Two Types of Indefinites: Hilbert & Russell

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Abstract

This paper compares Hilbert's ϵ-terms and Russell's approach to indefinite descriptions, Russell's indefinites for short. Despite the fact that both accounts are usually taken to express indefinite descriptions, there is a number of dissimilarities. Specifically, it can be shown that Russell indefinites—expressed in terms of a logical ρ-operator—are not directly representable in terms of their corresponding ϵ-terms. Nevertheless, there are two possible translations of Russell indefinites into epsilon logic. The first one is given in a language with classical ϵ-terms. The second translation is based on a refined account of epsilon terms, namely *indexed* ϵ-terms. In what follows we briefly outline these approaches both syntactically and semantically and discuss their respective connections; in particular, we establish two equivalence results between the (indexed) epsilon calculus and the proposed ρ-term approach to Russell's indefinites.

Keywords: Indefinite descriptions, epsilon terms, choice semantics, Hilbert, Russell

We would like to thank Richard Zach, Hannes Leitgeb, and the participants of the *Epsilon 2015* conference in Montpellier for helpful discussions. We are also grateful to two anonymous referees for their comments on earlier drafts of this paper.

*Research on this paper was supported by the Alexander von Humboldt Foundation.

†Research on this paper was supported by the Austrian Science Fund (Grant number P 27718-G16) as well as by the German Research Foundation (Project "*Mathematics: Objectivity by Representation*").

1 Introduction

In linguistics and philosophy of language, one generally distinguishes between two ways to logically represent indefinite descriptions of the form

$$\text{An } A \text{ is a } B.$$

The first approach goes back to Russell and views the expression "an A" as non-referential. Indefinites of this form are taken to function semantically like existential quantifiers (or variables bound by an existential quantifier). According to this *quantificational* account, the logical form of the above sentence is best captured by the following existentially quantified statement:

$$\exists x(A(x) \land B(x))$$

The second approach has roots in work by Hilbert and has recently been further developed by von Heusinger and Egli.[1] This is to view the expression "an A" as a constant term that denotes a particular object. Specifically, an indefinite phrase so understood can be presented logically by an epsilon term $\epsilon_x A(x)$. Informally speaking, this term picks out an *arbitrary* object that satisfies formula A if such an object exists. Accordingly, the indefinite description stated above is presented logically not in terms of a quantified statement, but in terms of the following statement:[2]

$$B(\epsilon_x A(x))$$

These two logical reconstructions of indefinite descriptions seem to be based on two different ways to understand indefinites. Let us dub them *Russell* and *Hilbert* indefinites. The central conceptual difference between them is usually taken to be the fact that unlike Hilbert's indefinites, Russell indefinites are not referring expressions or terms with a fixed reference. Nevertheless, as we want to show in this paper, there exists a natural way to represent Russell's ambiguous descriptions in terms of a logical language with a term-forming operator. Thus, in analogy to the representation of free choice indefinites in an epsilon-term logic, we will outline here a operator-based logic for the expression of Russel indefinites.

The central aim in this paper is to compare the representation of indefinites in term of epsilon logic with a Russellian approach to indefinite descriptions in terms of a logic based on a ρ-operator. As we will show, there is a number of dissimilarities between the two accounts. Specifically, it can be shown that Russell's ambiguous descriptions—expressed in terms of a logical ρ-operator—are not directly

[1] See, in particular, [16] and [15]

[2] See [16] for a detailed discussion of both approaches and for further references.

representable in terms of their corresponding ϵ-terms. Nevertheless, there are two possible translations of Russell indefinites into epsilon logic. The first one is based on an embedding of a language with ρ-terms into a classical language with ϵ-terms. The second translation is based on a refined account of epsilon terms, namely *indexed* ϵ-terms first introduced by von Heusinger.[3] In what follows we briefly outline these approaches to represent indefinites both syntactically and semantically and discuss their respective connections; in particular, we establish two equivalence results between the classical and indexed epsilon calculus on the one hand and the ρ-term approach to Russell's account of indefinite descriptions on the other hand.

