The Multiple-Orientability Thresholds for Random Hypergraphs[†]

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A k-uniform hypergraph $H = (V, E)$ is called ℓ-orientable if there is an assignment of each edge $e \in E$ to one of its vertices $v \in e$ such that no vertex is assigned more than ℓ edges. Let $H_{n,m,k}$ be a hypergraph, drawn uniformly at random from the set of all k-uniform hypergraphs with n vertices and m edges. In this paper we establish the threshold for the ℓ-orientability of $H_{n,m,k}$ for all $k \geqslant 3$ and $\ell \geqslant 2$, that is, we determine a critical quantity $c^*_{k,\ell}$ such that with probability $1 - o(1)$ the graph $H_{n,cn,k}$ has an ℓ-orientation if $c < c^*_{k,\ell}$, but fails to do so if $c > c^*_{k,\ell}$.

Our result has various applications, including sharp load thresholds for cuckoo hashing, load balancing with guaranteed maximum load, and massive parallel access to hard disk arrays.

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1. Introduction

This paper studies the property of multiple orientability of random hypergraphs. For any integers $k \geqslant 2$ and $\ell \geqslant 1$, a k-uniform hypergraph is called *ℓ-orientable* if, for each edge, we can select one of its vertices so that all vertices are selected at most ℓ times. This definition generalizes the classical notion of orientability of graphs, where we want to orient the edges under the condition that no vertex has in-degree larger than ℓ. In

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this paper, we consider random k-uniform hypergraphs $H_{n,m,k}$, for $k \geqslant 3$, with n vertices and $m = \lfloor cn \rfloor$ edges. Our main result establishes the existence of a critical density $c^*_{k,\ell}$ (determined explicitly in Thorem 1.1) such that, when c crosses this value, the probability that the random hypergraph is ℓ-orientable drops abruptly from $1 - o(1)$ to $o(1)$ as the number of vertices n grows.

The case $k = 2$ and $\ell \geqslant 1$ is well understood. In fact, this case corresponds to the classical random graph $G_{n,m}$ drawn uniformly from the set of all graphs with n vertices and m edges. A result of Fernholz and Ramachandran [7] and Cain, Sanders and Wormald [3] implies that there is a constant $c^*_{2,\ell}$ such that as $n \to \infty$

$$\mathbb{P}(G_{n,\lfloor cn \rfloor} \text{ is } \ell\text{-orientable}) \to \begin{cases} 0 & \text{if } c > c^*_{2,\ell}, \\ 1 & \text{if } c < c^*_{2,\ell}. \end{cases}$$

In other words, there is a critical value such that when the average degree is below this, then with high probability an ℓ-orientation exists, and otherwise not. We want to remark at this point that the orientation can be found efficiently by solving a matching problem on a suitably defined bipartite graph, but we will not consider computational issues any further in this paper.

Similarly, the case $\ell = 1$ and $k \geqslant 3$ arbitrary is also well-understood. The threshold for the 1-orientability is known from the work of the first and the third author [9, 10], and Frieze and Melsted [11]. In particular, there is a constant $c^*_{k,1}$ such that as $n \to \infty$

$$\mathbb{P}(H_{n,\lfloor cn \rfloor, k} \text{ is 1-orientable}) \to \begin{cases} 0 & \text{if } c > c^*_{k,1}, \\ 1 & \text{if } c < c^*_{k,1}. \end{cases}$$

In this paper we consider the general case, *i.e.*, k and ℓ arbitrary. Our main result is summarized in the following theorem, and settles the threshold for the ℓ-orientability property of random hypergraphs for all k and ℓ.

Theorem 1.1. *For integers $k \geqslant 3$ and $\ell \geqslant 2$ let ξ^* be the unique solution of the equation*

$$k\ell = \frac{\xi^* Q(\xi^*, \ell)}{Q(\xi^*, \ell + 1)}, \quad \text{where} \quad Q(x, y) = 1 - e^{-x} \sum_{j < y} \frac{x^j}{j!}. \tag{1.1}$$

Let $c^*_{k,\ell} = \xi^*/kQ(\xi^*, \ell)^{k-1}$. Then, as $n \to \infty$

$$\mathbb{P}(H_{n,\lfloor cn \rfloor, k} \text{ is } \ell\text{-orientable}) \to \begin{cases} 0 & \text{if } c > c^*_{k,\ell}, \\ 1 & \text{if } c < c^*_{k,\ell}. \end{cases} \tag{1.2}$$

The work of Frieze and Melsted [11] is based on the analysis of the Karp–Sipser algorithm for matchings in bipartite graphs. More specifically, the bipartite graph that is considered is the incidence graph where the vertices of one part are the edges of the random hypergraph and the other part consists of its vertices. Each vertex that corresponds to a hyper-edge is adjacent to its incident vertices. A 1-orientation corresponds to a matching in this bipartite graph. It is not easy to see how and whether this approach can deal with the ℓ-orientability, for $\ell > 1$.

A similar result which uses different techniques was also shown in a slightly different context by Gao and Wormald [12], with the restriction that the product $k\ell$ is large. In particular, Gao and Wormald [12] consider what is called the (r, ℓ)-*orientability* of a hypergraph where each edge is 'oriented' to r vertices with the restriction that no vertex has more than ℓ edges oriented to it. Their analysis is based on the notion of the $(r, \ell + 1)$-core of the random hypergraph which is an analogue of the notion of the $(\ell + 1)$-core of a graph. This core is discovered through a special deletion process, which is analysed with the differential equations method. This demands that ℓ is large. So, our result fills the remaining gap, and treats especially the cases of small k and arbitrary ℓ, which are most interesting in practical applications. Further generalizations of the concept of orientability of hypergraphs were considered after our work in [15, 16], where tight asymptotic results were also obtained.

The present paper is a non-trivial extension of the approach followed in [9, 10]. We consider the $(\ell + 1)$-*core* of $H_{n,\lfloor cn \rfloor, k}$ and its subsets that have density greater than ℓ. We use a tedious first moment argument that bounds the expected number of such subsets. This yields that when c is below the critical value, these do not exist.

1.1. Applications

Cuckoo hashing. The paradigm of many choices has significantly influenced the design of efficient data structures and, most notably, hash tables. *Cuckoo hashing*, introduced by Pagh and Rodler [18], is a technique that extends this concept. We consider here a slight variation of the original idea (see also Fotakis, Pagh, Sanders and Spirakis [8]), where we are given a table with n locations, and we assume that each location can hold ℓ items. Each item to be inserted chooses randomly $k \geqslant 2$ locations and has to be placed in any one of them. How much load can cuckoo hashing handle before collisions make successful assignment of the available items to the chosen locations impossible? Practical evaluations of this method have shown that one can allocate a number of elements that is a large proportion of the size of the table, being very close to 1 even for small values of $k\ell$ such as 4 or 6. Our main theorem provides the theoretical foundation for this empirical observation: if the number of items is less than $c_{k,\ell}^* n$, then it is highly likely that they can be allocated; however, if their number is larger, then most likely every allocation will have an overfull bin. Our result thus proves a conjecture about the threshold loads of cuckoo hashing made in [5].

Load balancing. In a typical load balancing problem we are given a set of $m = \lfloor cn \rfloor$ identical jobs, and n machines on which they can be executed. Suppose that each job may choose randomly among k different machines. Is there any upper bound for the maximum load that can be guaranteed with high probability? Our main result implies that as long as $c < c_{k,\ell}^*$, then there is an assignment of the jobs to their preferred machines such that no machine is assigned more than ℓ different tasks.

Parallel access to hard disks. In our final application we are given n hard disks (or any other means of storing large amounts of information), which can be accessed in parallel and independently of each other. We want to store a data set redundantly in order to obtain some degree of fault tolerance, while at the same time we aim at minimizing

the number of I/O steps needed to retrieve the whole data; see [19] for more details. Theorem 1.1 guarantees the following property with high probability. If k randomly allocated copies of each data block are stored on n hard disks, then $m = \lfloor cn \rfloor$ different data blocks can be read with at most ℓ queries to each disk, provided that $c < c_{k,\ell}^*$.

2. Proof strategy and the upper bound

Our main result follows immediately from the two theorems below. The first statement says that $H_{n,m,k}$ has a subgraph of density $> \ell$ (*i.e.*, the ratio of the number of edges to the number of vertices in this subgraph is greater than ℓ) if $c > c_{k,\ell}^*$. The $(\ell + 1)$-*core* of a hypergraph is its maximum subgraph that has minimum degree at least $\ell + 1$.

Theorem 2.1. *Let $c_{k,\ell}^*$ be defined as in Theorem 1.1. If $c > c_{k,\ell}^*$, then with probability $1 - o(1)$ the $(\ell + 1)$-core of $H_{n,cn,k}$ has density greater than ℓ.*

Note that this implies the statement in the first line of (1.2), as by the pigeonhole principle it is impossible to orient the edges of a hypergraph with density larger than ℓ so that each vertex has in-degree at most ℓ.

The above theorem is not very difficult to prove, as the core of random hypergraphs and its structural characteristics have been studied quite extensively in recent years; see *e.g.* the results by Cooper [4], Molloy [17] and Kim [14]. However, it requires some technical work, which is accomplished in Section 2.1. The heart of this paper is devoted to the 'subcritical' case, where we show that the above result is essentially tight.

Theorem 2.2. *Let $c_{k,\ell}^*$ be defined as in Theorem 1.1. If $c < c_{k,\ell}^*$, then with probability $1 - o(1)$ all subgraphs of $H_{n,cn,k}$ have density smaller than ℓ.*

Proof of Theorem 1.1. Let us construct an auxiliary bipartite graph $B = (\mathcal{E}, \mathcal{V}; E)$, where $\mathcal{E}$ represents the m edges and $\mathcal{V} = \{1, \ldots, n\} \times \{1, \ldots, \ell\}$ represents the n vertices of $H_{n,m,k}$. Also, $\{e, (i, j)\} \in E$ if the eth edge contains vertex i, and $1 \leqslant j \leqslant \ell$. Note that $H_{n,m,k}$ is ℓ-orientable if and only if B has a left-perfect matching, and by Hall's theorem such a matching exists if and only if for all $\mathcal{I} \subseteq \mathcal{E}$ we have that $|\mathcal{I}| \leqslant |\Gamma(\mathcal{I})|$, where $\Gamma(\mathcal{I})$ denotes the set of neighbours of the vertices in $\mathcal{I}$ in $\mathcal{V}$.

Observe that $\Gamma(\mathcal{I})$ is precisely the set of ℓ copies of the vertices that are contained in the hyperedges corresponding to items in $\mathcal{I}$. So, if $c < c_{k,\ell}^*$, Theorem 2.2 guarantees that with high probability for all $\mathcal{I}$ we have $|\mathcal{I}| \leqslant |\Gamma(\mathcal{I})|$, and therefore B has a left-perfect matching. On the other hand, if $c > c_{k,\ell}^*$, then with high probability there is a set $\mathcal{I}$ such that $|\mathcal{I}| > |\Gamma(\mathcal{I})|$; choose for example $\mathcal{I}$ to be the set of items that correspond to the edges in the $(\ell + 1)$-core of $H_{n,m,k}$. Hence a matching does not exist in this case, and the proof is completed. $\square$

In the rest of the paper we prove Theorem 2.1 and Theorem 2.2. The main line of the argument is similar to what was performed for the special case $\ell = 1$ in [10]. However, resolving the problem for general values of ℓ is technically much more involved, and also

several generalizations of all intermediate steps are required; this is the main contribution of the present work.

2.1. Proof of Theorem 2.1 and the value of $c_{k,\ell}^*$

The aim of this section is to determine the value $c_{k,\ell}^*$ and prove Theorem 2.1. Moreover, we will introduce some known facts and tools that will turn out to be very useful in the study of random hypergraphs, and will be used later on in the proof of Theorem 2.2 as well. In what follows we will be referring to a hyperedge of size k as a $(k\text{-})$edge and we will be calling a hypergraph with all its hyperedges of size k a *k-graph*.

