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CW-COMPLEXES AND EUCLIDEAN SPACES

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Non è difficile dare esempi di complessi CW che hanno anche la struttura di varietà differenziale; in questi casi, tali complessi cellulari possono essere immersi in spazi euclidiani, tramite il teorema di immersione dovuto a H. Whitney. Purtroppo, è anche facile dare esempi di complessi cellulari che non sono varietà; alcuni di questi possono ancora essere immersi in spazi euclidiani. In questo lavoro studiamo una classe di complessi CW di dimensione m che possono essere immersi in $\mathbf{R}^{2m+1}$ e caratterizziamo i complessi CW metrizzabili. I nostri teoremi principali sono presentati in un modo unificato.

1. Introduction

The playground of CW-complexes is filled with euclidean balls which we are supposed to judiciously glue together. The glueing machine is the categorical process of "push-out". In what follows we indicate by B^n the set of all vectors $\mathbf{x} \in \mathbf{R}^n$ with lenght $|\mathbf{x}| \leq 1$; this set is the *n-dimensional (unit) ball.* The boundary of B^n is the *(n -1)-dimensional (unit) sphere* S^{n-1}; this is the set of all vectors $\mathbf{x} \in \mathbf{R}^n$ such that $|\mathbf{x}| = 1$.

A CW-complex X is defined inductively as follows: take a discrete space X^0; let us suppose that we have defined spaces $X^1 \subseteq X^2 \subseteq \ldots \subseteq$

X^{n-1} and construct the space X^n by a pushout diagram

(*)

$$\begin{array}{ccc} \coprod\limits_{\lambda\in\Lambda(n)} B^n_\lambda & \xrightarrow{\bar{c}} & X^n \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle \bar{i}} \\ \coprod\limits_{\lambda\in\Lambda(n)} S^{n-1}_\lambda & \xrightarrow{c} & X^{n-1} \end{array}$$

where S^{n-1}_λ (respectively, B^n_λ) is a copy of the sphere S^{n-1} (respectively, of the ball B^n) for each element λ of the set $\Lambda(n)$, i is the inclusion of the coproduct (the topological sum) $\coprod_{\lambda\in\Lambda(n)} S^{n-1}_\lambda$ into the coproduct $\coprod_{\lambda\in\Lambda(n)} B^n_\lambda$ and c is an arbitrary map. The definition of pushout implies that X^n has the final topology with respect to the maps $\bar{c}$ and $\bar{i}$; moreover, the map $\bar{i}$ is the inclusion of a closed space, more precisely, is a closed cofribation. In these circumstances, it is customary to say that X^n is obtained from X^{n-1} by *adjunction* of n-cells; the map c is called an *attaching map*, while $\bar{c}$ is a *characteristic map* for the adjunction.

Now consider the expanding sequence of topological spaces

$$X^0 \subseteq X^1 \subseteq \ldots \subseteq X^{n-1} \subseteq X^n \ldots$$

and let X be its *union space* that is to say, $X = U_{n\in N} X^n$ with the *topology determined by the family*[1] $\{X^n : n \in \mathbf{N}\} : C \subseteq X$ is closed if, and only if, $C \cap X^n$ is closed in X^n, for every $n \in \mathbf{N}$. For a given natural number n, the space X^n is called the *n-skeleton* of the CW-complex X.

The map $\bar{c}$ induces an embedding of the open n-ball $B^n_\lambda \setminus S^{n-1}_\lambda$, for every $\lambda \in \Lambda(n)$; its image e_λ is called *open n-cell* of X. Thus the open n-cells of X are the connected components of $X^n \setminus X^{n-1}$. Any map

[1] We adopt the terminology *topology determined by a family* in contrast to the more or less widespread *weak topology* on advice of Ernest Michael, whom we thank; for a variety of reasons, we find this terminology more appropriate.

$\overline{c}_\lambda : B^n_\lambda \longrightarrow X$ inducing a homeomorphism $B^n_\lambda \setminus S^{n-1}_\lambda \longrightarrow e_\lambda$ is called a *characteristic map* for e_λ. The points of X^0 are the open 0-cells.

