwichtenberg [15]. In some sense it may be considered as a supplement to those; but nevertheless it is completely selfcontained.

2. The idea of using terms (built up from constants and function symbols with a welldefined semantical meaning) as notations for infinite derivations is nothing more than a slight (and rather obvious) generalization of Schütte's approach to systems of ordinal notations as presented in Chap. V of his "Proof Theory" [14].

3. We are indebted to G. Minc, W. Pohlers, and W. Sieg for substantial comments on an earlier version of this paper.

1 The formal system $Z_{\prec}$

Preliminaries

1.1 Syntax

In the following L denotes a fixed 1^{st}-order language consisting of the following symbols: a constant 0 (*zero*), a unary function constant $'$ (*successor*), and some predicate symbols. We distinguish two sorts of individual variables, free variables (denoted by u, v) and bound variables (denoted by x, y). The closed L-terms, $0, 0', 0'', \ldots$ are called *numerals*; we identify numerals and natural numbers and denote them by i, j, k, m, n. An expression of the shape $pt_1\ldots t_n$ or $\sim pt_1\ldots t_n$, where p is a n-ary predicate symbol of L and $t_1, \ldots, t_n$ are arbitrary L-terms, is called a *prime formula*. *Formulas* are built up from prime formulas by means of $\wedge$, $\vee$, $\forall x$, $\exists x$. The *negation* $\neg A$ of a formula A is defined by

$$\neg p\vec{t} :\equiv \sim p\vec{t}, \quad \neg(\sim p\vec{t}) :\equiv p\vec{t}, \quad \neg(A \overset{\wedge}{\vee} B) :\equiv (\neg A) \overset{\vee}{\wedge} (\neg B),$$
$$\neg \overset{\forall}{\exists} x F(x) :\equiv \overset{\exists}{\forall} x \neg F(x).$$

The length $\ell(A)$ of a formula A is defined by

$$\ell(p\vec{t}) :\equiv \ell(\sim p\vec{t}) := 0, \quad \ell(A \wedge B) := \ell(A \vee B) := \max\{\ell(A), \ell(B)\} + 1,$$
$$\ell(\forall x F(x)) := \ell(\exists x F(x)) := \ell(F(x)) + 1.$$

Note that $\ell(\neg A) = \ell(A)$ and $\ell(A(x)) = \ell(A(t))$.

In the whole paper we are working with Tait's sequent calculus (cf. [17]), where *sequents* are finite sets of formulas denoted by $\Gamma, \Delta, \ldots$. The intended meaning of a sequent $\{A_1, \ldots, A_k\}$ is the disjunction $A_1 \vee \ldots \vee A_k$. We use the following *syntactical variables*: A, B, C for formulas, F for 1-place nominal forms (in the sense of Schütte [14]), Γ, Δ for sequents, t for L-terms. If Θ is a term, formula, nominal form or squent then $FV(\Theta)$ denotes the set of all free variables occurring in Θ. Θ is called *closed* iff $FV(\Theta) = \emptyset$.

1.2 Assumptions

(i) We choose a standard Gödel-numbering $\Theta \mapsto \ulcorner\Theta\urcorner$ of all syntactic objects (formulas, sequents, finite derivations, etc.) occurring in the paper. A set $\mathfrak{X}$ of syntactic objects is called primitive recursive if the set $\{\ulcorner\Theta\urcorner;\ \Theta \in \mathfrak{X}\}$ has this property. An analogous agreement is made with respect to functions.

(ii) For each n-ary predicate symbol $p \in L$ we choose some fixed primitive recursive relation $\mathbf{p} \subseteq \mathbb{N}^n$. The set of all closed prime formulas which are true under this interpretation is denoted by $TRUE$, i.e.

$$TRUE := \{pk_1\ldots k_n : p \in L \text{ and } (k_1, \ldots, k_n) \in \mathbf{p}\}$$
$$\cup \{\sim pk_1\ldots k_n : p \in L \text{ and } (k_1 \ldots k_n) \notin \mathbf{p}\}.$$

We assume that $TRUE$ is primitive recursive.

(iii) We assume that L contains some distinguished binary predicate symbol p_0 such that $\mathbf{p}_0$ is a wellordering of its field $\mathscr{A}$. In the following we always write $\prec$ for $\mathbf{p}_0$. Moreover we assume that there are primitive recursive functions $\oplus : \mathbb{N}^2 \to \mathbb{N}$,

$\exp:\mathbb{N}\to\mathbb{N}$ satisfying the following conditions:

($\prec$1) $\exp(0)=1$; and 0, 1 are the first two elements of $(\mathcal{A},\prec)$

($\prec$2) $\forall a,b\in\mathcal{A}(a\oplus b\in\mathcal{A}\ \&\ \exp(a)\in\mathcal{A}\ \&\ a\oplus 0=0\oplus a=0)$

($\prec$3) $\forall a,b,c\in\mathcal{A}(b\prec a\Rightarrow c\oplus b\prec c\oplus a\ \&\ \exp(b)\prec\exp(a))$

($\prec$4) $\forall a,b,c\in\mathcal{A}(b,c\prec\exp(a)\Rightarrow b\oplus c\prec\exp(a))$.

1.3 Abbreviations

$m\nprec n :\Leftrightarrow$ not $m\prec n$

$s\prec t:\equiv p_0 st$

$A\to B:\equiv\neg A\vee B$.

$Prog_{\prec}(F):\equiv\forall x(\forall y(y\prec x\to F(y))\to F(x))$.

$\Gamma,A:=\Gamma\cup A:=\Gamma\cup\{A\}$; $\Gamma\backslash A:=\Gamma\backslash\{A\}$.

$FOR:=$ set of all L-formulas

$SEQ:=$ set of all sequents.

$\bigwedge\!\!\bigwedge\text{-}FOR:=\{A\in FOR$: A is of shape $A_0\wedge A_1$ or $\forall xF(x)\}$.

$\bigvee\!\!\bigvee\text{-}FOR:=\{A\in FOR$: A is of shape $A_0\vee A_1$ or $\exists xF(x)\}=\{\neg B$: $B\in\bigwedge\!\!\bigwedge\text{-}FOR\}$.

For $A\in\bigwedge\!\!\bigwedge\text{-}FOR\cup\bigvee\!\!\bigvee\text{-}FOR$ we set

$$A[n]:=\begin{cases}F(n), & \text{if } A\equiv{}^{\forall}_{\exists}xF(x)\\ A_1, & \text{if } A\equiv A_0{}^{\wedge}_{\vee}A_1 \text{ and } n=1\\ A_0, & \text{if } A\equiv A_0{}^{\wedge}_{\vee}A_1 \text{ and } n\neq 1\end{cases}$$

$\bigvee\!\!\bigvee_0\text{-}FOR:=\bigvee\!\!\bigvee\text{-}FOR\cup\{\neg A$: $A\in TRUE\}$

$\mathbb{N}^{<\omega}:=\{\langle n_0,\dots,n_{k-1}\rangle : k,n_0,\dots,n_{k-1}\in\mathbb{N}\}$;

$\langle m_0,\dots,m_{\ell-1}\rangle*\langle n_0,\dots,n_{k-1}\rangle:=\langle m_0,\dots,m_{\ell-1},n_0,\dots,n_{k-1}\rangle$.

$\langle\,\rangle:=$ empty sequence.

$$\alpha\dot{-}1:=\begin{cases}\alpha-1, & \text{if } 0<\alpha<\omega\\ \alpha, & \text{if } \alpha\in\{0,\omega\}\end{cases}.$$

1.4 The formal system $Z+TI_{\prec\restriction}$

$Z+TI_{\prec\restriction}$ is the extension of classical arithmetic Z (formulated in L) by the axioms of transfinite induction along every proper initial segment of $(\mathcal{A},\prec)$.

The axioms of $Z+TI_{\prec\restriction}$ (in sequent form) are:

(i) Basic axioms:

a) Logical axioms: $\{\neg A,A\}$, for every prime formula A.

b) Arithmetical axioms: Peano-axioms for 0 and $'$, defining axioms for primitive recursive relations (in sequent form). [It is not necessary to display these axioms in detail; we only need to know that they are closed under substitution of L-terms, and that $\Gamma\cap TRUE\neq\emptyset$ for each closed arithmetical axiom Γ.]

(ii) Complete Induction: $\{\neg F(0),\ \neg\forall x(F(x)\to F(x')),\ \forall xF(x)\}$, for each F.

(iii) Transfinite Induction: $\{\neg Prog_{\prec}(F),\ \forall x\prec mF(x)\}$, for each F and each $m\in\mathcal{A}$.

The inference rules of $Z+TI_{\prec\restriction}$ are:

$$(\wedge)\ \frac{\Gamma,A_0\quad \Gamma,A_1}{\Gamma,A_0\wedge A_1},\qquad (\forall)\ \frac{\Gamma,F(v)}{\Gamma,\forall xF(x)}\ \text{ with }\ v\notin FV(\Gamma,F),$$

$$(\vee)\ \frac{\Gamma,A_i}{\Gamma,A_0\vee A_1},\qquad (\exists)\ \frac{\Gamma,F(t)}{\Gamma,\exists xF(x)},\qquad (\text{Cut})\ \frac{\Gamma,A\quad \Gamma,\neg A}{\Gamma}.$$

Usually one defines the derivations of a system like $Z+TI_{\prec\restriction}$ as finite trees of sequents built up from axioms by the inference rules $(\wedge)$, $(\vee)$, $(\forall)$, $(\exists)$, (Cut). But for the purpose of this paper we need a somewhat sharper notion of derivation, where at each node σ of the tree also some special instance $\mathbf{r}$ of the above rules is specified by which the sequent Γ at node σ can be derived from its premises $\ldots\Gamma_n\ldots$. Moreover we require that every free varible v occurring in some premise of an inference $\mathbf{r}$ has to occur also in the conclusion, except $\mathbf{r}$ is an instance of $(\forall)$ and v is its eigenvariable. Further we deviate a little bit from common practice in so far as we consider $\dfrac{\ldots\Gamma_n\ldots(n<\tau)}{\Gamma}$ as shorthand for

"If, for each $n<\tau$, d_n is a derivation with endsequent $End(d_n)\subseteq\Gamma_n$, then

$$d:=\begin{cases}\ldots d_n\ldots\\ \quad\backslash|/ \\ \quad\Gamma\end{cases}\text{ is a derivation with } End(d)=\Gamma."$$

while usually one requires $End(d_n)=\Gamma_n$.

Now we are going to give the "official" definition of the set $Z_{\prec\restriction}$ of all $(Z+TI_{\prec\restriction})$-derivations. The elements of $Z_\prec$ will be finite trees of pairs $(\mathbf{r},\Gamma)$ represented in linear notation, i.e. every $d\in Z_\prec$ is of the form $\langle\mathbf{r},\Gamma,(d_i)_{i<\tau}\rangle$ with $End(d):=\Gamma\in SEQ$, $\tau\leqq 2$ and $d_i\in Z_\prec$. When writing down the defining rules for $Z_\prec$ we use the following abbreviation:

$$d\Vdash\Gamma :\Leftrightarrow d\in Z_\prec \;\&\; End(d)\subseteq\Gamma\,.$$

1.4.1 Inductive definition of the set $Z_\prec$

(Z.1) For every basic Axiom Γ: $\langle\mathsf{Ax},\Gamma,\emptyset\rangle\in Z_\prec$

(Z.2) For each nominal form F: $\langle(\mathsf{Ind},F),\ \{\neg F(0),\ \neg\forall x(F(x)\to F(x')),\ \forall xF(x)\},\emptyset\rangle\in Z_\prec$

(Z.3) For each nominal form F and $m\in\mathscr{A}$: $\langle(\mathsf{TI},F,m),\ \{\neg Prog_\prec(F),\ \forall x\prec mF(x)\},\emptyset\rangle\in Z_\prec$

(Z.4) $A_0\wedge A_1\in\Gamma \;\&\; d_i\Vdash\Gamma\cup A_i(i=0,1)\Rightarrow\langle(\bigwedge\!\!\!\bigwedge, A_0\wedge A_1),\Gamma,(d_0,d_1)\rangle\in Z_\prec$

(Z.5) $A_0\vee A_1\in\Gamma \;\&\; d_0\Vdash\Gamma\cup A_k(k=0\text{ or }1)\Rightarrow\langle(\bigvee\!\!\!\bigvee_k, A_0\vee A_1),\Gamma,d_0\rangle\in Z_\prec$

(Z.6) $\forall xF(x)\in\Gamma \;\&\; d_0\Vdash\Gamma\cup F(v)$ with $v\notin FV(\Gamma)\Rightarrow\langle(v,\forall xF(x)),\Gamma,d_0\rangle\in Z_\prec$

