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Edited by

MAURICE BOFFA
Université de l'Etat à Mons,
Belgium

DIRK VAN DALEN
Rijksuniversiteit Utrecht
The Netherlands

KENNETH MCALOON
Université Paris VII
France

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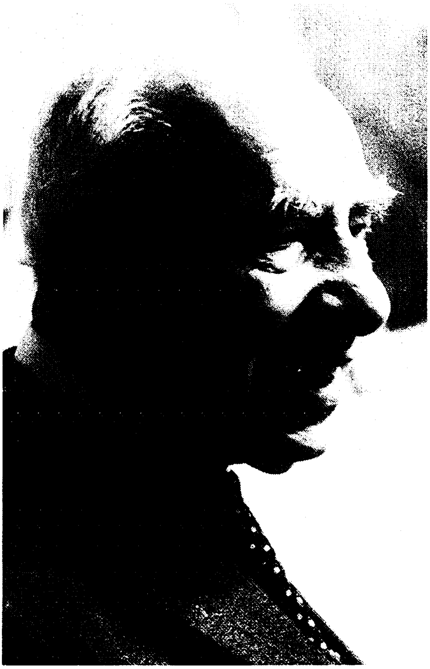
Dedicated to

PAUL BERNAYS

and

KURT GÖDEL



PAUL BERNAYS (1888 - 1977)

KURT GÖDEL (1906 - 1978)

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LOGIC COLLOQUIUM 78
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LOGIC AND THE AXIOM OF CHOICE

H. Schwichtenberg (Munich)

We shall prove the following:

(1) $\forall x \exists y\, \varphi(x,y) \rightarrow \exists f \forall x\, \varphi(x,fx)$ is conservative over classical (first order) logic.

(2) $\forall x \exists y\, \varphi(x,y) \rightarrow \exists f \forall x\, \varphi(x,fx)$ is conservative over intuitionistic logic without equality.

(3) $\forall x \exists y\, \varphi(x,y) \rightarrow \exists f \forall x\, \varphi(x,fx)$ is conservative over intuitionistic logic with decidable equality.

(4) $\forall x \exists y\, \varphi(x,y) \rightarrow \exists f \forall x\, \varphi(x,fx)$ is not conservative over intuitionistic logic.

(5) $$\bigwedge_n \forall x_1 \exists y_1 \ldots \forall x_n \exists y_n \Big[\bigwedge_i \varphi(x_i,y_i) \wedge \bigwedge_{i,j} (x_i = x_j \rightarrow y_i = y_j)\Big] \rightarrow$$
$$\rightarrow \exists f \forall x\, \varphi(x,fx)$$
is conservative over classical and intuitionistic logic.

More precisely: Addition of finitely many instances of the respective schema with all (number and function) parameters generalized is conservative over any first order theory in the respective logic.

None of these results is new. (1) is already in Hilbert-Bernays 1939 (p. 141 in the second edition). (2) - (4) are due to Minc 1966, 1974, 1966; note that (3) is an immediate consequence of (5). As to (4), a simpler counterexample is in Osswald 1975 and probably the simplest (which is reproduced here) in Smorynski 1978. (5) is due to Minc and Smorynski; it was first announced in Minc 1977. Proofs are in Smorynski 1978 and (in a generalized form dealing with "simultaneous Skolem functors") in Luckhardt A.

The proofs given here are relatively simple. For (5) the proof consists in a procedure which transform a derivation of a first order formula involving the axiom of choice into a derivation not involving it. The main technical tool is the use of a new type of

function variables: Whenever terms $r_1,\dots,r_n$, $s_1,\dots,s_n$ are introduced, then $f^{\underset{\sim}{s}}_{\underset{\sim}{r}}$ is a function variable. The intended meaning of $f^{\underset{\sim}{s}}_{\underset{\sim}{r}}$ is that it should range over all functions mapping $\underset{\sim}{r}$ in $\underset{\sim}{s}$ (provided $\underset{\sim}{r}$, $\underset{\sim}{s}$ determine a finite function, i.e. $\bigwedge_{i,j}(r_i = r_j \to s_i = s_j)$).
As already noted, (3) and also (1) are easy consequences of (5), since the premiss of the implication in (5) is under the assumption $\forall x,y\ (x = y \vee x \neq y)$ equivalent to $\forall x \exists y\ \varphi(x,y)$. So we start with a proof of (2), then give Smorynski's counterexample to prove (4), and finally extend the method for proving (2) to a proof of (5); only this last step involves the function variables $f^{\underset{\sim}{s}}_{\underset{\sim}{r}}$.