From the set theoretical point of view, a CW-complex is just the disjoint union of its open cells; for every $x \in X$, the unique open cell $e \subseteq X$ which contains x is the *carrier* of x. The closure $\overline{e}_\lambda$ of any open cell e_λ of X is called a *closed cell* of X; it is equal to $\overline{c}_\lambda(B^n_\lambda)$ for any characteristic map $\overline{c}_\lambda$ of e_λ. While closed cells are closed, and indeed compact subsets of X, in general open cells of X are not open subsets of X. In fact, an open cell of X is not an open subset of X if it meets the boundary of a cell of higher dimension.

If in the expanding sequence defining X, the inclusion $X^{m-1} \subseteq X^m$ is strict, but for every $n \geq m, X^n = X^m$, the union space coincides with X^m and the CW-complex is said to be of *finite dimension* m.

A CW-complex X is said to be *countable* (respectively, *finite*) if $X^0 \cup \{U_{n\in \mathbf{N}\setminus\{0\}} \Lambda(n)\}$ is countable (respectively, finite).

In this paper we discuss the following embedding theorem:

THEOREM A. *Every countable and locally compact CW-complex of dimension m can be embedded in* $\mathbf{R}^{2m+1}$.

The question of the embeddability of CW-complex into Euclidean spaces is not a trivial one; indeed, we now give an example of a countable CW-complex that cannot be embedded in any Euclidean space. Let X be the CW-complex of dimension 2 having 0 and 1 as 0-cells, the open interval $]\,0, 1\,[$ as the only 1-cell and, for every natural number $n \neq 0$, X has an open 2-cell e_n such that $\overline{e}_n \setminus e_n = \{1/n\}$. We now prove that X is not a Fréchet space by showing that no sequence of $X \setminus X^1$ converges to the point 0, altought $0 \in \overline{X \setminus X^1}$. In fact, suppose that (x_n) is a sequence of $X \setminus X^1$ which converges to a point of X^1; then, there exists $m \in \mathbf{N}\setminus\{0\}$ such that the 2-cell e_m contains a subsequence (y_n) of (x_n) (otherwise (x_n) would be closed in X and could not converge to a point outside $X\setminus X^1$). This implies that $1/m = \lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n \neq 0$. Now, if X could be embedded in an Euclidean space $\mathbf{R}^p$ it would satisfy the First Axiom of Countability and therefore, it would be Fréchet (see [4, Chapter 4, Theorem 1.1]).

Theorem A implies that countable and locally compact CW-complexes of finite dimension are metrizable; this, of course, brings up the question of characterizing the CW-complexes which are metrizable. The temptation of treading the classical path of the Metrization Theorems by Urysohn or Nagata-Smirnov is clearly great, particularly once one discovers that CW-complexes are paracompact and normal spaces. Would countability alone guarantee the existence of a countable basis (and hence of a compatible metric, via Urysohn's Metrization Theorem)? The countable CW-complex just described shows that the answer to this question is: no! In fact, the following characterization theorem shows that we are indeed closer to the Nagata-Smirnov Theorem.

THEOREM B. *Let X be a CW-complex; the following conditions on X are equivalent:*

(i) X is locally compact;

(ii) X is metrizable;

(iii) X satisfies the First Axiom of Countability.

If we let drop the dimension hypothesis in Theorem A we still obtain an interesting result:

THEOREM C. *A locally compact and countable CW-complex embeds in the Hilbert cube.*

Theorems A and C have a converse:

THEOREM D. *The following two results hold true for a CW-complex X:*

(i) If X can be embedded in the Hilbert cube, then X is countable and locally compact;

(ii) If X can be embedded in the Euclidean space $\mathbf{R}^m$, then X is countable, locally compact and has dimension $\leq m$.

The theorems stated above are, in one form or another, well-known to the experts. In this paper we construct a way conducing directly to their proof and present a unified treatment of the subject.

2. Local compactness

CW-complexes have a rich topological structure; for example, as stated before, they are paracompact and normal. They are also locally connected and furthermore, are Locally Equiconnected Spaces (LEC). The reader can find the proofs of these properties in [3]. Not all CW-complexes are compact or even locally compact; the aim of this section is to recall a characterization of local compactness for CW-complexes in terms of its cells.

LEMMA 2.1. *The topology of a CW-complex is determined by the family of its closed cells.*

Proof. Let X be a CW-complex with skeleta X^n and let $U \subseteq X$ be a set whose intersection with all closed cells of X is closed; then we have to prove that $U \cap X^n$ is closed, for every $n \in \mathbf{N}$.

