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COMPLEXITY OF NORMALIZATION IN THE PURE TYPED LAMBDA - CALCULUS

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By the pure typed λ-calculus we mean as usual the system of terms built up from typed variables $x^\tau, y^\tau, \dots$ and maybe typed constants $a^\tau, b^\tau, \dots$ by means of application $(t^{\sigma\to\tau}s^\sigma)$ and λ-abstraction $(\lambda x^\sigma t^\tau)^{\sigma\to\tau}$. Here the types $\tau, \sigma, \dots$ are inductively generated from a ground type 0 by means of $(\sigma\to\tau)$. It is well-known (cf. e.g. Troelstra [T]) that any such term has a uniquely determined normal form with respect to so-called β-reductions $\dots(\lambda xt)s\dots \to \dots t_x[s]\dots$, and that this normal form will eventually be reached no matter which sequence of reduction steps one chooses [1].

In this paper we will be concerned with estimates for the number of reduction steps necessary to reach the normal form. We will give an $\mathcal{E}^4$ lower bound in §1 by writing down terms t_n of length [2] $3n$ and showing that it takes at least $2_{n-2} - n$ reduction steps (with $2_0 := 1$, $2_{n+1} := 2^{2_n}$) to bring t_n into its normal form. In §2 we describe a particular normalization procedure and give an $\mathcal{E}^4$ upper bound (in terms of $\max(lh(t), L(t))$, where $lh(t)$ denotes the length of t and $L(t)$ denotes the inner type level of t, i.e. the maximum type level [3] of a subterm of t) for the number of reduction steps this procedure will carry out.

The result of §1 also follows from Statman [S], where it is shown more generally that the problem whether two terms t_1 and t_2 have the same normal form is not elementary recursive. However, for the more specific question we are interested in here it is possible to give our much simpler proof. Also, the mere result of §2, namely that for some specific normalization procedure there is an $\mathcal{E}^4$ upper

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[1] We make use here of the following conventions. (1) Type superscripts will be omitted whenever they are clear from the context or inessential. (2) Terms that differ only in the bound variables used are identified. (3) Substitution is denoted by $t_x[s]$. (4) Brackets will be omitted whenever possible; we will write tsr for (ts)r.

[2] By the length of a term t we mean the number of occurrences of variables or constants in t except those immediately behind a λ-symbol.

[3] The type level $|\tau|$ of a type τ is defined inductively by $|0| := 0$, $|\sigma\to\tau| = \max(|\sigma|+1, |\tau|)$.

bound on the number of reduction steps, is certainly not new to any expert in the field. However, it seems that the simple explicit description of the bounding function obtained below is of some interest.

It should also be noted that, by combining the results proved here with those of Gandy [G], one can obtain a universal $\mathcal{E}^4$ upper bound for the number of reduction steps with respect to any normalization procedure. This can be seen as follows. For any term t of type τ, by Gandy's method one can define a closed type-0-term $|t|$ with the property that its numerical value is a bound on the number of reduction steps, where it does not matter in which way the reduction steps are choosen. Now to obtain a bound for the numerical value of $|t|$, we first note that by the argument of §2 we have an $\mathcal{E}^4$ bound on the number of reduction steps the specific normalization procedure given there will carry out to produce the normal form of $|t|$; this bound is in terms of $\max(lh|t|), L(|t|))$. Since, by Gandy's construction of $|t|$, $lh(|t|)$ depends only linearly on $lh(t)$, and $L(|t|) = L(t)$, we also have an $\mathcal{E}^4$ upper bound on the number of reduction steps in terms of $\max(lh(t), L(t))$. Next note that any reduction step at most squares the length of the original term. So we have an $\mathcal{E}^4$ upper bound on the length (and hence on the numerical value) of the normal form of $|t|$, again in terms of $\max(lh(t), L(t))$. This gives the desired result. (The fact that one can obtain an $\mathcal{E}^4$ upper bound from Gandy's work in [G] has been mentioned to me by G.E. Minc and R. Statman).

§1. The <u>pure types</u> k are defined inductively by $0 := 0$, $k+1 := k \to k$. We define <u>iteration functionals</u> I_n of pure type $k+2$ by

$$I_n :\equiv \lambda f \lambda x\ f(f(\dots f(fx)\dots)),$$

with n occurrences of f after $\lambda f \lambda x$; here f,x are variables of type $k+1, k$, respectively. Let $f \circ g$ be an abbreviation for $\lambda x\ f(gx)$, and let $t = s$ mean that t and s have the same normal form. With this notation we can write

$$I_n = \lambda f\ f \circ f \circ \dots \circ f$$

with n occurrences of f after λf.

The main point of our argument is the following simple lemma, which can be traced back to Rosser (cf. Church [C, p. 30]).

<u>LEMMA</u>.
$$(I_m f) \circ (I_n f) = I_{m+n} f$$
$$I_m \circ I_n = I_{m \cdot n}$$
$$I_m I_n = I_{n^m}$$

As an immediate consequence we have

$$t_n :\equiv I_2 I_2 \ldots I_2 = I_{2_n}$$

(with $2_0 := 1$, $2_{n+1} := 2^{2_n}$). Now consider any sequence of reduction steps transforming t_n into its normal form, and let S_n denote the total number of reduction steps in this sequence.

THEOREM. $S_n \geqslant 2_{n-2} - n$.

Proof. The length of t_n is $3n$. Note that any reduction step $\ldots(\lambda x t)s\ldots \to \ldots t_x[s]\ldots$ can at most square the length of the original term. Hence we have

$$\begin{aligned} 2_n &< \text{length of } I_{2_n} \quad (\text{the normal form of } t_n) \\ &\leqslant (\text{length of } t_n)^{2^{S_n}} \\ &= (3n)^{2^{S_n}} \\ &\leqslant 2^{2^{n+S_n}} \qquad (\text{since } 3n \leqslant 2^{2^n}), \end{aligned}$$

and the theorem is proved.

