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ON BAR RECURSION OF TYPES 0 AND 1

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For general information on bar recursion the reader should consult the papers of Spector [8], where it was introduced, Howard [2] and Tait [11]. In this note we shall prove that the terms of Gödel's theory T (in its extensional version of Spector [8]) are closed under the rule $\mathrm{BR}_{0,1}$ of bar recursion of types 0 and 1. Our method of proof is based on the notion of an infinite term introduced by Tait [9]. The main tools of the proof are (i) the normalization theorem for (notations for) infinite terms and (ii) valuation functionals. Both are elaborated in [6]; for brevity some familiarity with this paper is assumed here. Using (i) and (ii) we reduce $\mathrm{BR}_{0,1}$ to ξ-recursion with $\xi < \varepsilon_0$. From this the result follows by work of Tait [10], who gave a reduction of 2^ξ-recursion to ξ-recursion at a higher type. At the end of the paper we discuss a perhaps more natural variant of bar recursion introduced by Kreisel in [4].

Related results are due to Kreisel (in his appendix to [8]), who obtains results which imply, using the reduction given by Howard [2] of the constant of bar recursion of type τ to the rule of bar recursion of type $(0 \to \tau) \to \tau$, that T is not closed under the rule of bar recursion of a type of level ≥ 2, to Diller [1], who gave a reduction of $\mathrm{BR}_{0,1}$ to ξ-recursion with ξ bounded by the least ω-critical number, and to Howard [3], who gave an ordinal analysis of the constant of bar recursion of type 0. I am grateful to H. Barendregt, W. Howard and G. Kreisel for many useful comments and discussions.

Recall that a functional F of type $0 \to (0 \to \tau) \to \sigma$ is said to be defined by (the rule of) bar recursion of type τ from Y and functionals G, H of the proper types if

$$(\mathrm{BR}_\tau) \qquad F(n, \alpha) = \begin{cases} G(n, \alpha) & \text{if } Y\alpha_n < n, \\ H(\lambda z.\ F(n+1, \alpha|_n^z), n, \alpha) & \text{otherwise,} \end{cases}$$

where $\alpha|_n^z m := \alpha m$ for $m \neq n$ and $:= z$ for $m = n$, and $\alpha_n m := \alpha m$ for $m < n$ and $:= \mathbf{0}$ for $m \geq n$ ($\mathbf{0}$ is the type τ object $\lambda x_1 \cdots x_n.\ 0$). We shall show that, for $\tau = 0$ and $\tau = 1 := 0 \to 0$, the functional F defined by BR_τ is primitive recursive if Y, G, H are.

We first deal with the case $\tau = 0$. So let Y be a primitive recursive functional of type $(0 \to 0) \to 0$. Y can be canonically represented by an (infinite) term t_Y (cf. [6, §2.4]). Let x be a variable of type $0 \to \tau$, i.e. $0 \to 0$. Then $t_Y x$ has type 0. By λ-conversions $t_Y x$ can be reduced to a normal form $(t_Y x)^*$ with rank $R(t_Y x)^* = 0$ and depth $|(t_Y x)^*| < \varepsilon_0$ (cf. [6, §2.10]). Clearly $(t_Y x)^*$ contains at most the variable x free.

We now consider in general (infinite) terms of type 0 in normal form (i.e. $Rt = 0$)

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containing at most the (fixed) variable x free; they will be denoted by $t, t_0, t_1, \ldots$. These terms have a particularly simple build-up: they can only be of the form xt or $\langle t_i \rangle_{i<\omega} t$ or $\bar{n}$ (the nth numeral). For any such t we define inductively a predicate S_t (S from "secured") by

$$S_{xt}(n, \alpha) : \leftrightarrow S_t(n, \alpha) \wedge n > \mathrm{Val}_x^\alpha t,$$
$$S_{\langle t_i \rangle t}(n, \alpha) : \leftrightarrow S_t(n, \alpha) \wedge S_{t_j}(n, \alpha), \qquad j = \mathrm{Val}_x^\alpha t,$$
$$S_{\bar{k}}(n, \alpha) : \leftrightarrow 0 = 0.$$

The (obvious) definition of $\mathrm{Val}_x^\alpha t$ is written out in [6, §2.2].