(Z.7) $\exists xF(x)\in\Gamma \;\&\; d_0\Vdash\Gamma\cup F(t)$ with $FV(t)\subseteq FV(\Gamma)\Rightarrow\langle(\bigvee\!\!\!\bigvee_t,\exists xF(x)),\Gamma,d_0\rangle\in Z_\prec$

(Z.8) $d_0\Vdash\Gamma\cup A \;\&\; d_1\Vdash\Gamma\cup\neg A$ with $FV(A)\subseteq FV(\Gamma)\Rightarrow\langle(\mathsf{Cut},A),\Gamma,(d_0,d_1)\rangle\in Z_\prec$

1.4.2 Definition of the cut-rank $deg(d)\in\mathbb{N}$ for $d\in Z_\prec$. Let $d=\langle\mathbf{r},\Gamma,(d_i)_{i<\tau}\rangle\in Z_\prec$.

$$deg(d):=\max(\{deg(d_i):i<\tau\}\cup\{\ell_C(\mathbf{r})\}),$$

where

$$\ell_C(\mathbf{r}):=\begin{cases}\ell(A)+1, & \text{if }\ \mathbf{r}=(\mathsf{Cut},A)\\ 0, & \text{otherwise}\end{cases}.$$

2 The system Z^∞ and the cut-elimination operator $\mathscr{E}$

Combining work of Schütte [13], Tait [17], and Minc [10] we shall now describe the system Z^∞ of wellfounded infinitary derivations (of ω-arithmetic), and the operator $\mathscr{E}$ transforming each Z^∞-derivation with finite cut-rank m into a derivation of the same sequent but with cut-rank $\leqq m\dot{-}1$. Informally Z^∞ is defined as the set of all wellfounded trees (*derivations*) generated from initial sequents Γ

with $\Gamma \cap TRUE \neq \emptyset$ by the rules ($\wedge$), ($\vee$), (Cut), ($\exists$) [with witness $t \in \mathbb{N}$] and

$$(\forall)^\infty \ \frac{\ldots\Gamma, F(n)\ldots(n\in\mathbb{N})}{\Gamma, \forall x F(x)} \ (\omega\text{-rule}), \qquad (\text{Rep}) \ \frac{\Gamma}{\Gamma} \ (\text{repetition-rule}).$$

But as in 1.4 we will consider a modified notion of derivation, where to each node is assigned a *pair* $(\mathbf{r}, \Gamma)$ with $\Gamma \in SEQ$ and $\mathbf{r}$ an expression indicating some special instance of the above inference rules. The set of all these expressions will be called $RULE$. Then the inference rules can be written as *local correctness conditions* $(LC.1), \ldots, (LC.5)$ between a pair $(\mathbf{r}, \Gamma) \in RULE \times SEQ$ and a family $(\Gamma_n)_{n\in I}$ of sequents (the premises of Γ); and Z^∞ is defined as the set of all wellfounded ($RULE \times SEQ$)-trees which are locally correct with respect to $(LC1), \ldots, (LC.5)$.

2.1 Definition

$RULE := \{\mathsf{Ax}\} \cup \{(\bigwedge, A): A \in \bigwedge\text{-}FOR\} \cup \{(\bigvee_k, A): k \in \mathbb{N}, A \in \bigvee\text{-}FOR\}$
$\cup \{\mathsf{Rep}_n : n \in \mathbb{N}\} \cup \{(\mathsf{Cut}, A): A \in FOR\}$.

For $\mathbf{r} \in RULE$ we use the following abbreviations
$(\mathbf{r}, \Gamma) \in \mathsf{Ax} :\Leftrightarrow \mathbf{r} = \mathsf{Ax}$,
$\mathbf{r} \in \mathsf{Cut} :\Leftrightarrow \exists A[\mathbf{r} = (\mathsf{Cut}, A)]$, $\mathbf{r} \in \mathsf{Rep} :\Leftrightarrow \exists n[\mathbf{r} = \mathsf{Rep}_n]$,

$$\ell_C(\mathbf{r}) := \begin{cases} \ell_C(A)+1, & \text{if } \mathbf{r} = (\mathsf{Cut}, A) \\ 0, & \text{otherwise} \end{cases}$$

2.2 Definition. $LC((\mathbf{r}, \Gamma), (\Gamma_n)_{n\in\mathbb{N}})$ abbreviates the conjunction of $(LC.1), \ldots, (LC.5)$ below:

$(LC.1)$ $\mathbf{r} = \mathsf{Ax} \Rightarrow \Gamma \cap TRUE \neq \emptyset$
$(LC.2)$ $\mathbf{r} = (\bigwedge, A) \Rightarrow A \in \Gamma \ \& \ \forall n(\Gamma_n \subseteq \Gamma \cup A[n])$
$(LC.3)$ $\mathbf{r} = (\bigvee_k, A) \Rightarrow A \in \Gamma \ \& \ \Gamma_0 \subseteq \Gamma \cup A[k]$
$(LC.4)$ $\mathbf{r} = (\mathsf{Cut}, A) \Rightarrow \Gamma_0 \subseteq \Gamma \cup A \ \& \ \Gamma_1 \subseteq \Gamma \cup \neg A$
$(LC.5)$ $\mathbf{r} = \mathsf{Rep}_n \Rightarrow \Gamma_n \subseteq \Gamma$.

2.3 Definitions

a) Let $TREE$ be the set of all functions $\varphi: \mathbb{N}^{<\omega} \to RULE \times SEQ$ satisfying the condition

$$\forall \sigma \in \mathbb{N}^{<\omega} \forall n \in \mathbb{N}[\varphi(\sigma) \in \mathsf{Ax} \Rightarrow \varphi(\sigma * \langle n \rangle) = \varphi(\sigma)].$$

b) $\mathbf{0}_{(\mathbf{r},\Gamma)}$ denotes the constant tree φ with $\varphi(\sigma) = (\mathbf{r}, \Gamma)$ for all $\sigma \in \mathbb{N}^{<\omega}$.

c) For $\varphi \in TREE$ we define:

(i) $\varphi^0(\sigma) := \mathbf{r}$, $\varphi^1(\sigma) := \Gamma$, with $(\mathbf{r}, \Gamma) := \varphi(\sigma)$.
(ii) $Rule(\varphi) := \varphi^0(\langle\,\rangle)$, $End(\varphi) := \varphi^1(\langle\,\rangle)$.
(iii) $\varphi[n] := \lambda\sigma . \varphi(\langle n \rangle * \sigma) \in TREE$ (the n-th immediate subtree of φ).
$\varphi[\![\tau]\!] := \lambda\sigma . \varphi(\tau * \sigma) \in TREE$ (the subtree of φ determined by τ).
(iv) φ is *wellfounded*$:\Leftrightarrow \forall (n_i)_{i\in\mathbb{N}} \exists k(\varphi(\langle n_0, \ldots, n_{k-1} \rangle) \in \mathsf{Ax})$.
(v) φ is *locally correct*$:\Leftrightarrow \forall \sigma \in \mathbb{N}^{<\omega} LC(\varphi(\sigma), (\varphi^1(\sigma * \langle n \rangle))_{n\in\mathbb{N}})$.

Notation. We use φ as syntactical variable for elements of $TREE$.

2.4 Remark

a) $\varphi \in TREE \ \& \ \varphi(\langle\,\rangle) = (\mathsf{Ax}, \Gamma) \Rightarrow \varphi = \mathbf{0}_{(\mathsf{Ax},\Gamma)}$.
b) $\varphi \in TREE \Rightarrow [\forall n(\varphi[n] = \varphi) \Leftrightarrow \varphi = \mathbf{0}_{\varphi(\langle\,\rangle)}]$.

2.5 Definition

$WT := \{\varphi \in TREE: \varphi \text{ wellfounded}\}$
$Z^\infty := \{\varphi \in WT: \varphi \text{ locally correct}\}$.

2.6 Proposition (Inductive definition of WT and Z^∞)

a) *WT is the least subset of TREE satisfying*

(WT.0) $\mathbf{0}_{(\mathsf{Ax},\Gamma)} \in WT$, *for each* $\Gamma \in SEQ$.

(WT.1) $Rule(\varphi) \neq \mathsf{Ax}\ \&\ \forall n \in \mathbb{N}(\varphi[n] \in WT) \Rightarrow \varphi \in WT$.

b) *Z^∞ is the least subset of TREE satisfying*

(Z^∞.0) $\mathbf{0}_{(\mathsf{Ax},\Gamma)} \in Z^\infty$, *for each* Γ *with* $\Gamma \cap TRUE \neq \emptyset$,

(Z^∞.1) $Rule(\varphi) \neq \mathsf{Ax}\ \&\ \forall n \in \mathbb{N}(\varphi[n] \in Z^\infty)\ \&\ LC(\varphi(\langle\rangle), (End(\varphi[n]))_{n\in\mathbb{N}})$
$\Rightarrow \varphi \in Z^\infty$.

2.7 Definition of $deg(\varphi) \leqq \omega$ for $\varphi \in TREE$
$deg(\varphi) := \sup\{\ell_C(\varphi^0(\sigma)) : \sigma \in \mathbb{N}^{<\omega}\}$ $(= \sup\{\ell(A)+1 : \exists \sigma \in \mathbb{N}^{<\omega}(\varphi^0(\sigma) = (\mathsf{Cut}, A))\})$.

2.8 Definition of $\|\varphi\| \in On$ for $\varphi \in WT$

$$\|\varphi\| := \begin{cases} 0, & \text{if } Rule(\varphi) = \mathsf{Ax} \\ \sup\{\|\varphi[n]\|+1 : n \in \mathbb{N}\}, & \text{otherwise} \end{cases}.$$

Now we are going to define the cut-elimination operator $\mathscr{E}: WT \to WT$ which transforms every $\varphi \in Z^\infty$ into a derivation $\mathscr{E}(\varphi) \in Z^\infty$ with $End(\mathscr{E}(\varphi)) = End(\varphi)$ and $deg(\mathscr{E}(\varphi)) \leqq deg(\varphi) \dot{-} 1$. For the definition of $\mathscr{E}$ we need two families of auxiliary operators, namely the inversion operators $\mathscr{I}_{k,A}$ ($k \in \mathbb{N}$, $A \in \bigwedge\!\!\!\bigwedge$-$FOR$) and the reduction operators $\mathscr{R}_C$($C \in \bigvee\!\!\!\bigvee_0$-$FOR$). The operator $\mathscr{I}_{k,A}$ replaces in φ every sequent Γ by $(\Gamma \setminus A) \cup A[k]$, and every occurrence of $(\bigwedge\!\!\!\bigwedge, A)$ by Rep_k. The effect of $\mathscr{R}_C$ is to reduce a cut with cutformula C to cuts with cutformulas $C[k]$; namely if ψ, $\varphi \in Z^\infty$ are derivations with $End(\psi) \subseteq \Gamma \cup \neg C$, $End(\varphi) \subseteq \Gamma \cup C$ and $\max\{deg(\psi), deg(\varphi)\} \leqq \ell(C)$ then $\tilde{\varphi} := \mathscr{R}_C(\psi, \varphi) \in Z^\infty$ is a derivation with $End(\tilde{\varphi}) \subseteq \Gamma$ and $deg(\tilde{\varphi}) \leqq \ell(C)$. – The definitions of $\mathscr{I}_{k,A}(\varphi)$, $\mathscr{R}_C(\psi, \varphi)$, $\mathscr{E}(\varphi)$ proceed by transfinite recursion on $\|\varphi\|$ using the fact that every $\varphi \in TREE$ is uniquely determined by the data $End(\varphi)$, $Rule(\varphi)$, $(\varphi[n])_{n\in\mathbb{N}}$.

2.9 Definition of the inversion operator $\mathscr{I}_{k,A}$: $WT \to WT$ for $A \in \bigwedge\!\!\!\bigwedge$-$FOR$, $k \in \mathbb{N}$

(i) $End(\mathscr{I}_{k,A}(\varphi)) := (End(\varphi) \setminus A) \cup A[k]$

(ii) $Rule(\mathscr{I}_{k,A}(\varphi)) := \begin{cases} \mathsf{Rep}_k, & \text{if } Rule(\varphi) = (\bigwedge\!\!\!\bigwedge, A) \\ Rule(\varphi), & \text{otherwise} \end{cases}$

(iii) $\mathscr{I}_{k,A}(\varphi)[n] := \mathscr{I}_{k,A}(\varphi[n])$.