Note: (1) - (5) remain valid - with essentially the same proofs - when all variables x, y, f, ... are replaced by finite sequences $\underset{\sim}{x}$, $\underset{\sim}{y}$, $\underset{\sim}{f}$, ... of variables; $\underset{\sim}{f}\underset{\sim}{x}$ then means $f_1\underset{\sim}{x},\dots,f_n\underset{\sim}{x}$. However, for simplicity we only deal with single variables here.

Note: (2), (4) and (5) also hold for minimal logic. This is seen easily from the proofs.

Proof of (2):

In this section we only consider first order intuitionistic logic without equality. We shall work with a Gentzen sequent calculus as described in Kleene 1952, p. 481 (there it is called G3). For simplicity we modify it to include $\bot$ (falsum) as a propositional constant and treat $\neg\varphi$ as defined by $\varphi \to \bot$. First note that (2) can be reduced to

(2)' If $\vdash \forall x\ \varphi(x,fx), \Delta \to \Psi$ with Δ,Ψ of first order and without f, then $\vdash \forall x \exists y\ \varphi(x,y), \Delta \to \Psi$.

Proof of (2) from (2)': Let a cut-free derivation of $\Gamma,\Delta \to \Psi$ be given with Δ,Ψ of first order and Γ a list of generalizations of instances the schema $\forall x \exists y\ \varphi(x,y) \to \exists f \forall x\ \varphi(x,fx)$. By induction on the length of this derivation we construct a derivation of $\Delta \to \Psi$. It suffices to consider an inference

$$\frac{\forall x\exists y \to \exists f\forall x,\ \Gamma',\ \Theta \to \forall x\exists y\ \varphi(x,y) \qquad \exists f\forall x\ \varphi(x,fx),\ \forall x\exists y \to \exists f\forall x,\ \Gamma',\Theta \to \chi}{\forall x\exists y \to \exists f\forall x,\ \Gamma',\ \Theta \to \chi}$$

First the leftmost $\exists f$ in the right hand subderivation can be cancelled by an inversion lemma. Then the occurences of $\forall x\exists y \to \exists f\forall x$, Γ' in the antecedent of both subderivations can be cancelled by induction hypothesis. Then by (2)' $\forall x\ \varphi(x,fx)$ in the antecedent of the right hand subderivation can be replaced by $\forall x \exists y\ \varphi(x,y)$. A cut then gives the desired derivation of $\Theta \to \chi$.

Proof of (2)': Let a cut-free derivation of $\forall x\ \varphi(x,fx),\ \Delta \to \Psi$ with Δ,Ψ of first order and without f be given. By induction on the length of this derivation we construct a derivation of $\forall x \exists y\ \varphi(x,y),\ \Delta \to \Psi$. It suffices to consider

<u>Case 1</u> $$\frac{\varphi(t,ft),\ \forall x\ \varphi(x,fx),\ \Gamma \to \chi}{\forall x\ \varphi(x,fx),\ \Gamma \to \chi}$$

Replace all occurrences of ft in this derivation by a new variable w. This gives a derivation of $\varphi(t,w),\ \forall x\ \varphi(x,fx),\ \Gamma \to \chi$. By induction hypothesis we obtain a derivation of $\varphi(t,w)$, $\forall x \exists y\ \varphi(x,y),\ \Gamma \to \chi$. Application of $(\exists\to)$ and $(\forall\to)$ gives the desired derivation of $\forall x \exists y\ \varphi(x,y),\ \Gamma \to \chi$.