LEMMA 1. $S_t(n, \alpha) \wedge \forall m_{m<n} \alpha m = \beta m \rightarrow \mathrm{Val}_x^\alpha t = \mathrm{Val}_x^\beta t$.

The proof is by induction on t. Case xt. Assume $S_{xt}(n, \alpha)$ and $\forall m_{m<n} \alpha m = \beta m$. We have to show $\alpha(\mathrm{Val}_x^\alpha t) = \beta(\mathrm{Val}_x^\beta t)$. From $S_t(n, \alpha)$ we can conclude by ind. hyp. $\mathrm{Val}_x^\alpha t = \mathrm{Val}_x^\beta t$. With $n > \mathrm{Val}_x^\alpha t$ the above equation follows. Case $\langle t_i \rangle t$. Assume $S_{\langle t_i \rangle t}(n, \alpha)$ and $\forall m_{m<n} \alpha m = \beta m$. We have to show $(\mathrm{Val}_x^\alpha \langle t_i \rangle)(\mathrm{Val}_x^\alpha t) = (\mathrm{Val}_x^\beta \langle t_i \rangle) \cdot (\mathrm{Val}_\beta^x t)$. From $S_t(n, \alpha)$ we can conclude by ind. hyp. $\mathrm{Val}_x^\alpha t = \mathrm{Val}_x^\beta t = : \mathrm{j}$. Hence we have to show $\mathrm{Val}_x^\alpha t_j = \mathrm{Val}_x^\beta t_j$. This follows by ind. hyp. from $S_{t_j}(n, \alpha)$. Case $\bar{k}$. Trivial.

LEMMA 2. $S_t(n, \alpha) \wedge m > n \rightarrow S_t(m, \alpha)$.

The proof is by induction on t. Case xt. Assume $S_{xt}(n, \alpha)$ and $m > n$. From $S_t(n, \alpha)$ we can conclude by ind. hyp. $S_t(m, \alpha)$. Since $m > n > \mathrm{Val}_x^\alpha t$ we have $S_{xt}(m, \alpha)$. Case $\langle t_i \rangle t$. Assume $S_{\langle t_i \rangle t}(n, \alpha)$ and $m > n$. From $S_t(n, \alpha)$ and $S_{t_j}(n, \alpha)$, $j = \mathrm{Val}_x^\alpha t$, we can conclude by ind. hyp. $S_t(m, \alpha)$ and $S_{t_j}(m, \alpha)$, and hence $S_{\langle t_i \rangle t}(m, \alpha)$. Case $\bar{k}$. Trivial.

Let U_t (U from "unsecured") be the complement of S_t, i.e. $U_t(n, \alpha) \leftrightarrow \neg S_t(n, \alpha)$. By Lemma 2, U_t is a tree, i.e. $U_t(n, \alpha) \wedge m < n \rightarrow U_t(m, \alpha)$. We now define an order preserving embedding f_t from U_t in the ordinals $< 2^{\omega |t|}$ by induction on t, as follows. $f_t(n, \alpha) : = 0$ if $\neg U_t(n, \alpha)$. Otherwise,

$$f_{xt}(n, \alpha) : = \begin{cases} (\mathrm{Val}_x^\alpha t) - n & \text{if } S_t(n, \alpha), \\ (\omega + f_t(n, \alpha)) & \text{if } U_t(n, \alpha), \end{cases}$$

$$f_{\langle t_i \rangle t}(n, \alpha) : = \begin{cases} f_{t_j}(n, \alpha) \quad \text{with } j = \mathrm{Val}_x^\alpha t & \text{if } S_t(n, \alpha), \\ 2^{\omega |\langle t_i \rangle|} + f_t(n, \alpha) & \text{if } U_t(n, \alpha). \end{cases}$$

LEMMA 3. $f_t(n, \alpha) < 2^{\omega |t|}$.