Explanation. Let $\mathscr{I}$ abbreviate $\mathscr{I}_{k,A}$. If $Rule(\varphi) \neq \mathsf{Ax}$ then in defining $\mathscr{I}(\varphi)[n]$ we can refer to the previously defined $\mathscr{I}(\varphi[n])$, which is not possible when $Rule(\varphi) = \mathsf{Ax}$. In that case we have $\varphi = \mathbf{0}_{(\mathsf{Ax},\Gamma)}$, and (iii) becomes $\mathscr{I}(\varphi)[n] := \mathscr{I}(\varphi)$ which is circular. But according to 2.4b) we consider this as an abbreviation for $\mathscr{I}(\varphi) := \mathbf{0}_{(\mathbf{r},\Gamma')}$ with $\Gamma' := End(\mathscr{I}(\varphi))$, $\mathbf{r} := Rule(\mathscr{I}(\varphi))$ as defined under (i) and (ii). Moreover in that case by (ii) we have $\mathbf{r} = \mathsf{Ax}$ and thus $\mathscr{I}(\varphi) = \mathbf{0}_{(\mathsf{Ax},\Gamma')} \in WT$. An analogues remark applies to the definitions of $\mathscr{R}_C$ and $\mathscr{E}$ below.

2.10 Definition of $\mathscr{R}_C$: $WT \times WT \to WT$ for $C \in \bigvee\!\!\!\bigvee_0$-$FOR$
$End(\mathscr{R}_C(\psi, \varphi)) := (End(\psi) \setminus \neg C) \cup (End(\varphi) \setminus C)$

$$Rule(\mathscr{R}_C(\psi, \varphi)) := \begin{cases} (\mathsf{Cut}, C[k]), & \text{if } Rule(\varphi) = (\bigvee\!\!\!\bigvee_k, C) \\ Rule(\varphi), & \text{otherwise} \end{cases}$$

$$\mathscr{R}_C(\psi, \varphi)[n] := \begin{cases} \mathscr{I}_{k,\neg C}(\psi), & \text{if } Rule(\varphi) = (\bigvee\!\!\!\bigvee_k, C) \text{ and } n = 1 \\ \mathscr{R}_C(\psi, \varphi[n]), & \text{otherwise} \end{cases}$$

2.11 Definition of $\mathscr{E}$: $WT \to WT$

$End(\mathscr{E}(\varphi)) := End(\varphi)$.

$$Rule(\mathscr{E}(\varphi)) := \begin{cases} \mathsf{Rep}_0, & \text{if } Rule(\varphi) \in \mathsf{Cut} \\ Rule(\varphi), & \text{otherwise} \end{cases}$$

$$\mathscr{E}(\varphi)[n] := \begin{cases} \mathscr{R}_C(\mathscr{E}(\varphi[1]), \mathscr{E}(\varphi[0])), \\ \quad \text{if } Rule(\varphi) = (\mathsf{Cut}, C) \text{ with } C \in \mathbb{W}_0\text{-}FOR \\ \mathscr{R}_C(\mathscr{E}(\varphi[0]), \mathscr{E}(\varphi[1])), \\ \quad \text{if } Rule(\varphi) = (\mathsf{Cut}, \neg C) \text{ with } C \in \mathbb{W}_0\text{-}FOR \\ \mathscr{E}(\varphi[n]), \quad \text{otherwise} \end{cases}$$

2.12 Proposition. *For all* $\psi, \varphi \in Z^\infty$, $A \in \mathbb{M}\text{-}FOR$, $k \in \mathbb{N}$, $C \in \mathbb{W}_0\text{-}FOR$ *we have:*

a) $\mathscr{I}_{k,A}(\varphi) \in Z^\infty$, $deg(\mathscr{I}_{k,A}(\varphi)) \leqq deg(\varphi)$, $\|\mathscr{I}_{k,A}(\varphi)\| \leqq \|\varphi\|$

b) $\mathscr{R}_C(\psi, \varphi) \in Z^\infty$, $deg(\mathscr{R}_C(\psi, \varphi)) \leqq \max\{\ell(C), deg(\psi), deg(\varphi)\}$, $\|\mathscr{R}_C(\psi, \varphi)\| \leqq \|\psi\| + \|\varphi\|$

c) $\mathscr{E}(\varphi) \in Z^\infty$, $deg(\mathscr{E}(\varphi)) \leqq deg(\varphi) \dot{-} 1$, $\|\mathscr{E}(\varphi)\| \leqq \omega^{\|\varphi\|}$.

Proof by induction on $\|\varphi\|$.

2.13 Definition *of* $Z^0_\prec$. $Z^0_\prec := \{d \in Z_\prec : End(d) \text{ closed}\}$.

2.14 Proposition. *There is a canonical embedding* $Z^0_\prec \to Z^\infty$, $d \mapsto d^\infty$ *such that* $End(d^\infty) = End(d)$ *and* $deg(d^\infty) = deg(d)$. d^∞ *is obtained from d essentially by replacing in d every axiom of complete or transfinite induction by its (cutfree) ω-derivation.*

3 The notation system $Z^*_\prec$

Our goal is to show that every provably recursive function of $Z + TI_{\prec\restriction}$ can be represented as $\mathfrak{f}(n) = \mathfrak{g}(n, \min\{k : \mathfrak{o}(n, k+1) \nprec \mathfrak{o}(n,k)\})$ with prim. rec. function $\mathfrak{g}$, $\mathfrak{o}$. Of course it suffices to show that for each $p \in L$ with $Z + TI_{\prec\restriction} \vdash \forall x \exists y pxy$ the Skolem-function $\mathfrak{f}_p(n) := \min\{m : \mathbf{p}(n,m)\}$ has such a representation. We want to do this by methods formalizable in $PRA + PRWO(\prec)$, i.e. primitive recursive arithmetic together with the axiom that there are no primitive recursive infinite descending sequences in $(\mathscr{A}, \prec)$. This will be achieved as follows: we introduce a primitive recursive system $Z^*_\prec$ of (finite) notations for all those $\varphi \in Z^\infty$ which can be obtained via cut-elimination from derivations d^∞ ($d \in Z^0_\prec$). Then we define primitive recursive functions $\mathfrak{e} : Z^*_\prec \to SEQ$, $\mathfrak{r} : Z^*_\prec \to RULE$, $\mathfrak{s} : \mathbb{N} \times Z^*_\prec \to Z^*_\prec$, $\mathfrak{d} : Z^*_\prec \to \mathbb{N}$ such that, for all $h \in Z^*_\prec$, $n \in \mathbb{N}$, $\mathfrak{e}(h) = End(\mathfrak{v}(h))$, $\mathfrak{r}(h) = Rule(\mathfrak{v}(h))$, $\mathfrak{v}(\mathfrak{s}(n,h)) = \mathfrak{v}(h)[n]$, $deg(\mathfrak{v}(h)) \leqq \mathfrak{d}(h)$, where $\mathfrak{v}(h)$ is the *value* of h, i.e. the Z^∞-derivation denoted by h.

Notation. $\mathfrak{s}_n h :\equiv \mathfrak{s}_n(h) :\equiv \mathfrak{s}(n,h)$.

We also define a prim. rec. function $\mathfrak{o} : Z^*_\prec \to \mathscr{A}$ such that $[\mathfrak{r}(h) \neq \mathsf{Ax} \Rightarrow \mathfrak{o}(\mathfrak{s}_n h) \prec \mathfrak{o}(h)]$, for all $h \in Z^*_\prec$, $n \in \mathbb{N}$. (Thus $\mathfrak{o}(h)$ corresponds to $\|\mathfrak{v}(h)\|$).

Now assume that $Z + TI_{\prec\restriction} \vdash A$, where $A \equiv \forall x \exists y pxy$. Then there exists some $d \in Z^0_\prec$ with $End(d) \subseteqq \{A\}$ and $v := deg(d) < \omega$.

Let

$$\varphi_{n,0} := \mathscr{I}_{n,A}\mathscr{E}^{(v)}(d^\infty),$$

$$\varphi_{n,k+1} := \begin{cases} \varphi_{n,k}[i], & \text{if } Rule(\varphi_{n,k}) = \mathsf{Rep}_i, \\ \varphi_{n,k}[0], & \text{otherwise}, \end{cases} \qquad \Gamma_{n,k} := End(\varphi_{n,k}).$$

Then by induction on k we get: $\Gamma_{n,k} \subseteq \{\exists y p n y, pn0, pn1, \ldots\}$, and thus

$$Rule(\varphi_{n,k}) = \mathsf{Ax} \Rightarrow \exists m \leq \mathsf{u}(\Gamma_{n,k})\mathfrak{p}(n,m), \tag{1}$$

with

$$\mathsf{u}(\Gamma) := \sup\{m\colon m \text{ occurs in } \Gamma\}.$$

Using $\mathfrak{r}$, $\mathfrak{s}$ we can define a prim. rec. function $\mathsf{s}: \mathbb{N}^2 \to Z^*_\prec$ such that $\mathfrak{v}(\mathsf{s}(n,k)) = \varphi_{n,k}$.

Then $\mathsf{s}(n,k+1) \in \{\mathfrak{s}_i \mathsf{s}(n,k)\colon i \in \mathbb{N}\}$ and therefore

$$\mathfrak{r}(\mathsf{s}(n,k)) \neq \mathsf{Ax} \Rightarrow \mathsf{o}(n,k+1) \prec \mathsf{o}(n,k), \tag{2}$$

with

$$\mathsf{o}(n,k) := \mathfrak{o}(\mathsf{s}(n,k)).$$

Using $\mathfrak{r}(\mathsf{s}(n,k)) = Rule(\varphi_{n,k})$, $\mathfrak{e}(\mathsf{s}(n,k)) = \Gamma_{n,k}$ and (1), (2) we obtain

$$\mathsf{o}(n,k+1) \not\prec \mathsf{o}(n,k) \Rightarrow \mathsf{f}_p(n) = \mathsf{g}(n,k), \tag{3}$$

with

$$\mathsf{g}(n,k) := \min\{m \leq \mathsf{u}(\mathfrak{e}(\mathsf{s}(n,k)))\colon \mathfrak{p}(n,m)\}.$$

Since by $PRWO(\prec)$ we have $\forall n \exists k(\mathsf{o}(n,k+1) \not\prec \mathsf{o}(n,k))$, this yields

$$\mathsf{f}_p(n) = \mathsf{g}(n, \min\{k: \mathsf{o}(n,k+1) \not\prec \mathsf{o}(n,k)\}).$$

Remark. The above argument cannot directly be formalized in PRA, since there we cannot define the interpretation $\mathfrak{v}: Z^*_\prec \to Z^\infty$. But as we will see below this is really not necessary. It will suffice to prove in PRA the Π^0_1-statement

$$\forall h \in Z^*_\prec [LC((\mathfrak{r}(h), \mathfrak{e}(h)), (\mathfrak{e}(\mathfrak{s}_n h))_{n\in\mathbb{N}}) \,\&\, \ell_C(\mathfrak{r}(h)) \leq \mathfrak{d}(h) \,\&\, \forall n(\mathfrak{d}(\mathfrak{s}_n(h) \leq \mathfrak{d}(h)) \,\&\, [\mathfrak{r}(h) \neq \mathsf{Ax} \Rightarrow \forall n(\mathfrak{o}(\mathfrak{s}_n h) \prec \mathfrak{o}(h))]].$$

3.1 Definition of the set $Z^*_\prec$. Since we want that $Z^*_\prec$ contains notations just for all $\varphi \in Z^\infty$ obtainable from derivations d^∞ ($d \in Z^0_\prec$) by applications of the operators $\mathscr{I}_{k,A}$, $\mathscr{R}_C$, $\mathscr{E}$, we now introduce function symbols $I_{k,A}$, R_C, E ($k \in \mathbb{N}$, $A \in \bigwedge$-FOR, $C \in \bigvee_0$-FOR) and define

$$Z^*_\prec := \begin{cases} \text{the set of all (finite) terms } h \text{ generated from} \\ \text{“constants” } d \in Z^0_\prec \textit{ by the function symbols } I_{k,A}, R_C, E. \end{cases}$$

3.2 Definition of the value $\mathfrak{v}(h) \in Z^\infty$ for each $h \in Z^*_\prec$

(i) $\mathfrak{v}(h) := h^\infty$, if $h \in Z^0_\prec$;

(ii) $\mathfrak{v}(I_{k,A}h) := \mathscr{I}_{k,A}(\mathfrak{v}(h))$; $\mathfrak{v}(R_C(h_0 h_1) := \mathscr{R}_C(\mathfrak{v}(h_0), \mathfrak{v}(h_1))$; $\mathfrak{v}(Eh) := \mathscr{E}(\mathfrak{v}(h))$.