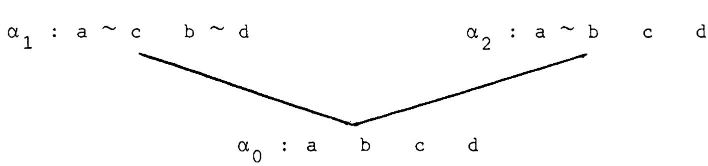
<u>Case 2</u> $$\frac{\forall x\ \varphi(x,fx),\ \Gamma \to\ \chi(r(ft_1,\dots,ft_n))}{\forall x\ \varphi(x,fx),\ \Gamma \to\ \exists z\ \chi(z)}$$

where $ft_1,\dots,ft_n$ are all outermost occurrences of f-terms in $r(ft_1,\dots,ft_n)$. Replace again all outermost occurrences of $ft_1,\dots,ft_n$ in this derivation by new variables $w_1,\dots,w_n$. This gives a derivation of
$\varphi(t_1,w_1),\dots,\varphi(t_n,w_n),\ \forall x\ \varphi(x,fx),\ \Gamma \to \chi(r(w_1,\dots,w_n))$. Then apply the induction hypothesis, $(\to\exists)$, n times $(\exists\to)$ and n times $(\forall\to)$.

<u>Proof of (4)</u>:

In this section we only consider first order intuitionistic logic (with equality). It suffices to prove
$\vdash \forall x \exists y\ (x \neq y) \to \forall x_1 \exists y_1 \forall x_2 \exists y_2\ (x_1 \neq y_1 \wedge x_2 \neq y_2 \wedge (x_1 = x_2 \to y_1 = y_2))$.

Consider the Kripke model



This is obviously a model of the equality axioms. Furthermore, $\alpha_0 \Vdash \forall x \exists y\ (x \neq y)$ since $\alpha_i \Vdash a \neq d,\ b \neq c$ for all i. But $\alpha_0 \Vdash \forall x_1 \exists y_1 \forall x_2 \exists y_2\ (x_1 \neq y_1 \wedge x_2 \neq y_2 \wedge\ (x_1 = x_2 \to y_1 = y_2))$. To see this assume the contrary. Choose a for x_1. Then y_1 must be d. Choose b for x_2. Then y_2 must be c. Hence $\alpha_0 \Vdash (a = b \to c = d)$. But this is a contradiction, since $\alpha_2 \Vdash a = b$, $\alpha_2 \Vdash c = d$. (This counterexample is due to Smorynski 1978)

<u>Proof of (5)</u>:

We only consider first order intuitionistic logic; it is easily seen that the same proof also applies to classical logic. First note that (5) can be reduced to

(5)' If $\vdash \forall x\, \varphi(x,fx), \Delta \to \Psi$ with Δ,Ψ of first order and without f, then there is an n such that $\vdash \forall x_1 \exists y_1 \ldots \forall x_n \exists y_n [\bigwedge_i \varphi(x_i,y_i) \wedge Fct(x;y)], \Delta \to \Psi$,where $Fct(x,y)$ abbreviates $\bigwedge_{i,j} (x_i = x_j \to y_i = y_j)$.

(5) can be proved from (5)' exactly as we proved (2) from (2)' above. To prove (5)' we cannot proceed as simply as in the proof of (2)'. For, the replacement of ft by a new variable w in case 1 would not lead to a derivation anymore, since an equality axiom t=s ft=fs would be transformed into an underivable formula t=s w=fs. The idea now is to replace f by a new variable f_w^t with the intended meaning that it should range over functions mapping w into t. To make this precise we first extend our language. Variables and terms and now generated simultaneously with the additional clause

> If $r_1,\ldots,r_n$ $s_1,\ldots,s_n$ (short: r, s) are terms, then f_r^s is a function variable (where f is any of the countably many symbols reserved for function variables).

Corresponding to the intended meaning of f_r^s we add the following axioms to our logical formalism:

$$Fct(r;s) \quad f_r^s r_i = s_i \quad \text{for all } i.$$

We now formulate a generalization of (5)' involving these new function variables, which can then be proved by induction.

(5)" If $\vdash \forall x\, \varphi(x,f_{\tilde{r}}^{\tilde{s}}x), Fct(\tilde{r};\tilde{s}), \Delta \to \Psi$ with Δ,Ψ of first order and without $f_{\tilde{r}}^{\tilde{s}}$, then there is an n such that $\vdash \forall x_1 \exists y_1 \ldots \forall x_n \exists y_n [\bigwedge_i \varphi(x_i,\tilde{y}_i) \wedge Fct(r,x;s,y)], \Delta \to \Psi$.