Since X^0 is discrete, $U \cap X^0$ is closed in X^0. Assume, by induction, that $U \cap X^{n-1}$ is closed in X^{n-1}. Take a pushout diagram $(*)$; we have to prove that $\bar{c}^{-1}(U \cap X^n)$ is closed in $\coprod_{\lambda \in \Lambda(n)} B^n_\lambda$. The map $\bar{c}$ induces a family $\{\bar{c}_\lambda : \lambda \in \Lambda(n)\}$ of characteristic maps for the n-cells of X; the hypothesis implies that $\bar{c}_\lambda^{-1}(U \cap X^n) = \bar{c}_\lambda^{-1}(U \cap \bar{e}_\lambda)$ is closed in B^n_λ for every $\lambda \in \Lambda(n)$. Consequently, $\bar{c}^{-1}(U \cap X^n) = \cup_{\lambda \in \Lambda(n)} \bar{c}_\lambda^{-1}(U \cap X^n)$ is closed in $\coprod_{\lambda \in \Lambda(n)} B^n_\lambda$. □

The next result is standard in virtually any work describing CW-complexes; its proof is given here for the sake of completeness.

LEMMA 2.2. *Let K be a compact subset of a CW-complex X. Then, K is contained in a finite union of open cells of X.*

Proof. Let $S \subseteq K$ be a set formed by taking a point $x_e \in e \cap K$, for each open cell e of X which intersects K. We are going to prove, by induction, that the set S intersects any skeleton of X in only finitely many points; thus S is a discrete closed subset of X and also, of K. Since any discrete closed subset of a compact space is finite, it follows that S is finite.

Clearly, $S \cap X^0 = K \cap X^0$ is a discrete, closed subset of the compact space K and thus, it is finite. Assume that $S \cap X^{n-1}$ is finite. For every closed n-cell $\bar{e}$, the intersection $S \cap \bar{e}$ consists at most of x_e and the finitely many elements of $S \cap X^{n-1}$, altogether, a finite number of points. Since X^n is itself a CW-complex and therefore, determined by the family of its closed cells (see Lemma 2.1), $S \cap X^n$ is a closed subset of X^n which is discrete and contained in the compact space K, thus, finite. □

COROLLARY 2.3. *A CW-complex is finite if, and only if, it is compact.* □

The next result gives a characterization of local compactness of CW-complexes in terms of open and closed cells.

PROPOSITION 2.4. *A CW-complex X is locally compact if, and only if, every open cell of X meets only finitely many closed cells of X.*

Proof $\Rightarrow$: Let e be an open cell of the locally compact CW-complex X. The hypothesis implies that every point of $\bar{e}$ has a compact neighborhood; since $\bar{e}$ is itself compact, the open cell e is covered by finitely many of these compact neighborhoods and hence, there is a compact neighborhood V of e in X. Using Lemma 2.2 we conclude that V intersects only a finite number of open cells of X; on the other hand, e does not intersect the closure of any open cell of X contained in $X \setminus V$. Together, these two facts imply the desired conclusion.

$\Leftarrow$: Take $x \in X$ arbitrarily and let the m-cell e_x be its carrier. Let

Ω be the finite set of all closed cells of X which meet e_x; the union

$$W = \bigcup_{\bar{e}\in\Omega} \bar{e}$$

is a compact set such that $x \in e_x \subseteq W$. We are going to prove that W is a neighborhood of x.

To this end, let Ω' be the set of all closed cells of X which meet W. Lemma 2.2 and the hypothesis imply that the set Ω' is also finite; thus, the union

$$C = \bigcup_{\overline{e'}\in\Omega'\setminus\Omega} \overline{e'}$$

is again compact and in particular, is a closed subset of X. It follows that the set $U = X \setminus C$ is an open subset of X containing the point x and its carrier e_x. Now, for each $n \geq m$ fix a map

$$\bar{c}^n : \coprod_{\lambda\in\Lambda(n+1)} B_\lambda^{n+1} \longrightarrow X^{n+1}$$

which describes the adjunction of the $(n+1)$-balls to X^n generating X^{n+1}; next, starting with $V_m = e_x$, define inductively the sets V_n by taking

$$V_{n+1} = V_n \cup \bar{c}^n(\{ts : s \in (\bar{c}^n)^{-1}(V_n), 1/2 < t \leq 1\}) \setminus C;$$

it follows by induction that each set V_n is open[2] and that $V_{n+1} \cap X^n = V_n$. Taking $V = \bigcup_{n\geq m} V_n$ we obtain

$$V \cap X^n = \begin{cases} \emptyset, & 0 \leq n < m, \\ V_n, & n \geq m, \end{cases}$$

showing that V is an open set of X. On the other hand, induction also yields that each V_n is contained in W implying that $V \subseteq W$ and thus, the