The proof is by induction on t. Case xt. If $S_t(n, \alpha)$, then $f_{xt}(n, \alpha) < \omega < 2^{\omega |xt|}$. If $U_t(n, \alpha)$, then using the ind. hyp. we have $f_{xt}(n, \alpha) = \omega + f_t(n, \alpha) < \omega + 2^{\omega |t|} \leq 2^{\omega |t| + 1} < 2^{\omega |xt|}$. Case $\langle t_i \rangle t$. If $S_t(n, \alpha)$, then by ind. hyp. $f_{\langle t \rangle t_i}(n, \alpha) = f_{t_j}(n, \alpha) < 2^{\omega |t_j|} < 2^{\omega |\langle t_i \rangle|}$. If $U_t(n, \alpha)$, then again by ind. hyp. $f_{\langle t_i \rangle t}(n, \alpha) = 2^{\omega |\langle t_i \rangle|} + f_t(n, \alpha) < 2^{\omega |\langle t_i \rangle|} + 2^{\omega |t|} \leq 2^{\omega(\max(|\langle t_i \rangle|, |t|)+1)} = 2^{\omega |\langle t_i \rangle t|}$. Case $\bar{k}$. Trivial.

LEMMA 4. $U_t(n, \alpha) \wedge n > m \rightarrow f_t(n, \alpha) < f_t(m, \alpha)$.

The proof is by induction on t. Case xt. Assume $U_{xt}(n, \alpha)$ and $n > m$. If $S_t(m, \alpha)$, then by Lemma 2, $S_t(n, \alpha)$ and hence, since $n > m$, $f_{xt}(n, \alpha) < f_{xt}(m, \alpha)$. If $U_t(m, \alpha)$ and $S_t(n, \alpha)$, then we have $f_{xt}(n, \alpha) < \omega \leq f_{xt}(m, \alpha)$. If $U_t(m, \alpha)$ and $U_t(n, \alpha)$, then by ind. hyp. $f_t(n, \alpha) < f_t(m, \alpha)$ and hence $f_{xt}(n, \alpha) < f_{xt}(m, \alpha)$. Case $\langle t_i \rangle t$. Assume $U_{\langle t_i \rangle t}(n, \alpha)$ and $n > m$. If $S_t(m, \alpha)$, then again by Lemma 2, $S_t(n, \alpha)$. Hence we have $U_{t_j}(n, \alpha)$ with $j = \mathrm{Val}_x^\alpha t$ and from this by ind. hyp. $f_{t_j}(n, \alpha) < f_{t_j}(m, \alpha)$, hence $f_{\langle t_i \rangle t}(n, \alpha) < f_{\langle t_i \rangle t}(m, \alpha)$. If $U_t(m, \alpha)$ and $S_t(n, \alpha)$, then by Lemma 3, $f_{\langle t_i \rangle t}(n, \alpha) = f_{t_j}(n, \alpha) < 2^{\omega|t_j|} < 2^{\omega|\langle t_i \rangle|} \leq 2^{\omega|\langle t_i \rangle|} + f_t(m, \alpha) = f_{\langle t_i \rangle t}(m, \alpha)$. If $U_t(n, \alpha)$, then by ind. hyp. $f_t(n, \alpha) < f_t(m, \alpha)$ and hence $f_{\langle t_i \rangle t}(n, \alpha) < f_{\langle t_i \rangle t}(m, \alpha)$. Case k. Trivial.

