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A CATEGORY'S QUOTIENT CATEGORY OF ISOMORPHISM TYPES VERSUS ITS SKELETON

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Abstract. It is shown — in geometric language — that a category $\mathscr{C}$ and its quotient category $[\mathscr{C}]$ of isomorphism types have different fundamental groups. This contrasts with the fact any skeleton of $\mathscr{C}$ is a strong deformation retract of $\mathscr{C}$.

Given a category $\mathscr{C}$ one sometimes looks for another category $\mathscr{C}'$, which should be quite similar to $\mathscr{C}$ but has the additional property that the isomorphism classes of objects consist of one object only. Categorists take a skeleton while »working« mathematicians tend to form the »quotient category $[\mathscr{C}]$ of isomorphism types« just by formal identification of isomorphic objects[1]. The aim of this note is to point out that the quotient category $[\mathscr{C}]$ differs from $\mathscr{C}$ much more strongly than any skeleton of $\mathscr{C}$, namely that there is not even a pair of adjoint functors connecting $\mathscr{C}$ and $[\mathscr{C}]$ while any skeleton of $\mathscr{C}$ is equivalent to $\mathscr{C}$ itself [7; p. 91].

To be precise, let $\mathscr{C}$ be a category and $\mathscr{A}$ its class of objects. Then there are a class $\mathscr{B}$ and a surjective map $[\ ] : \mathscr{A} \to \mathscr{B}$ such that

$$[A] = [C] \Leftrightarrow A \equiv C \text{ for all } A, C \in \mathscr{A}$$

([4, p. 72]). Consider $\mathscr{A}$ as discrete subcategory of $\mathscr{C}$, $\mathscr{B}$ also as a discrete category and $[\ ]$ as a functor in the evident manner. Then form the pushout

$$\begin{array}{ccc} & \mathscr{A} & \subset & \mathscr{C} \\ [\ ] & \downarrow & & \downarrow \\ & \mathscr{B} & \subset & [\mathscr{C}] \end{array}$$

to obtain $[\mathscr{C}]$, the *quotient category of isomorphism types of* $\mathscr{C}$.

It has the class $\mathscr{B}$ as class of objects and as non-identity arrows all lists

$$(f_k, \ldots, f_2, f_1)$$

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[1] I found this tendency in many oral and by letter discussions in contrast to the opinion of a referee that nobody »would really expect the quotient category to do the job proposed«. The main reasons for it are that quotient constructions appear everywhere in the mathematical world and that the choice of a skeleton requires a suitable axiom of choice.

of arrows in $\mathscr{C}$ which satisfy

(*i*) f_i is a non-identity arrow, $1 \leqslant i \leqslant k$,

(*ii*) $\operatorname{dom} f_i \neq \operatorname{cod} f_{i-1}$, $1 < k$, but

(*iii*) $[\operatorname{dom} f_i] = [\operatorname{cod} f_{i-1}]$, $1 < i \leqslant k$

(cf [3; Theorem 4, p. 73] and [1, p. 263/264]).

Next take the groupoids $\mathrm{Gr}\,\mathscr{C}$ and $\mathrm{Gr}\,[\mathscr{C}]$ associated with $\mathscr{C}$, $[\mathscr{C}]$ respectively (in the sense of [2, p. 10]; the smallness condition there again can be avoided by the set-theoretical trick used for obtaining $\mathscr{B}$). For an object C of $\mathscr{C}$ define the *fundamental group* $\pi(\mathscr{C}, C)$ of $\mathscr{C}$ based at C as the object group $\mathrm{Gr}\,\mathscr{C}\,\{C\}$ (= group of automorphisms of C) in the groupoid $\mathrm{Gr}\,\mathscr{C}$. Analogously we have the fundamental group $\pi([\mathscr{C}], [C])$ of $[\mathscr{C}]$ based at $[C]$. This notion allows to formulate the main result of this note:

THEOREM. *Let $\mathscr{C}$ be a category and let $[\mathscr{C}]$ be its quotient category of isomorphism type. Then there is for all $C \in \mathrm{Ob}\,\mathscr{C}$ an (in general large) free group $\mathscr{F}_C$ such that*

$$\pi([\mathscr{C}], [C]) \cong \pi(\mathscr{C}, C) * \mathscr{F}_C$$

where $[C]$ denotes the isomorphism type of C and $*$ denotes the free product (of groups). Moreover $\mathscr{F}_C$ depends (up to isomorphism) only on the connected component of C in $\mathscr{C}$.

