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AN APPROXIMATION THEOREM FOR MAPS INTO KAN FIBRATIONS

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In this note we prove that a semisimplicial map into the base of a Kan fibration having a continuous lifting to the total space also has a semisimplicial lifiting, very "close" to a given continuous lifting. As a special case we obtain a new proof of the famous Milnor-Lamotke theorem that a Kan set is a strong deformation retract of the singular set of its geometric realization.

First we state our main

THEOREM. *Let*

$$(*)\qquad \begin{array}{ccc} X & \xrightarrow{f} & E \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow[h]{} & B \end{array}$$

be a commutative square in the category of semisimplicial sets with i an inclusion and p a Kan fibration. Further, suppose given a continuous $\bar{g}\colon |Y| \to |E|$ with $\bar{g} \circ |i| = |f|$ and $|p| \circ \bar{g} = |h|$. Then there exists a homotopy $\bar{g} \cong g'$ rel. $|X|$ and over $|B|$ so that $g' = |g|$ for some semisimplicial g.

This theorem has an interesting special case. Take $X = E$ a Kan set, $Y = S|E|$, B a point, p, h the unique constant maps, $f = idE$, i the natural inclusion and $\bar{g}$ the natural retraction. What comes out is the famous Milnor-Lamotke theorem saying E is strong deformation retract of $S|E|$. Thus we get a new proof of this theorem which in contrast to the original one [4] avoids any reference to J.H.C. Whitehead's theorems.

On the other hand, if B is a point, the statement is a trivial consequence of the Milnor-Lamotke theorem. An elementary proof for this case—avoiding the Milnor-Lamotke theorem—has been given by B. J. Sanderson [7] whose techniques are also important for our proceeding.

Proof of theorem. (For the technical details we use the notation explained in §0 of [1].) By an induction over skeletons, it is enough to prove the theorem in the case y is an n-simplex $\Delta[n]$ with $n > 0$ and X is its boundary $\dot{\Delta}[n]$. Let ι be the generating simplex of $\Delta[n]$, $y = h\iota \in B$ and $\bar{y} = S\bar{g}(\iota) \in S|E|$. We have to prove that $\bar{y}$ is $S|B|$-equivalent ([3] p. 123) to a simplex in E[1].

Decompose $y = y^+ y^0$ with y^+ nondegenerate and y^0 surjective. We perfom a further induction, over a (partial) ordering of the set of the possible y^0, that is the set D_n of surjective monotone maps with domain $[n]$. Choose[2] an ordering of this set satisfying (i) and (ii):

(i) $\beta\alpha \leqq \alpha$ if $\alpha, \beta\alpha \in D_m$; and

(ii) each nonconstant $\alpha \in D_n$ admits an $\alpha' < \alpha$ so that α' is the surjective part of $\alpha\sigma_i\delta_j$ for some suitable pair i, j.

Evidently the constant map is the minimum of D_n with respect to this ordering.

First, assume y^0 is constant. Denote by F the fibre over y which is Kan. Now comes Sanderson's idea. Since the boundary of $\bar{y}$ belongs to F we can choose the zeroth vertex * of $\bar{y}$ for base point of F. Then, form the path fibration $q: W(F) \to F$ ([5] p. 196) and lift $\bar{y}$ to a filling $\bar{u}$ in $S|W(F)|$ of the horn $(-, \bar{y}\delta_1\sigma_0, \cdots, \bar{y}\delta_n\sigma_0)$ in $W(F) \subset S|W(F)|$. By induction, $\bar{u}\delta_0$ is $S|F|$ – equivalent to an $u \in W(F)$. That gives a $\bar{z} \in S|W(F)|$ with boundary $(u, \bar{u}\delta_0, u\sigma_0\delta_2, \cdots, u\sigma_0\delta_n)$ and $S|q|\bar{z} = \bar{y}\delta\sigma_0 \in F$ ([5] p. 25). Next we use that every sphere in $W(F)$ can be filled ([5] p. 196) and also every sphere in $S|W(F)|$ since $W(F)$ is contractible. Take a filling $v \in W(F)$ of the sphere $(u, \bar{y}\delta_1\sigma_0, \cdots, \bar{y}\delta_n\sigma_0)$ and finally a filling $\bar{v} \in S|W(F)|$ of the sphere $(\bar{z}, v, \bar{u}, \bar{z}\sigma_0\delta_3, \cdots, \bar{z}\sigma_0\delta_{n+1})$. Then $S|q|\bar{v}$ is an $S|B|$-equivalence between $\bar{y}$ and $qv \in F \subset E$.