3.3 Definition of the length $\ell^*(h) \in \mathbb{N}$ for each $h \in Z^*_\prec$

$$\ell^*(h) := 0, \quad \text{if} \quad h \in Z^0_\prec; \qquad \ell^*(Eh) := \ell^*(I_{k,A}h) := \ell^*(h) + 1,$$
$$\ell^*(R_C h_0 h_1) := \max\{\ell^*(h_0), \ell^*(h_1)\} + 1.$$

Now we are going to define the above mentioned functions

$$\mathfrak{e}: Z^*_\prec \to SEQ, \quad \mathfrak{r}: Z^*_\prec \to RULE, \quad \mathfrak{s}: \mathbb{N} \times Z^*_\prec \to Z^*_\prec,$$
$$\mathfrak{d}: Z^*_\prec \to \mathbb{N}, \quad \mathfrak{o}: Z^*_\prec \to \mathscr{A}.$$

The definition proceeds by (primitive) recursion on $\ell^*(h)$. First we define $\mathfrak{e}(d)$, $\mathfrak{r}(d), \ldots$ for all $d \in Z^0_{\prec}$, and then we define $\mathfrak{e}(I_{k,A}h_1)$, $\mathfrak{e}(R_C h_0 h_1)$, $\mathfrak{e}(Eh_1)$, $\mathfrak{r}(I_{k,A}h_1), \ldots$ under the induction hypothesis that $\mathfrak{e}(h_i)$, $\mathfrak{r}(h_i), \ldots$ are already defined. The defining equations for $\mathfrak{e}$, $\mathfrak{r}$, $\mathfrak{s}$ in the induction step are obtained by a simple rewriting from the corresponding equations for $\mathscr{I}_{k,A}$, $\mathscr{R}_C$, $\mathscr{E}$ on pp. 283, 284. But of course both definitions are fundamentally different with respect to the underlying recursion principle.

3.4 Definition of $\mathfrak{e}(d)$, $\mathfrak{r}(d)$, $\mathfrak{s}_n(d)$ for $d \in Z_{\prec}$. Let $d = \langle \mathbf{r}, \Gamma, (d_i)_{i<\tau} \rangle \in Z_{\prec}$.

$\mathfrak{e}(d) := \Gamma = End(d)$

$$\mathfrak{r}(d) := \begin{cases} (\bigwedge, \forall x F(x)), & \text{if } \mathbf{r} = (v, \forall x F(x)) \text{ or } (\mathsf{Ind}, F) \\ (\bigwedge, \forall x \prec m F(x)), & \text{if } \mathbf{r} = (\mathsf{TI}, F, m) \\ \mathbf{r}, & \text{otherwise} \end{cases}$$

$$\mathfrak{s}_n(d) := \begin{cases} d, & \text{if } \mathbf{r} = \mathsf{Ax} \\ c^F_n, & \text{if } \mathbf{r} = (\mathsf{Ind}, F) \\ c^F_{m,n}, & \text{if } \mathbf{r} = (\mathsf{TI}, F, m) \\ d_0[v/n], & \text{if } \mathbf{r} = (v, \forall x F(x)) \\ d_n, & \text{if } \tau = 2 \text{ and } n < 2 \\ d_0, & \text{otherwise} \end{cases}$$

c^F_n and $c^F_{m,n}$ are defined below. For each $n \in \mathbb{N}$, $d_0[v/n]$ denotes the result of substituting n for every occurrence of v in d_0 which is "linked" to the root of d_0.

Definition of $\kappa[A]$. For $A \in FOR$ let $\kappa[A] \in Z_{\prec}$ be the canonical cutfree derivation of $\{\neg A, A\}$.

Definition of c^F_n. Let $G :\equiv \neg \forall x(F(x) \to F(x')) \equiv \exists x(F(x) \wedge \neg F(x'))$.

$$c^F_0 := \kappa[F(0)]$$

$$c^F_{n+1} := \langle (\bigvee_n, G), \Gamma, \langle (\bigwedge, F(n) \wedge \neg F(n')), \Gamma_n, (c^F_n, \kappa[F(n')]) \rangle \rangle$$

with

$$\Gamma := \{\neg F(0), G, F(n')\}, \qquad \Gamma_n := \{\neg F(0), F, F(n) \wedge \neg F(n'), F(n')\}.$$

To improve readability we repeat the definition of c^F_{n+1} in familiar notation:

$$c^F_{n+1} := \left\{ \dfrac{\dfrac{\dfrac{c^F_n}{\neg F(0), G, F(n)} \quad \dfrac{\kappa[F(n')]}{\neg F(n'), F(n')}}{\neg F(0), G, F(n) \wedge \neg F(n'), F(n')} (\wedge)}{\neg F(0), G, F(n')} (\exists) \right.$$

Definition of $c^F_{m,n}$

(i) $n \not\prec m$: $c^F_{m,n} := \langle (\bigvee_0, n \prec m \to F(n)), \{n \prec m \to F(n)\}, \langle \mathsf{Ax}, \{\neg n \prec m\}, \emptyset \rangle \rangle$

(ii) $n \prec m$: Let $G :\equiv \neg Prog_{\prec}(F)$, i.e. $G \equiv \exists x(\forall y \prec x F(y) \wedge \neg F(x))$.

$$c^F_{m,n} := \langle (\bigvee_1, \neg n \prec m \vee F(n)), \Delta, \langle (\bigvee_n, G), \Delta', \langle (\bigwedge, \forall y \prec n F(y) \wedge \neg F(n)), \Delta'', (e, \kappa[F(n)]) \rangle \rangle \rangle$$

with

$$e := \langle (\mathsf{TI}, F, n), \{G, \forall y \prec n F(y)\}, \emptyset \rangle$$

and

$$\Delta := \{G, n \prec m \to F(n)\}, \qquad \Delta' := \{G, F(n)\},$$
$$\Delta'' := \{G, \forall y \prec n F(y) \wedge \neg F(n), F(n)\}.$$

In familiar notation:

$$c^F_{m,n} := \begin{cases} \dfrac{\dfrac{\dfrac{G, \forall y \prec n F(y) \quad \overset{\kappa[F(n)]}{\neg F(n), F(n)}}{G, \forall y \prec n F(y) \wedge \neg F(n), F(n)}\,(\wedge)}{G, F(n)}\,(\exists)}{G, n \prec m \to F(n)}\,(\vee) \end{cases}$$

3.5 Lemma. *$d \in Z^0_\prec \Rightarrow \mathfrak{r}(d) \in RULE$ and $\mathfrak{s}_n d \in Z^0_\prec$.*

Proof. Straightforward, using the fact that for $d = \langle \mathbf{r}, \Gamma, (d_i)_{i<\tau} \rangle \in Z^0_\prec$ one has $FV(End(d_i)) \subseteqq \begin{cases} \{v\}, & \text{if } \mathbf{r} = (v, \forall x F(x)) \\ \emptyset, & \text{otherwise} \end{cases}$ and $[\mathbf{r} = (\bigvee_t, \exists x F(x)) \Rightarrow t \in \mathbb{N}]$.

3.6 Definition of $\mathfrak{d}(d) \in \mathbb{N}$ and $\mathfrak{o}(d) \in \mathscr{A}$ for $d \in Z_\prec$. Let $d = \langle \mathbf{r}, \Gamma, (d_i)_{i<\tau} \rangle$.

$$\mathfrak{d}(d) := deg(d),$$

$$\mathfrak{o}(d) := \begin{cases} \mathsf{exp}(1), & \text{if } \mathbf{r} = (\mathsf{Ind}, F) \\ \prec -\max\{\mathsf{exp}(m), 1 \oplus 1\}, & \text{if } \mathbf{r} = (\mathsf{TI}, F, m) \\ \prec -\sup\{\mathfrak{o}(d_i) \oplus 1 : i < \tau\}, & \text{otherwise}. \end{cases}$$

3.7 Definition of $\mathfrak{e}(h)$, $\mathfrak{r}(h)$, $\mathfrak{s}_n(h)$, $\mathfrak{d}(h)$, $\mathfrak{o}(h)$ for $h \in Z^*_\prec \backslash Z^0_\prec$

$\mathfrak{e}(I_{k,A}h) := (\mathfrak{e}(h) \backslash A) \cup A[k]$, $\mathfrak{e}(R_C h_0 h_1) := (\mathfrak{e}(h_0) \backslash \neg C) \cup (\mathfrak{e}(h_1) \backslash C)$, $\mathfrak{e}(Eh) := \mathfrak{e}(h)$.

$$\mathfrak{r}(I_{k,A}h) := \begin{cases} \mathsf{Rep}_k, & \text{if } \mathfrak{r}(h) = (\bigwedge, A) \\ \mathfrak{r}(h), & \text{otherwise}. \end{cases}$$

$$\mathfrak{r}(R_C h_0 h_1) := \begin{cases} (\mathsf{Cut}, C[k]), & \text{if } \mathfrak{r}(h_1) = (\bigvee_k, C) \\ \mathfrak{r}(h_1) & \text{otherwise} \end{cases}$$

$$\mathfrak{r}(Eh) := \begin{cases} \mathsf{Rep}_0, & \text{if } \mathfrak{r}(h) \in \mathsf{Cut} \\ \mathfrak{r}(h), & \text{otherwise} \end{cases}$$

$$\mathfrak{s}_n(I_{k,A}h) := I_{k,A}\mathfrak{s}_n(h)$$

$$\mathfrak{s}_n(R_C h_0 h_1) := \begin{cases} I_{k,\neg C} h_0, & \text{if } \mathfrak{r}(h_1) = (\bigvee_k, C) \text{ and } n = 1 \\ R_C h_0 \mathfrak{s}_n(h_1), & \text{otherwise} \end{cases}$$

$$\mathfrak{s}_n(Eh) := \begin{cases} R_C E\mathfrak{s}_1(h) E\mathfrak{s}_0(h), & \text{if } \mathfrak{r}(h) = (\mathsf{Cut}, C), C \in \bigvee_0\text{-}FOR \\ R_C E\mathfrak{s}_0(h) E\mathfrak{s}_1(h), & \text{if } \mathfrak{r}(h) = (\mathsf{Cut}, \neg C), C \in \bigvee_0\text{-}FOR \\ E\mathfrak{s}_n(h), & \text{otherwise}. \end{cases}$$

$\mathfrak{d}(I_{k,A}h) := \mathfrak{d}(h)$, $\mathfrak{d}(R_C h_0 h_1) := \max\{\ell(C), \mathfrak{d}(h_0), \mathfrak{d}(h_1)\}$, $\mathfrak{d}(Eh) := \mathfrak{d}(h) \dot{-} 1$,

$\mathfrak{o}(I_{k,A}h) := \mathfrak{o}(h)$, $\mathfrak{o}(R_C h_0 h_1) := \mathfrak{o}(h_0) \oplus \mathfrak{o}(h_1)$, $\mathfrak{o}(Eh) := \mathsf{exp}(\mathfrak{o}(h))$.

3.8 Theorem. *For all $h \in Z^*_\prec$ the following holds:*

a) $\mathfrak{e}(h) \in SEQ \;\&\; \mathfrak{r}(h) \in RULE \;\&\; \mathfrak{s}_n h \in Z^*_\prec \;\&\; (\mathfrak{r}(h) = \mathsf{Ax} \Rightarrow \mathfrak{s}_n h = h)$,

b) $LC((\mathfrak{r}(h), \mathfrak{e}(h)), (\mathfrak{e}(\mathfrak{s}_n h))_{n \in \mathbb{N}})$,

c) $\ell_C(\mathfrak{r}(h)) \leqq \mathfrak{d}(h)$ & $\mathfrak{d}(\mathfrak{s}_n h) \leqq \mathfrak{d}(h)$,

d) $\mathfrak{r}(h) \neq \mathsf{Ax} \Rightarrow \mathfrak{o}(\mathfrak{s}_n h) \prec \mathfrak{o}(h)$.

Proof by induction on $\ell^*(h)$.

The proof of a) is trivial (cf. Lemma 3.5) and one easily verifies that b) and c) hold for all $h \in Z^0_{\prec}$. The induction step for b) and c) is treated in Sect. 5. Here we only carry out the proof of d).