The paper is organized as follows: Section 2 will introduce the extensional epsilon calculus EC as well as a suitable choice semantics for (closed) epsilon terms. Section 3 will then present a logic for ρ-terms based on Russell's remarks on indefinite descriptions. Section 4 will then give a closer comparison between the two logical representations of ambiguous descriptions. Specifically, we present a translation of the ρ-term presentation of "An A is a B" in classical epsilon logic (4.1) as well as in a language of indexed epsilon terms (4.2). Finally, section 5 will contain some concluding remarks and suggestions for future research.

2 Hilbert's ϵ-terms

A natural logical representation of indefinites (or indefinite descriptions) can be given in terms of epsilon terms, that is, terms formed with the help of an epsilon operator.[4] As understood by Hilbert, the ϵ-operator functions as a logical term-forming operator: given a first-order formula $A(x)$ with variable x occurring free in it, $\epsilon_x A(x)$ is a closed term in which all occurrences of x are bound. Informally speaking, this term refers to an arbitrary object satisfying the formula A if there exists such an object.[5]

Different epsilon calculi have been proposed in the literature since Hilbert to describe the logical behaviour of such terms. The *extensional* EC usually consists of two axiom schemes (in addition to the standard axioms and deduction rules of

[3] See, in particular, [16] and [9].

[4] Epsilon terms were originally introduced in Hilbert's proof-theoretic work on the foundations of mathematics in the 1920s. See, in particular, [14] and [17] for detailed historical discussions of the development of the epsilon calculus as well as of Hilbert's epsilon substitution method in his syntactic consistency proofs. Compare also [1] for a first systematic study of the epsilon calculus.

[5] The following discussion of epsilon logic follows closely the presentation given in [18]. See also [13] for a similar discussion of epsilon terms and their choice semantics.

first-order logic), namely

$$A(t) \to A(\epsilon_x A(x)) \qquad \text{(Critical formulas)}$$

$$\forall x(A(x) \leftrightarrow B(x)) \to \epsilon_x A(x) = \epsilon_x B(x) \qquad \text{(Extensionality)}$$

The second axiom expresses an extensionality principle for epsilon logic: if two formulas are equivalent, then their respective ϵ-representatives are identical.[6]

Hilbert's original motivation for the introduction of a calculus for epsilon terms was to show that one can explicitly define the first-order quantifiers in terms of epsilon terms in the following way:

$$\exists x A(x) :\leftrightarrow A(\epsilon_x A(x)) \qquad \text{(Def}\exists\text{)}$$

$$\forall x A(x) :\leftrightarrow A(\epsilon_x \neg A(x)) \qquad \text{(Def}\forall\text{)}$$

It is a well known fact that first-order predicate logic is embeddable in EC. This is based on a translation function $(.)^\epsilon$ that maps expressions of the a first-order language $\mathcal{L}$ to expressions of the language with epsilon-terms $\mathcal{L}_\epsilon$ (see [10]):

1. $e^\epsilon = e$, for e a variable or constant symbol
2. $P(t_1, \ldots, t_n)^\epsilon = P(t_1^\epsilon, \ldots, t_n^\epsilon)$
3. $f(t_1, \ldots, t_n)^\epsilon = f(t_1^\epsilon, \ldots, t_n^\epsilon)$
4. $(\neg A)^\epsilon = \neg A^\epsilon$
5. $(A \wedge B)^\epsilon = A^\epsilon \wedge B^\epsilon$
6. $(A \vee B)^\epsilon = A^\epsilon \vee B^\epsilon$
7. $(\exists x(A(x)))^\epsilon = A^\epsilon(\epsilon_x A(x)^\epsilon)$
8. $(\forall x(A(x)))^\epsilon = A^\epsilon(\epsilon_x \neg A(x)^\epsilon)$

As a consequence of this, any first-order formula can be represented as a quantifier-free formula in $\mathcal{L}_\epsilon$, a result which was of central importance in Hilbert's proof theoretic work, in particular, in his two ϵ-theorems.[7]

[6] It should be noted here that the extensionality axiom was already mentioned in Hilbert's work, but not used in the proofs of his famous epsilon theorems. The axiom is discussed again in Asser's study of the epsilon calculus [2] as well as in [6]. See [18] and [8] for modern presentations of intensional and extensional epsilon calculi.