If y^0 is not constant, we choose i and j such that the surjective part of $y^0\sigma_i\delta_j$ is less than y^0. Set $\varepsilon = 0$ if $j < i$ and $\varepsilon = 1$ if $j > i + 1$. Lift y to $u \in E$ with $u\delta_k = \bar{y}\delta_k$ if $k \neq j - \varepsilon$ and lift $y\sigma_i$ to $\bar{u} \in S|E|$ with $\bar{u}\delta_i = \bar{y}$, $\bar{u}\delta_{i+1} = u$, $\bar{u}\delta_k = \bar{y}\sigma_i\delta_k$ if $k \neq i, i+1, j$. By induction, $\bar{u}\delta_j$ is $|B|$-equivalent to a $v \in E$ and there is a $\bar{v} \in S|E|$ with boundary $(v\sigma_{i+\varepsilon}\delta_0, \cdots, v, \bar{u}\delta_j, \cdots, v\sigma_{i+\varepsilon}\delta_{n+1})$ and $S|p|\bar{v} = y\sigma_i\sigma_{i+1}\delta_{j+\varepsilon}$. Next, lift $y\sigma_i$ to $w \in E$ with $w\delta_{i+1} = u$, $w\delta_j = v$, $w\delta_k = \bar{y}\sigma_i\delta_k$ if $k \neq i, i+1, j$ and lift $y\sigma_i\sigma_{i+1}$ to $\bar{w}$ with $\bar{u}\delta_{i+1} = w$, $\bar{w}\delta_{i+2} = \bar{u}$, $\bar{w}\delta_{j+\varepsilon} = \bar{v}$, $\bar{w}_k = w\sigma_{i+1}\delta_k$ if $k \neq i, i+1, i+2, j+\varepsilon$. Then $\bar{w}\delta_i$ is an $S|B|$-equivalence between $\bar{y}$ and $w\delta_i \in E$.

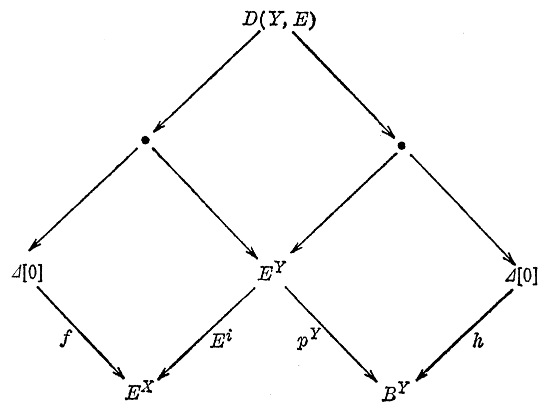
This finishes the proof. As an application, we'll derive a streng-

[1] Note that $S|p|$ is also a Kan fibration, by Quillen's result **[6]**.

[2] Cf. the proof of Lemma 4 in [2].

thening of this result which is based on the cartesian closedness of the category of semisimplicial sets. Roughly speaking, it states the semisimplicial set of semisimplicial diagonals of a square as in the theorem is a strong deformation retract of the semisimplicial set of its continuous diagonals.

To make this precise, we define the *semisimplicial set* $D(Y, E)$ *of* (semisimplicial) *diagonals* of a square (*) by means of the following diagram where the sqares involved are pullbacks



Further, the semisimplicial set of continuous diagonals of (*) is defined to be the semisimplicial set $D(Y, S|E|)$ of semisimplicial diagonals of the square

$$\begin{array}{ccc} X & \xrightarrow{i_E \circ f} & S|E| \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle S|p|} \\ Y & \xrightarrow[i_B \circ h]{} & S|B| \end{array}$$

The following lemma gives another description of $D(Y, S|E|)$.

LEMMA. *Let*

$$\begin{array}{ccc} \bar{E} & \longrightarrow & S|E| \\ {\scriptstyle \bar{p}}\downarrow & & \downarrow{\scriptstyle S|p|} \\ B & \xrightarrow[i_B]{} & S|\bar{B}| \end{array}$$

be a pullback. Then the semisimplicial set $D(Y, \bar{E})$ of diagonals of the induced square

$$\begin{array}{ccc} X & \xrightarrow{\bar{f}} & E \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \bar{p}} \\ Y & \xrightarrow[h]{} & B \end{array}$$

is isomorphic to $D(Y, S|E|)$.

The proof of this lemma is evident. Note the universal property of $\bar{E}$: The continuous $\bar{g}: |Y| \to |E|$ so that $|p| \circ \bar{g}$ is realized correspond bijectively to the semisimplicial maps $Y \to \bar{E}$. If B is a point, this is the adjunction between geometric realization and singular functor.

With these definitions we have the

COROLLARY. *Under the assumptions of the theorem on the square* (*) $D(Y, E)$ *is an strong deformation retract of* $D(Y, \bar{E})$.

Proof. The map $|\bar{E}| \to |E|$ corresponding to $\mathrm{id}\bar{E}$ is a continuous diagonal of the square

$$\begin{array}{ccc} E & \xrightarrow{\mathrm{id}E} & E \\ \downarrow & & \downarrow{\scriptstyle p} \\ \bar{E} & \longrightarrow & B \end{array}$$

Thus, the theorem implies E is a strong deformation retract of $\bar{E}$. Let $G: \bar{E} \times \Delta[1]$ be a suitable deformation. Further, let e denote the evalution $Y \times \bar{E}^Y \to \bar{E}$ and $\mathrm{id}\bar{E}$. Then, by adjointness $G \circ e$ corresponds to a map $K: \bar{E}^Y \times \Delta[1] \to \bar{E}^Y$. Its restriction to $D(Y, \bar{E}) \times \Delta[1]$ factors through $D(Y, \bar{E})$ and induces a deformation of the desired kind.

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