Models of random hypergraphs. For the sake of convenience we will carry out our calculations in the $H_{n,p,k}$ model of random k-graphs. This is the 'higher-dimensional' analogue of the well-studied $G_{n,p}$ model, where each possible $(k\text{-})$edge is included independently with probability p. More precisely, given $n \geqslant k$ vertices we obtain $H_{n,p,k}$ by including each k-tuple of vertices with probability p, independently of every other k-tuple.

Standard arguments show that if we adjust p suitably, then the $H_{n,p,k}$ model is essentially equivalent to the $H_{n,cn,k}$ model. Let us be more precise. Suppose that $\mathcal{P}$ is a *convex* hypergraph property, that is, whenever we have three hypergraphs H_1, H_2, H_3 such that $H_1 \subseteq H_2 \subseteq H_3$ and $H_1, H_3 \in \mathcal{P}$, then also $H_2 \in \mathcal{P}$. We also assume that $\mathcal{P}$ is closed under automorphisms. Note that any monotone property is also convex. The following proposition is a generalization of Proposition 1.15 from [13, p. 16] and its proof is very similar, so we omit it.

Proposition 2.3. *Let $\mathcal{P}$ be a convex property of hypergraphs, and let $p = ck/\binom{n-1}{k-1}$, where $c > 0$. If $\mathbb{P}(H_{n,p,k} \in \mathcal{P}) \to 1$ as $n \to \infty$, then $\mathbb{P}(H_{n,\lfloor cn\rfloor,k} \in \mathcal{P}) \to 1$ as well.*

Working on the $(\ell+1)$-core of $H_{n,p,k}$: the cloning model. Recall that the $(\ell+1)$-core of a hypergraph is its maximum subgraph that has minimum degree (at least) $\ell+1$. At this point we introduce the main tool for our analysis. The *cloning model* with parameters (N, D, k), where N and D are integer-valued random variables, is defined as follows. We generate a graph in three stages.

(1) We expose the value of N.
(2) If $N \geqslant 1$ we expose the degrees $\mathbf{d} = (d_1, \ldots, d_N)$, where the d_i are independent samples from the distribution D.
(3) For each $1 \leqslant v \leqslant N$ we generate d_v copies, which we call *v-clones* or simply *clones*. Then we choose uniformly at random a matching from all perfect k-matchings on the set of all clones, that is, all partitions of the set of clones into sets of size k. Note that such a matching may not exist; in this case we choose a random matching that leaves less than k clones unmatched. Finally, we construct the k-graph $H_{\mathbf{d},k}$ by contracting the clones to vertices, that is, by projecting the clones of v onto v itself for every $1 \leqslant v \leqslant N$.

Note that the last stage in the above procedure is equivalent to the *configuration model* [2, 1] $H_{\mathbf{d},k}$ for random hypergraphs with degree sequence $\mathbf{d} = (d_1, \ldots, d_n)$. In other words, $H_{\mathbf{d},k}$ is a random multigraph where the ith vertex has degree d_i.

One particular case of the cloning model is the so-called *Poisson cloning model* $\widetilde{H}_{n,p,k}$ for k-graphs with n vertices and parameter $p \in [0, 1]$, which was introduced by Kim [14]. There, we choose $N = n$ with probability 1, and the distribution D is the Poisson distribution with parameter $\lambda := p\binom{n-1}{k-1}$. Note that D is essentially the vertex degree distribution in the binomial random graph $H_{n,p,k}$, so we would expect the two models to behave similarly. The following statement confirms this, and is implied by Theorem 1.1 in [14].

Theorem 2.4. *Let $\mathcal{P}$ be any property of hypergraphs. If $\mathbb{P}(\widetilde{H}_{n,p,k} \in \mathcal{P}) \to 0$ as $n \to \infty$, then $\mathbb{P}(H_{n,p,k} \in \mathcal{P}) \to 0$ as well.*

One big advantage of the Poisson cloning model is that it provides a rather precise description of the $(\ell + 1)$-core of $\widetilde{H}_{n,p,k}$. In particular, Theorem 6.2 in [14] implies the following statement, where we write '$x \pm y$' for the interval of numbers $(x - y, x + y)$.

Theorem 2.5. *Let*

$$\lambda_{k,\ell+1} := \min_{x > 0} \frac{x}{Q(x, \ell)^{k-1}}.$$

Assume that $ck = p\binom{n-1}{k-1} > \lambda_{k,\ell+1}$. Moreover, let $\bar{x}$ be the largest solution of the equation $x = Q(xck, \ell)^{k-1}$, and set $\xi := \bar{x}ck$. Then, for any $0 < \delta < 1$ the following is true with probability $1 - n^{-\omega(1)}$. If $\tilde{N}_{\ell+1}$ denotes the number of vertices in the $(\ell + 1)$-core of $\widetilde{H}_{n,p,k}$, then

$$\tilde{N}_{\ell+1} = Q(\xi, \ell + 1)n \pm \delta n.$$

Furthermore, the $(\ell + 1)$-core itself is distributed like the cloning model with parameters

$$(\tilde{N}_{\ell+1}, \mathrm{Po}_{\geqslant \ell+1}(\Lambda_{c,k,\ell}), k),$$

where $\mathrm{Po}_{\geqslant \ell+1}(\Lambda_{c,k,\ell})$ denotes a Poisson random variable conditioned on being at least $(\ell + 1)$ and parameter $\Lambda_{c,k,\ell}$, where $\Lambda_{c,k,\ell} = \xi + \beta$, for some β satisfying $|\beta| \leqslant \delta$.

In what follows, we say that a random variable is an ℓ-*truncated* Poisson variable if it is distributed like a Poisson variable, conditioned on being at least ℓ. The following theorem, which is a special case of Theorem II.4.I in [6] from large deviation theory, bounds the sum of i.i.d. random variables. We apply the result to the case of i.i.d. $(\ell + 1)$-truncated Poisson random variables, which are simply the degrees of the vertices of the $(\ell + 1)$-core. As an immediate corollary we obtain tight bounds on the number of edges in the $(\ell + 1)$-core of $\widetilde{H}_{n,p,k}$. Moreover, it also serves as our main tool in counting the expected number of subsets (with some density constraints) of the $(\ell + 1)$-core, assuming that the degree sequence has been exposed. Such estimates are required for the proof of Theorem 2.2 and will be presented in the next section.

Theorem 2.6. *Let X be a random variable taking real values and set $c(t) = \log \mathbb{E}(e^{tX})$, for any $t \in \mathbb{R}$. For any $z > 0$ we define $I(z) = \sup_{t \in \mathbb{R}}\{zt - c(t)\}$. If $X_1, \ldots, X_s$ are i.i.d. random*

variables distributed as X, then for $s \to \infty$

$$\mathbb{P}\left(\sum_{i=1}^{s} X_i \leqslant sz\right) = \exp(-s\inf\{I(x) : x \leqslant z\}(1 + o(1))).$$

The function $I(z)$ is non-negative and convex.

The function $c(t)$ is called the *free energy function of X* and the following holds.

Proposition 2.7 (Proposition VII.1.1 of [6]). *$c(t)$ is a convex function on $\mathbb{R}$.*

The function $I(z)$ (also known as the *rate function* of the random variable X) in the above theorem measures the discrepancy between z and the expected value of the sum of the i.i.d. random variables in the sense that $I(z) \geqslant 0$ with equality if and only if z equals the expected value of X. The following lemma applies Theorem 2.6 to $(\ell + 1)$-truncated Poisson random variables.

Lemma 2.8. *Let $X_1, \ldots, X_s$ be i.i.d. $(\ell + 1)$-truncated Poisson random variables with parameter Λ. For any $z > \ell + 1$, let T_z be the unique solution of*

$$z = T_z \cdot \frac{Q(T_z, \ell)}{Q(T_z, \ell + 1)}$$

and

$$I_\Lambda(z) = \begin{cases} z(\log T_z - \log \Lambda) - T_z + \Lambda - \log Q(T_z, \ell + 1) + \log Q(\Lambda, \ell + 1) & \text{if } z > \ell + 1, \\ \log(\ell + 1)! - (\ell + 1)\log \Lambda + \Lambda + \log Q(\Lambda, \ell + 1) & \text{if } z = \ell + 1. \end{cases}$$
(2.1)

Then $I_\Lambda(z)$ is continuous for all $z > \ell + 1$ and right continuous at $z = \ell + 1$. Furthermore, it is convex. It has a unique minimum at

$$z = \mu = \Lambda \cdot \frac{Q(\Lambda, \ell)}{Q(\Lambda, \ell + 1)},$$

where $I_\Lambda(\mu) = 0$. Moreover, uniformly for any z such that $\ell + 1 \leqslant z \leqslant \mu$, we have as $s \to \infty$

$$\mathbb{P}\left(\sum_{i=1}^{s} X_i \leqslant sz\right) \leqslant \exp(-sI_\Lambda(z)(1 + o(1))).$$

Proof. We shall first determine $c(t) = \log \mathbb{E}(e^{tX})$, where X is an $(\ell + 1)$-truncated Poisson random variable with parameter Λ. We note that

$$\begin{aligned} \exp(c(t)) &= \frac{\sum_{j \geqslant \ell + 1} e^{tj} \cdot \frac{e^{-\Lambda}\Lambda^j}{j!}}{Q(\Lambda, \ell + 1)} \\ &= e^{-\Lambda} \cdot e^{\Lambda e^t} \cdot \frac{\sum_{j \geqslant \ell + 1} \frac{e^{-\Lambda e^t}(e^t\Lambda)^j}{j!}}{Q(\Lambda, \ell + 1)} \\ &= e^{\Lambda e^t - \Lambda} \cdot \frac{Q(\Lambda e^t, \ell + 1)}{Q(\Lambda, \ell + 1)}. \end{aligned}$$

Differentiating $zt - c(t)$ with respect to t, we obtain

$$(zt - c(t))' = z - \log\left(e^{\Lambda e^t - \Lambda} \cdot \frac{Q(\Lambda e^t, \ell + 1)}{Q(\Lambda, \ell + 1)}\right)'$$

$$= z - \Lambda e^t - (\log Q(\Lambda e^t, \ell + 1))'$$

$$= z - \Lambda e^t + \frac{\Lambda e^t \cdot (Q(\Lambda e^t, \ell + 1) - Q(\Lambda e^t, \ell))}{Q(\Lambda e^t, \ell + 1)}.$$

Substituting $T = \Lambda e^t$, we get

$$(zt - c(t))' = z - T + \frac{T \cdot (Q(T, \ell + 1) - Q(T, \ell))}{Q(T, \ell + 1)}$$

$$= z - T \cdot \frac{Q(T, \ell)}{Q(T, \ell + 1)}.$$

Setting this expression to zero and solving for T gives the value of T_z as in the statement of the lemma. The uniqueness of the solution for $z > \ell + 1$ follows from the fact that the function

$$x \cdot \frac{Q(x, \ell)}{Q(x, \ell + 1)}$$

is strictly increasing with respect to x (see Claim 3.14) and, as x approaches 0, it tends to $\ell + 1$. Letting t_z be such that $T_z = \Lambda e^{t_z}$, we obtain

$$-c(t_z) = -T_z - \log Q(T_z, \ell + 1) + \Lambda + \log Q(\Lambda, \ell + 1)$$

and

$$t_z z = z(\log T_z - \log \Lambda).$$

The function $-c(t)$ is concave with respect to t, by Proposition 2.7, and therefore the addition of the linear term zt does preserve concavity. So t_z is the point where the unique maximum of $zt - c(t)$ is attained over $t \in \mathbb{R}$. Combining the above we obtain $I_\Lambda(z)$ as stated in the lemma. For

$$z = \frac{\Lambda Q(\Lambda, \ell)}{Q(\Lambda, \ell + 1)}$$

we have $T_z = \Lambda$, which yields $I_\Lambda(\mu) = 0$. As far as $I_\Lambda(\ell + 1)$ is concerned, note that strictly speaking this is not defined, as there is no positive solution of the equation

$$\ell + 1 = T \cdot \frac{Q(T, \ell)}{Q(T, \ell + 1)}.$$

We will express $I_\Lambda(\ell + 1)$ as a limit as $T \to 0$ from the right, and show that

$$\mathbb{P}\left(\sum_{i=1}^{s} X_i \leqslant s(\ell + 1)\right) = \exp(-sI_\Lambda(\ell + 1)).$$