[2] Cf. the section on collaring in [3].

desired result. For the induction step, consider a point $y = \overline{c}^n(ts) \in V_{n+1}$ with carrier e_y. By the induction hypothesis, $\overline{c}^n(s) \in (W\backslash C)\cap\overline{e_y}$. Thus, the closed cell $\overline{e}_y$ belongs to the set Ω' but not to the set $\Omega' \setminus \Omega$. This shows that $\overline{e_y} \in \Omega$, that is to say, $y \in \overline{e_y} \subseteq W$, thus completing the argument. □

In view of Lemma 2.2. we have the following immediate consequence of this proposition:

COROLLARY 2.5. *A CW-complex X is locally compact if, and only if, the closed cells of X form a locally finite (closed) covering of X.* □

A CW-complex satisfying the property that each one of its open cells intersects only finitely many closed cells is said to be *locally finite.* In view of Prposition 2.4, we could use "locally finite" and "locally compact" interchangeably, whenever dealing with CW-complexes.

3. Subcomplexes

Let Ω be a set of open cells of a CW-complex X. The set $A = \cup_{e\in\Omega} e$ is called a *subcomplex* of X if, for every $e \in \Omega, \overline{e} \subseteq A$. This definition shows clearly that arbitrary unions and intersections of subcomplexes of a CW-complex X are subcomplexes of X. A subcomplex of X is a CW-complex on its own right.

The following is an important class of subcomplexes of a CW-complex X: for every open cell e of X, $X(e)$ is the intersection of all subcomplexes of X containing e. The interest of these subcomplexes lies on the fact that they are compact spaces, i.e., finite CW-complexes (this is an easy consequence of Lemma 2.2).

Another interesting class of subcomplexes of X is given by taking, for every open cell e of X, the space

$$St(e) = \cup_{\overline{e\prime}\cap e\neq\emptyset} X(e\prime).$$

The subcomplexes $St(e)$ in general, are not finite; however, they are finite (i.e. compact) whenever X is locally finite (i.e. locally compact): this is an immediate consequence of the definition.

To complete this list of examples of subcomplexes we observe that the path-components of a CW-complex X are subcomplexes of X (see[3, Proposition 1.4.11]).

The following two results are needed for the proof of Theorem B.

LEMMA 3.1 *A locally finite and countable CW-complex is the union space of an expanding sequence of finite subcomplexes X_n such that, for every n, X_n is contained in the interior of X_{n+1} (the interior taken with respect to the topology of X).*

Proof Let X be a CW-complex satisfying the hypothesis of the lemma. We wish to prove that X is the union space of a family $\{X_n : n \in \mathbf{N}\}$ of finite subcomplexes of X. Let $\{e_n : n \in \mathbf{N}\}$ be the countable set of open cells of X. Define X_0 as the empty space and assume that X_n has been defined. Take the integer $i = \min\{j : e_j \not\subset X_n\}$ and define

$$X_{n+1} = St(e_i) \cup \cup_{e \in \Omega} St(e),$$

where Ω is the finite set of all open cells contained in X_n. Notice that, as a finite union of finite subcomplexes of X, X_{n+1} is also a finite subcomplex of X. Clearly, as sets, X and $\cup_{n \in \mathbf{N}} X_n$ coincide.