[7] Compare again [17] and [10] for further details.

Turning to the semantic interpretation of (extensional) EC, we saw that epsilon terms of the form $\epsilon_x A(x)$ were understood by Hilbert and subsequent logicians to function as referring indefinite expressions, i.e. as terms that pick out *any* object which satisfies the formula $A(x)$ under the condition that there are such objects. Compare, for instance, Hilbert & Bernays' informal description of the semantic interpretation of such terms in the second volume of *Grundlagen der Mathematik* (1939):

> Syntactically, [the ϵ-symbol] provides a function of a variable predicate, which–besides the argument to which the variable bound by the ϵ-symbol refers–may contain free variables as arguments ("parameters"). The value of this function for a given predicate A (for fixed values of the parameters) is an object of the universe for which–according to the semantical translation of the formula (ϵ_0)–the predicate A holds, provided that A holds for any object of the universe at all. [5, p.12]

A natural model-theoretic formalization of this understanding of epsilon terms is given today in terms of a choice-functional semantics. A choice semantics for the extensional EC can be characterized as follows:[8] an interpretation $\mathfrak{M}$ of the language $\mathcal{L}_\epsilon$ has the form $\langle D, I\rangle$ with D a domain and I an interpretation function for the signature of $\mathcal{L}_\epsilon$. We further hold that $s : Var \rightarrow D$ is an assignment function on $\mathfrak{M}$. The ϵ-operator is interpreted by an extensional choice function of the form $\delta : \wp(D) \rightarrow D$ such that, for any $X \subseteq D$:

$$\delta(X) = \begin{cases} x \in X, & \text{if } X \neq \emptyset; \\ x \in D & \text{otherwise.} \end{cases}$$

Such a choice function assigns a "representative" object to any non-empty subset of D. It gives an arbitrary object from domain D in case the set X is empty.

Based on this notion of extensional choice functions, one can give a choice-functional semantics for EC. Valuation rules for terms of $\mathcal{L}_\epsilon$ not containing epsilon terms are specified as for standard first-order logic. In addition, the semantic evaluation of ϵ-terms is specified relative to a structure $\mathfrak{M}$, assignment function s, and a choice function δ on $\mathfrak{M}$ based on the following valuation rule:[9]

$$val^{\mathfrak{M},\delta,s}(\epsilon_x A(x)) = \delta(\{d \in D \mid \mathfrak{M}, s[x/d] \models A(x)\}).$$

[8] Early formulations of a choice-functional semantics for the epsilon calculus were given in [2] and in [6]. See also [8] and [18] for modern presentations of a choice semantics for extensional EC. Both [6] and [18] contain a proof of the completeness of EC with respect to this semantics.

[9] The following rule applies only to closed epsilon terms. For a more comprehensive discussion of the semantics of extensional and intensional epsilon logics, including valuation rules for open epsilon terms see [8].

The semantic value of a given term $\epsilon_x A(x)$ is thus the object that the choice function δ picks out from the truth-set of formula A. If the set defined by A is empty, then δ picks out any object in D otherwise. Based on this, the semantic notions of satisfaction of formulas of $\mathcal{L}_\epsilon$ can then be specified in the usual way and in direct analogy with first-order logic.

This choice-functional semantics for EC can be seen as a way to make precise the particular *indefinite* character of epsilon terms. As is argued in [13], the reason to view such terms as indefinites lies precisely in their semantic character, more specifically, in the kind of "arbitrary reference" usually associated with such terms. This mode of reference (typical also for instantial terms in logic and mathematical reasoning) has recently been described by Magidor and Breckenridge in the following way:

> *Arbitrary Reference* (AR): It is possible to fix the reference of an expression arbitrarily. When we do so, the expression receives its ordinary kind of semantic-value, though we do not and cannot know which value in particular it receives. [3, p.378]

This kind of undetermined reference is also characterstic for Hilbert's understanding of epsilon terms as indefinite expressions. Thus, in Hilbert's account of indefinite phrases, indefiniteness is explained best in the sense that such phrases refer *arbitrarily* to objects. Moreover, one can view the valuation rule for ϵ-terms stated above as a way to make precise this very notion of arbitrary reference. We can thus paraphrase the epsilon-term representation $B(\epsilon_x(A(x))$ of the indefinite description stated in the introduction as "An *arbitrary* A is a B." With this in mind, let us now turn to Russell's account of indefinite descriptions.