Proof of 3.8d). Let $Z^-_{\prec} := \{d \in Z_{\prec}: d$ is built up by the rules (Z.1), (Z.4), ..., (Z.7)$\}$. Obviously $\mathfrak{o}(d) \prec \exp(1)$, for all $d \in Z^-_{\prec}$, and $\kappa[A] \in Z^-_{\prec}$, for all $A \in$ FOR.
(0.1) $h = \langle(\mathsf{Ind}, F), \Gamma, \emptyset\rangle$: Then $\mathfrak{s}_n h = c^F_n \in Z^-_{\prec}$ and thus $\mathfrak{o}(c^F_n) \prec \exp(1) = \mathfrak{o}(h)$.
(0.2) $h = \langle(\mathsf{TI}, F, m), \Gamma, \emptyset\rangle$:
(0.2.1) $n \prec m$: Then $\mathfrak{s}_n h = \langle(\bigvee_1, \cdot), \cdot, \langle(\bigvee_n, \cdot), \cdot, \langle(\bigwedge, \cdot), \cdot, (e, \kappa[F(n)])\rangle\rangle\rangle$ with $e = \langle(\mathsf{TI}, F, n), \cdot, \emptyset\rangle$, $\mathfrak{o}(\kappa[F(n)]) \prec \exp(m)$, $\mathfrak{o}(e) = \prec -\max\{\exp(n), 1 \oplus 1\} \prec \exp(m)$. Hence $\mathfrak{o}(\mathfrak{s}_n h) = \prec -\max\{\mathfrak{o}(e) \oplus 1, \mathfrak{o}(\kappa[F(n)]) \oplus 1\} \oplus 1 \oplus 1 \prec \exp(m) = \mathfrak{o}(h)$.
(0.2.1) $n \not\prec m$: Then $\mathfrak{s}_n h = \langle(\bigvee_0, \cdot), \cdot, \langle \mathsf{Ax}, \cdot, \emptyset\rangle\rangle$ and therefore $\mathfrak{o}(\mathfrak{s}_n h) = 1 \prec \oplus \leqq \mathfrak{o}(h)$.
(0.3) $h = \langle(v, \forall x F(x)), \cdot, d_0\rangle$: Then $\mathfrak{s}_n h = d_0[v/n]$ and $\mathfrak{o}(d_0[v/n]) = \mathfrak{o}(d_0) \prec \mathfrak{o}(d_0) \oplus 1 = \mathfrak{o}(h)$.
(0.4) otherwise: trivial.
(1) $h = I_{k,A} g$: We have $\mathfrak{o}(h) = \mathfrak{o}(g)$, $\mathfrak{o}(\mathfrak{s}_n h) = \mathfrak{o}(I_{k,A} \mathfrak{s}_n g) = \mathfrak{o}(\mathfrak{s}_n g)$, and by I.H. $\mathfrak{o}(\mathfrak{s}_n g) \prec \mathfrak{o}(g)$.
(2) $h = R_C h_0 h_1$: We have $\mathfrak{o}(h) = \mathfrak{o}(h_0) \oplus \mathfrak{o}(h_1)$, ($\mathfrak{s}_n h = I_{k,\neg C} h_0$ or $\mathfrak{s}_n h = R_C h_0 \mathfrak{s}_n h_1$), $\mathfrak{o}(I_{k,\neg C} h_0) = \mathfrak{o}(h_0) \leqq \mathfrak{o}(h_0) \oplus \mathfrak{o}(\mathfrak{s}_n h_1) = \mathfrak{o}(R_C h_0 \mathfrak{s}_n h_1)$ and (by I.H.) $\mathfrak{o}(\mathfrak{s}_n h_1) \prec \mathfrak{o}(h_1)$.
(3) $h = Eg$: It suffices to consider the case where $\mathfrak{s}_n h = R_A(E\mathfrak{s}_1 g)\,(E\mathfrak{s}_0 g)$.
Then we have $\mathfrak{o}(\mathfrak{s}_n h) = \exp(\mathfrak{o}(\mathfrak{s}_1 g)) \oplus \exp(\mathfrak{o}(\mathfrak{s}_0 g))$ and (by I.H.) $\mathfrak{o}(\mathfrak{s}_0 g)$, $\mathfrak{o}(\mathfrak{s}_1 g) \prec \mathfrak{o}(g)$. This yields $\mathfrak{o}(\mathfrak{s}_n h) \prec \exp(\mathfrak{o}(g)) = \mathfrak{o}(h)$. □

4 Proof theoretic analysis of $Z + TI_{\prec\upharpoonright}$

Now we carry out the proof theoretic analysis of $Z + TI_{\prec\upharpoonright}$ sketched at the beginning of Sect. 3. Let $p \in L$ be a fixed binary predicate symbol and $\mathbf{p} \subseteq \mathbb{N}^2$ the primitive recursive relation assigned to p.
Abbreviation: $A :\equiv \forall x \exists y pxy$, $\mathsf{f}_p(n) := \min\{m: \mathbf{p}(n, m)\}$.

4.1 Definition of prim. rec. functions $\mathsf{sub}: Z^*_{\prec} \to Z^*_{\prec}$, $\wedge: Z^*_{\prec} \to Z^*_{\prec}$

$$\mathsf{sub}(h) := \begin{cases} \mathfrak{s}_n h, & \text{if } \mathfrak{r}(h) = \mathsf{Rep}_n \\ \mathfrak{s}_0 h, & \text{otherwise} \end{cases}; \qquad \hat{h} := \underbrace{E \ldots E}_{n} h \quad \text{with} \quad n := \mathfrak{d}(h).$$

4.2 Lemma. *For every $h \in Z^*_{\prec}$ holds:*

a) $\mathfrak{d}(\hat{h}) = 0$.

b) $\mathfrak{r}(h) \neq \mathsf{Ax} \Rightarrow \mathfrak{o}(\mathsf{sub}(h)) \prec \mathfrak{o}(h)$, $\mathfrak{r}(h) = \mathsf{Ax} \Rightarrow \mathsf{sub}(h) = h$.

Proof. Trivial. (cf. definition of $\mathfrak{d}$ and Theorem 3.7a), d)). □

4.3 Definition of prim. rec. functions $\mathsf{s}: Z^0_{\prec} \times \mathbb{N}^2 \to Z^*_{\prec}$, $\mathsf{u}: SEQ \to \mathbb{N}$
$\mathsf{s}(d, n, k) := \mathsf{sub}^{(k)}(I_{n,A}\hat{d})$ (where $\mathsf{sub}^{(k)}$ is the kth iterate of sub).
$\mathsf{u}(\Gamma) := \sup\{m: m$ occurs in $\Gamma\}$.

4.4 Lemma
$d \in Z^0_{\prec}$ & $\mathfrak{e}(d) \subseteq \{A\}$ & $\mathfrak{o}(\mathsf{s}(d,n,k+1)) \not\prec \mathfrak{o}(\mathsf{s}(d,n,k))$
$\Rightarrow \exists m \leq \mathsf{u}(\mathfrak{e}(\mathsf{s}(d,n,k)))\mathbf{p}(n,m)$

Proof. Abb.: $h_{n,k} := \mathsf{s}(d,n,k)$, $\Gamma(n) := \{\exists ypny\} \cup \{pnm \colon m \in \mathbb{N}\}$.
Then we obtain

(1) $h_{n,0} \in Z^*_{\prec} \wedge \mathfrak{d}(h_{n,0}) = 0 \wedge \mathfrak{e}(h_{n,0}) \subseteq \Gamma(n)$ [by definition of $\mathfrak{d}$ and $\mathfrak{e}$]
(2) $h_{n,k} \in Z^*_{\prec} \wedge \mathfrak{d}(h_{n,k}) = 0$ [(1), Theorem 3.7a), c), induction on k]
(3) $\mathfrak{e}(h_{n,k}) \subseteq \Gamma(n) \Rightarrow \mathfrak{r}(h_{n,k}) \in \{\mathsf{Ax}, \mathsf{Rep}_i, (\bigvee_i, \exists ypny)\}$ [(2), Theorem 3.7b)]
(4) $\mathfrak{e}(h_{n,k}) \subseteq \Gamma(n) \Rightarrow \mathfrak{e}(h_{n,k+1}) \subseteq \Gamma(n)$ [(3), Theorem 3.7b)]
(5) $\mathfrak{e}(h_{n,k}) \subseteq \Gamma(n)$ [(4), induction on k]
(6) $\mathfrak{r}(h_{n,k}) = \mathsf{Ax} \Rightarrow \exists m(pnm \in \mathfrak{e}(h_{n,k})$ & $pnm \in TRUE))$ [(5), Theorem 3.7b)]
(7) $\mathfrak{r}(h_{n,k}) = \mathsf{Ax} \Rightarrow \exists m \leq \mathsf{u}(\mathfrak{e}(h_{n,k}))\mathbf{p}(n,m)$ [(6)]

From (7) the assertion follows by Lemma 4.2b). □

4.5 Theorem. *If* $Z + TI_{\prec\restriction} \vdash \forall x \exists ypxy$ *then there are prim. rec. functions* $\mathsf{g}, \mathsf{o} \colon \mathbb{N}^2 \to \mathbb{N}$, $\mathsf{q} \colon \mathbb{N} \to Z^*_{\prec}$, $\mathsf{g}' \colon Z^*_{\prec} \to \mathbb{N}$ *and an* $\mathsf{a} \in \mathscr{A}$, *such that, for all* $n \in \mathbb{N}$:

a) $\mathsf{f}_p(n) = \mathsf{g}(n, \min\{k \colon \mathsf{o}(n,k+1) \not\prec \mathsf{o}(n,k)\})$ and $\forall k(\mathsf{o}(n,k) \preceq a)$.
b) $\mathsf{f}_p(n) = \mathscr{F}(\mathsf{q}(n))$, *where*

$$\mathscr{F}(h) := \begin{cases} \mathscr{F}(\mathsf{sub}(h)), & \text{if } \mathfrak{o}(\mathsf{sub}(h)) \prec \mathfrak{o}(h) \preceq a \\ \mathsf{g}'(h), & \text{otherwise} \end{cases}$$

Proof. By assumption there exists a $d \in Z^0_{\prec}$ with $\mathfrak{e}(d) \subseteq \{A\}$.

a) We set $\mathsf{o}(n,k) := \mathfrak{o}(\mathsf{s}(d,n,k))$, $\mathsf{g}(n,k) := \min\{m \leq \mathsf{u}(\mathfrak{e}(\mathsf{s}(d,n,k))) \colon \mathbf{p}(n,m)\}$, $\mathsf{a} := \mathfrak{o}(\hat{d})$. By definition of $\mathfrak{o}$ and Lemma 4.2b) we get $\forall n, k(\mathsf{o}(n,k) \preceq \mathsf{o}(n,0) = \mathsf{a})$. Since $\prec$ is wellfounded, we have $\forall n \exists k(\mathsf{o}(n,k+1) \not\prec \mathsf{o}(n,k))$. By Lemma 4.4 this yields the assertion.

b) Let $\mathsf{q}(n) := \mathsf{s}(d,n,0) = I_{n,A}\hat{d}$ and $\mathsf{g}'(h) := \min\{m \leq \mathsf{u}(\mathfrak{e}(h)) \colon \mathbf{p}(\mathfrak{n}(h), m)\}$, where

$$\mathfrak{n}(h) := \begin{cases} n, & \text{if } h = I_{n,A}h_0 \\ 0, & \text{if } h \text{ has not the form } I_{n,A}h_0 \end{cases}.$$

Obviously $\mathscr{F}(\mathsf{q}(n)) = \mathsf{g}'(h^*_n)$ with $h^*_n = \mathsf{s}(d', n, k_n)$, $k_n := \min\{k \colon \mathsf{o}(n,k+1) \not\prec \mathsf{o}(n,k)\}$. Since $\forall i \in \mathbb{N}, h \in Z^*_{\prec}(\mathfrak{s}_i I_{n,A} h = I_{n,A}\mathfrak{s}_i h)$, we have $\mathfrak{n}(h^*_n) = n$ and thus

$$\mathsf{g}'(h^*_n) = \min\{m \leq \mathsf{u}(\mathfrak{e}(h^*_n)) \colon \mathbf{p}(n,m)\} = \mathsf{g}(n, k_n).$$

But $\mathsf{g}(n,k_n) = \mathsf{f}_p(n)$ by a). □

Now we assume that the elementary properties of $(\mathscr{A}, \prec, \oplus, \mathsf{exp})$, i.e. ($\prec$.1)–($\prec$.4) on p. 280 and the transitivity of $\prec$, are provable in PRA. Then the proof of Lemma 4.4 can be formalized in PRA, and we obtain the following theorem.

4.6 Theorem ($PRA \vdash PRWO(\prec) \to \Pi^0_2$-Reflection$(Z + TI_{\prec\restriction})$). *There exists a primitive recursive function* $\tilde{\mathsf{o}} \colon \mathbb{N}^2 \to \mathbb{N}$ *such that*

$$PRA \vdash WF(\tilde{\mathsf{o}}, \prec) \to (Prov_{Z+TI_{\prec\restriction}}(\ulcorner C \urcorner) \to C),$$

for each Π^0_2*-sentence* C *of* L, *where*

$$WF(\tilde{\mathsf{o}}, \prec) :\equiv \forall x \exists y(\tilde{\mathsf{o}}(x, y+1) \not\prec \tilde{\mathsf{o}}(x,y)),$$

and $Prov_{Z+TI_{\prec\restriction}}(\ulcorner C \urcorner)$ *formalizes the statement* "$\exists d \in Z^0_{\prec}(End(d) \subseteq \{C\})$".