In order to prove that X is indeed the union space of $\{X_n : n \in \mathbf{N}\}$, we proceed as follows. We first observe that the proof of Proposition 2.4 can be used "ipsis litteris" to show that, for every open cell e of X, the space $St(e)$ is a neighborhood of e; hence, each X_n is contained in the interior of X_{n+1}. This shows that the set X is equal to the set $\cup_{n \in \mathbf{N}} \overset{\circ}{X}_n$. Now take a set $W \subseteq X$ such that, for every $n \in \mathbf{N}$, $W \cap X_n$ is open in X_n. Then, $W \cap \overset{\circ}{X}_n$ is open in $\overset{\circ}{X}_n$ and hence, in X. It follows that

$$W = W \cap X = W \cap \cup_{n \in \mathbf{N}} \overset{\circ}{X}_n = \cup_{n \in \mathbf{N}} W \cap \overset{\circ}{X}_n$$

is open in X. $\square$

LEMMA 3.2. *Let X be a locally finite and path-connected CW-complex; then X is countable.*

Proof. Let Ω be the set of all open cells of X and let $e_\star$ be a fixed open cell of X; for each $n \in \mathbf{N}$, define the set

$$A_n = \{(e_0, e_1, \ldots, e_n) : e_\star = e_0, e_i \in \Omega \text{ and } e_i \subseteq St(e_{i-1}) \\ \text{for } i = 1, \ldots, n\}$$

Since the subcomplexes $St(e_{i-1})$ are finite, it follows that the sets A_n are all finite and so, the set $A = \bigcup_{n=0}^{\infty} A_n$ is countable.

Now take the function $\alpha : A \longrightarrow \Omega$ defined by

$$\alpha \mid A_n((e_0, \ldots, e_n)) = e_n$$

for every $n \in \mathbf{N}$. The desired countability of Ω follows from the fact that the function α is onto; this, in turn, is a consequence of the path-connectivity of X. To prove that α is onto, consider an arbitrary open cell e of X and take a path $w : [0,1] \longrightarrow X$ connecting a point of $e_\star$ with a point of e. Let Ω_0 denote the set of all open cells that meet this path. The set Ω_0 is finite, in view of Lemma 2.2. Starting with $t_0 = 0$ and $e_0 = e_\star$, define inductively $t_{i+1} = \max\{t \in]t_i, 1] : w(t) \in St(e_i)\}$ and take e_{i+1} as the carrier of $w(t_{i+1})$ which is contained in $St(e_i)$. The process stops with $i = n$ if $t_n = 1$; in this case, we have $e = e_n = \alpha(e_0, \ldots, e_n)$. It remains to show that there is such an n. To this end, define $\Omega_i = \{e \in \Omega_{i-1} : e \cap w(]t_i, 1]) \neq \emptyset\}$ and $C_i = \cup_{e \in \Omega_i} \overline{e}$ for $i > 0$, as long as $t_i < 1$. Since C is a closed set containing $w(]t_i, 1])$ we have $w(t_i) \in C_i$ and thus, we conclude that $e \in \Omega_i$ and $w(t_i) \in \overline{e}$; hence, $e \subseteq St(e_i)$. This forces $t_{i+1} > t_i$ and $\Omega_{i+1} \subseteq \Omega_i \setminus \{e\}$. Therefore the sequence

$$\Omega_0 \supset \Omega_1 \supset \ldots \supset \Omega_i \supset \ldots$$

is strictly decreasing and must stop because Ω_0 is finite. □

4. Proofs of the Main Theorems

We begin by proving the following crucial result.

THEOREM 4.1. *Let X be a locally finite and countable CW-complex of dimension m; then, there exists an embedding*

$$f : X \longrightarrow \mathbf{R}^{k(m)},$$

with $k(m) = \frac{(m+1)(m+2)}{2}$.

Proof. We are going to construct the map f inductively over the skeleta of X. To begin with, we enumerate the 0-cells of X and define $f_0 : X^0 \to \mathbf{R}^1$ as the function which sends the only point of the j^{th} 0-cell of X into $2j \in \mathbf{R}$. Suppose that $f_n : X^n \to \mathbf{R}^{k(n)}$ has been defined. Let $e_0, e_1, \ldots, e_j, \ldots$ be an enumeration of the open $(n+1)$-cells of X; for each $j \in \mathbf{N}$, let

$$\overline{c}_j : B_j^{n+1} \longrightarrow X^{n+1}$$

be a characteristic map for the open cell e_j and let

$$c_j : S_j^n \longrightarrow X^n$$

be its restriction to the boundary. Then define the injection $f_{n+1}(x) =$

$$\begin{cases} (f_n(x), 0), & x \in X^n \\ 2j(1-t)\mathbf{e}_{k(n)+1} + [tf_n(c_j(\mathrm{s})), (1-t)t\mathrm{s}, 1-t], & x = \overline{c}_j(t\mathrm{s}) \in e_j, \end{cases}$$

where $\mathbf{e}_{k(n)+1} \in \mathbf{R}^{k(n+1)}$ is the unit vector with the $(k(n)+1)^{th}$ coordinate equal to 1. Finally, set $f = f_m$.