We define

$$I_\Lambda(\ell + 1) := \lim_{T \to 0^+} ((\ell + 1)\log T - T - \log Q(T, \ell + 1))$$

$$- (\ell + 1)\log \Lambda + \Lambda + \log Q(\Lambda, \ell + 1).$$

But

$$\lim_{T \to 0^+} ((\ell+1)\log T - T - \log Q(T, \ell+1)) = \lim_{T \to 0^+} \log \frac{T^{\ell+1}}{e^T Q(T, \ell+1)}$$

$$= \lim_{T \to 0^+} \log \frac{T^{\ell+1}}{\frac{T^{\ell+1}}{(\ell+1)!} + \frac{T^{\ell+2}}{(\ell+2)!} + \cdots}$$

$$= \lim_{T \to 0^+} \log \frac{1}{\frac{1}{(\ell+1)!} + \frac{T}{(\ell+2)!} + \cdots}$$

$$= \log(\ell+1)!,$$

and therefore

$$I_\Lambda(\ell+1) = \log(\ell+1)! - (\ell+1)\log\Lambda + \Lambda + \log Q(\Lambda, \ell+1).$$

On the other hand, the independence of the X_i guarantees that

$$\mathbb{P}\left(\sum_{i=1}^s X_i \leqslant s(\ell+1)\right) = [\mathbb{P}(X_1 = \ell+1)]^s = \left(\frac{\frac{e^{-\Lambda}\Lambda^{\ell+1}}{(\ell+1)!}}{Q(\Lambda, \ell+1)}\right)^s = \exp(-sI_\Lambda(\ell+1)).$$

Also, according to Theorem 2.6 the function $I_\Lambda(z)$ is non-negative and convex on its domain. So if $z \leqslant \mu$ then $\inf\{I_\Lambda(x) : x \leqslant z\} = I_\Lambda(z)$, and the second part of the lemma follows. □

Theorem II.3.3 in [6] along with the above lemma then implies the following corollary.

Corollary 2.9. *Let $X_1, \ldots, X_s$ be i.i.d. $(\ell+1)$-truncated Poisson random variables with parameter Λ and set $\mu = \mathbb{E}(X_1)$. For any $\varepsilon > 0$ there exists a constant $C = C(\varepsilon) > 0$ such that, for any s sufficiently large,*

$$\mathbb{P}\left(\left|\sum_{i=1}^s X_i - s\mu\right| \geqslant s\varepsilon\right) \leqslant e^{-Cs}.$$

With the above results in hand we are ready to prove the following corollary about the density of the $(\ell+1)$-core.

Corollary 2.10. *Let $\tilde{N}_{\ell+1}$ and $\tilde{M}_{\ell+1}$ denote the number of vertices and edges in the $(\ell+1)$-core of $\tilde{H}_{n,p,k}$. Also let $ck = p\binom{n-1}{k-1}$. Then, for any $0 < \delta < 1$, with probability $1 - n^{-\omega(1)}$,*

$$\tilde{N}_{\ell+1} = Q(\xi, \ell+1)n \pm \delta n, \tag{2.2}$$

$$\tilde{M}_{\ell+1} = \frac{\xi Q(\xi, \ell)}{kQ(\xi, \ell+1)}\tilde{N}_{\ell+1} \pm \delta n, \tag{2.3}$$

where $\xi := \bar{x}ck$ and $\bar{x}$ is the largest solution of the equation $x = Q(xck, \ell)^{k-1}$.

Proof. The statement about $\tilde{N}_{\ell+1}$ follows immediately from the first part of Theorem 2.5. To see the second statement, we condition on certain values of $\tilde{N}_{\ell+1}$ and $\Lambda_{c,k,\ell}$ that lie in the intervals stated in Theorem 2.5. In particular, we can assume that the total degree of

the core of $\tilde{H}_{n,p,k}$ is the sum of independent $(\ell+1)$-truncated Poisson random variables $d_1, \ldots, d_{\tilde{N}_{\ell+1}}$ with parameter $\Lambda_{c,k,\ell} = \xi + \beta$ for $|\beta| < \delta^2$. Let D be the sum of the d_i. For any $\varepsilon > 0$ and a constant $C(\varepsilon) > 0$, Corollary 2.9 yields

$$\mathbb{P}(|D - \mathbb{E}(D)| \geqslant \varepsilon \tilde{N}_{\ell+1}) \leqslant e^{-C(\varepsilon)\tilde{N}_{\ell+1}}.$$

Note that

$$\mathbb{E}(D) = \frac{\Lambda_{c,k,\ell} Q(\Lambda_{c,k,\ell}, \ell)}{Q(\Lambda_{c,k,\ell}, \ell+1)}.$$

The claim then follows by choosing ε, δ sufficiently small and from the continuity of the above expression as a function of $\Lambda_{c,k,\ell}$. $\qquad\square$

We proceed with the proof of Theorem 2.1, that is, we will show that the $(\ell+1)$-core of $\tilde{H}_{n,p,k}$ has density at least ℓ if $p = ck/\binom{n-1}{k-1}$ and $c > c^*_{k,\ell}$.

Proof of Theorem 2.1. Let $0 < \delta < 1$, and let $\tilde{N}_{\ell+1}$ and $\tilde{M}_{\ell+1}$ denote the number of vertices and edges in the $(\ell+1)$-core of $\tilde{H}_{n,p,k}$. Applying Corollary 2.10, we obtain that with probability $1 - n^{-\omega(1)}$

$$\tilde{N}_{\ell+1} = Q(\xi, \ell+1)n \pm \delta n \quad \text{and} \quad \tilde{M}_{\ell+1} = \frac{\xi Q(\xi, \ell)}{kQ(\xi, \ell+1)} \tilde{N}_{\ell+1} \pm \delta n,$$

where $\xi = \bar{x}ck$ and $\bar{x}$ is the largest solution of the equation $x = Q(xck, \ell)^{k-1}$. The value of $c^*_{k,\ell}$ is obtained by taking $\tilde{M}_{\ell+1} = \ell \tilde{N}_{\ell+1}$, and ignoring the additive error terms. The above values imply that the critical ξ^* is given by the equation

$$\xi^* \frac{Q(\xi^*, \ell)}{kQ(\xi^*, \ell+1)} = \ell \implies k\ell = \xi^* \frac{Q(\xi^*, \ell)}{Q(\xi^*, \ell+1)}. \tag{2.4}$$

This is identical to (1.1). So the product $k\ell$ determines ξ^* and $\bar{x}$ satisfies

$$\bar{x} = Q(\bar{x}ck, \ell)^{k-1} = Q(\xi^*, \ell)^{k-1}.$$

Therefore, the critical density is

$$c^*_{k,\ell} = \frac{\xi^*}{\bar{x}k} = \frac{\xi^*}{kQ(\xi^*, \ell)^{k-1}}. \tag{2.5}$$

The above calculations imply that uniformly for any $0 < \delta < 1$, with probability $1 - o(1)$,

$$\frac{\tilde{M}_{\ell+1}}{\tilde{N}_{\ell+1}} = \frac{1}{k} \frac{\xi Q(\xi, \ell)}{Q(\xi, \ell+1)} \pm \Theta(\delta).$$

In particular, if $c = c^*_{k,\ell}$, then $\tilde{M}_{\ell+1}/\tilde{N}_{\ell+1} = \ell \pm \Theta(\delta)$. To complete the proof it is therefore sufficient to show that the ratio

$$\frac{\xi Q(\xi, \ell)}{Q(\xi, \ell+1)}$$

is an increasing function of c. Note that this is the expected value of an $(\ell+1)$-truncated Poisson random variable with parameter ξ, which is increasing in ξ (see Corollary 3.15). Recall that $\xi = \bar{x}ck$. We conclude the proof by showing the following claim.

Claim 2.11. *The quantity $\xi = \bar{x}ck$ is increasing with respect to c. So, for some fixed c, with probability $1 - o(1)$*

$$\frac{\tilde{M}_{\ell+1}}{\tilde{N}_{\ell+1}} < \ell \quad \text{if } c < c^*_{k,\ell} \quad \text{and} \quad \frac{\tilde{M}_{\ell+1}}{\tilde{N}_{\ell+1}} > \ell \quad \text{if } c > c^*_{k,\ell}.$$

Indeed, recall that $\bar{x}$ satisfies $\bar{x} = Q(\bar{x}ck, \ell)^{k-1}$. Equivalently, $\bar{x}ck = ck \cdot Q(\bar{x}ck, \ell)^{k-1}$. We have

$$ck = \frac{\xi}{Q(\xi, \ell)^{k-1}}. \tag{2.6}$$

The derivative of the function $F(\xi) := \xi/Q(\xi, \ell)^{k-1}$ with respect to ξ is given by

$$Q(\xi, \ell)^{-k}\big(Q(\xi, \ell) - (k-1)\xi \cdot \mathbb{P}(\text{Po}(\xi) = \ell - 1)\big).$$

An easy calculation shows that $F'(\xi)$ is positive when ξ satisfies the inequality

$$\sum_{i \geq \ell} \frac{\xi^{i-\ell}}{i!} > \frac{k}{(\ell-1)!},$$

and negative otherwise. We therefore conclude that $F(\xi)$ is a convex function. Moreover, by the assumption in Theorem 2.5 we have $ck > \min_{x>0}(x/Q(x, \ell)^{k-1})$. This implies that the function $\xi \cdot Q(\xi, \ell)^{-(k-1)}$ is strictly increasing in the domain of interest. Note that by (2.6) the first derivative of ξ with respect to c is given by $k/F'(\xi)$, which is positive by the above discussion, thus proving our claim. □

3. Proof of Theorem 2.2

Let us begin by introducing some notation. For a hypergraph H we will let V_H denote its vertex set and E_H its set of edges. Further, we write $v_H = |V_H|$ and $e_H = |V_H|$. For $U \subset V_H$ we let v_U, e_U denote the number of vertices in U and the number of edges joining vertices only in U. Finally, d_U is the total degree in U, that is, the sum of the degrees in H of all vertices in U. We say that a subset U of the vertex set of a hypergraph is ℓ-*dense* if $e_U/v_U \geq \ell$. By a *maximal* ℓ-dense subset we mean that whenever we add a vertex to such a set, then its density drops below ℓ.

To prove Theorem 2.2 we will show that whenever $c < c^*_{k,\ell}$, the random graph $H_{n,\lfloor cn \rfloor, k}$ does not contain any ℓ-dense subset with probability $1 - o(1)$. We will accomplish this by proving that such a hypergraph does not contain any maximal ℓ-dense subset with probability $1 - o(1)$. Note that this is sufficient, as any ℓ-dense subset will be contained in some maximal ℓ-dense subset. We shall use the following property.