(One gets an impression of the size of the free group $\mathscr{F}_C$ by exhibiting a system of free generators: Let $\hat{\mathscr{C}}$ be the component of C in $\mathscr{C}$ and let $\{C_\mu\}$ be a class of objects in $\hat{\mathscr{C}}$ containing exactly one representative for every isomorphism type; then the class $\mathrm{Ob}\,\hat{\mathscr{C}} \setminus \{C_\mu\}$ forms such a system.)

This allows the final conclusion: *In general the fundamental groups $\pi([\mathscr{C}], [C])$ and $\pi(\mathscr{C}, C)$ will be non-isomorphic, while a pair of adjoint functors between $\mathscr{C}$ and $[\mathscr{C}]$ would induce an isomorphism between these groups.*

For small categories $\mathscr{C}$ this can be expressed more geometrically, using the notion of the classifying space $\mathrm{B}\,\mathscr{C}$ of $\mathscr{C}$ [7; p. 106]. The discussion of strong homotopy in [8; p. 199] shows that $\mathrm{B}\,\mathscr{C}'$ is a strong deformation retract of $\mathrm{B}\,\mathscr{C}$ for every skeleton C' of $\mathscr{C}$ and that any adjoint functor induces a homotopy equivalence between the corresponding classifying spaces. Since the fundamental groups of a category $\mathscr{C}$ as defined before are just the usual fundamental groups of the classifying space $\mathrm{B}\,\mathscr{C}$, the discussion shows that in general $\mathrm{B}\,\mathscr{C}$ and $\mathrm{B}\,[\mathscr{C}]$ have already different fundamental groups so that there is no chance for a homotopy equivalence between them.

The key to the theorem lies in the two following propositions.

PROPOSITION 1. — *Let* $\mathscr{B}$ *and* $\mathscr{C}$ *be groupoids and assume the free product* [1; p. 269/270]

$$\mathscr{D} = \mathscr{B} * \mathscr{C}$$

to be connected. Then, for any object $D \in \operatorname{Ob} \mathscr{D}$ *the object group* $\mathscr{D}\{D\}$ *is of the form*

$$\mathscr{D}\{D\} = \mathscr{F} * \mathscr{G} * \mathscr{H}$$

where

F *is a free group,*

G *is a free product of object groups of* $\mathscr{B}$*, exactly one for every component of* $\mathscr{B}$*, and*

H *is a free product of object groups of* $\mathscr{C}$*, exactly one for every component of* $\mathscr{C}$*.*

This can be proved by the same method as Kurosch's Theorem is proved in [3; p. 118 f.].

PROPOSITION 2. *Let*

$$\begin{array}{ccc} \mathscr{A} & \subset & \mathscr{C} \\ \mathscr{P}\downarrow & & \downarrow\mathscr{Q} \\ \mathscr{B} & \subset & \mathscr{D} \end{array}$$

be a pushout of categories such that $\mathscr{A}$ *and* $\mathscr{B}$ *are discrete, p is surjective and is connected. Then there is a free group* $\mathscr{F}$ *such that*

$$\pi(\mathscr{D}, q\, C) \cong \pi(\mathscr{C}, C) * \mathscr{F} \text{ for all } C \in \operatorname{Ob} \mathscr{C}.$$

Moreover $\mathscr{F}$ *is trivial iff* $\mathscr{P}$ *is injective.*

To prove Proposition 2 one can assume $\mathscr{C}$ and $\mathscr{D}$ to be groupoids; then a careful analysis of the pushout in question yields the result by means of an application of Proposition 1 in the special case, where $\mathscr{B}$ is totally disconnected.

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ODNOS KVOCIJENTNE KATEGORIJE TIPOVA IZOMORFNOSTI PREMA SKELETU KATEGORIJE

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Sadržaj

Neka $\mathscr{C}$ označava kategoriju a [C] njenu kvocijentnu kategoriju tipova izomorfnosti (tj. kategoriju u kojoj se formalno identificiraju izomorfni objekti). Pokazano je da te kategorije (izraženo geometrijskim jezikom) imaju različite fundamentalne grupe. To je u oprečnosti sa činjenicom da je svaki skelet od $\mathscr{C}$ strogi deformacioni rektrakt od $\mathscr{C}$.