Proof. Let

$$\tilde{o}(j,k) := \begin{cases} o(\mathrm{sub}^{(k)}(h)), & \text{if } j \text{ is the Gödelnumber of } h \in Z^*_{\prec} \\ 0, & \text{otherwise} \end{cases}.$$

Without loss of generality we may assume that $C \equiv \forall x \exists y pxy$.
Then by Lemma 4.4 we have

(*) $\forall j \exists k (\tilde{o}(j,k+1) \not\prec \tilde{o}(j,k)) \,\&\, \exists d \in Z^0_{\prec}(End(d) \subseteq \{C\}) \Rightarrow \forall n \exists m p(n,m)$,

and therefore the assertion is obtained by formalizing in PRA the proof of 4.4. □

Corollary. $Z + TI_{\prec\restriction} \nvdash WF(\tilde{o}, \prec)$.

Proof. By 4.6

$$Z + TI_{\prec\restriction} \vdash WF(\tilde{o}, \prec) \rightarrow (Prov_{Z+TI_{\prec\restriction}}(\ulcorner 0 = 1 \urcorner) \rightarrow 0 = 1),$$

i.e.

$$Z + TI_{\prec\restriction} \vdash WF(\tilde{o}, \prec) \rightarrow CON(Z + TI_{\prec\restriction}).$$

Hence $Z + TI_{\prec\restriction} \nvdash WF(\tilde{o}, \prec)$ by Gödel's Second Incompleteness Theorem. □

5 Arbitrary notation systems for derivations

In this section we analyze the construction of the system $(Z^*_{\prec}, \mathfrak{e}, \mathfrak{r}, \mathfrak{s})$ in a somewhat more general context. We introduce the notion of an arbitrary *notation system* (for elements of $TREE$) and define the *$*$-extension* $\mathscr{H}^*$ of an arbitrary notation system $\mathscr{H}$ literally in the same way as we have defined $(Z^*_{\prec}, \mathfrak{e}, \mathfrak{r}, \mathfrak{s})$ from $(Z^0_{\prec}, \mathfrak{e}, \mathfrak{r}, \mathfrak{s})$ in Sect. 3. This will then be used in Sect. 6 to give an alternative description of Minc's continuous cut-elimination operator $\mathscr{E}'$ for arbitrary (not necessarily wellfounded) proof-figures.

5.1 Definition. A *notation system* consists of a nonempty set H and functions $\mathfrak{q}: H \rightarrow RULE \times SEQ$, $\mathfrak{s}: \mathbb{N} \times H \rightarrow H$ such that $\forall (n,h) \in \mathbb{N} \times H(\mathfrak{q}(h) \in \mathsf{Ax} \Rightarrow \mathfrak{s}_n h = h)$.

5.2 Definitions. Let $\mathscr{H} = (H, \mathfrak{q}, \mathfrak{s})$ be a notation system and $h \in H$:

a) $\mathfrak{r}(h) := \mathfrak{q}^0(h) := \mathbf{r}$, $\mathfrak{e}(h) := \mathfrak{q}^1(h) := \Gamma$, where $(\mathbf{r}, \Gamma) := \mathfrak{q}(h)$.
b) $[h, \langle n_0, \ldots, n_{k-1} \rangle] := \mathfrak{s}_{n_{k-1}} \ldots \mathfrak{s}_{n_0} h$
c) $\{h\}: \mathbb{N}^{<\omega} \rightarrow RULE \times SEQ$, $\{h\}(\sigma) := \mathfrak{q}([h, \sigma])$
d) $\mathscr{H}$ is *correct*$:\Leftrightarrow (\forall h \in H) LC_{\mathscr{H}}(h)$, with $LC_{\mathscr{H}}(h) :\Leftrightarrow LC(\mathfrak{q}(h), (\mathfrak{e}(\mathfrak{s}_n h))_{n \in \mathbb{N}})$.
e) $\mathfrak{d}: H \rightarrow \mathbb{N}$ is a *cut-bound for* $\mathscr{H}$ $:\Leftrightarrow \forall h \in H \forall n [\ell_C(\mathfrak{r}(h)) \leqq \mathfrak{d}(h) \,\&\, \mathfrak{d}(\mathfrak{s}_n h) \leqq \mathfrak{d}(h)]$.

5.3 Theorem. *Let $\mathscr{H} = (H, \mathfrak{q}, \mathfrak{s})$ be a notation system.*

a) *For all $h \in H$, $n \in \mathbb{N}$, $\sigma \in \mathbb{N}^{<\omega}$ we have $\{h\} \in TREE$, $End(\{h\}) = \mathfrak{e}(h)$, $Rule(\{h\}) = \mathfrak{r}(h)$, $\{\mathfrak{s}_n h\} = \{h\}[n]$, $\{[h, \sigma]\} = \{h\} [\![\sigma]\!]$.*
b) *$\mathscr{H}$ is correct $\Leftrightarrow \forall h \in H(\{h\}$ is locally correct)*
c) *If $\mathfrak{d}$ is a cut-bound for $\mathscr{H}$, then $\forall h \in H(deg(\{h\}) \leqq \mathfrak{d}(h))$.*
d) *$h \in H \,\&\, \mathfrak{q}(h) \in \mathsf{Ax} \Rightarrow \{h\} = \mathbf{0}_{\mathfrak{q}(h)}$.*

Proof. a)

(1) $\{h\}(\sigma) \in \mathsf{Ax} \Rightarrow \mathfrak{q}([h,\sigma]) \in \mathsf{Ax} \Rightarrow \forall n(\mathfrak{s}_n[h,\sigma] = [h,\sigma])$

$$\Rightarrow \forall n(\{h\}(\sigma * \langle n \rangle) = \mathfrak{q}([h, \sigma * \langle n \rangle]) = \mathfrak{q}(\mathfrak{s}_n(h,\sigma])$$

$$= \mathfrak{q}([h,\sigma]) = \{h\}(\sigma)). \quad \text{Therefore } \{h\} \in TREE.$$

(2) $(Rule(\{h\}), End(\{h\})) = \{h\}(\langle\rangle) = \mathfrak{q}(h) = (\mathfrak{r}(h), \mathfrak{e}(h))$.
(3) $\{\mathfrak{s}_n h\}(\sigma) = \mathfrak{q}([\mathfrak{s}_n h, \sigma]) = \mathfrak{q}([h, \langle n\rangle * \sigma]) = \{h\}(\langle n\rangle * \sigma) = (\{h\}[n])(\sigma)$.
(4) $\{[h,\sigma]\} = \{h\}[\![\sigma]\!]$ follows from (3) by induction on the length of σ.
b) Using $\mathfrak{e}(\mathfrak{s}_n[h,\sigma]) = \{h\}^1(\sigma * \langle n\rangle)$ we obtain

$$\forall h . LC_{\mathscr{H}}(h) \Leftrightarrow \forall h . LC(\mathfrak{q}(h), (\mathfrak{e}(\mathfrak{s}_n h))_{n\in\mathbb{N}})$$
$$\Leftrightarrow \forall h \forall \sigma . LC(\mathfrak{q}([h,\sigma]), (\mathfrak{e}(\mathfrak{s}_n Hh, \sigma]))_{n\in\mathbb{N}})$$
$$\Leftrightarrow \forall h \forall \sigma . LC(\{h\}(\sigma), (\{h\}^1(\sigma * \langle n\rangle))_{n\in\mathbb{N}})$$
$$\Leftrightarrow \forall h(\{h\} \text{ locally correct}).$$

c) $\{h\}^0(\sigma) = (\mathsf{Cut}, A) \Rightarrow \mathfrak{r}([h,\sigma]) = (\mathsf{Cut}, A) \Rightarrow \ell(A) < \mathfrak{d}([h,\sigma]) \leqq \mathfrak{d}(h)$.
d) $\{h\}(\sigma) = \mathfrak{q}([h,\sigma]) = \mathfrak{q}(h)$. □

5.4 Definition and Remark. Let $\mathscr{H} = (H, \mathfrak{q}, \mathfrak{s})$ be a notation system.
a) The mapping $\{\cdot\}: H \to TREE$, $h \mapsto \{h\}$ is called the *canonical interpretation* for $\mathscr{H}$. If

$$\mathbf{j}: H \to TREE \quad \text{with} \quad \forall h \in H \forall n (\mathbf{j}(h)(\langle\rangle) = \mathfrak{q}(h) \;\&\; \mathbf{j}(\mathfrak{s}_n h) = \mathbf{j}(h)[n])$$

then

$$\mathbf{j}(h)(\sigma) = (\mathbf{j}(h)[\![\sigma]\!])(\langle\rangle) = \mathbf{j}([h,\sigma])(\langle\rangle) = \mathfrak{q}([h,\sigma]) = \{h\}(\sigma)$$

and therefore $\mathbf{j} = \{\cdot\}$.
b) $\mathscr{H}$ is called *wellfounded* iff $\forall h \in H(\{h\} \in WT)$. If $\mathscr{H}$ is wellfounded then $\|\{h\}\| \in On$ is defined for all $h \in H$, and by 5.3a), we have $\mathfrak{r}(h) \neq \mathsf{Ax} \Rightarrow \forall n(\|\{\mathfrak{s}_n h\}\| < \|\{h\}\|)$.

5.5 Definition of the notation system $\mathscr{H}^*$. Let $\mathscr{H} = (H, \mathfrak{q}, \mathfrak{s})$ be a notation system.

$$H^* := \begin{cases} \text{set of all terms } h \text{ generated from constants } d \in H \text{ by the} \\ \text{function symbols } I_{k,A}, R_C, E \ (k \in \mathbb{N}, A \in \bigwedge\!\!\!\bigwedge\text{-}FOR, C \in \mathbb{W}_0\text{-}FOR). \end{cases}$$

For $h \in H^*$ we define $\ell^*(h)$ as in 3.3.
Then the functions $\mathfrak{e} = \mathfrak{q}^1: H \to SEQ$, $\mathfrak{r} = \mathfrak{q}^0: H \to RULE$, $\mathfrak{s}: \mathbb{N} \times H \to H$ are extended to functions $\mathfrak{e}: H^* \to SEQ$, $\mathfrak{r}: H^* \to RULE$, $\mathfrak{s}: \mathbb{N} \times H^* \to H^*$ precisely as in Sect. 3, i.e. by Def. 3.7.

$\mathscr{H}^* := (H^*, \mathfrak{q}, \mathfrak{s})$ with the extended functions $\mathfrak{q} = (\mathfrak{r}, \mathfrak{e})$ and $\mathfrak{s}$.

We call $\mathscr{H}^*$ the *$*$-extension* of $\mathscr{H}$.

5.6. Remark. The canonical interpretation $\{\cdot\}^*: \mathscr{H}^* \to TREE$, $\{h\}^* := \lambda\sigma . \mathfrak{q}([h,\sigma])$ is of course an extension of the canonical interpretation $\{\cdot\}: \mathscr{H} \to TREE$. So we may drop the superscript * as we have already done for $\mathfrak{e}$, $\mathfrak{r}$, $\mathfrak{s}$.

5.7 Theorem
a) *$\mathscr{H}$ correct $\Rightarrow$ $\mathscr{H}^*$ correct*
b) *Every cut-bound $\mathfrak{d}$ for $\mathscr{H}$ can be extended to a cut-bound for $\mathscr{H}^*$ by*

$$\mathfrak{d}(I_{k,A}h) := \mathfrak{d}(h),\ \mathfrak{d}(R_C h_0 h_1) := \max\{\ell(C), \mathfrak{d}(h_0), \mathfrak{d}(h_1)\},\ \mathfrak{d}(Eh) := \mathfrak{d}(h) \dot{-} 1.$$

c) *$\mathscr{H}$ wellfounded $\Rightarrow$ $\mathscr{H}^*$ wellfounded.*

Proof. a) and b) are proved by (quantifierfree) induction on $\ell^*(h)$. The proof is just a straightforward verification. Nevertheless we carry it out in full detail, since it is one of the main proofs of this paper, and in addition we want to demonstrate that even such a detailed exposition is possible without excessive effect.

a) Let $\mathfrak{A}(h,n)$ abbreviate the conjunction of (1)–(5).