3 Russell ρ-terms

We want to motivate our presentation of Russell's account of indefinite descriptions with the following quote:

> The definition is as follows: The statement that an object having the property ϕ has the property ψ means: The joint assertion of ϕx and ψx is not always false. So far as logic goes, this is the same proposition as might be expressed by some ϕ's are ψ's; but rhetorically there is a difference, because in the one case there is a suggestion of singularity, and in the other case of plurality. [12, p.171]

Given this quote, an ambiguous description in this sense is the occurrence of an indefinite phrase "an A" in a context B, viz. "an A is a B".[10] As we saw in the Introduction, the standard formalization of this in first-order logic is:

$$\exists x(A(x) \wedge B(x))$$

where both A and B are unary predicates or formulas.[11] Russell's indefinites can alternatively be expressed in terms of a term-forming operator that is in several ways similar to Hilbert's ϵ-operator. For a given formula A with x occurring free in it, let $\rho_x A(x)$ be a term standing for "an x, such that x has A". This ρ-operator can then be defined in the following way (relative to some context):[12]

$$B(\rho_x A(x)) :\leftrightarrow \exists x(A(x) \wedge B(x)) \qquad \text{(Def } \rho\text{)}$$

In this paper, we assume that for Russell indefinite descriptions can function (at least on the surface) as singular terms – as the following quote shows:

> The identity in 'Socrates is a man' is identity between an object named (accepting 'Socrates' as a name, subject to qualifications explained later) and an object ambiguously described. [12, p.172]

Again, nowadays we are more inclined to view a sentence as 'Socrates is a man' as an atomic sentence in which 'is a man' is predicated from (a singular term) 'Socrates'.

With respect to (Def ρ) Russell's famous theory of definite description can be seen as an extension of his account of indefinite descriptions by adding a uniqueness condition to the existential condition already present in (Def ρ). A contextual definition of an indefinite description can also be constructed from the definite description "*the* A is a B", expressed by $\exists x(A(x) \wedge \forall y(A(y) \rightarrow x = y) \wedge B(x))$, simply by dropping the uniqueness clause.[13]

For reasons belonging to Russell's particular approach to proper names, neither definite nor indefinite descriptions belong to the class of proper singular terms.

[10] For a closer discussion of the relation between ambiguous and definite descriptions, see also Russell's classical paper [12].

[11] A problem with this existential reading of indefinite descriptions in the formalization of natural language discourse is that $A(x)$ and $B(x)$ are treated symmetrically in $\exists x(A(x) \wedge B(x))$. In the natural language sentence above, this is not necessarily the case. We would like to thank one of the reviewers for bringing this point to our attention.

[12] It should noted here that this operator-based interpretation of indefinite descriptions was not given by Russel himself, but has been developed by the second author of the present article.

[13] Compare Russell's own presentation of his theory of descriptions in [11].

Instead, his stance on both types of descriptions is that they are incomplete symbols, i.e. that the meaning of the (in)definite description is constituted by some context.

In analogy to Russell's considerations for definite descriptions ρ-terms are not given a direct interpretation; rather these terms are interpreted in a *contextual* way, or are defined contextually. If we follow Russell in his understanding of both definite and indefinite descriptions, then there is no need of extending the semantical framework of first order predicate logic since the semantical conditions for formulas containing a ρ-term (or a definite description) can be directly read off the righthand side of (Def ρ).

