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An upper bound for reduction sequences in the typed λ-calculus

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It is well known that the full reduction tree for any term of the typed λ-calculus is finite. However, it is not obvious how a reasonable estimate for its height might be obtained.

Here we note that the head reduction tree has the property that the number of its nodes with conversions bounds the length of any reduction sequence*. The height of that tree, and hence also the number of its nodes, can be estimated using a technique due to Howard [3], which in turn is based on work of Sanchis [4] and Diller [1]. This gives the desired upper bound.

The method of Gandy [2] can also be used to obtain a bound for the length of arbitrary reduction sequences; this is carried out in [5]. However, the bound derived here, apart from being more intelligible, is also better.

Let r, s, t denote terms of the typed λ-calculus. The *level* $\mathrm{lev}(r)$ of r is defined to be the level $\mathrm{lev}(\varrho)$ of its type ϱ, where ground types have level 0 and $\mathrm{lev}(\varrho \to \sigma) = \max(\mathrm{lev}(\varrho)+1, \mathrm{lev}(\sigma))$. For r of level 0 we define $\vdash_m^a r$ inductively by

– *β-Rule.* If $\vdash_m^a r_x[s]\vec{t}$, then $\vdash_m^{a+1} (\lambda x r)s\vec{t}$.

– *Variable Rule.* If $\vdash_m^a t_i \vec{y}_i$ for $i = 1, \dots, n$, then $\vdash_m^{a+1} x t_1 \dots t_n$. In particular, $\vdash_m^{a+1} x$ for any a and m.

– *Cut Rule.* If $\vdash_m^a r y_1 \dots y_n$ with $n \geqq 1$ and $\vdash_m^a t_i \vec{y}_i$ and $\mathrm{lev}(t_i) < m$ for $i = 1, \dots, n$, then $\vdash_m^{a+1} r t_1 \dots t_n$.

Note that $\vdash_0^a r$ is generated by a uniquely determined rule. Hence the generation tree (with the a's stripped off) is uniquely determined; we call it the *head reduction tree* of r.

Variable Lemma. *If* $\mathrm{lev}(x) < a$, *then* $\vdash_m^a x\vec{y}$.

Proof. By induction on $\mathrm{lev}(x)$. By induction hypothesis $\vdash_m^{a-1} y_i \vec{z}_i$, hence $\vdash_m^a x\vec{y}$ by the Variable Rule. □

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* This is not quite true, but only for so-called $\lambda - I$-terms, where any variable bound by λ actually occurs in the kernel. But the general case can be easily reduced to this one by introducing dummy variables; this is carried out below

Substitution Lemma. *If* $\vdash_m^a r$ *and* $\vdash_m^b s_j\vec{y}_j$ *and* $\mathrm{lev}(s_j)\leq m$, *then* $\vdash_m^{b+a} r_{\vec{x}}[\vec{s}]$.

Proof. By induction on $\vdash_m^a r$. We write t^* for $t_{\vec{x}}[\vec{s}]$.

β-Rule. By induction hypothesis $\vdash_m^{b+a} r^*[s^*]\vec{t}^*$, hence $\vdash_m^{b+a+1}(\lambda x r^*)s^*\vec{t}^*$ by the β-Rule.

Variable Rule. By induction hypothesis $\vdash_m^{b+a} t_i^*\vec{y}_i$, hence $\vdash_m^{b+a+1} x t_1^* \dots t_n^*$ by the Variable Rule. Now if x is one of the variables x_j to be substituted by s_j, we must use the Cut Rule instead of the Variable Rule. This is possible since $\mathrm{lev}(s_j)\leq m$ by hypothesis and hence $\mathrm{lev}(t_i)<m$, and since $\vdash_m^{b+a} s_j\vec{y}_j$ also by hypothesis. Then (if $n>0$) the Cut Rule yields $\vdash_m^{b+a+1} s_j t_1^* \dots t_n^*$, as required. In case $n=0$ there are no t_i's and we have used the Variable Rule to generate $\vdash_m^{a+1} x_j$. But then $\vdash_m^{b+a+1} s_j$ by hypothesis.

Cut Rule. By induction hypothesis $\vdash_m^{b+a} r^*\vec{y}$ and $\vdash_m^{b+a} t_i^*\vec{y}_i$, hence $\vdash_m^{b+a+1} r^*\vec{t}^*$ by the Cut Rule. □

Cut Elimination Lemma. *If* $\vdash_{m+1}^a r$, *then* $\vdash_m^{2^a} r$.

Proof. By induction on $\vdash_{m+1}^a r$.