§2. Our aim here is to set up a specific normalization procedure for which an $\mathcal{E}^4$ upper bound on the number of reduction steps can be obtained easily. So let an arbitrary term be given. Our normalization procedure is an obvious one: we search for redexes of maximal type level, and among those we take the rightmost one and convert it. Here by a redex we mean as usual an occurrence of a subterm $(\lambda x^\sigma t^\tau)^{\sigma\to\tau} s^\sigma$, and to convert it means to replace it by $t_x[s]$. Its type level is the type level of $\sigma\to\tau$.

In order to get an estimate for the number of reduction steps needed, we associate a number with any given term and show that this number decreases with any reduction step. To obtain such a number, we first assign to any term t a sequence $a_i(t)$ of numbers, as follows: $a_i(t)$ is the number of redexes or of variables in t with type level $i+1$. Obviously only finitely many $a_i(t)$ will be different from 0. Now let us consider a reduction step and see how the assigned sequence will change:

$$\begin{array}{lcl}
\ldots(\lambda x^{\sigma} t^{\tau})^{\sigma\to\tau} s^{\sigma}\ldots & \to & \ldots t_x^{\tau}[s^{\sigma}]\ldots \\
\vdots & & \vdots \\
0 & & 0 \\
c_m + a_m(t) + 1 & & c_m + a_m(t) \\
c_{m-1} + a_{m-1}(t) + a_{m-1}(s) & \leqslant & c_{m-1} + a_{m-1}(t) + a_{m-1}(s) a_{n-1}(t) \\
\vdots & & \vdots \\
c_0 + a_0(t) + a_0(s) & \leqslant & c_0 + a_0(t) + a_0(s) a_{n-1}(t).
\end{array}$$

Here $m+1$ is the level of the type $\sigma\to\tau$ of λxt, and n is the level of the type σ of s. (Note that, if $n=0$, we have $a_i(s)=0$ for all i). c_i is the contribution to our sequence of that part of the terms above that is denoted by ... . More precisely, if those terms are written as $r_y[(\lambda xt)s]$ and $r_y[t_x[s]]$, respectively, then $c_i = a_i(r) - 1$.

Now denote the sequence associated with the original term by $\mathfrak{A}$ and the sequence associated with the reduced term (for which we only gave an estimate) by $\mathfrak{A}'$. We want to have numbers $|\mathfrak{A}|$, $|\mathfrak{A}'|$ assigned to $\mathfrak{A},\mathfrak{A}'$ such that $|\mathfrak{A}| > |\mathfrak{A}'|$. This can be done as follows. Let $\mathfrak{D} = (d_i)$ be a sequence with only finitely many entries different from 0. Let m be maximal with $d_m > 0$. Then define

$$|\mathfrak{D}| = g(m, d_m, d) \quad \text{with } d = \max(d_0, \ldots, d_{m-1}),$$

where

$$g(m, a+1, b) = g(m, a, b^2) + 1$$

$$g(m+1, 0, b) = g(m, b, b)$$

$$g(0, 0, b) = b.$$

Note that g belongs to the class $\mathcal{E}^4$ of Grzegorczyk [Gr], and that for any fixed m the function $g(m,.,.)$ belongs to $\mathcal{E}^3$, i.e. is elementary recursive.

It is easy to check that $g(m,a,b)$ is monotone in a and b for any fixed m. Using this, let us show that $|\mathfrak{A}| > |\mathfrak{A}'|$. Case 1: $c_m + a_m(t) > 0$.
Then we have

$$\begin{aligned}
|\mathfrak{A}| &= g(m, c_m + a_m(t) + 1, m) \quad \text{with } M := \max_{0\leqslant i<m} (c_i + a_i(t) + a_i(s)) \\
&= g(m, c_m + a_m(t), M^2) + 1 \\
&\geqslant g(m, c_m + a_m(t), \max_{0\leqslant i<m} (c_i + a_i(t) + a_i(s) a_{n-1}(t))) + 1. \\
&\geqslant |\mathfrak{A}'| + 1
\end{aligned}$$

Case 2: $c_m + a_m(t) = 0$ and $M = 0$. Obvious. Case 3: $c_m + a_m(t) = 0$, i maximal such that $c_i + a_i(t) + a_i(s) = 0$. Then we have

$$\begin{aligned} |\mathfrak{A}| &= g(m,1,M) \\ &= g(m,0,M^2) + 1 \\ &= g(m-1,M^2,M^2) + 1 \\ &> g(i,M^2,M^2) \\ &\geq |\mathfrak{A}'|. \end{aligned}$$

Here we have made use of the obvious fact that $g(m,b,b)$ is monotone in m. This concludes the proof of $|\mathfrak{A}| > |\mathfrak{A}'|$. We can summarize our argument as follows.

THEOREM. For any given term t, the number of reduction steps for the procedure described above is $\leq g(m,a_m(t),a(t))$. Here $m+1$ is the maximal type level of a redex in t, $a(t) := \max_{0 \leq i \leq m} a_i(t)$ and $a_i(t)$ is the number of redexes or of variables in t whose type level is $i+1$.

COROLLARY. (1) There is an $\mathcal{E}^4$ function f such that for all closed type-0-terms t the above normalization procedure terminates in $\leq f(\max(lh(t),L(t)))$ steps. (2) For all m there is an elementary recursive function g_m such that for all closed type-0-terms t with $L(t) \leq m$ the above normalization procedure terminates in $\leq g_m(lh(t))$ steps.

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