However, if we wish to to give ρ-terms special semantical considerations, we outline an approach for doing so. This approach is presented rather informally here: given a model $\mathfrak{M}$, let $\mathcal{A} \subseteq dom(\mathfrak{M})$ be the set of objects defined by formula A, and let $\mathcal{B} \subseteq dom(\mathfrak{M})$ be the set defined by formula B. Intuitively speaking, the operator ρ picks out one element in $\mathcal{A}$ that is also in $\mathcal{B}$ (assuming that their intersection is non-empty). The central conceptual idea underlying this Russellian account of indefinites is a kind of semantic *context dependency*, that is the fact that the specification of an A-representative picked out by the operator depends on the particular sentential context in which formula A occurs. In terms of the informal semantics underlying the ρ-operator, this point is given by the constraint that the selection of a particular ρ-representative of set $\mathcal{A}$ is specified only relative to a given 'context' set $\mathcal{B}$ in which the ρ-representative also occurs. Thus, in a slogan, one can say that the reference of a given term is a function of its particular sentential context. Clearly, this only works if neither $\mathcal{A}$ nor $\mathcal{B}$ is empty. In the case that $\mathcal{A}$ is empty, one could think of a solution familiar from the treatment of definite descriptions as done by Carnap. In this case we would require a *chosen object* which is by fiat in the extension of every predicate.

4 Russell and Hilbert indefinites

The relationship between Hilbert's and Russell's accounts of indefinite descriptions can be studied in a precise way by comparing the two underlying logics and their respective term-forming operators. In particular, it can be shown that the two logical representations of "An A is a B" in terms of an epsilon and a rho operator do not coincide.[14] Thus, given two first-order formulas A, B, we can show that:

$$B(\epsilon_x A(x)) \nleftrightarrow B(\rho_x A(x))$$

[14] The following discussion follows closely the presentation of this result in [13].

Proof sketch: To see that the left-to-right implication does not hold, consider a model $\mathfrak{M}$ where $\mathcal{A} = \emptyset$ as well as a choice-function δ interpreting the ϵ-operator such that $\delta(\mathcal{A}) \in \mathcal{B}$. Relative to $\mathfrak{M}$ and δ, the antecedent will turn out true. However, the consequent will be false given that $\mathcal{A} \cap \mathcal{B} = \emptyset$. It follows from this that $\exists x(A(x) \wedge B(x))$ is false and therefore also the right-hand side of the above equivalence. In order to show the right-to-left direction to be non-valid, consider a model where $\mathcal{A} \cap \mathcal{B} \neq \emptyset$ and $\mathcal{A} \not\subseteq \mathcal{B}$. Consider a choice function interpreting the ϵ-operator such that $\delta(\mathcal{A}) = x \notin \mathcal{B}$. The right hand side formula will clearly be true in this model. Nevertheless, the epsilon formula on the left-hand side will be false relative to the particular choice function δ.[15]

This shows that *Hilbert* and *Russell* indefinites are not identical. The main conceptual reason for this fact lies in their different semantic nature. As we saw, Hilbert indefinites are characterized by a specific mode of reference, that is, by the fact that the terms representing such indefinites refer *arbitrarily* to objects in the domain. The indefinite nature of such terms is thus best explained in terms of their arbitrary reference. By contrast, indefiniteness in Russell's sense primarily means *non-uniqueness* of reference (in opposition to his account of definite descriptions). Moreover, another central semantic feature of ρ-terms is, as we saw, the fact that their reference is not specified in isolation, but *contextually*, that is, relative to a given sentential context. This semantic *context dependency* is clearly missing in the choice semantic treatment of the extensional ϵ-logic presented in Section 2. As we saw above, the semantic value of an epsilon term is specified in a model relative to a specific choice function. Given such a choice-functional interpretation, the semantic value of $\epsilon_x A(x)$ remains *stable* under changes of sentential contexts in which the term might occur.

The question remains whether Russell's and Hilbert's accounts of indefinite descriptions, if expressed by means of rho-terms and epsilon-terms respectively, are inter-translatable. The answer to this is positive. In fact, we will present two different ways in which Russell's account of indefinite descriptions can be expressed in a language containing an epsilon operator.