β-Rule. By induction hypothesis $\vdash_m^{2^a} r_x[s]\vec{t}$, hence $\vdash_m^{2^a+1}(\lambda x r)s\vec{t}$.

Variable Rule. By induction hypothesis $\vdash_m^{2^a} t_i\vec{y}_i$, hence by the Variable Rule $\vdash_m^{2^a+1} x\vec{t}$.

Cut Rule. By induction hypothesis $\vdash_m^{2^a} r\vec{y}$ and $\vdash_m^{2^a} t_i\vec{y}_i$. Since $\mathrm{lev}(t_i)<m+1$, we get $\vdash_m^{2^a+2^a} r\vec{t}$ from the Substitution Lemma. □

Embedding Lemma. *If all subterms of* r *have levels* $\leq m$, *then* $\vdash_m^a r\vec{y}$ *where* $a=m+\mathrm{height}(r)$.

Proof. By induction on r.

Case x. The claim follows from the Variable Lemma with $a:=\mathrm{lev}(x)+1$.

Case $\lambda x r$. By induction hypothesis $\vdash_m^a r\vec{y}$ where $a=m+\mathrm{height}(r)$, hence $\vdash_m^{a+1}(\lambda x r)x\vec{y}$ by the β-Rule.

Case ts. By induction hypothesis $\vdash_m^a ty\vec{y}$ and $\vdash_m^a s\vec{z}$ and $\vdash_m^a y_i\vec{y}_i$ where a is the maximum of $m+\mathrm{height}(t)$ and $m+\mathrm{height}(s)$. Hence $\vdash_m^{a+1} ts\vec{y}$ by the Cut Rule. □

It now follows that the head reduction tree of any given term r has the

$$\mathrm{height}\leq 2_m(m+\mathrm{height}(r)),$$

where m is a bound for the levels of subterms of r and $2_m(x)$ is defined by $2_0(x)=x$, $2_{m+1}(x)=2^{2_m(x)}$.

Our key observation is that, under a slight additional hypothesis, the number $\#r$ of conversions in the head reduction tree of r bounds the length of any reduction sequence. More precisely, $\#r$ is defined for any term r of level 0 by induction on $\vdash_0^a r$:

1. $\#((\lambda x r)s\vec{t})=\#(r_x[s]\vec{t})+1$
2. $\#(xt_1 \dots t_n):=\sum_{i=1}^{n}\#(t_i\vec{y}_i)$.

Note first that it is easy to estimate $\#r$ in terms of the height of the head reduction tree for r:

Estimate Lemma. *If* $\vdash_0^a r$ *and if any variable* x *free in* r *has arity* $\leq k$ *where* $k\geq 1$, *then* $\#r\leq k^a$.

Proof. By induction on $\vdash_0^a r$

β-Rule. $\#((\lambda xr)s\vec{t}) = \#(r_x[s]\vec{t}) + 1 \leqq k^a + 1 \leqq k^{a+1}$.

Variable-Rule. $\#(x\vec{t}) = \sum \#(t_i\vec{y}_i) \leqq \sum k^a \leqq k \cdot k^a \leqq k^{a+1}$. □

A term is called a $\lambda - I$-term if for any subterm of the form λxs we have $x \in \mathrm{vars}(s)$.

Main Lemma. *Let r be a $\lambda - I$-term of level* 0. *Then $r \to^1 r'$ implies that $\# r > \# r'$.*

Proof. We show more generally that for any $\lambda - I$-term r with $z \in \mathrm{vars}(r)$ we have $\# r_z[(\lambda xp)q] > \# r_z[p_x[q]]$. For brevity we write t^* for $t_z[(\lambda xp)q]$ and t' for $t_z[p_x[q]]$. The proof is by induction on $\# r^*$.

$$\begin{aligned}\#((\lambda xr)s\vec{t})^* &= \#(r_x[s]\vec{t})^* + 1\\ &> \#(r_x[s]\vec{t})' + 1\\ &= \#((\lambda xr)s\vec{t})',\end{aligned}$$

where the $>$ follows by induction hypothesis. Note that for the application of the induction hypothesis here we have used $x \in \mathrm{vars}(r)$, which follows from our assumption that we are dealing with $\lambda - I$-terms.

$$\begin{aligned}\#(y\vec{t})^* &= \sum_i \# t_i^* \vec{y}_i\\ &> \sum_i \# t_i' \vec{y}_i\\ &= \#(y\vec{t})'\\ \#(z\vec{t})^* &= \#((\lambda xp)q\vec{t}^*)\\ &= \#(p_x[q]\vec{t}^*) + 1\\ &\geqq \#(p_x[q]\vec{t}') + 1\\ &> \#(p_x[q]\vec{t}')\\ &= \#(z\vec{t})'.\ \square\end{aligned}$$