Proof of (5)": For simplicity we only write out the case for $\mathfrak{r}$, $\mathfrak{s}$ empty. The general case can be dealt with in exactly the same manner. We use induction on the length of the given derivation, which we may assume to be cut-free. It suffices to consider

Case 1 $$\frac{\varphi(t,ft),\ \forall x\varphi(x,fx),\ \Gamma \rightarrow \chi}{\forall x\varphi(x,fx\ ,\ \Gamma \rightarrow \chi}$$

We first describe the well-known technique of "extracting f-subterms from ft". Let $ft_1,\ldots,ft_n = ft$ be all f-subterms of ft ordered by increasing depth of nesting of f. Let $w_1,\ldots,w_n$ be new variables. For any subterm s of ft denote by s* the result of replacing all outermost occurrences of f-subterms $ft_{i_1},\ldots,ft_{i_k}$ in s by $w_{i_1},\ldots,w_{i_k}$. Using the new axioms on $f^{\underset{\sim}{s}}_{\underset{\sim}{r}}$ one can prove easily by induction on s

$$(*)\qquad Fct(\underset{\sim}{t}^*;\underset{\sim}{w}) \rightarrow s(f^{\underset{\sim}{w}}_{\underset{\sim}{t}^*}) = s^*.$$

$\underset{\sim}{t}^*$ denotes of course $t_1,\ldots,t_n$; note t_i contains only w_j with $j < i$. - Now replace all occurrences of f in the given derivation by f^{w}_{t} . Writing t(f) for t one obtains a derivation of

$$\varphi(t(f^{\underset{\sim}{w}}_{\underset{\sim}{t}^*})\ ,\ f^{\underset{\sim}{w}}_{\underset{\sim}{t}^*}t(f^{\underset{\sim}{w}}_{\underset{\sim}{t}^*}))\ ,\ \forall x\varphi(x,f^{\underset{\sim}{w}}_{\underset{\sim}{t}^*}x)\ ,\Gamma \rightarrow \chi$$

Using (*) this derivation can easily be transformed into a derivation of

$$\varphi(t^*,w_n)\ ,\ \forall x\varphi(x,f^{\underset{\sim}{w}}_{\underset{\sim}{t}^*}x)\ ,\ Fct(\underset{\sim}{t}^*;\underset{\sim}{w}),\Gamma \rightarrow \chi$$

of the same lenght (it is necessary her to allow as axioms all quasi-tautologics, i.e. all tautological consequences of the equality axioms including the new axioms on $f^{\underset{\sim}{s}}_{\underset{\sim}{t}}$).

By induction hypothesis we then obtain a derivation of

$$\varphi(t^*,w_n)\ ,\ \forall x_1\exists y_1\ldots\forall x_m\exists y_m\ [\ _i\ \varphi(x_i,y_i) \wedge Fct(\underset{\sim}{t}^*,\underset{\sim}{x};\underset{\sim}{w},\underset{\sim}{y})],\Gamma \rightarrow \chi.$$

Now $\varphi(t^*,w_n)$ can be cancelled since it follows from the second member of the antecedent (we may assume $m \geqslant 1$). Then using the rules $(\exists\rightarrow)$, $(\forall\rightarrow)$ we obtain

$$\forall u_1\exists w_1\ldots\forall u_n\exists w_n\forall x_1\exists y_1\ldots\forall x_m\exists y_m\ [\ _i\ \varphi(x_i,y_i) \wedge Fct(\underset{\sim}{u},\underset{\sim}{x};\underset{\sim}{w},\underset{\sim}{y})],\Gamma \rightarrow \chi.$$

Case 2 $$\frac{\forall x\ \varphi(x,fx)\ ,\ \Gamma \rightarrow \chi(t)}{\forall x\ \varphi(x,fx)\ ,\ \Gamma \rightarrow \exists z\chi(z)}$$

Here again we extract the f-subterms from t and then replace f by $f^{\tilde{w}}_{\tilde{t}*}$. This gives as above a derivation of

$$\forall x \varphi(x, f^{\tilde{w}}_{\tilde{t}*}x) \quad Fct(\tilde{t}^*;\tilde{w}) \;, \Gamma \to \chi(t^*) \;.$$

By induction hypothesis we then obtain a derivation of

$$\forall x_1 \exists y_1 \ldots \forall x_m \exists y_m \; [\bigwedge_i \varphi(x_i, y_i) \wedge Fct(\tilde{t}^*, \tilde{x}; \tilde{w}, \tilde{y}) \;, \Gamma \to \chi(t^*) \;.$$

Now apply (→∃) and then proceed as in case 1 above.

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