(1) $\mathfrak{r}(h)=\mathsf{Ax} \Rightarrow \mathfrak{e}(h)\cap TRUE \neq \emptyset$,

(2) $\mathfrak{r}(h)=(\bigwedge, A) \Rightarrow A\in \mathfrak{e}(h)\ \&\ \mathfrak{e}(\mathfrak{s}_n h)\subseteq \mathfrak{e}(h)\cup A[n]$,

(3) $\mathfrak{r}(h)=(\bigvee_k, A) \Rightarrow A\in \mathfrak{e}(h)\ \&\ \mathfrak{e}(\mathfrak{s}_0 h)\subseteq \mathfrak{e}(h)\cup A[k]$,

(4) $\mathfrak{r}(h)=(\mathsf{Cut}, C) \Rightarrow \mathfrak{e}(\mathfrak{s}_0 h)\subseteq \mathfrak{e}(h)\cup C\ \&\ \mathfrak{e}(\mathfrak{s}_1 h)\subseteq \mathfrak{e}(h)\cup \neg C$,

(5) $\mathfrak{r}(h)=\mathsf{Rep}_n \Rightarrow \mathfrak{e}(\mathfrak{s}_n h)\subseteq \mathfrak{e}(h)$.

Obviously

$$\forall h\in H[LC_{\mathscr{H}}(h) \Leftrightarrow \forall n \mathfrak{A}(h,n)]$$

and

$$\forall h\in H^*[LC_{\mathscr{H}^*}(h) \Leftrightarrow \forall n \mathfrak{A}(h,n)].$$

So we have to prove $\forall h\in H^*\mathfrak{A}(h,n)$ under the premise $\forall h\in H\mathfrak{A}(h,n)$.

This will be done by induction on $\ell^*(h)$.

If $\ell^*(h)=0$ then $h\in H$, and we are done. – Now let $\ell^*(h)>0$.

I. $h=I_{m,B}h_1$. – Then $\mathfrak{e}(h)=(\mathfrak{e}(h_1)\backslash B)\cup B[m]$ and $\mathfrak{s}_n h=I_{m,B}\mathfrak{s}_n h_1$.

I.1 $\mathfrak{r}(h_1)=(\bigwedge, B)$: Then $\mathfrak{r}(h)=\mathsf{Rep}_m$ and (1)–(4) are trivially true. To prove (5) we assume that $\mathfrak{r}(h)=\mathsf{Rep}_n$. Then $n=m$ and (by I.H.) $\mathfrak{e}(\mathfrak{s}_n h_1)\subseteq \mathfrak{e}(h_1)\cup B[n]$. This yields

$$\mathfrak{e}(\mathfrak{s}_n h)=(\mathfrak{e}(\mathfrak{s}_n h_1)\backslash B)\cup B[m]\subseteq (\mathfrak{e}(h_1)\backslash B)\cup B[m]=\mathfrak{e}(h).$$

I.2 $\mathfrak{r}(h_1)=(\bigwedge, A)$ with $A\not\equiv B$: Then $\mathfrak{r}(h)=\mathfrak{r}(h_1)$ and (by I.H.)

$$A\in \mathfrak{e}(h_1)\ \&\ \mathfrak{e}(\mathfrak{s}_n h_1)\subseteq \mathfrak{e}(h_1)\cup A[n].$$

Hence

$$\begin{aligned}\mathfrak{e}(\mathfrak{s}_n h)&=(\mathfrak{e}(\mathfrak{s}_n h_1)\backslash B)\cup B[m]\subseteq (\mathfrak{e}(h_1)\backslash B)\cup B[m]\cup A[n]\\&=\mathfrak{e}(h)\cup A[n],\end{aligned}$$

and $A\in \mathfrak{e}(h_1)\backslash B\subseteq \mathfrak{e}(h)$.

I.3 $\mathfrak{r}(h_1)=(\bigvee_k, A)$, (Cut, A), Rep_k: as I.2.

II. $h=R_C h_0 h_1$. By definition $\mathfrak{e}(h)=\Gamma_0\cup(\mathfrak{e}(h_1)\backslash C)$ with $\Gamma_0:=\mathfrak{e}(h_0)\backslash \neg C$.

II.1 $\mathfrak{r}(h_1)=(\bigvee_k, C)$: Then $\mathfrak{r}(h)=(\mathsf{Cut}, C[k])$, $\mathfrak{s}_0 h=R_C h_0 \mathfrak{s}_0 h_1$, $\mathfrak{s}_1 h=I_{k,\neg C}h_0$, and (by I.H.) $\mathfrak{e}(\mathfrak{s}_0 h_1)\subseteq \mathfrak{e}(h_1)\cup C[k]$. Hence

$$\begin{aligned}\mathfrak{e}(\mathfrak{s}_0 h)&=\mathfrak{e}(R_C h_0 \mathfrak{s}_0 h_1)=\Gamma_0\cup(\mathfrak{e}(\mathfrak{s}_0 h_1)\backslash C)\subseteq \Gamma_0\cup(\mathfrak{e}(h_1)\backslash C)\cup C[k]\\&=\mathfrak{e}(h)\cup C[k]\end{aligned}$$

and

$$\mathfrak{e}(\mathfrak{s}_1 h)=\mathfrak{e}(I_{k,\neg C}h_0)=(\mathfrak{e}(h_0)\backslash\neg C)\cup\neg C[k]\subseteq \mathfrak{e}(h)\cup\neg C[k].$$

II.2 $\mathfrak{r}(h_1)=(\bigwedge, A)$: Then $\mathfrak{r}(h)=(\bigwedge, A)$, $A\not\equiv C$, $\mathfrak{s}_n h=R_C h_0 \mathfrak{s}_n h_1$ and (by I.H.)

$$A\in \mathfrak{e}(h_1)\ \&\ \mathfrak{e}(\mathfrak{s}_n h_1)\subseteq \mathfrak{e}(h_1)\cup A[n].$$

Hence $A\in \mathfrak{e}(h)$ and

$$\mathfrak{e}(\mathfrak{s}_n h)=\Gamma_0\cup(\mathfrak{e}(\mathfrak{s}_n h_1)\backslash C)\subseteq \Gamma_0\cup(\mathfrak{e}(h_1)\backslash C)\cup A[n]=\mathfrak{e}(h)\cup A[n].$$

II.3 $\mathfrak{r}(h_1)=(\mathsf{Cut}, A)$ or Rep_k: Then $\mathfrak{r}(h)=\mathfrak{r}(h_1)$, and $\mathfrak{A}(h,n)$ follows immediately from the I.H.

II.4 $\mathfrak{r}(h_1)=\mathsf{Ax}$: Then also $\mathfrak{r}(h)=\mathsf{Ax}$, and by I.H. we have $\mathfrak{e}(h_1)\cap TRUE\neq\emptyset$. From this we obtain $\mathfrak{e}(h)\cap TRUE\neq\emptyset$, since $C\notin TRUE$.

III.1 $h=Eh_1$ and $\mathfrak{r}(h_1)=(\mathsf{Cut},C)$: w.l.o.g. $C\in \mathbb{W}_0\text{-}FOR$. Then $\mathfrak{r}(h)=\mathsf{Rep}_0$, $\mathfrak{s}_0 h=R_C(E\mathfrak{s}_1 h_1)(E\mathfrak{s}_0 h_1)$ and

$$\mathfrak{e}(\mathfrak{s}_0 h_1)\subseteq \mathfrak{e}(h_1)\backslash C \;\&\; \mathfrak{e}(\mathfrak{s}_1 h_1)\subseteq \mathfrak{e}(h_1)\backslash\neg C.$$

Hence

$$\mathfrak{e}(\mathfrak{s}_0 h)=(\mathfrak{e}(\mathfrak{s}_1 h_1)\backslash\neg C)\cup(\mathfrak{e}(\mathfrak{s}_0 h_1)\backslash C)\subseteq \mathfrak{e}(h_1)=\mathfrak{e}(h).$$

III.2 $h=Eh_1$ and $\mathfrak{r}(h_1)\notin\mathsf{Cut}$: Then $\mathfrak{r}(h)=\mathfrak{r}(h_1)$ and $\mathfrak{s}_n h=E\mathfrak{s}_n h_1$. This together with $\mathfrak{e}(h)=\mathfrak{e}(h_1)$, $\mathfrak{e}(\mathfrak{s}_n h)=\mathfrak{e}(\mathfrak{s}_n h_1)$ and the I.H. yields the assertion $\mathfrak{A}(h,n)$.

b) We prove

$$\mathfrak{d}(\mathfrak{s}_n h)\leqq\mathfrak{d}(h)\;\&\;(\mathfrak{r}(h)=(\mathsf{Cut},A)\Rightarrow \ell(A)<\mathfrak{d}(h))$$

by induction on $\ell^*(h)$.

I. $h=I_{m,B}h_1$. By I.H. we obtain $\mathfrak{d}(\mathfrak{s}_n h)=\mathfrak{d}(\mathfrak{s}_n h_1)\leqq\mathfrak{d}(h_1)=\mathfrak{d}(h)$ and

$$\mathfrak{r}(h)=(\mathsf{Cut},A)\;\Rightarrow\;\mathfrak{r}(h_1)=(\mathsf{Cut},A)\;\Rightarrow\;\ell(A)<\mathfrak{d}(h_1)=\mathfrak{d}(h).$$

II. $h=R_C h_0 h_1$. Then $\mathfrak{d}(h)=\max\{\ell(C),\mathfrak{d}(h_0),\mathfrak{d}(h_1)\}$ and (by I.H.) $\mathfrak{d}(\mathfrak{s}_n h_1)\leqq\mathfrak{d}(h_1)$.

II.1 $\mathfrak{r}(h_1)=(\mathbb{W}_k,C)$: Then $\mathfrak{r}(h)=(\mathsf{Cut},C[k])$ and $\ell(C[k])<\ell(C)\leqq\mathfrak{d}(h)$.

II.1.1 $n=1$: Then $\mathfrak{s}_n h=I_{k,\neg C}h_0$ and therefore $\mathfrak{d}(\mathfrak{s}_n h)=\mathfrak{d}(h_0)\leqq\mathfrak{d}(h)$

II.1.2 $n\neq 1$: Then $\mathfrak{s}_n h=R_C h_0\mathfrak{s}_n h_1$ and therefore

$$\mathfrak{d}(\mathfrak{s}_n h)=\max\{\ell(C),\mathfrak{d}(h_0),\mathfrak{d}(\mathfrak{s}_n h_1)\}\leqq\max\{\ell(C),\mathfrak{d}(h_0),\mathfrak{d}(h_1)\}=\mathfrak{d}(h).$$

II.2 $\mathfrak{r}(h_1)\neq(\mathbb{W}_k,C)$: Then also $\mathfrak{s}_n h=R_C h_0\mathfrak{s}_n h_1$ and thus $\mathfrak{d}(\mathfrak{s}_n h)\leqq\mathfrak{d}(h)$ as in II.1.2. Moreover:

$$\mathfrak{r}(h)=(\mathsf{Cut},A)\Rightarrow\mathfrak{r}(h_1)=(\mathsf{Cut},A)\Rightarrow\ell(A)<\mathfrak{d}(h_1)\leqq\mathfrak{d}(h).$$

III.1 $h=Eh_1$ and $\mathfrak{r}(h_1)=(\mathsf{Cut},C)$: w.l.o.g. $C\in\mathbb{W}_0\text{-}FOR$. Then $\mathfrak{r}(h)=\mathsf{Rep}_0$, $\mathfrak{s}_n h=R_C(E\mathfrak{s}_1 h_1)(E\mathfrak{s}_0 h_1)$ and (by I.H.) $\ell(C)<\mathfrak{d}(h_1)$, $\mathfrak{d}(\mathfrak{s}_i h_1)\leqq\mathfrak{d}(h_1)$ for $i=0,1$. Hence

$$\begin{aligned}\mathfrak{d}(\mathfrak{s}_n h)&=\max\{\ell(C),\mathfrak{d}(E\mathfrak{s}_1 h_1),\mathfrak{d}(E\mathfrak{s}_0 h_1)\}\\&=\max\{\ell(C),\mathfrak{d}(\mathfrak{s}_1 h_1)\dot{-}1,\mathfrak{d}(\mathfrak{s}_0 h_1)\dot{-}1\}\\&\leqq\max\{\ell(C),\mathfrak{d}(h_1)\dot{-}1)\}=\mathfrak{d}(h_1)\dot{-}1=\mathfrak{d}(h).\end{aligned}$$

III.2 $h=Eh_1$ and $\mathfrak{r}(h_1)\notin\mathsf{Cut}$: Then $\mathfrak{r}(h)=\mathfrak{r}(h_1)$, $\mathfrak{s}_n h=E\mathfrak{s}_n h_1$, and (by I.H.) $\mathfrak{d}(\mathfrak{s}_n h_1)\leqq\mathfrak{d}(h_1)$. Hence

$$\mathfrak{d}(\mathfrak{s}_n h)=\mathfrak{d}(\mathfrak{s}_n h_1)\dot{-}1\leqq\mathfrak{d}(h_1)\dot{-}1=\mathfrak{d}(h).$$

c) By Remark 5.4b we have

(1) $\forall h\in H\forall n(\|\{h\}\|\in On\;\&\;(\mathfrak{r}(h)\neq\mathsf{Ax}\Rightarrow\|\{\mathfrak{s}_n h\}\|<\|\{h\}\|))$.