In this proof we have made use of the hypothesis that r is a $\lambda - I$-term in order to conclude $\# r > \# r'$ from $r \to^1 r'$. This hypothesis is certainly necessary, since in non-$\lambda - I$-terms subterms can disappear by means of conversions, and hence the head reduction tree may not show any trace of a conversion inside the term. An example is $(\lambda xy)((\lambda xp)q)$ and $(\lambda xy)(p_x[q])$, both of which have the same head reduction tree (consisting of one additional node labeled y).

However, we can easily reduce the general case to the case of $\lambda - I$-terms. To achieve this we just introduce dummy variables which turn the given term r into a $\lambda - I$-term r^* (a variant of r, as we shall say), and note that the length of any reduction sequence for r is bounded by the length of a reduction sequence for r^*.

By an *immediate variant* of a term r of type $\vec{\varrho} \to \iota$ we mean a term

$$r' \equiv \lambda\vec{y} \cdot ut(r\vec{y}),$$

where t is any term of some type σ with $\vec{y} \notin \mathrm{vars}(t)$ and u is a new variable of type $\sigma, \iota \to \iota$; the variables $\vec{y}$ are supposed to have types $\vec{\varrho}$. Note that r' has the same type $\vec{\varrho} \to \iota$ as r. Call a term $r^{(m)}$ an *m-fold immediate variant* of r if there are terms $r^{(0)}, r^{(1)}, \ldots, r^{(m-1)}$ such that $r^{(0)} \equiv r$ and $r^{(i+1)}$ is an immediate variant of $r^{(i)}$. Finally a term r^* is called a *variant* of r if it is obtained from r by taking possibly multiple

immediate variants of all of its subterms. More precisely, we define inductively
1. $x^{(m)}$ is a variant of x.
2. If r^* is a variant of r, then $(\lambda x r^*)^{(m)}$ is a variant of $\lambda x r$.
3. If t^*, s^* are variants of t, s, then $(t^* s^*)^{(m)}$ is a variant of ts.

Note that, if r^*, s^* are variants of r, s, then $r^*[s^*]$ is a variant of $r_x[s]$. This can be proved easily by induction on r.

Variant Lemma. *If $r \to^1 r_1$ and r^* is a variant of r, then we can find a variant r_1^* of r_1 such that $r^* \to^+ r_1^*$, where $\to^+$ is defined just as $\to^*$ except that reflexivity is not allowed.*

Proof. Note first that any $t's$ converts into some $(ts)'$, since

$$(\lambda y \vec{y} \cdot ur(ty\vec{y}))s \text{ converts into } \lambda \vec{y} \cdot ur(ts\vec{y}). \tag{1}$$

We restrict ourselves to the case $(\lambda x r)s \to^1 r_x[s]$; the other cases are similar or immediate by induction hypothesis. Now by (1)

$$((\lambda x r^*)^{(m)} s^*)^{(n)} \to^* ((\lambda x r) s^*)^{(m+n)} \to^1 r_x^*[s^*]^{(m+n)}.$$

By the note above we can take $r_x^*[s^*]^{(m+n)}$ as the required variant of $r_x[s]$. □

To summarize, we get the following result.

Theorem. *Let r be a term of the typed λ-calculus of level 0. Let m be a bound for the levels of subterms of r and $k \geqq 2$ be a bound for the arities of subterms of r. Then the length of an arbitrary reduction sequence for r with respect to $\to^1$ is bounded by*

$$k^{2_m(m+2\cdot \text{height}(r)+2k+2)}.$$

Proof. Let r^* be a variant of r which is a $\lambda - I$-term. By the Main Lemma, the length of any reduction sequence for r^* is $\leqq \# r^*$. Since the head reduction tree of r^* has height $\leqq 2_m(m + \text{height}(r^*))$ and any variable free in r^* has arity $\leqq k$, the Estimate Lemma gives

$$\# r^* \leqq k^{2_m(m+\text{height}(r^*))}.$$

Hence by the Variant Lemma it suffices to show $\text{height}(r^*) \leqq 2 \cdot \text{height}(r) + 2k + 2$. This can be achieved easily: just replace each variable z in r by its variant $\lambda y \cdot u(v\vec{x})(z\vec{y})$, where $\vec{x}$ consists of all variables x_i such that some $\lambda x_i t$ with $x_i \notin \text{vars}(t)$ is a subterm of r. □

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