4.1 Russell indefinites in EC

The first way to translate the Russellian account of definition descriptions into epsilon logic is based on the classical language of epsilon terms outlined in section 2. Recall that first-order predicate logic can be embedded into EC based on a translation function specified above. Given that RC is embeddable in predicate logic, it follows that RC must also be interpretable in (classical) EC. In particular, we can

[15] Compare, again, [13] for a discussion of this result.

translate the ρ-term representation of the indefinite description "An A is a B" into a quantifier-free statement of $\mathcal{L}_\epsilon$ of the following from:

$$\begin{aligned} B(\rho_x A(x)) &\leftrightarrow (\exists x(A(x) \wedge B(x)))^\epsilon \\ &\leftrightarrow A(\underbrace{\epsilon_x(A(x) \wedge B(x))}_{e1}) \wedge B(\underbrace{\epsilon_x(A(x) \wedge B(x))}_{e1}) \end{aligned}$$

Intuitively speaking, the context dependency of the Russell indefinite is reflected here on the right-hand side by the epsilon term built from the conjunction of formulas $A(x)$ and $B(x)$. One could say that context dependency of the semantic interpretation of the ρ-term in the sentential context $B(\rho_x A(x))$ is *internalized* here in a more complex ϵ-term. Notice, moreover, the defining formula on the right-hand side contains two identical epsilon terms $\epsilon_x(A(x) \wedge (B(x))$.[16] The fact that the term occurs in both conjuncts on the right-hand side is central for the translation of Russell indefinite descriptions into EC. Put differently, it can easily be shown that the weaker equivalence statement

$$B(\rho_x A(x)) \leftrightarrow B(\epsilon_x(A(x) \wedge B(x)))$$

is not generally true. To see this, consider a model in which $\mathcal{A} \cap \mathcal{B} = \emptyset$ and a choice function δ interpreting the ϵ-operator such that $\delta(\mathcal{A} \cap \mathcal{B}) \in \mathcal{B}$. (The choice function picks out an arbitrary member of the model domain here that happens to be in $\mathcal{B}$.). The right-hand side of the formula would then be true, but the left-hand side would clearly be false (by definition of the ρ-operator). Counterexamples of this form are explicitly ruled out in the above stronger equivalence statement by the fact that any possible semantic value of the term $\epsilon_x(A(x) \wedge B(x))$ is forced to be a member of *both* $\mathcal{A}$ and $\mathcal{B}$ if the right-hand side of formula is to be true.

4.2 Russell indefinites in indexed EC

While the translation of Russell's indefinite descriptions in the classical language of extensional EC is somewhat cumbersome, it turns out that they can be formulated in a natural extension of it, namely in a language of *indexed* epsilon terms. Indexed epsilon terms have been subject to recent investigation, both in the logical and semantic literature.[17]

[16] Thus, for this simple example of indefinite descriptions we do not have to concern us here with nested epsilon terms and the *rank* and *degree* of epsilon terms. See [10] for further details on this topic.

[17] Such ϵ-terms were first discussed in work by Egli and von Heusinger (e.g. [4], [16]) and subsequently (more systematically) by Mints & Sarenac in [9].

Roughly, the language $\mathcal{L}_{\epsilon_i}$ of an indexed epsilon-calculus (IEC) contains ϵ-operators $\epsilon_i x$ indexed by context variables $i, j, \ldots$ or context constants $c, c', \ldots$. If $A(x)$ is a formula with a free variable x, and i a context variable, then $\epsilon_i x A(x)$ is a term of the language. The context variables occurring in such a term can also be bound by existential and universal quantifiers and allow the formulation of sentences such as $\exists i \exists j B(\epsilon_i x A(x), \epsilon_j x A(x))$.[18] Intuitively speaking, the context-indices of an epsilon-symbol represent different contexts (or situations) in which the ϵ-term may occur. The epsilon operator thus picks out one particular A-representative relative to a particular context i, and possibly a different object relative to another context j. This context variability can be expressed semantically in terms of indexed choice functions that are intended to represent such contexts.