We now prove by induction that each $f_n, n = 0, \ldots, m$, is an embedding. Clearly this is so for $n = 0$. Assume that f_n is an embedding; we wish to prove that $f_{n+1} : X^{n+1} \to \mathbf{R}^{k(n+1)}$ is also an embedding.

Since f_{n+1} is continuous and one-to-one, we have only to show that it takes open sets of X^{n+1} into open sets of $f(X^{n+1})$. Let e_j be an open cell of dimension $n+1$ (in the fixed enumeration of the $(n+1)$-cells); notice that $f_{n+1}(e_j) = f(X^{n+1}) \cap V_j$, where V_j is the set of all the elements $z = (z_1, \ldots, z_{k(n+1)}) \in \mathbf{R}^{k(n+1)}$ such that

$$(2j-1)z_{k(n+1)} < z_{k(n)+1} < (2j+1)z_{k(n+1)}$$

for each $j \in \mathbf{N}$. Because V_j is an open set of $\mathbf{R}^{k(n+1)}$, it follows that $f_{n+1}(e_j)$ is open in $f(X^{n+1})$. Now let V be an arbitrary open set of X^{n+1}; we are going to prove that for every $x \in V$, $f_{n+1}(x)$ is an interior point of $f_{n+1}(V)$ with respect to $f_{n+1}(X^{n+1})$.

Case 1 - Suppose that $x \in X^{n+1} \setminus X^n$; let e_j be the open cell of dimension $n+1$ which is the carrier of x. Because $f_{n+1}|\overline{e}_j$ is an embedding, $f_{n+1}(V \cap e_j)$ is open in $f_{n+1}(\overline{e}_j)$; thus, $f_{n+1}(V \cap e_j)$ is open in $f_{n+1}(e_j)$ and therefore, in $f_{n+1}(X^{n+1})$.

Case 2 Now suppose that $x \in X^n$. In view of corollary 2.5 we can assume that V meets only finitely many closed $(n+1)$-cells, say $\overline{e}_{j_0}, \ldots, \overline{e}_{j_r}$. It suffices to prove that no sequence in $f_{n+1}(X^{n+1}) \setminus f_{n+1}(V)$ converges to $f_{n+1}(x) = f_n(x)$. Assume the contrary, i.e., suppose that there is a sequence $\{x_i : i \in \mathbf{N}\}$ in $X^{n+1} \setminus V$ such that

$$\lim_{i \to \infty} f_{n+1}(x_i) = f_{n+1}(x) = f_n(x).$$

By the induction hypothesis $f_{n+1}(V \cap X^n)$ is open in $f_{n+1}(X^n)$ and therefore the sequence $\{x_i\}$ cannot have a subsequence contained in X^n. Hence, one may assume that $\{x_i : i \in \mathbf{N}\} \subset X^{n+1} \setminus X^n$; this means that each x_i is of the form

$$x_i = \overline{c}_{j_{p(i)}}(t_i \mathbf{s}_i)$$

for some $p(i) \in \mathbf{N}, t_i \in [0,1)$ and $\mathbf{s}_i \in S^n$. Considering that the last coordinate of $f_{n+1}(x_i)$ is $1 - t_1$ and the last coordinate of $f_n(x)$ is 0, it follows that

$$\lim_{i \to \infty}(1 - t_i) = 0$$

that is to say,

$$\lim_{i\to\infty} t_i = 1.$$

This implies that

$$f_n(x) = \lim_{i\to\infty} f_{n+1}(c_{j_{p(i)}}(\mathbf{s}_i)) = f_n(\lim_{i\to\infty}(c_{j_{p(i)}}(\mathbf{s}_i))).$$