From U_t we define a somewhat bigger tree $\bar{U}_t$ by $\bar{U}_t(n, \alpha) : \leftrightarrow U_t(n, \alpha) \vee \mathrm{Val}_x^\alpha t \geq n$. Hence outside of $\bar{U}_t$, i.e. for n, α with $\neg \bar{U}_t(n, \alpha)$, we have $\mathrm{Val}_x^\alpha t < n$. By Lemma 2, we know that $\bar{U}_t$ is a tree, i.e. $\bar{U}_t(n, \alpha) \wedge m < n \to \bar{U}_t(m, \alpha)$. Furthermore, $\bar{U}_t$ can be embedded by the following $\bar{f}_t$ in the ordinals $< \omega + 2^{\omega|t|}$: $\bar{f}_t(n, \alpha) := 0$, if $\neg \bar{U}_t(n, \alpha)$. Otherwise,

$$\bar{f}_t(n, \alpha) := \begin{cases} (\mathrm{Val}_x^\alpha t) - n & \text{if } S_t(n, \alpha), \\ \omega + f_t(n, \alpha) & \text{if } U_t(n, \alpha). \end{cases}$$

By Lemmas 3 and 4 we then have immediately $\bar{f}_t(n, \alpha) < \omega + 2^{\omega|t|}$ and $\bar{U}_t(n, \alpha) \wedge m < n \to \bar{f}_t(m, \alpha) < \bar{f}_t(n, \alpha)$.

Now we come back to $(t_Y x)^* =: r$ constructed above. Outside of $\bar{U}_r$, i.e. in the case $\neg U_r(n, \alpha)$, we then have $\mathrm{Val}_x^\alpha r < n$ and $S_r(n, \alpha)$. With Lemma 1, we can conclude $\mathrm{Val}_x^\alpha r = \mathrm{Val}_x^{\alpha_n}(t_Y x)^* = \mathrm{Val}_x^{\alpha_n} t_Y x = Y\alpha_n$, so $Y\alpha_n < n$, i.e. outside of $\bar{U}_r$ we are in the initial case of BR_0. Hence BR_0 can be considered as a recursion on the tree $\bar{U}_r$, and, since we have an order preserving embedding $\bar{f}_r$ of $\bar{U}_r$ in the ordinals $< \omega + 2^{\omega|r|} < \varepsilon_0$, also as a recursion on a section $< \varepsilon_0$ of the ordinals.

Hence it suffices to find analogs of $\bar{U}_r$ and $\bar{f}_r$ definable in T. For this we use term numbers as in [6, §3]. Sufficiently big bounds $\xi < \varepsilon_0$ for all depth bounds occurring in the term numbers and M for the set of all types in the term numbers can be fixed in advance (cf. [6, §3.1]). From the definitions of S_t, f_t etc. it is immediately clear how one can define correspondingly $\lambda un\alpha.\ S_u(n, \alpha)$, $\lambda un\alpha.\ f_u(n, \alpha)$ etc. in T_ξ (i.e. $T_\prec$ as explained in [6, §4.1], where $\prec$ is a standard wellordering of order type ξ) and hence also on T. By the same proofs one then obtains analogs to the properties of S_t, f_t etc. proved above, e.g. for Lemma 3: $u \in \mathrm{Num} \to f_u(n, \alpha) < \ulcorner 2^{\omega \cdot 0(|u|)} \urcorner$. Now from this we can conclude that BR_0 is reducible to a ξ-recursion and hence (Tait [10]) also to primitive recursions of higher types.

The formalizability of this proof in HA_ξ^ω (cf. [6, §4.1]) is immediately clear. But then we also have the formalizability in T, since HA_ξ^ω is a conservative extension of T (cf. Tait [10]).

For the case $\tau = 1$ only minimal changes are necessary. In the definitions and proofs by induction on t one has to replace xt by xts. Everything else remains unchanged.