The semantic interpretation of $\mathcal{L}_{\epsilon_i}$ differs from that of $\mathcal{L}_{\epsilon}$ only in one point.[19] In the case of classical epsilon terms, choice functions interpreting the ϵ-operator are considered as *external* to a model. In the case of indexed epsilon terms, a family of possible choice functions is incorporated in the model. A *choice structure* interpreting $\mathcal{L}_{\epsilon_i}$ is thus a triple $\mathfrak{M} = \langle D, I, \mathbb{F} \rangle$ where D and I are interpreted as before and $\mathbb{F}$ is a non-empty set of choice functions. As pointed out in [9], one can understand the context variables of an ϵ-operator as ranging over this collection of choice functions. An assignment function s on $\mathfrak{M}$ then maps individual variables to elements in the domain D and context variables to elements in $\mathbb{F}$. A valuation rule for indexed ϵ-terms can then be specified analogously to the above case:[20]

$$val^{\mathfrak{M},s}(\epsilon_i x A(x)) = s(i)(\{d \in D \mid \mathfrak{M}, s[x/d] \models A(x)\}),$$

where $s(i) \in \mathbb{F}$.

A number of axioms have been introduced in [9] to describe the logical behaviour of indexed epsilon terms. These contain, in particular, the following variants of the axioms of classical EC:

$$A(t) \rightarrow A(\epsilon_a x A(x)), \text{ with } a \text{ a context term.} \qquad \text{(Critical formulas)}$$

$$\forall x(A(x) \leftrightarrow B(x)) \rightarrow \forall i(\epsilon_i x A(x) = \epsilon_i x B(x)) \qquad \text{(Extensionality)}$$

$$\varphi[i/a] \rightarrow \exists i \varphi, \text{ with } a \text{ and } i \text{ context terms.} \qquad \text{(EI for context variables)}$$

The second axiom of extensionality states that equivalent formulas have the same ϵ-representative in all possible contexts. The third axiom states that if a formula

[18] Quantifiers can be defined in the language of indexed epsilon terms in a number of equivalent ways. For instance, the existential quantifier can be specified by $\exists x A(x) :\leftrightarrow \exists i A(\epsilon_i x A(x))$ or by $\exists x A(x) :\leftrightarrow \forall i A(\epsilon_i x A(x))$. See [9, p.619].

[19] The following discussion is based on the presentation given in [9].

[20] See again [9] for a more detailed presentation.

contains an epsilon term with a context constant a, then a can be substituted by an existentially bound context variable i.[21]

Russell's indefinite descriptions turn out to be embeddable in the language of IEC. In particular, it can be shown that the following equivalence holds:[22]

$$B(\rho_x A(x)) \leftrightarrow \exists x(A(x)) \wedge \exists i B(\epsilon_i x A(x))$$

The right hand side of the formula states that (i) set $\mathcal{A}$ is nonempty and (ii) there exists at least one context in which the element picked out from $\mathcal{A}$ by the corresponding choice function also lives in set $\mathcal{B}$. This is precisely the claim also expressed on the left hand side of the formula. Thus, both sides are true if and only if there exists an element in the intersection of $\mathcal{A}$ and $\mathcal{B}$. A proof of this theorem can be given in a combined indexed epsilon and rho-calculus:

Proof sketch: Consider first the left-to-right direction: assume $B(\rho_x A(x))$. Then, by definition (Def ρ), it follows that $\exists x(A(x) \wedge B(x))$ and therefore also $\exists x A(x)$. A theorem in IEC is $\exists x(A(x) \wedge B(x)) \rightarrow \exists i B(\epsilon_i x A(x))$ (see [9, p.622]) Together, these results give us $B(\rho_x A(x)) \rightarrow \exists x(A(x)) \wedge \exists i B(\epsilon_i x A(x))$.
The other direction follows directly from another theorem in IEC, namely $\exists x(A(x)) \wedge \exists i B(\epsilon_i x(A(x))) \rightarrow \exists x(A(x) \wedge B(x))$ and (Def ρ) (see again [9, p.622]).