Proposition 3.1. *Let H be a k-uniform hypergraph with density less than ℓ, and let U be a maximal ℓ-dense subset of V_H. Then there is a $0 \leq \theta < \ell$ such that $e_U = \ell \cdot v_U + \theta$. Also, for each vertex $v \in V_H \setminus U$ the corresponding degree d in U, that is, the number of edges in H that contain v and all other vertices only from U, is less than $\ell - \theta$.*

Proof. If $\theta \geqslant \ell$, then we have $e_U \geqslant \ell \cdot (v_U + 1)$. Let $U' = U \cup \{v\}$, where v is any vertex in $V_H \setminus U$. Note that such a vertex always exists, as $U \neq V_H$. Let d be the degree of v in U. Then

$$\frac{e_{U'}}{v_{U'}} = \frac{e_U + d}{v_U + 1} \geqslant \frac{e_U}{v_U + 1} \geqslant \ell,$$

which contradicts the maximality of U in H. Similarly, if there exists a vertex $v \in V_H \setminus U$ with degree $d \geqslant \ell - \theta$ in U, then we could obtain a larger ℓ-dense subset of V_H by adding v to U. $\qquad\square$

We begin by showing that whenever $c < \ell$, the random graph $H_{n,cn,k}$ does not contain small maximal ℓ-dense subsets. In particular, the following lemma considers subsets of size at most $0.6n$.

Lemma 3.2. *Let $c < \ell$ and $k \geqslant 3$, $\ell \geqslant 2$. With probability $1 - o(1)$, $H_{n,\lfloor cn \rfloor,k}$ contains no maximal ℓ-dense subset with less than $0.6n$ vertices.*

Proof. We first prove the lemma for all $k \geqslant 3$ and $\ell \geqslant 2$ except for the case $(k, \ell) = (3, 2)$ by using a rough first moment argument. The probability that an edge of $H_{n,cn,k}$ is contained completely in a subset U of the vertex set is given by

$$\binom{|U|}{k} \bigg/ \binom{n}{k} \leqslant \left(\frac{|U|}{n} \right)^k.$$

Let $k/n \leqslant u \leqslant 0.6$, and for $x \in (0,1)$ let $H(x) = -x \log x - (1-x) \log(1-x)$ denote the entropy function. Then

$$\mathbb{P}(\exists \ell\text{-dense subset with } un \text{ vertices}) \leqslant \binom{n}{un} \cdot \binom{cn}{\ell un} (u^k)^{\ell un} \leqslant \binom{n}{un} \cdot \binom{\ell n}{\ell un} (u^k)^{\ell un}$$
$$\leqslant e^{n((\ell+1)H(u) + k\ell u \log u)}. \tag{3.1}$$

We first show that the exponent attains its maximum at $u = k/n$ or $u = 0.6$. Let $u_{\max} = 1 - (\ell+1)/k\ell$. We note that the second derivative of the exponent in (3.1) equals

$$(k\ell(1-u) - (\ell+1))/(u(1-u)),$$

which is positive for $k \geqslant 3, \ell \geqslant 2$ and $u \in (0, u_{\max}]$. Hence the exponent is convex for $u \leqslant u_{\max}$, implying that it attains a global maximum at $u = k/n$ or at $u = (k\ell - (\ell+1))/k\ell$. Moreover, for any $k \geqslant 4, \ell \geqslant 2$ we have $u_{\max} > 0.6$. The case $k = 3$ and $\ell \geqslant 3$ is slightly more involved. Note that $u_{\max} \geqslant 5/9$ in this case. The second derivative of the exponent is negative for $u \in (u_{\max}, 1)$, implying that the function is concave in the specified range. But the first derivative of the exponent is $(\ell+1) \log((1-u)/u) + 3\ell(1 + \log(u))$, which is at least $2.8\ell - 0.41 > 0$ for $u = 0.6$. Hence, the exponent is increasing at $u = 0.6$.

We can now infer that for $k = 3$, $\ell \geqslant 3$ and $k \geqslant 4$, $\ell \geqslant 2$, the exponent is either maximized at $u = k/n$ or at $u = 0.6$. Note that for any fixed positive k and ℓ we have

$$(\ell+1) H\left(\frac{k}{n} \right) + \frac{k^2 \ell}{n} \log\left(\frac{k}{n} \right) = -\frac{(k^2\ell - (\ell+1)k) \log n}{n} + O\left(\frac{1}{n} \right).$$

Also, for $k \geqslant 4$ and $\ell \geqslant 2$ we obtain

$$(\ell + 1)H(0.6) + k\ell \cdot 0.6 \log(0.6) \leqslant (\ell + 1)H(0.6) + 4\ell \cdot 0.6 \log(0.6)$$
$$\leqslant H(0.6) - 0.56\ell \leqslant -0.44,$$

and for $k = 3$ and $\ell \geqslant 3$

$$(\ell + 1)H(0.6) + k\ell \cdot 0.6 \log(0.6) \leqslant (\ell + 1)H(0.6) + 3\ell \cdot 0.6 \log(0.6)$$
$$\leqslant H(0.6) - 0.24\ell \leqslant -0.04.$$

So the maximum is obtained at $u = k/n$ for n sufficiently large, and we conclude the case in which $(k, \ell) \neq (3, 2)$ with

$$\mathbb{P}(\exists\ \ell\text{-dense subset with} \leqslant 0.6n \text{ vertices}) \leqslant \sum_{u=k/n}^{0.6} n^{-k^2\ell+(\ell+1)k} = O(n^{-8}).$$

For the case $(k, \ell) = (3, 2)$, a counting argument as above involving the 2-dense sets does not work, and we will use the property that the considered sets are *maximal* 2-dense. By (2.5) we obtain $c_{3,2}^* < 1.97$. Let $p = c'/\binom{n-1}{2}$, where $c' = 3 \cdot c \leqslant 3 \cdot c_{3,2}^* \leqslant 5.91$. A simple application of Stirling's formula reveals

$$\mathbb{P}(H_{n,p,3} \text{ has exactly } cn \text{ edges}) = (1 + o(1))(2\pi cn)^{-1/2}.$$

Let U be a maximal 2-dense subset of $H_{n,cn,3}$. As the distribution of $H_{n,cn,3}$ is the same as the distribution of $H_{n,p,3}$ conditioned on the number of edges being precisely cn, we infer that

$$\mathbb{P}(H_{n,cn,3} \text{ contains a maximal 2-dense subset } U \text{ with at most } 0.6n \text{ vertices}) =$$

$$O(\sqrt{n}) \cdot \mathbb{P}(H_{n,p,3} \text{ contains a maximal 2-dense subset } U \text{ with at most } 0.6n \text{ vertices}).$$

To complete the proof it is therefore sufficient to show that the latter probability is $o(n^{-1/2})$. By Proposition 3.1 the event that $H_{n,p,3}$ contains a maximal 2-dense subset U implies that there exists a $\theta \in \{0, 1\}$ such that $e_U = 2 \cdot v_U + \theta$ and all vertices in $V_H \setminus U$ have degree less than $2 - \theta$ in U. We will show that the expected number of such sets with at most $0.6n$ vertices is $o(1)$. We accomplish this in two steps. Note that if a subset U is maximal 2-dense, then certainly $|U| \geqslant 5$. Let us begin with the case $s := |U| \leqslant n^{1/3}$. There are at most n^s ways to choose the vertices in U, and at most $s^{3(2s+\theta)}$ ways to choose the edges that are contained in U. Hence, for large n the probability that $H_{n,p,3}$ contains such a subset with at most $\lfloor n^{1/3} \rfloor$ vertices is bounded by

$$\sum_{s=5}^{\lfloor n^{1/3} \rfloor} \sum_{\theta=0}^{1} n^s s^{6s+3\theta} p^{2s+\theta} < \sum_{s=5}^{\lfloor n^{1/3} \rfloor} 2n^s s^{6s+3} p^{2s} = \sum_{s=5}^{\lfloor n^{1/3} \rfloor} 2\left(ns^6 \left(\frac{c'}{\binom{n-1}{2}}\right)^2\right)^s \cdot s^3$$

$$\leqslant n \sum_{s=5}^{\lfloor n^{1/3} \rfloor} 2(c'^2 n^{(1+6/3)-4})^s \leqslant n \sum_{s=5}^{\lfloor n^{1/3} \rfloor} (n^{-1+o(1)})^s = n^{-4+o(1)}.$$

Let us now consider the case $n^{1/3} \leqslant |U| \leqslant 0.6n$. We note that

$$\log p = \log\left(\frac{c'}{\binom{n-1}{2}}\right) = \log \frac{2c'}{n^2} + \Theta\left(\frac{1}{n}\right).$$

Also, there are $\binom{n}{un} \leqslant e^{nH(u)}$ ways to select U. Moreover, the number of ways to choose the $2un + \theta$ edges that are completely contained in U is

$$\binom{\binom{un}{3}}{2un + \theta} \leqslant \left(\frac{e(un)^3}{6(2un + \theta)}\right)^{2un} = \exp\left\{2un \log\left(\frac{e(un)^2}{12}\right) + O(1)\right\}.$$

Finally, the probability that a vertex outside U has a degree less than $2 - \theta$ in $|U|$ is at most

$$(1-p)^{\binom{un}{2}} + \binom{un}{2}p(1-p)^{\binom{un}{2}-1} = e^{-u^2c'}(1 + u^2c')(1 + O(1/n)).$$

Combining the above facts we obtain that the probability P_u that $H_{n,p,3}$ contains a maximal 2-dense subset U with $2un$ vertices is

$$P_u \leqslant \sum_{\theta=0}^{1} \binom{n}{un}\binom{\binom{un}{3}}{2un + \theta}p^{2un+\theta}(1-p)^{\binom{un}{3}-2un-\theta}(e^{-u^2c'}(1 + u^2c')(1 + O(1/n)))^{(1-u)n}$$

$$\leqslant 2 \cdot \exp\left\{n\left(H(u) + 2u\log\left(\frac{eu^2n^2}{12}\right) + 2u\log p\right) - p\left(\binom{un}{3} - 2un - 1\right)\right.$$

$$\left. + (1-u)n(-u^2c' + \log(1 + u^2c')) + O(1/n)\right\}$$

$$\leqslant 2 \cdot \exp\left\{n\left(H(u) + 2u\log\left(\frac{ec'u^2}{6}\right) - \frac{u^3c'}{3} + (1-u)(-u^2c' + \log(1 + u^2c'))\right)\right.$$

$$\left. + O(1/n)\right\}.$$

If we fix u, the derivative of the exponent with respect to c' is given by

$$\frac{2u}{c'} - \frac{u^3}{3} + (1-u)\left(-u^2 + \frac{u^2}{1 + u^2c'}\right) \overset{c' \leqslant 5.91}{\geqslant} \frac{2u}{6} - \frac{u^3}{3} + (1-u)\left(-u^2 + \frac{u^2}{1 + 6u^2}\right)$$

$$= u\left(\frac{1}{3} - \frac{u^2/3 + 6u^3 - 4u^4}{1 + 6u^2}\right) \overset{u \leqslant 0.6}{\geqslant} u\left(\frac{1}{3} - 0.29\right) \overset{u > 0}{>} 0,$$

thus implying that for all $u \in (0, 0.6]$ the exponent is increasing with respect to c'. Therefore, it is sufficient to consider only the case when $c' = 5.91$.

