The Schreier Refinement Theorem for Categories

By

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We use the preceding note [1] to prove the following theorem.

Theorem. *Any two normal series of subobjects of an object G of a pointed category $\mathscr{C}$ have isomorphic refinements if $\mathscr{C}$ satisfies* Z2, *and the following condition.*

Z1*. *$\mathscr{C}$ has cokernels of kernels, and kernels of any composition $e'e$ of two cokernels e and e'.*

We note first that Z1* implies that every cokernel has a kernel, since every identity morphism is a cokernel of a zero morphism, and we have the following three lemmas.

Lemma 1. Z1* *is equivalent to the following statement: $\mathscr{C}$ has cokernels of kernels, and every possible pullback square*

$$\begin{array}{ccc} \cdot & \xrightarrow{m'} & \cdot \\ \downarrow{\scriptstyle e'} & & \downarrow{\scriptstyle e} \\ \cdot & \xrightarrow{m} & \cdot \end{array},$$

for a kernel m and a cokernel e with the same target, exists in $\mathscr{C}$.

Proof. If Z1* is valid, let e_1 be a cokernel of m and m' a kernel of $e_1 e$. Then $em' = me'$ for a morphism e', and this obviously is the desired pullback. For the converse, we observe first that a cokernel of an identity morphism (which is a kernel of a zero morphism) produces a zero object. If $e: A \to B$ and $m = 0: Z \to B$ for a zero object Z, then m is a kernel of id B, and a pullback produces a kernel m' of e. Now if $e_1 e$ is the composition of two cokernels and m a kernel of e, then a pullback produces a kernel m' of $e_1 e$. ▌

Definition. A morphism m of $\mathscr{C}$ will be called a *subkernel* if m is the composition in $\mathscr{C}$ of kernels in $\mathscr{C}$, and (A, a) will be called a *subnormal subobject* of G if $a: A \to G$ is a subkernel in $\mathscr{C}$.

Lemma 2. (i) *The intersection of any two subnormal subobjects (A, a) and (B, b) of an object G of $\mathscr{C}$ exists in $\mathscr{C}$ and is a subnormal subobject of $\mathscr{C}$.*

(ii) *If mm' is defined in $\mathscr{C}$ and m is a subkernel, then mm' is a subkernel if and only if m' is a subkernel.*

Proof. Let $a = a_{10}a_{20}\cdots a_{h0}$ and $b = b_{01}b_{02}\cdots b_{0k}$ for kernels a_{i0} and b_{0j}. If $a_{i,j-1}$ and $b_{i-1,j}$ are defined and kernels, then a pullback

$$\begin{array}{ccc} \cdot & \xrightarrow{a_{ij}} & \cdot \\ {\scriptstyle b_{ij}}\downarrow & & \downarrow{\scriptstyle b_{i-1,j}} \\ \cdot & \xrightarrow[a_{i,j-1}]{} & \cdot \end{array}$$

exists in $\mathscr{C}$, for kernels a_{ij} and b_{ij}, by the dual of 2.3 in [1]. This defines kernels a_{ij} and b_{ij} recursively for $1 \leqq i \leqq h$ and $1 \leqq j \leqq k$. If $a' = a_{1k}a_{2k}\cdots a_{hk}$ and $b' = b_{h1}\,b_{h2}\cdots b_{hk}$, then a' and b' are subkernels, and the commutative square

$$\begin{array}{ccc} \cdot & \xrightarrow{a'} & \cdot \\ {\scriptstyle b'}\downarrow & & \downarrow{\scriptstyle b} \\ \cdot & \xrightarrow[a]{} & \cdot \end{array}$$

in $\mathscr{C}$ is the composition of $h \cdot k$ pullback squares, and hence a pullback. Thus $a \cap b$ exists and is subnormal.

The "if" part of (ii) is trivial. For the "only if" part, put $a = mm'$ and $b = m$ in (i). Then we can carry out the construction of (i) if m and mm' are subkernels, and we obtain $a' \cong m'$. Thus m' is a subkernel. ▌

Lemma 3. *If f is a cokernel and m a subkernel so that fm is defined, then $f[m]$ and $f^{-1}[f[m]]$ exist in $\mathscr{C}$ and are subnormal, and $f[f^{-1}[f[m]]] \cong f[m]$ in $\mathscr{C}$.*

Proof. Let $f\colon A \to B$, and let $m = a_1 a_2 \cdots a_h$ for kernels a_i. Put $m_0 = \mathrm{id}\,A$ and $m_i = m_{i-1}a_i$ for $1 \leqq i \leqq h$, so that $m_h = m$. Define kernels b_i and cokernels e_i recursively in $\mathscr{C}$, using Z2, by putting $e_0 = f$, and $e_{i-1}a_i = b_i e_i$ for $1 \leqq i \leqq h$. If we put $m_0' = \mathrm{id}\,B$ and $m_i' = m_{i-1}' b_i$ for $1 \leqq i \leqq h$, then $m_i' e_i = f m_i$, and m_i' is a subkernel. Thus $f[m_i] \cong m_i'$, and $f[m] \cong m_h'$ exists and is subnormal.

Now we construct recursively the following diagram

$$\begin{array}{ccccc} \cdot & \xrightarrow{c_i} & \cdot & \xrightarrow{m_{i-1}''} & \cdot \\ {\scriptstyle e_i'}\downarrow & & \downarrow{\scriptstyle e_{i-1}'} & & \downarrow{\scriptstyle f} \\ \cdot & \xrightarrow[b_i]{} & \cdot & \xrightarrow[m_{i-1}']{} & \cdot \end{array}$$

beginning with $e_0' = f$, $m_0' = \mathrm{id}\,B$ and $m_0'' = \mathrm{id}\,A$. We assume that the righthand square is a pullback, with m_{i-1}'' a subkernel and e_{i-1}' a cokernel. This is satisfied for $i = 1$. By Lemma 1, the lefthand square can be constructed as a pullback, and then c_i is a kernel, the rectangle is a pullback, and $m_i'' = m_{i-1}'' c_i$ defines a subkernel m_i''. Thus the construction can go on if e_i' is a cokernel. To see that e_i' is a cokernel, put $b_i e_i' = e_{i-1}' c_i = b_i' e_i''$ for a cokernel e_i'' and a kernel b_i', using Z2. Then $b_i' \leqq b_i$ since (e_i'', b_i') is an image. On the other hand, $f m_i = m_i' e_i$, and thus $m_i = m_i'' u_i$, $e_i = e_i' u_i$, for a morphism u_i, and $b_i e_i = b_i' e_i'' u_i$ follows. But then $b_i \leqq b_i'$ since (e_i, b_i) is an image. Thus $b_i' = b_i x$, $e_i' = x e_i''$, for an isomorphism x, and e_i' is a cokernel.

Now $f^{-1}[m_i'] \cong m_i''$, and $f[m_i''] \cong m_i'$, and the lemma is proved. ∎

All results of [1] now become available, except for 4.2, if we restrict ourselves to subnormal subobjects. The last part of Lemma 3 replaces 4.2 in proofs in [1]; no other changes are needed. Subobjects occuring in a normal series are ipso facto subnormal. Thus the Zassenhaus Lemma is valid for these subobjects, and the usual proof of the Schreier Theorem goes through without any changes.

Reference

[1] O. WYLER, The Zassenhaus Lemma for categories. Arch. Math. 22, 561–569 (1971).

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