From the induction hypothesis we now obtain that

$$x = \lim_{i\to\infty}(c_{j_{p(i)}}(\mathbf{s}_i))$$

so that we may assume

$$\{c_{j_{p(i)}}(\mathbf{s}_i) : i \in \mathbf{N}\} \subset X^n \cap V;$$

hence, $\{j_{p(i)} : i \in \mathbf{N}\} \subset \{j_0, \dots, j_r\}$. This implies that the sequence $\{p(i)\}$ must contain a constant sequence, i.e., that we have a subsequence $\{y_k : k \in \mathbf{N}\}$ of the sequence $\{x_i\}$ which is contained in one open $(n+1)$-cell e_j, with $0 \leq s \leq r$. Finally this shows that

$$x = \lim_{k\to\infty} y_k$$

contradicting the fact that $\{y_k\} \subset \overline{e}_j \setminus V$. □

In the previous section we defined the dimension of a CW-complex in terms of its skeleta; we now show that for finite dimensional CW-complexes, such a definition coincides with the classical notion of *covering dimension* (for the definition of covering dimension and its properties see [2, Chapter 1,§6]). In the sequel, we shall need the following characterization of covering dimension for normal spaces (see [2, Theorem 3.2.10]):

LEMMA 4.2. *A normal space X has covering dimension $\leq n$ if, and only if, for each closed subset $C \subseteq X$, an arbitrary map $C \longrightarrow S^n$ can be extended to a map $X \longrightarrow S^n$.* □

LEMMA 4.3. *Let $X = \bigcup_{n\in \mathbf{N}} X^n$ be a CW-complex. Then, for $n \in \mathbf{N}$, the covering dimension of X^n is at most n; the covering dimension of X^n is just n if, and only if, there are n-cells.*

Proof. Since X^0 is discrete, its covering dimension is 0; assume that the covering dimension of X^{n-1} is $\leq n-1$.

If there are no n-cells then, $X^n = X^{n-1}$ and thus, the covering dimension of X^n is less than n. Assume that X has n-cells and make the following two observations: 1) the covering dimension of the n-ball is just n (see [2, Chapter 1, §8]) and 2), any closed subspace of a space of covering dimension $\leq n$ has itself covering dimension $\leq n$. Hence, the assumption that X^n has n-cells implies that the covering dimension of X^n is at least n: in fact, n-cells contain closed n-balls in their interior and therefore their covering dimension is at least n. It remains to prove that the covering dimension of X^n is $\leq n$. To this end, take a pushout diagram like $(*)$, a closed subset C of X^n and a map $k : C \to S^n$. The induction hypothesis and Lemma 4.2 imply that the map $k|C \cap X^{n-1} \to S^n$ can be extended to a map $g : X^{n-1} \to S^n$. Let $\overline{c} : \coprod_{\lambda\in\Lambda(n)} B^n_\lambda \to X^n$ be the map which describes the adjunction of n-balls to X^{n-1} generating X^n and let $c : \coprod_{\lambda\in\Lambda(n)} S^{n-1}_\lambda \to X^{n-1}$ be the induced map; furthermore let $\tilde{c} : \overline{c}^{-1}(C) \to C$ be the map induced by the restriction of $\overline{c}$ to $\overline{c}^{-1}(C)$. Now take the space $D = \overline{c}^{-1}(C) \cup \coprod_{\lambda\in\Lambda(n)} S^{n-1}_\lambda$; note that D is a closed subset of $\coprod_{\lambda\in\Lambda(n)} B^n_\lambda$, as the union of two closed subsets. The maps $k \circ \tilde{c}$ and $g \circ c$ fit together to produce a map $d : D \to S^n$ which we extend to a map $h : \coprod_{\lambda\in\Lambda(n)} B^n_\lambda \to S^n$ again using Lemma 4.2. The universal property of pushouts now gives rise to a map $h' : X^n \to S^n$, whose restriction to C is k. Then, the covering dimension of X^n is n, in view of Lemma 4.2. □

THEOREM 4.4. *A finite positive integer n is the covering dimension of a CW-complex X if, and only if, $n = min\{m \in \mathbf{N} : X = X^m\}$.*

Proof $\Rightarrow$: Notice that -as explained in the previous proof- the presence of m-cells in a CW-complex X forces it to have covering dimension at least m. Thus, if X has covering dimension n, there cannot be m-cells in X with $m > n$, i.e., $X = X^n$. If there is an integer $m < n$ such that $X = X^m$, then Lemma 4.3 would imply that the covering dimension of X should be less or equal than m i.e., strictly less than n. Thus, $n = \min\{m \in \mathbf{N} : X = X^m\}$.