We define $\nu(h)\in On$, for $h\in H^*$, by recursion on $\ell^*(h)$ as follows:

$$\nu(h):=\|\{h\}\|,\ \text{ if }\ h\in H;\qquad \nu(I_{k,A}h_1):=\nu(h_1);$$
$$\nu(R_C h_0 h_1):=\nu(h_0)+\nu(h_1);\qquad \nu(Eh_1):=\omega^{\nu(h_1)}.$$

Using (1) one obtains by induction on $\ell^*(h)$:

(2) $\forall h\in H^*\forall n(\mathfrak{r}(h)\neq\mathsf{Ax}\Rightarrow\nu(\mathfrak{s}_n h)<\nu(h))$ (cf. proof of 3.7d).

This gives

$$\forall(n_i)_{i\in\mathbb{N}}\exists k(\{h\}(\langle n_0,\dots,n_{k-1}\rangle)=\mathfrak{q}([h,\langle n_0,\dots,n_{k-1}\rangle])\in\mathsf{Ax}),$$

i.e. $\{h\}\in WT$. $\square$

5.8 Theorem. *If $\mathscr{H}$ (and thus also $\mathscr{H}^*$) is wellfounded then, for all $h_0, h_1 \in H^*$, we have*

$\{I_{k,A}h_1\} = \mathscr{I}_{k,A}(\{h_1\})$, $\{R_C h_0 h_1\} = \mathscr{R}_C(\{h_0\}, \{h_1\})$, $\{Eh_1\} = \mathscr{E}(\{h_1\})$.

Proof. We define $\mathfrak{v}: H^* \to WT$ by recursion on $\ell^*(h)$ (as in Sect. 3): $\mathfrak{v}(h) := \{h\}$, if $h \in H$; $\mathfrak{v}(I_{k,A}h_1) := \mathscr{I}_{k,A}(\mathfrak{v}(h_1))$: $\mathfrak{v}(R_C h_0 h_1) := \mathscr{R}_C(\mathfrak{v}(h_0), \mathfrak{v}(h_1))$; $\mathfrak{v}(Eh_1) := \mathscr{E}(\mathfrak{v}(h_1))$.
By Theorem 5.3a) we have, for all $h \in H$,

$$(*) \qquad \mathfrak{v}(h)(\langle\,\rangle) = \mathfrak{q}(h) \;\&\; \mathfrak{v}(\mathfrak{s}_n h) = \mathfrak{v}(h)[n].$$

But for $h \in H^* \backslash H$ we defined $\mathfrak{e}(h)$, $\mathfrak{r}(h)$, $\mathfrak{s}_n h$ just in such a way that $(*)$ also holds for all $h \in H^*$ (cf. 3.7, 5.5). So by 5.4a) we obtain $\mathfrak{v} = \{\cdot\}$ and thus

$$\{I_{k,A}h_1\} = \mathfrak{v}(I_{k,A}h_1) = \mathscr{I}_{k,A}(\mathfrak{v}(h_1)) = \mathscr{I}_{k,A}(\{h_1\})$$

etc. □

6 Continuous cut-elimination

In this section we define an extension $\mathscr{E}': TREE \to TREE$ of $\mathscr{E}: WT \to WT$ such that $\forall \varphi \in TREE$ (φ locally correct $\Rightarrow \mathscr{E}'(\varphi)$ locally correct & $deg(\mathscr{E}'(\varphi)) \leqq deg(\varphi) \dot{-} 1$). Moreover the functional $\lambda\varphi\lambda\sigma . \mathscr{E}'(\varphi)(\sigma)$ will turn out primitive recursive so that $\mathscr{E}'$ is automatically continuous. In fact we will prove a somewhat sharper result, namely

$$\forall \varphi, \psi \in TREE \,\forall \sigma \in \mathbb{N}^{<\omega}(\varphi \restriction N_\sigma = \psi \restriction N_\sigma \Rightarrow \mathscr{E}'(\varphi)(\sigma) = \mathscr{E}'(\psi)(\sigma)),$$

with

$$N_{\langle n_0, \ldots, n_{k-1}\rangle} := \{\langle m_0, \ldots, m_{\ell-1}\rangle : \ell \leqq k \;\&\; \{m_0, \ldots, m_{\ell-1}\} \subseteq \{0, 1, n_0, \ldots, n_{k-1}\}\}.$$

6.1 Definition. $H := \mathbb{N}^{<\omega}$ and, for each $\varphi \in TREE$, $\mathscr{H}_\varphi := (H, \varphi, \mathfrak{s})$ with $\mathfrak{s}: \mathbb{N} \times H \to H$, $\mathfrak{s}_n(\sigma) := \sigma * \langle n\rangle$. Obviously $\mathscr{H}_\varphi$ is a notation system. Let $\mathscr{H}_\varphi^* = (H^*, \mathfrak{q}^\varphi, \mathfrak{s}^\varphi)$ be the $*$-extension of $\mathscr{H}_\varphi$ as defined in 5.5. Let $\{\cdot\}^\varphi: H^* \to TREE$ be the canonical interpretation for $\mathscr{H}_\varphi^*$. Then we define

$$\mathscr{E}': TREE \to TREE, \qquad \mathscr{E}'(\varphi) := \{E\langle\,\rangle\}^\varphi.$$

6.2 Lemma. $\{\tau\}^\varphi = \varphi[\![\tau]\!]$, *for each* $\tau \in H$.

Proof. For every $\sigma = \langle n_0, \ldots, n_{k-1}\rangle \in H$: $\{\tau\}^\varphi(\sigma) = \varphi([\tau, \sigma]) = \varphi(\mathfrak{s}_{n_{k-1}} \ldots \mathfrak{s}_{n_0}(\tau)) = \varphi(\tau * \sigma) = \varphi[\![\tau]\!](\sigma)$. □

6.3 Theorem. $\varphi \in WT \Rightarrow \mathscr{E}'(\varphi) = \mathscr{E}(\varphi) \in WT$.

Proof. Using Lemma 6.2 and Theorem 5.8 we obtain: $\varphi \in WT \Rightarrow \forall \sigma(\{\sigma\}^\varphi = \varphi[\![\sigma]\!] \in WT) \Rightarrow \mathscr{H}_\varphi$ wellfounded $\Rightarrow \mathscr{E}'(\varphi) = \{E\langle\,\rangle\}^\varphi = \mathscr{E}(\{\langle\,\rangle\}^\varphi) = \mathscr{E}(\varphi)$. □

6.4 Theorem. *φ locally correct $\Rightarrow \mathscr{E}'(\varphi)$ locally correct and $deg(\mathscr{E}'(\varphi)) \leqq deg(\varphi) \dot{-} 1$.*

Proof. By assumption we have $\forall \sigma . LC(\varphi(\sigma), (\varphi^1(\sigma * \langle n\rangle))_{n \in \mathbb{N}})$, i.e. $\mathscr{H}_\varphi$ is correct. Now by 5.7a) and 5.3b) we obtain that $\mathscr{H}_\varphi^*$ is correct and thus $\mathscr{E}'(\varphi) = \{E\langle\,\rangle\}^\varphi$ is locally correct. Let $m := deg(\varphi) \in \mathbb{N}$. Then $\mathfrak{d}(h) := m$ is a cut-bound for $\mathscr{H}_\varphi$. By 5.7b) this can be extended to a cut-bound $\mathfrak{d}$ for $\mathscr{H}_\varphi^*$ with $\mathfrak{d}(E\langle\,\rangle) = \mathfrak{d}(\langle\,\rangle) \dot{-} 1 = m \dot{-} 1$. By 5.3c) we have $deg(\mathscr{E}'(\varphi)) \leqq \mathfrak{d}(E\langle\,\rangle)$. □

6.5. *Remark.* One easily verifies that the functions $\mathfrak{q}^\varphi : H^* \to RULE \times SEQ$ and $\mathfrak{s}^\varphi : \mathbb{N} \times H^* \to H^*$ are uniformly primitive recursive in φ. Together with $\mathscr{E}'(\varphi)(\langle n_0, \ldots, n_{k-1}\rangle) = \mathfrak{q}^\varphi \mathfrak{s}^\varphi_{n_{k-1}} \ldots \mathfrak{s}^\varphi_{n_0}(E\langle\,\rangle)$ this yields that the functional $(\varphi, \sigma) \mapsto \mathscr{E}'(\varphi)(\sigma)$ is primitive recursive and therefore continuous, i.e. $\forall \varphi, \sigma \exists M \subseteq \mathbb{N}^{<\omega}$ (M finite & $\forall \psi(\varphi \restriction M = \psi \restriction M \Rightarrow \mathscr{E}'(\varphi)(\sigma) = \mathscr{E}'(\psi)(\sigma))$ (∗). In the following we establish (∗) by a more direct argument. In fact we will prove more, namely

6.6 Theorem. $\forall \varphi, \psi \in TREE \, \forall \sigma \in \mathbb{N}^{<\omega}(\varphi \restriction N_\sigma = \psi \restriction N_\sigma \Rightarrow \mathscr{E}'(\varphi)(\sigma) = \mathscr{E}'(\psi)(\sigma))$, *with the above defined* N_σ.

Proof (informal). Let $\sigma = \langle n_0, \ldots, n_{k-1}\rangle$ and $\sigma_i := \langle n_0, \ldots, n_{i-1}\rangle$ $(i = 0, \ldots, k)$. Then $\mathscr{E}'(\varphi)(\sigma) = \mathfrak{q}^\varphi \mathfrak{s}^\varphi_{n_{k-1}} \ldots \mathfrak{s}^\varphi_{n_0}(E\langle\,\rangle)$. So $\mathscr{E}'(\varphi)(\sigma)$ is obtained by computing successively $h_1 := \mathfrak{s}^\varphi_{n_0}(E\langle\,\rangle)$, $h_2 := \mathfrak{s}^\varphi_{n_1}\mathfrak{s}^\varphi_{n_0}(E\langle\,\rangle), \ldots, h_k := \mathfrak{s}^\varphi_{n_{k-1}} \ldots \mathfrak{s}^\varphi_{n_0}(E\langle\,\rangle)$, and $\mathfrak{q}^\varphi(h_k)$. – One easily verifies that for computing $\mathfrak{q}^\varphi(h)$ and $\mathfrak{s}^\varphi_n(h)$ $(h \in H^*)$ one only needs to know $\varphi \restriction K(h)$, where $K(h) := \{\sigma \in H : \sigma \text{ occurs in } h\}$. In addition $K(h_i) \subseteq N_{\sigma_i} \subseteq N_0$ for $i = 0, \ldots, k$. Hence for computing $\mathfrak{q}^\varphi(h_k)$ (i.e. $\mathscr{E}'(\varphi)(\sigma)$) we only need to know $\varphi \restriction N_\sigma$. □

Formally the theorem is an immediate consequence of Lemmata 6.8, 6.9 below.

6.7 Definitions

(i) $M * N := \{\sigma * \tau : \sigma \in M, \tau \in N\}$ $\quad (M, N \subseteq H)$

(ii) $K(h) := \{\sigma \in H : \sigma \text{ occurs in } h\}$ $\quad (h \in H^*)$

(iii) $[h, \langle n_0, \ldots, n_{k-1}\rangle]^\varphi := \mathfrak{s}^\varphi_{n_{k-1}} \ldots \mathfrak{s}^\varphi_{n_0} h$ $\quad (h \in H^*)$

6.8 Lemma

a) $h \in H^* \Rightarrow K(\mathfrak{s}^\varphi_n h) \subseteq K(h) * N_{\langle n\rangle}$

b) $h \in H^* \;\&\; \varphi \restriction K(h) = \omega \restriction K(h) \Rightarrow \mathfrak{q}^\varphi(h) = \mathfrak{q}^\psi(h) \;\&\; \mathfrak{s}^\varphi_n h = \mathfrak{s}^\psi_n h$.

Proof by induction on $\ell^*(h)$.

6.9 Lemma

a) $K([E\langle\,\rangle, \sigma]^\varphi) \subseteq N_\sigma$

b) $\varphi \restriction N_\sigma = \psi \restriction N_\sigma \Rightarrow [E\langle\,\rangle, \sigma]^\varphi = [E\langle\,\rangle, \sigma]^\psi$.

Proof by induction on length(σ) using Lemma 6.8.

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