Variants of bar recursion. Let us consider again the general rule of bar recursion

$$(\mathrm{BR}_\tau) \qquad F(n, \alpha) = \begin{cases} G(n, \alpha) & \text{if } Y\alpha \leq n, \\ H(\lambda x.\ F(n+1, \alpha|_n^x)) & \text{if } Y\alpha > n. \end{cases}$$

It is natural to ask whether, given G, H and Y, there will always be an F satisfying BR_τ. Now already Spector answered this in the affirmative, provided one asumes extensionality and Y satisfies

$$(*) \qquad \forall \alpha \exists n \forall \beta (\bar{\alpha} n = \bar{\beta} n \to Y\alpha = Y\beta)$$

(this is true e.g. for continuous Y). The argument goes as follows. Obviously BR_τ can be considered as a recursion on the partial ordering given by

$$(n, \alpha) \prec (m, \beta) \leftrightarrow n > m \text{ and } \bar{\alpha}m = \bar{\beta}m$$

with field $\{(n, \alpha) | Y\alpha > n\}$. Now it suffices to show that there are no infinite descending sequences w.r.t. $\prec$. Assume there would be one, i.e. $(n_{i+1}, \alpha_{i+1}) \prec (n_i, \alpha_i)$ for all i. Define α by $\alpha m = \alpha_i m$ if $n_i > m$; this clearly does not depend on i. Chose n by $(*)$ such that $Y\alpha$ only depends on $\bar{\alpha}n$. Chose i such that $n_i > n$ and $n_i \geq Y\alpha$. Then we have $Y\alpha_i = Y\alpha \leq n_i$ (since $\bar{\alpha}_i n = \bar{\alpha}n$) and hence (n_i, α_i) is not in the field of our ordering $\prec$, a contradiction.

Here we needed the condition $(*)$ to ensure that there will always be a solution to BR_τ. It seems to be natural to look for variants of BR_τ which make this condition somewhat more explicit, e.g. by requiring that there is a modulus of continuity M_Y for Y satisfying

$$(**) \qquad \bar{\alpha}(M_Y\alpha) = \bar{\beta}(M_Y\alpha) \rightarrow Y\alpha = Y\beta;$$

the rule (BR_τ) with this condition $(**)$ added has been called natural bar recursion by Kreisel in [4]. He also mentions yet another variant there where, in addition, the condition $Y\alpha \leq n$ in BR_τ is replaced by $M_Y\alpha \leq n$.

Now for which types does a modulus of continuity M_Y for a Y definable in T and of type $(0 \rightarrow \tau) \rightarrow 0$ exist? It is known that for $\tau = 0, 1$ such a M_Y can be defined within T such that $(**)$ becomes provable in T. This was first proved by Kreisel in lectures 1971/72; other proofs are in [5], [7] and [12]. The proof in [5] uses the present method of infinite terms and goes as follows. For any t define M_t inductively by

$$\begin{aligned} M_{xt}\alpha &= \max(M_t\alpha, (\mathrm{Val}_x^\alpha t) + 1), \\ M_{\langle t_i \rangle t}\alpha &= \max(M_t\alpha, M_{t_j}\alpha), \qquad \text{where } j := \mathrm{Val}_x^\alpha t, \\ M_{\bar{k}}\alpha &= 0. \end{aligned}$$

One can prove easily by induction on t that M_t is in fact a modulus of continuity for the functional $\lambda\alpha\ \mathrm{Val}_x^\alpha t$, i.e.

$$\bar{\alpha}(M_t\alpha) = \bar{\beta}(M_t\alpha) \rightarrow \mathrm{Val}_x^\alpha t = \mathrm{Val}_x^\beta t.$$

As above, one can then formalize this proof in T. However, for $\tau = 2$ there are functionals Y definable in T, e.g. $Y_0 = \lambda\alpha^{0\rightarrow 2}.\ \alpha 0\,(\lambda n.\ \alpha(n+1)0^1)$, which do not even possess a continuous modulus of continuity. This result is due, independently, to W. Howard, M. Hyland and H. Vogel; it answers a question asked previously by Kreisel. Now this situation gives rise to another natural question, also asked by Kreisel: Is T closed under natural bar recursion? Or, more explicitly: Assume Y has a modulus of continuity M_Y such that $(**)$ is provable in T. Is T closed under BR_τ for such Y? At present I do not know the answer.

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