The derivative of the exponent with respect to u equals

$$\log(c'^2u^3(1-u)) + 6 - \log 6 - \log(1 + u^2c') - (1-u)\frac{2u^3c'^2}{1 + u^2c'}$$

$$= \log(c'u^3) + \frac{2u^4c'^3}{1 + u^2c'} + \log(1-u) - \log(1 + u^2c') - \frac{2u^3c'^2}{1 + u^2c'} + 6 - \log 6.$$

As the function

$$\log(c'u^3) + 2c'^2 \cdot \frac{u^4}{(1 + u^2c')}$$

is increasing and

$$\log(1-u) - \log(1 + u^2c') - 2c'^2 \cdot \frac{u^3}{(1 + u^2c')}$$

is decreasing in u, there is at most one $n^{-2/3} \leqslant u_0 \leqslant 0.6$ where the derivative of the exponent vanishes. Moreover the derivative of the exponent at $u = 0.6$ is positive. Therefore, u_0 is a global minimum, and the bound on P_u is maximized at either $u = n^{-2/3}$ or $u = 0.6$. Elementary algebra then yields that the left point is the right choice, giving the estimate $P_u = o(2^{-n^{1/3}})$, and the proof concludes by adding up this expression for all admissible $n^{-2/3} \leqslant u \leqslant 0.6$. □

In order to deal with larger subsets we switch to the Poisson cloning model. Let C denote the $(\ell + 1)$-core of $\widetilde{H}_{n,p,k}$, where $p = ck/\binom{n-1}{k-1}$, and note that Theorem 2.4 and Proposition 2.3 guarantee that $\widetilde{H}_{n,p,k}$ and $H_{n,cn,k}$ are sufficiently similar. Observe that any minimal ℓ-dense set in $\widetilde{H}_{n,p,k}$ is always a subset of C, as otherwise, by removing vertices of degree at most ℓ, the density would not decrease. In other words, C contains all minimal ℓ-dense subsets, and so it is enough to show that the core does not contain any ℓ-dense subset. Therefore, from now on we will restrict our attention to the study of C.

Assume that the degree sequence of C is given by $\mathbf{d} = (d_1, \ldots, d_{\widetilde{N}_{\ell+1}})$, where again we let $\widetilde{N}_{\ell+1}$ denote the number of vertices in C. The number of edges in C is

$$\widetilde{M}_{\ell+1} = k^{-1} \sum_{i=1}^{\widetilde{N}_{\ell+1}} d_i.$$

For $q, \beta \in [0, 1]$, let

$$X_{q,\beta} = X_{q,\beta}(C) = X_{q,\beta}(\mathbf{d})$$

denote the number of subsets of C with $\lfloor \beta \widetilde{N}_{\ell+1} \rfloor$ vertices and total degree $\lfloor q k \widetilde{M}_{\ell+1} \rfloor$.

Let $\xi^* = \bar{x}^* c^*_{k,\ell} k$, where $\bar{x}^*$ is the largest solution of the equation $x = Q(x c^*_{k,\ell} k, \ell)^{k-1}$, and note that ξ^* satisfies (2.4). Moreover, let ξ be given by $\xi = \bar{x} c k$, where $\bar{x}$ is the largest solution of the equation $x = Q(x c k, \ell)^{k-1}$. As ξ is increasing with respect to c (see Claim 2.11), there exists a $\delta > 0$ and a $\gamma = \gamma(\delta) > 0$ such that $c = c^*_{k,\ell} - \gamma$ and $\xi = \xi^* - \delta$. Also, $\gamma \to 0$ as $\delta \to 0$ by continuity of the largest solution of $x = Q(x c k, \ell)^{k-1}$.

We will assume that $\delta > 0$ is fixed (and sufficiently small for all our estimates to hold), and we will choose $c < c^*_{k,\ell}$ such that $c = c^*_{k,\ell} - \gamma$ and $\xi = \xi^* - \delta$. Set

$$n_{\ell+1} = Q(\xi, \ell + 1)n \quad \text{and} \quad m_{\ell+1} = \frac{\xi Q(\xi, \ell)}{k Q(\xi, \ell + 1)} n_{\ell+1}. \tag{3.2}$$

By applying Corollary 2.10 (and using δ^3 instead of δ) we obtain that with probability $1 - n^{-\omega(1)}$

$$\widetilde{N}_{\ell+1} = n_{\ell+1} \pm \delta^3 n \quad \text{and} \quad \widetilde{M}_{\ell+1} = m_{\ell+1} \pm \delta^3 n. \tag{3.3}$$

Moreover, by applying Theorem 2.5 we infer that C is distributed like the cloning model with parameters $\widetilde{N}_{\ell+1}$ and vertex degree distribution $\mathrm{Po}_{\geqslant \ell+1}(\Lambda_{c,k,\ell})$, where

$$\Lambda_{c,k,\ell} = \xi \pm \delta^3 = \xi^* - \delta \pm \delta^3. \tag{3.4}$$

Recall that the definition of ξ^* implies that

$$k\ell = \frac{\xi^* Q(\xi^*, \ell)}{Q(\xi^*, \ell+1)}.$$

Let $e_{k,\ell}$ denote the value of the first derivative of

$$\frac{xQ(x,\ell)}{k\ell Q(x,\ell+1)}$$

with respect to x at $x = \xi^*$. By applying Taylor's theorem to

$$\frac{xQ(x,\ell)}{Q(x,\ell+1)}$$

around $x = \xi^*$, we obtain

$$m_{\ell+1} = (1 - e_{k,\ell} \cdot \delta + \Theta(\delta^2))\ell \cdot n_{\ell+1}, \quad \text{where } \frac{\xi Q(\xi,\ell)}{Q(\xi,\ell+1)} = k\ell(1 - e_{k,\ell} \cdot \delta + \Theta(\delta^2)). \quad (3.5)$$

Recall that $H_{\mathbf{d},k}$ is a random hypergraph where the ith vertex has degree d_i. We start by bounding the probability that a given subset of the vertices in $H_{\mathbf{d},k}$ is maximal ℓ-dense. In particular, we will work on stage 3 of the exposure process, that is, when the number of vertices and degree sequence of the core have already been exposed. We will show the following.

Lemma 3.3. *Let $k \geqslant 3, \ell \geqslant 2$ and $\mathbf{d} = (d_1, \ldots, d_N)$ be a degree sequence and $U \subseteq \{1, \ldots, N\}$ such that $|U| = \lfloor \beta N \rfloor$. Moreover, set $M = k^{-1}\sum_{i=1}^{N} d_i$ and $q = (kM)^{-1}\sum_{i \in U} d_i$. Assume that $M < \ell \cdot N$. If $\mathbb{P}_{\mathbf{d},k}$ denotes the probability measure on the space of k-uniform hypergraphs with degree sequence given by $\mathbf{d}$, $\mathcal{B}(\beta, q)$ denotes the event that U is a maximal ℓ-dense set in $H_{\mathbf{d},k}$, and $H(x) = -x \log x - (1-x)\log(1-x)$ denotes the entropy function, then*

$$\mathbb{P}_{\mathbf{d},k}(\mathcal{B}(\beta,q)) \leqslant O(M^{\ell+0.5})\binom{M}{\ell|U|}e^{-kMH(q)}(2^k - 1)^{M-\ell|U|}.$$

Proof. Recall that $H_{\mathbf{d},k}$ is obtained by beginning with d_i clones for each $1 \leqslant i \leqslant N$ and by choosing uniformly at random a perfect k-matching on this set of clones. This is equivalent to throwing kM balls into M bins such that every bin contains k balls. In order to estimate the probability for $\mathcal{B}(\beta, q)$, assume that we colour the kqM clones of the vertices in U red, and the remaining $k(1-q)M$ clones blue. Let θ be an integer such that $0 \leqslant \theta < \ell$. So, by applying Proposition 3.1 we are interested in the probability of the event that there are exactly $B_\theta = \ell|U| + \theta$ bins with k red balls. We estimate the above probability as follows. We begin by putting into each bin k black balls, labelled with the numbers $1, \ldots, k$. Let $\mathcal{K} = \{1, \ldots, k\}$, and let $X_1, \ldots, X_M$ be independent random sets such that for $1 \leqslant i \leqslant M$

$$\forall \mathcal{K}' \subseteq \mathcal{K} : \mathbb{P}(X_i = \mathcal{K}') = q^{|\mathcal{K}'|}(1-q)^{k-|\mathcal{K}'|}.$$

Note that $|X_i|$ follows the binomial distribution $\text{Bin}(k,q)$. We then recolour the balls in the ith bin that are in X_i with red, and all others with blue. So, the total number of red

balls is $X = \sum_{i=1}^{M} |X_i|$. Note that $\mathbb{E}(X) = kqM$, and that X is distributed as $\mathrm{Bin}(kM, q)$. A straightforward application of Stirling's formula then gives

$$\mathbb{P}(X = kqM) = \mathbb{P}(X = \mathbb{E}(X)) = (1 + o(1))(2\pi q(1-q)kM)^{-1/2}.$$

Let R_j be the number of X_i that contain j elements. Then

$$\mathbb{P}_{\mathbf{d}, k}(\mathcal{B}(\beta, q)) \leqslant \sum_{\theta=0}^{\ell-1} \mathbb{P}(R_k = B_\theta | X = kqM) = \sum_{\theta=0}^{\ell-1} \frac{\mathbb{P}(X = kqM \wedge R_k = B_\theta)}{\mathbb{P}(X = kqM)}$$

$$= O(\sqrt{M}) \sum_{\theta=0}^{\ell-1} \mathbb{P}(X = kqM \wedge R_k = B_\theta). \tag{3.6}$$

Let

$$p_j = \mathbb{P}(|X_i| = j) = \binom{k}{j} q^j (1-q)^{k-j}.$$

Moreover, define the set of integer sequences

$$\mathcal{A} = \left\{ (b_0, \ldots, b_{k-1}) \in \mathbb{N}^k : \sum_{j=0}^{k-1} b_j = M - B_\theta \text{ and } \sum_{j=0}^{k-1} j b_j = kqM - kB_\theta \right\}.$$

Then

$$\mathbb{P}(X = kqM \wedge R_k = B_\theta) \leqslant \sum_{\theta=0}^{\ell-1} \sum_{(b_0, \ldots, b_{k-1}) \in \mathcal{A}} \binom{M}{b_0, \ldots, b_{k-1}, B_\theta} \cdot \left(\prod_{j=0}^{k-1} p_j^{b_j} \right) \cdot p_k^{B_\theta}.$$

Now observe that the summand can be rewritten as

$$\binom{M}{B_\theta} q^{kqM} (1-q)^{k(1-q)M} \cdot \binom{M - B_\theta}{b_0, \ldots, b_{k-1}} \prod_{j=0}^{k-1} \binom{k}{j}^{b_j}.$$

Also,

$$\sum_{(b_0, \ldots, b_{k-1}) \in \mathcal{A}} \binom{M - B_\theta}{b_0, \ldots, b_{k-1}} \prod_{j=0}^{k-1} \binom{k}{j}^{b_j} \leqslant \left(\sum_{j=0}^{k-1} \binom{k}{j} \right)^{M - B_\theta} = (2^k - 1)^{M - B_\theta}.$$

Thus, we have

$$\mathbb{P}(X = kqM \wedge R_k = B_\theta) \leqslant \sum_{\theta=0}^{\ell-1} \binom{M}{B_\theta} q^{kqM} (1-q)^{k(1-q)M} (2^k - 1)^{M - B_\theta}$$

$$\leqslant \sum_{\theta=0}^{\ell-1} M^\theta \binom{M}{\ell|U|} e^{-kMH(q)} (2^k - 1)^{M - \ell|U|} \cdot (2^k - 1)^{-\theta}$$

$$\leqslant \ell M^\ell \binom{M}{\ell|U|} (2^k - 1)^{M - \ell|U|} e^{-kMH(q)}.$$

The claim then follows by substituting the above bound into (3.6). $\qquad \square$

As already mentioned, the above lemma gives us a bound on the probability that a subset of the $(\ell + 1)$-core with a given number of vertices and total degree is maximal

ℓ-dense, assuming that the degree sequence is given. In particular, we work on the probability space of stage 3 of the exposure process. In order to show that the $(\ell + 1)$-core contains no ℓ-dense subset, we will estimate the number of such subsets. Recall that $X_{q,\beta}(\mathbf{d})$ denotes the number of subsets of $H_{\mathbf{d},k}$ with $\lfloor \beta \tilde{N}_{\ell+1} \rfloor$ vertices and total degree $\lfloor q \cdot k \tilde{M}_{\ell+1} \rfloor$. Also, let $X_{q,\beta}^{(\ell)}$ denote the number of these sets that are maximal ℓ-dense. As an immediate consequence of Markov's inequality we obtain the following corollary.