$\Leftarrow$: If $X = X^m$, the covering dimension of X is equal to the covering dimension of X^m, the latter one being less or equal to m by Lemma 4.3. Our hypothesis implies that the covering dimension of $X \leq n$. Since $X \neq X^{n-1}$, X must contain m-cells with $m \geq n$ and thus, the covering dimension of $X \geq n$. This implies that actually the covering dimension of X is just n. □

Proof of Theorem A. From Theorem 4.1 we conclude that a countable and locally finite m-dimensional CW-complex is metrizable and satisfies the Second Axiom of Countability; now use Theorem 4.4 and the Theorem of Menger-Nöbeling: "A metrizable space of covering dimension m satisfying the Second Axiom of Countability embeds in $\mathbf{R}^{2m+1}$" (see [2, Theorem 1.11.4]). □

Proof of Theorem B. $(i) \Rightarrow (ii)$: The path-components of X are subcomplexes and in view of (i), are locally finite; then, because of Lemma 3.2, every path-component of X is a locally finite and countable CW-complex. Let Y be a path-component of X; because of Lemma 3.1, we can write Y as the union space of an expanding sequence $\{Y_n : n \in \mathbf{N}\}$ of finite CW-complexes. Now, each of these CW-complexes Y_n can be embedded in a convenient Euclidean Space (see Theorem 4.1). Using Cantor's diagonal process we construct a countable basis for Y; this and the normality of Y show that Y is metrizable by Urysohn's Metrization Theorem. Then, X is metrizable because it is the topological sum of its path-components.

$(ii) \Rightarrow (iii)$: Trivial

$(iii) \Rightarrow (i)$: We prove this implication by contradiction. Suppose that X is not locally finite; then there is an open cell $e \subset X$

meeting infinitely many closed cells. Choose a sequence $\{e_j : j \in \mathbf{N}\}$ of pairwise distinct open cells such that $e \cap \overline{e}_j \neq \emptyset$ and, for every $j \in \mathbf{N}$, choose a point $x_j \in e \cap \overline{e}_j$. Because $\overline{e}$ is compact, the sequence $\{x_j : j \in \mathbf{N}\}$ has a cluster point $x \in X$ (see [1, Chapter I, Section 9.1, Definition 1]).

Let $U_0 \supset U_1 \supset \ldots \supset U_n \supset \ldots$ be an open basis for the neighborhood system of x -recall that X satisfies the First Axiom of Countability! Because each U_n meets infinitely many points x_j and therefore, infinitely many open cells e_j we can define a sequence $\{j_n : n \in \mathbf{N}\}$ of natural numbers by taking

$$j_0 = \min\{j : U_0 \cap e_j \neq \emptyset\},$$
$$j_{n+1} = \min\{j : j > j_n, U_{n+1} \cap e_j \neq \emptyset\}.$$

For every $n \in \mathbf{N}$, choose a point $z_n \in U_n \cap e_{j_n}$ and observe that the sequence $\{z_n : n \in \mathbf{N}\}$ is closed: in fact, every closed cell of X contains at most finitely many elements z_n (use Lemma 2.2 to prove this). Every neighborhood U of x contains one U_n and thus, U contains all the points z_i such that $i \geq n$. This fact implies that $x = \lim_{n\to\infty} z_n$, contradicting the fact that $\{z_n : n \in \mathbf{N}\}$ is a discrete subset of X. □

Proof of Theorem C. Suppose that X is a locally finite and countable CW-complex. By Theorem B, X is metrizable; on the other hand, Lemma 3.1 permits us to construct a countable basis for the open sets of X i.e., X satisfies the Second Axiom of Countability. Now use [1, Chapter IX, Section 2.8, Theorem 12]. □

Proof of Theorem D. (i) As a subspace of the Hilbert cube, X satisfies both axioms of countability; in particular, the First Axiom of Countability and Theorem B show that X is locally compact. Lemma 3.2 now implies that the path-components of X are countable. Then, if X is not countable it cannot have a countable number of path-components and therefore, X cannot satisfy the Second Axiom of Countability.

(ii) By the Theorem of Invariance of Domain $\mathbf{R}^m$ cannot contain open cells of dimension $> m$. □

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