This result shows that Russell's account of indefinite descriptions (and the *contextual* principle of indefinites implicit in it) can also be represented in terms of Hilbert's epsilon terms if one allows the generalization of the language of EC to include context indices. The main reason for this is that the extended language of IEC allows one to capture also *syntactically* (that is, by means of context variables and context quantifiers) the kind of semantic context sensitivity that is already expressible *metatheoretically* for standard EC in terms of the quantification over choice functions.[23]

[21] Given this choice semantics, Mints & Sarenac also present a Henkin-style proof of the completeness of IEC. It should be noted here that Hans Leiß (LMU Munich) has recently argued in his talk at EPSILON2015 in Montpellier that the completeness proof in [9] may contain a gap. He has also suggested that a completeness theorem can be proven if context equality in IEC is dropped. See [7] for further details.

[22] The stronger and, arguably, more intuitive equivalence $B(\rho_x A(x)) \leftrightarrow \exists i B(\epsilon_i x A(x))$ does not hold. One can think of a model with $\mathcal{A} = \emptyset$ and an assignment of choice function δ to variable i such that $\delta(\mathcal{A}) \in \mathcal{B}$. It follows that $\exists i B(\epsilon_i x A(x))$ is true and $B(\rho_x A(x))$ is false.

[23] This point has first been stressed by von Heusinger with respect to indefinite noun phrases. As he puts it, IEC allows a "a uniform representation of indefinite [noun phrases] by means of indexed epsilon terms" [16, 261].

5 Conclusion

What did we achieve in this paper? We presented two accounts of indefinite descriptions. The first one was based on Hilbert's ϵ-terms, the second one was Russell's account of indefinite descriptions formulated here in terms of what we coined ρ-terms. Both accounts can be seen as formal investigations of indefinites. There are, however, significant differences between them. In the case of Hilbert's epsilon terms, the indeterminacy comes from the inherent understanding of ϵ-terms themselves; a fact that relates directly to the semantic conditions outlined in section 2. In the semantic framework that we have outlined here (i.e. a semantics based on choice functions), the reference of an ϵ-term is always ensured even if the predicate on which the ϵ-term depends on is empty. We then related this choice-functional treatment of epsilon logic to a notion from the philosophy of language, namely arbitrary reference; we claimed that this type of undetermined reference to objects is characteristic for Hilbert's understanding of ϵ-terms as indefinite expressions.

The second approach described in the paper was in the spirit of Russell's conception of indefinite descriptions. The semantic indeterminacy of Russell indefinites can be qualified in two ways: (a) as we have said in section 3, Russell's account of ambiguous descriptions can be seen as a restriction of his view of definite descriptions (simply be dropping the uniqueness condition). Interesting here is the fact that indeterminacy enters by its formal interpretation of formulas containing ρ-terms as existentially quantified sentences. However, there is more, i.e. (b) there is also a context dependency which is expressed by the fact that indefinite descriptions have to occur in a certain context, otherwise they would be meaningless – we are following here Russell's doctrine of *incomplete symbols*. Semantically speaking, there is no real need to extend the usual framework of first order predicate logic. Nonetheless, we have discussed a possibly promising alternative semantical route briefly. A more considerate model-theoretic study of ρ-terms will be subject to future research.

We have then addressed the question concerning interrelations between the two approaches. As was pointed out, there is no direct connection between these two calculi. If we, however, employ the $(.)^\epsilon$-translation (outlined in section 2) to the righthand side of (Def. ρ) we have found that there is a connection of two formal approaches after all. Another interconnection between Hilbert and Russell indefinites has been established by exploiting the fact that Russell's indefinites always rely on some (sentential) context. This was the main thought behind why we have chosen to turn to an indexed ϵ-calculus for the representation of Russell's ambiguous descriptions. The indexed ϵ-calculus has the syntactical means to express both the context dependency of Russell's indefinites (by hand of an index) and furthermore allows us to quantify over indices which is possible in Hilbert's ϵ-calculus only on the semantic side.

The present paper contains a philosophical outline of a rather formal investigation of Russell and Hilbert indefinites. A more rigorous presentation of the material covered here will have to be given in a future paper.

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Received \jreceived