Corollary 3.4. *Let* $\mathcal{B}(q, \beta)$ *be defined as in Lemma 3.3, and let* $\mathbf{d}$ *be the degree sequence of the core of* $\tilde{H}_{n,p,k}$. *Then*

$$\mathbb{P}\big(X_{q,\beta}^{(\ell)} > 0 \mid \mathbf{d}\big) \leqslant X_{q,\beta}(\mathbf{d})\mathbb{P}_{\mathbf{d},k}(\mathcal{B}(q, \beta)).$$

By applying Lemma 3.2 we obtain that $H_{n,cn,k}$ does not obtain any ℓ-dense set with less that $0.6n$ vertices. In particular, this is also true for C, and so it remains to prove Theorem 2.2 for sets of size bigger than $0.6n \geqslant 0.6\tilde{N}_{\ell+1}$. We also observe that it is sufficient to argue about subsets of size up to, say, $(1 - e_{k,\ell}\delta/2)\tilde{N}_{\ell+1}$, as (3.5) implies that for small δ all larger subsets have density smaller than ℓ. Moreover, the total degree D of any ℓ-dense subset with $\beta \tilde{N}_{\ell+1}$ vertices is at least $k\ell \cdot \beta \tilde{N}_{\ell+1}$, that is,

$$D = k \cdot q \tilde{M}_{\ell+1} \Rightarrow k\ell \cdot \beta \tilde{N}_{\ell+1} \leqslant k \cdot q \tilde{M}_{\ell+1}.$$

By (3.3) and (3.5) we infer $\tilde{M}_{\ell+1} = \ell(1 - \Theta(\delta))$. Combined with the above inequality this implies that $q \geqslant (1 + \Theta(\delta))\beta$. Note that as each of the vertices in C has degree at least $\ell + 1$, the total degree of the $(\ell + 1)$-core with a ℓ-dense subset with $\beta \tilde{N}_{\ell+1}$ vertices and degree $q \cdot k \tilde{M}_{\ell+1}$ satisfies

$$k\tilde{M}_{\ell+1} \geqslant q \cdot k \tilde{M}_{\ell+1} + (\ell + 1)(\tilde{N}_{\ell+1} - \beta \tilde{N}_{\ell+1})$$

$$\Rightarrow q \leqslant 1 - \frac{(\ell + 1)(1 - \beta)\tilde{N}_{\ell+1}}{k\tilde{M}_{\ell+1}} \overset{(3.3),(3.5)}{\leqslant} 1 - \frac{(\ell + 1)(1 - \beta)}{k\ell},$$

where the last inequality holds for any sufficiently small δ. Therefore, we fix β and q as follows:

$$0.6 \leqslant \beta < 1 - e_{k,\ell}\delta/2 \quad \text{and} \quad \ell(1 + \Theta(\delta))\beta \leqslant q \leqslant 1 - \frac{(\ell + 1)(1 - \beta)}{k\ell}. \tag{3.7}$$

With Lemma 3.3 and Corollary 3.4 in hand, we are ready to show the following.

Lemma 3.5. *Let* $m_{\ell+1}$ *and* $n_{\ell+1}$ *be as defined in* (3.2) *and let* $\mathcal{E}$ *be the event that* (3.3) *holds. Then*

$$\mathbb{P}\big(X_{q,\beta}^{(\ell)} > 0\big) \leqslant \mathbb{E}(X_{q,\beta}|\mathcal{E})(2^k - 1)^{m_{\ell+1} - \ell\beta n_{\ell+1}} \cdot e^{\ell n_{\ell+1}H(\beta) - km_{\ell+1}H(q) + O(\delta^3 n)} + O(n^{-3}).$$

Proof. Let $\mathcal{E}_1$ be the event that $X_{q,\beta} \leqslant n^3 \mathbb{E}(X_{q,\beta} \mid \mathcal{E})$. Markov's inequality immediately implies that $\mathbb{P}(\mathcal{E}_1 \mid \mathcal{E}) \geqslant 1 - n^{-3}$. If $\vec{d}$ is a vector, we write $\vec{d} \in \{\mathcal{E} \cap \mathcal{E}_1\}$ to denote that $\vec{d}$ is

a possible degree sequence of C where the events $\mathcal{E}$ and $\mathcal{E}_1$ are realized. We have

$$
\begin{aligned}
\mathbb{P}\big(X_{q,\beta}^{(\ell)} > 0\big) &\leqslant \mathbb{P}\big(X_{q,\beta}^{(\ell)} > 0 \mid \mathcal{E}_1 \cap \mathcal{E}\big) + \mathbb{P}(\overline{\mathcal{E}_1}) + \mathbb{P}(\overline{\mathcal{E}}) \\
&= \sum_{\vec{d} \in \{\mathcal{E} \cap \mathcal{E}_1\}} \mathbb{P}\big(X_{q,\beta}^{(\ell)} > 0 \mid \mathcal{E}_1 \cap \mathcal{E} \text{ and } \mathbf{d} = \vec{d}\big) \cdot \mathbb{P}(\mathbf{d} = \vec{d} \mid \mathcal{E}_1 \cap \mathcal{E}) + O(n^{-3}) \\
&= \sum_{\vec{d} \in \{\mathcal{E} \cap \mathcal{E}_1\}} \mathbb{P}\big(X_{q,\beta}^{(\ell)} > 0 \mid \mathbf{d} = \vec{d}\big) \cdot \mathbb{P}(\mathbf{d} = \vec{d} \mid \mathcal{E}_1 \cap \mathcal{E}) + O(n^{-3}).
\end{aligned}
$$

By applying Corollary 3.4 we infer that

$$
\begin{aligned}
\mathbb{P}\big(X_{q,\beta}^{(\ell)} > 0\big) &\leqslant \sum_{\vec{d} \in \{\mathcal{E} \cap \mathcal{E}_1\}} X_{q,\beta}(\vec{d}) \mathbb{P}_{\vec{d},k}(\mathcal{B}(q,\beta)) \cdot \mathbb{P}(\mathbf{d} = \vec{d} \mid \mathcal{E}_1 \cap \mathcal{E}) + O(n^{-3}) \\
&\leqslant n^3 \; \mathbb{E}(X_{q,\beta} \mid \mathcal{E}) \cdot \sum_{\vec{d} \in \{\mathcal{E} \cap \mathcal{E}_1\}} \mathbb{P}_{\vec{d},k}(\mathcal{B}(q,\beta)) \mathbb{P}(\mathbf{d} = \vec{d} \mid \mathcal{E}_1 \cap \mathcal{E}) + O(n^{-3}).
\end{aligned}
$$

Note that the assumption $\vec{d} \in \{\mathcal{E} \cap \mathcal{E}_1\}$ implies that the number of vertices $\tilde{N}_{\ell+1}$ of $\vec{d}$ is $n_{\ell+1} \pm \delta^3 n$ and the number of edges $\tilde{M}_{\ell+1}$ is $m_{\ell+1} \pm \delta^3 n$, by $\mathcal{E}$. Further note that for sufficiently small δ,

$$
\tilde{M}_{\ell+1} \leqslant m_{\ell+1} + \delta^3 n \leqslant (1 - \Theta(\delta))\ell n_{\ell+1} + \delta^3 n \leqslant \ell \tilde{N}_{\ell+1} - \Theta(\delta)n.
$$

Using Stirling's formula, we obtain

$$
\binom{\tilde{M}_{\ell+1}}{\ell \beta \tilde{N}_{\ell+1}} < \binom{\ell \tilde{N}_{\ell+1}}{\ell \beta \tilde{N}_{\ell+1}} = \exp(\ell n_{\ell+1} H(\beta) + O(\delta^3 n)).
$$

Thus, applying Lemma 3.3 we obtain uniformly for all $\vec{d} \in \{\mathcal{E} \cap \mathcal{E}_1\}$ that

$$
\mathbb{P}_{\vec{d},k}(\mathcal{B}(q,\beta)) = (2^k - 1)^{m_{\ell+1} - \beta n_{\ell+1}} \cdot e^{\ell n_{\ell+1} H(\beta) - k m_{\ell+1} H(q) + O(\delta^3 n)}.
$$

The claim follows. $\qquad\square$

The following lemma bounds the expected value of $X_{q,\beta}$ conditional on $\mathcal{E}$.

Lemma 3.6. *There exists $\delta_0 > 0$ such that, whenever $\delta < \delta_0$,*

$$
\begin{aligned}
&\mathbb{E}(X_{q,\beta} \mid \mathcal{E}) \\
&\qquad < \exp\left(n_{\ell+1} H(\beta) - n_{\ell+1}(1 - \beta) I_{\xi^*}\left(\frac{k\ell(1-q)}{1-\beta}\right) + 0.4 \cdot \frac{k\ell}{\xi^*} \cdot n_{\ell+1}\delta + O(\delta^2 n) \right),
\end{aligned}
$$

where I_{ξ^*} is the rate function as defined in (2.1).

Proof. Let $t = \lfloor \beta \tilde{N}_{\ell+1} \rfloor$. Conditional on $\mathcal{E}$ there are

$$
\binom{\tilde{N}_{\ell+1}}{t} = e^{n_{\ell+1} H(\beta) + O(\delta^3 n)}
$$

ways to select a set with t vertices. We shall next calculate the probability that one of them has the claimed property, and the statement will follow from the linearity of expectation. Let U be a fixed subset of the vertex set of C that has size t. We label the vertices as

$1,\ldots,\tilde{N}_{\ell+1}$ so that the vertices which are not in U are indexed from $t+1$ to $\tilde{N}_{\ell+1}$. Let the random variable d_i denote the degree of vertex i. We recall that $d_1, d_2, \ldots, d_{\tilde{N}_{\ell+1}}$ are i.i.d. $(\ell+1)$-truncated Poisson variables with parameter $\Lambda = \Lambda_{c,k,\ell} = \xi \pm \delta^3$ and mean

$$\mu_\Lambda = \Lambda \frac{Q(\Lambda,\ell)}{Q(\Lambda,\ell+1)}.$$

By Taylor's expansion of μ_λ around ξ we obtain

$$\mu_\Lambda = \xi \frac{Q(\xi,\ell)}{Q(\xi,\ell+1)} \pm \Theta(\delta^3).$$

(Here and below, the Landau notation Θ indicates a positive term.) We will calculate the probability of the event $\sum_{i=1}^{t} d_i = qk\tilde{M}_{\ell+1}$ conditional on $\mathcal{E}$. This is equivalent to calculating the probability of the event $\sum_{i=t}^{\tilde{N}_{\ell+1}} d_i = k(1-q)\tilde{M}_{\ell+1}$ conditional on $\mathcal{E}$, which by using (3.2) is the same as the event

$$\sum_{i=t+1}^{\tilde{N}_{\ell+1}} \frac{d_i}{\tilde{N}_{\ell+1} - t} = \xi \frac{Q(\xi,\ell)}{Q(\xi,\ell+1)} \cdot \frac{1-q}{1-\beta} \pm \Theta(\delta^3).$$

Let us abbreviate:

$$z = \xi \frac{Q(\xi,\ell)}{Q(\xi,\ell+1)} \cdot \frac{1-q}{1-\beta} \pm \Theta(\delta^3).$$

Using the lower bound of q from (3.7), we obtain

$$z - \mu_\Lambda = \xi \frac{Q(\xi,\ell)}{Q(\xi,\ell+1)} \cdot \frac{\beta}{1-\beta} \Theta(\delta) \pm \Theta(\delta^3) > 0.$$

As $I_\Lambda(x)$ is a non-negative convex function and $I_\Lambda(\mu_\Lambda) = 0$, then $I_\Lambda(x)$ is a decreasing function for $x < \mu_\Lambda$. Therefore, by Lemma 2.8,

$$\mathbb{P}\left(\sum_{i=t+1}^{\tilde{N}_{\ell+1}} d_i = z(\tilde{N}_{\ell+1} - t) \mid \mathcal{E}\right) = \exp\left(-n_{\ell+1}(1-\beta) \cdot I_\Lambda(z)(1 + o(1))\right)$$

and

$$I_\Lambda(z) = z(\log T_z - \log \Lambda) - T_z + \Lambda - \log Q(T_z, \ell+1) + \log Q(\Lambda, \ell+1),$$

where T_z is the unique solution of

$$z = T_z \cdot \frac{Q(T_z,\ell)}{Q(T_z,\ell+1)}.$$

(By Lemma 2.8, the function $I_\Lambda(z)$ is strictly positive for $z \neq \mu_\Lambda$.) Note that

$$\frac{\partial I_\Lambda(z)}{\partial \Lambda} = -\frac{z}{\Lambda} + 1 + \frac{e^{-\Lambda}\Lambda^\ell/\ell!}{Q(\Lambda,\ell+1)} = -\frac{z}{\Lambda} + \frac{Q(\Lambda,\ell)}{Q(\Lambda,\ell+1)} = \frac{\mu_\Lambda - z}{\Lambda}.$$

But recall that $\Lambda = \xi \pm \delta^3 = \xi^* - \delta \pm \delta^3$. Using Taylor's expansion around ξ^* to write $I_\Lambda(z)$ in terms of $I_{\xi^*}(z)$, we obtain

$$I_\Lambda(z) = I_{\xi^*}(z) - \left(\frac{\mu_{\xi^*} - z}{\xi^*}\right)(\delta \pm \delta^3) \pm O(\delta^2) = I_{\xi^*}(z) - \frac{\mu_{\xi^*}}{\xi^*} \cdot \frac{q-\beta}{1-\beta} \delta \pm O(\delta^2).$$

The last equality holds as

$$z = \mu_{\xi^*} \frac{1-q}{1-\beta}(1 - e_{k,\ell}\delta + \Theta(\delta^2)).$$

Since $\beta \geqslant 0.6$ we have $q - \beta < 0.4$. Also, $\mu_{\xi^*} = k\ell$. Therefore,

$$I_{\Lambda}(z) \geqslant I_{\xi^*}(z) - \frac{k\ell}{\xi^*} \cdot \frac{0.4}{1-\beta} \, \delta \pm O(\delta^2). \tag{3.8}$$

We will now approximate $I_{\xi^*}(z)$ in terms of

$$I_{\xi^*}\left(k\ell \frac{1-q}{1-\beta}\right).$$

Note that

$$\frac{\partial I_{\xi^*}(z)}{\partial z} = \log T_z - \log \xi^*.$$

By Taylor's expansion of $I_{\xi^*}(z)$ around

$$z_0 := k\ell \frac{1-q}{1-\beta}$$

we obtain

$$I_{\xi^*}(z) = I_{\xi}^*\left(k\ell \frac{1-q}{1-\beta}\right) + \delta \cdot e_{k,\ell}\left(k\ell \frac{1-q}{1-\beta}\right)\left(\log \frac{\xi^*}{T_{z_0}}\right) \pm O(\delta^2). \tag{3.9}$$

By Claim 3.14 the function μ_t is increasing with respect to t. This implies that $T_{z_0} < \xi^*$ as $z_0 < k\ell$, whereby $\log(\xi^*/T_{z_0}) > 0$. Also recall that $e_{k,\ell}$ denotes the value of the partial derivative of

$$\frac{1}{k\ell} \cdot \frac{tQ(t,\ell)}{Q(t,\ell+1)}$$

with respect to t at $t = \xi^*$. Again, Claim 3.14 implies that this is positive. We therefore obtain

$$I_{\xi^*}(z) > I_{\xi^*}\left(k\ell \frac{1-q}{1-\beta}\right) - \Theta(\delta^2). \tag{3.10}$$

Combining (3.8), (3.9) and (3.10) we obtain

$$I_{\Lambda}(z) > I_{\xi^*}\left(k\ell \frac{1-q}{1-\beta}\right) - \frac{k\ell}{\xi^*} \cdot \frac{0.4}{1-\beta} \, \delta - O(\delta^2).$$

The proof is then completed by using the fact that $\mathbb{P}(\mathcal{E}) = 1 - n^{-\omega(1)}$. □

Lemma 3.5 along with Lemmas 3.3 and 3.6 yield the following estimate.

Corollary 3.7. *There exists $\delta_0 > 0$ such that, whenever $\delta < \delta_0$,*

$$\mathbb{P}\big(X_{q,\beta}^{(\ell)} > 0\big) < O(n^{-3}) + F(\beta, q; \ell),$$

where

$$F(\beta, q; \ell) = (2^k - 1)^{m_{\ell+1} - \ell \beta n_{\ell+1}}$$

$$\times \exp\left((\ell + 1) n_{\ell+1} H(\beta) - k m_{\ell+1} H(q) - n_{\ell+1}(1 - \beta) I_{\xi^*}\left(\frac{k\ell(1-q)}{1-\beta} \right) \right.$$

$$\left. + 0.4 \cdot \frac{k\ell}{\xi^*} \cdot n_{\ell+1} \cdot \delta + O(\delta^2 n) \right).$$

Let us abbreviate:

$$f(\beta, q) := (\ell + 1)H(\beta) + \ell \cdot (1 - \beta)\log(2^k - 1) - k\ell \cdot H(q) - (1 - \beta) I_{\xi^*}\left(\frac{k\ell(1-q)}{1-\beta} \right).$$

By using Corollary 3.7 we infer that

$$\frac{1}{n_{\ell+1}} \log F(\beta, q; \ell) \leqslant f(\beta, q) + e_{k,\ell} \cdot \delta \cdot k\ell\left(H(q) - \frac{\log(2^k - 1)}{k} + \frac{0.4}{e_{k,\ell} \cdot \xi^*} \right) + O(\delta^2).$$

We bound the product $e_{k,\ell} \cdot \xi^*$, and therefore the function $\log F(\beta, q; \ell)$, using the following technical result, whose proof we postpone until later, where we bound $f(\beta, q)$.

Claim 3.8. *Let $e_{k,\ell}$ be the value of derivative of*

$$\frac{xQ(x, \ell)}{k\ell \cdot Q(x, \ell + 1)}$$

with respect to x at $x = \xi^*$. Then $e_{k,\ell} \cdot \xi^* > 0.77$.

By the above claim, $e_{k,\ell} > 0.77/\xi^*$. So

$$\frac{1}{n_{\ell+1}} \log F(\beta, q; \ell) \leqslant f(\beta, q) + e_{k,\ell} \cdot \delta \cdot k\ell\left(H(q) - \frac{\log(2^k - 1)}{k} + 0.52 \right) + O(\delta^2). \quad (3.11)$$

We will now prove the main tool for the proof of Theorem 2.2.

Lemma 3.9. *There exists $\hat{\delta} = \hat{\delta}(k, \ell) > 0$ such that if $0 < \delta < \hat{\delta}$ the following holds. With probability $1 - n^{-\omega(1)}$, for any $0.6 \leqslant \beta < 1 - e_{k,\ell}\delta/2$ and $\beta < q \leqslant 1 - (\ell + 1)(1 - \beta)/k\ell$, we have $X^{(\ell)}_{q,\beta} = 0$.*

Proof of Theorem 2.2. First, note that it is enough to argue that with probability $1 - o(1)$ the $(\ell + 1)$-core does not contain any maximal ℓ-dense subset; this follows from the discussion after Lemma 3.2. Moreover, by Theorem 2.4 and Proposition 2.3, it is enough to consider the $(\ell + 1)$-core C of $\tilde{H}_{n,p,k}$, where $p = ck/\binom{n-1}{k-1}$. The proof is completed by applying Lemma 3.9, as we can choose $\delta > 0$ as small as we please. $\quad \square$

To deduce Lemma 3.9 our main tool is the following assertion.

Claim 3.10. *For any $k \geqslant 3$ and $\ell \geqslant 2$, there exists $C > 0$ such that for any $\varepsilon < 1/e$ the following holds. For any $0.6 < \beta \leqslant 1 - \varepsilon$, and q as in Lemma 3.9, we have*

$$f(\beta, q) \leqslant -C\varepsilon.$$

Proof of Lemma 3.9.   We show that for δ small enough the rest of the right-hand side of (3.11) is negative. First, let $\delta_1 = \delta_1(k, \ell)$ be such that for any $\delta < \delta_1$ we have $1 - e_{k,\ell}\delta/2 > 0.999$. We will consider a case distinction according to the value of q.

If $q < 0.99$, then $\beta < 0.99$ as well, and Claim 3.10 implies that $f(\beta, q) \leqslant -0.01 \cdot C$, where $C > 0$ depends on k and ℓ. Then let $\delta_2 = \delta_2(k, \ell) > 0$ be such that for $\delta < \delta_2$, we have

$$e_{k,\ell} \cdot \delta \cdot k\ell \left(H(0.6) - \frac{\log(2^k - 1)}{k} + 0.52 \right) + O(\delta^2) < 0.005 \cdot C.$$

Here recall that $\beta \geqslant 0.6$. So for any $\delta < \min\{\delta_0, \delta_1, \delta_2\}$, (3.11) implies that

$$\frac{1}{n_{\ell+1}} \log F(\beta, q; \ell) \leqslant -0.005 \cdot C.$$

Assume now that $q \geqslant 0.99$. The monotonicity of the entropy function implies that

$$H(q) - \frac{\log(2^k - 1)}{k} + 0.52 \leqslant H(0.99) - \frac{\log(2^k - 1)}{k} + 0.52 \overset{k \geqslant 3}{<} -0.072.$$

Now with $0.6 \leqslant \beta \leqslant 1 - e_{k,\ell} \cdot \delta/2$ as in Lemma 3.9, the bound of Claim 3.10 substituted in (3.11) yields

$$\frac{1}{n_{\ell+1}} \log F(\beta, q; \ell) \leqslant -C e_{k,\ell} \cdot \delta/2 + O(\delta^2).$$

In turn, this is at most $-C e_{k,\ell} \cdot \delta/4$, if $\delta < \delta_3 = \delta_3(k, \ell)$. The above cases imply that if $\delta < \min\{\delta_0, \delta_1, \delta_2, \delta_3\} =: \hat{\delta}$, then with probability $1 - e^{-\Omega(n_{\ell+1})} - O(n^{-3})$ we have $X_{q,\beta}^{(\ell)} = 0$, for all β and q as in Lemma 3.9. $\square$

The rest of the paper is devoted to the (rather technical and analytical) proof of Claim 3.10 and contains a detailed analysis of the function f. We proceed as follows. We will fix arbitrarily a β and we will consider $f(\beta, q)$ solely as a function of q. Then we will show that if $q_0 = q_0(\beta)$ is a point where the partial derivative of f with respect to β vanishes, then $f(\beta, q_0) \leqslant -C_1 \varepsilon$. Additionally, we will show that this holds for $f(\beta, \beta)$ and $f(\beta, 1 - (\ell+1)(1 - \beta)/k\ell)$.

Bounding $f(\beta, q)$ at its critical points. Let β be fixed. We will evaluate $f(\beta, q)$ at a point where the partial derivative with respect to q vanishes. To calculate the partial derivative with respect to q, we first need to determine the derivative of $I(z)$ with respect to z. According to Lemma 2.8,

$$I_{\xi^*}'(z) = z(\log T_z - \log \xi^*) - \log Q(T_z, \ell+1) - T_z + \log Q(\xi^*, \ell+1) + \xi^*,$$

where T_z is the unique solution of

$$z = T_z \cdot \frac{Q(T_z, \ell)}{Q(T_z, \ell+1)}.$$

Differentiating this with respect to z, we obtain

$$
\begin{aligned}
I'_{\xi^*}(z) &= \log T_z - \log \xi^* + \frac{z}{T_z} \frac{dT_z}{dz} - \frac{dT_z}{dz} - \frac{Q(T_z, \ell) - Q(T_z, \ell+1)}{Q(T_z, \ell+1)} \frac{dT_z}{dz} \\
&= \log T_z - \log \xi^* + \frac{z}{T_z} \frac{dT_z}{dz} - \frac{Q(T_z, \ell)}{Q(T_z, \ell+1)} \frac{dT_z}{dz} \\
&= \log T_z - \log \xi^*.
\end{aligned}
\tag{3.12}
$$

Note that in the differentiation of f we need to differentiate $I_{\xi^*}(k\ell(1-q)/(1-\beta))$ with respect to q. Using (3.12), we obtain

$$
\frac{\partial I_{\xi^*}\left(\frac{k\ell(1-q)}{1-\beta}\right)}{\partial q} = -\frac{k\ell}{1-\beta}(\log H_q - \log \xi^*),
$$

where H_q is the unique solution of the equation

$$
\frac{k\ell(1-q)}{1-\beta} = \frac{H_q \cdot Q(H_q, \ell)}{Q(H_q, \ell+1)}.
$$

Observe that the choice of the range of q is such that the left-hand side of the above equation is at least $\ell + 1$. So, H_q is well-defined. Also, an elementary calculation shows that the derivative of the entropy function, $H'(q)$, is given by $\log((1-q)/q)$. All the above facts together yield the derivative of $f(\beta, q)$ with respect to q:

$$
\frac{\partial f(\beta, q)}{\partial q} = k\ell\left(-\log\left(\frac{1-q}{q}\right) + \log \frac{H_q}{\xi^*}\right).
$$

Therefore, if q_0 is a critical point, that is, if

$$
\left.\frac{\partial f(\beta, q)}{\partial q}\right|_{q=q_0} = 0,
$$

then with $T_0 = H_{q_0}$, q_0 satisfies

$$
T_0 = \xi^* \frac{1-q_0}{q_0} \quad \text{and} \quad \frac{k\ell(1-q_0)}{1-\beta} = \frac{T_0 Q(T_0, \ell)}{Q(T_0, \ell+1)}.
\tag{3.13}
$$

At this point, we have the main tool that will allow us to evaluate $f(\beta, q_0)$. We will use (3.13) in order to eliminate T_0 and express $f(\beta, q_0)$ solely as a function of q_0.

Claim 3.11. *For any given $\beta \in (0.6, 1)$, if $q_0 = q_0(\beta)$ satisfies* (3.13), *then*

$$
f(\beta, q_0) = \log\left(e^{(\ell+1)H(\beta)} q_0^{k\ell}\left(\frac{(2^k-1)(1-q_0)}{q_0}\right)^{\ell(1-\beta)} \cdot \left(\frac{(1-\beta)(k\ell-\xi^*)}{k\ell q_0 - \xi^*(1-\beta)}\right)^{1-\beta}\right).
\tag{3.14}
$$

Proof. Note that

$$
\begin{aligned}
I_{\xi^*}\left(\frac{k\ell(1-q_0)}{1-\beta}\right) &= \frac{k\ell(1-q_0)}{1-\beta}\log \frac{T_0}{\xi^*} + \log\left(\frac{e^{\xi^*}Q(\xi^*, \ell+1)}{e^{T_0}Q(T_0, \ell+1)}\right) \\
&\overset{(3.13)}{=} \frac{k\ell(1-q_0)}{1-\beta}\log\left(\frac{1-q_0}{q_0}\right) + \log\left(\frac{e^{\xi^*}Q(\xi^*, \ell+1)}{e^{T_0}Q(T_0, \ell+1)}\right).
\end{aligned}
$$

Therefore,

$$-(1-\beta)I_{\xi^*}\left(\frac{k\ell(1-q_0)}{1-\beta}\right) = -k\ell(1-q_0)\log\left(\frac{1-q_0}{q_0}\right) + (1-\beta)\log\left(\frac{e^{T_0}Q(T_0,\ell+1)}{e^{\xi^*}Q(\xi,\ell+1)}\right)$$

$$= -k\ell(1-q_0)\log(1-q_0) + k\ell\ \log(q_0) - k\ell q_0\log(q_0)$$

$$+ (1-\beta)\log\left(\frac{e^{T_0}Q(T_0,\ell+1)}{e^{\xi}Q(\xi,\ell+1)}\right).$$

Also, the definition of the entropy function implies that

$$-k\ell H(q_0) = k\ell q_0\log(q_0) + k\ell(1-q_0)\log(1-q_0).$$

Thus

$$-(1-\beta)I_{\xi^*}\left(\frac{k\ell(1-q_0)}{1-\beta}\right) - k\ell H(q_0) = \log\left(q_0^{k\ell}\left(\frac{e^{T_0}Q(T_0,\ell+1)}{e^{\xi}Q(\xi^*,\ell+1)}\right)^{1-\beta}\right). \tag{3.15}$$

Let

$$z_0 := \frac{k\ell(1-q_0)}{1-\beta}.$$

Now we will express $e^{T_0}Q(T_0,\ell+1)$ as a rational function of T_0 and z_0. Solving (3.13) with respect to $e^{T_0}Q(T_0,\ell+1)$ yields

$$e^{T_0}Q(T_0,\ell+1) = e^{T_0}\frac{T_0Q(T_0,\ell)}{z_0} = \frac{e^{T_0}T_0}{z_0}\left(Q(T_0,\ell+1) + e^{-T_0}\frac{T_0^{\ell}}{\ell!}\right).$$

Therefore,

$$e^{T_0}Q(T_0,\ell+1) = \frac{T_0^{\ell}}{\ell!}\left(\frac{z_0}{T_0}-1\right)^{-1}.$$

Note that

$$z_0 - T_0 = \frac{k\ell(1-q_0)}{1-\beta} - \frac{\xi^*(1-q_0)}{q_0} = \frac{(1-q_0)(k\ell q_0 - \xi^*(1-\beta))}{(1-\beta)q_0}.$$

Thus we obtain

$$\log(e^{T_0}Q(T_0,\ell+1)) = \log\left(\frac{T_0^{\ell+1}}{(z-T_0)\ell!}\right)$$

$$\stackrel{(3.13)}{=} \log\left(\left(\frac{\xi^*(1-q_0)}{q_0}\right)^{\ell+1}\cdot\frac{(1-\beta)q_0}{(1-q_0)(k\ell q_0 - \xi^*(1-\beta))\ell!}\right)$$

$$= \log\left(\frac{(\xi^*)^{\ell+1}}{\ell!}\left(\frac{1-q_0}{q_0}\right)^{\ell}\cdot\frac{1-\beta}{k\ell q_0 - \xi^*(1-\beta)}\right).$$

Also, by definition of ξ^* we have

$$k = \frac{\xi^*Q(\xi^*,\ell)}{\ell Q(\xi^*,\ell+1)},$$

which is equivalent to

$$k\ell = \xi^*\left(\frac{1+e^{-\xi^*}(\xi^*)^{\ell}/\ell!}{Q(\xi^*,\ell+1)}\right)$$

and implies

$$e^{\xi^*} Q(\xi^*, \ell + 1) = \frac{(\xi^*)^{\ell+1}/\ell!}{k\ell - \xi^*}.$$

Substituting this into (3.15) and adding the remaining terms, we obtain (3.14). □

We will now treat q_0 as a free variable lying in the interval containing q, and we will study $f(\beta, q_0)$ for a fixed β as a function of q_0. In particular, we will show that for any fixed β in the domain of interest, $f(\beta, q_0)$ is increasing. Thereafter, we will evaluate $f(\beta, q_0)$ at the largest possible value that q_0 can take, which is

$$1 - \frac{(\ell + 1)(1 - \beta)}{k\ell},$$

and show that this value is negative.

Claim 3.12. *For any $k \geqslant 3, \ell \geqslant 2$ and for any $\beta > 0.6$, we have $\partial f(\beta, q_0)/\partial q_0 > 0$.*

Proof. The partial derivative of $f(\beta, q_0)$ with respect to q_0 is

$$\frac{\partial f(\beta, q_0)}{\partial q_0} = \frac{k\ell}{q_0} - \ell\frac{1 - \beta}{1 - q_0} - \ell\frac{1 - \beta}{q_0} - \frac{k\ell(1 - \beta)}{k\ell q_0 - \xi^*(1 - \beta)}.$$

Since

$$q_0 \leqslant 1 - \frac{(\ell + 1)(1 - \beta)}{k\ell},$$

we obtain

$$-\frac{1 - \beta}{1 - q_0} \geqslant -\frac{k\ell}{\ell + 1}.$$

Also, $q_0 \geqslant \beta$ and $\xi < k\ell$. Therefore,

$$k\ell q_0 - \xi(1 - \beta) > k\ell\beta - k\ell(1 - \beta) = 2\beta k\ell - k\ell = k\ell(2\beta - 1).$$

Substituting these bounds into

$$\frac{\partial f(\beta, q_0)}{\partial q_0}$$

yields

$$\frac{\partial f(\beta, q_0)}{\partial q_0} > \frac{k\ell}{q_0} - \frac{k\ell^2}{\ell + 1} - \frac{\ell(1 - \beta)}{q_0} - \frac{1 - \beta}{2\beta - 1} = \frac{k\ell - \ell(1 - \beta)}{q_0} - \frac{k\ell^2}{\ell + 1} - \frac{1 - \beta}{2\beta - 1}$$

$$\geqslant k\ell\frac{k\ell - \ell(1 - \beta)}{k\ell - (\ell + 1)(1 - \beta)} - \frac{k\ell^2}{\ell + 1} - \frac{1 - \beta}{2\beta - 1} \geqslant k\left(\ell - \frac{\ell^2}{\ell + 1} - \frac{1 - \beta}{k(2\beta - 1)}\right)$$

$$= k\left(\frac{\ell}{\ell + 1} - \frac{1 - \beta}{k(2\beta - 1)}\right).$$

But

$$\frac{\ell}{\ell + 1} > \frac{1 - \beta}{k(2\beta - 1)},$$

as $k\ell(2\beta - 1) > (\ell + 1)(1 - \beta)$, which is equivalent to $\beta > (k\ell + \ell + 1)/(2k\ell + \ell + 1)$. Elementary algebra then yields that $(k\ell + \ell + 1)/(2k\ell + \ell + 1)$ is a decreasing function in k and ℓ. In particular, its maximum is 0.6 for $k = 3$ and $\ell = 2$. Since $\beta > 0.6$, the above holds. $\qquad\square$

We begin by setting

$$q_0 := 1 - \frac{(\ell + 1)(1 - \beta)}{k\ell}$$

into $f(\beta, q_0)$ and obtain a function which depends only on β, namely

$$h(\beta) := \log\left(\beta^{-(\ell+1)\beta} \cdot \left(\left(\frac{(2^k - 1)(\ell + 1)}{k\ell - (\ell + 1)(1 - \beta)}\right)^{\ell} \frac{k\ell - \xi^*}{k\ell - (1 + \ell + \xi^*)(1 - \beta)}\right)^{1-\beta}\right.$$
$$\left. \times \left(1 - \frac{(\ell + 1)(1 - \beta)}{k\ell}\right)^{k\ell}\right).$$

Bounding $f(\beta, q)$ globally. To conclude the proof of Claim 3.10 it suffices to show that there exist ε_0 and $C > 0$ such that for any $\varepsilon < \varepsilon_0$ the following bounds hold:

$$h(\beta), f(\beta, 1 - (\ell + 1)(1 - \beta)/k\ell), f(\beta, \beta) \leqslant -C\varepsilon, \tag{3.16}$$

for all $0.6 \leqslant \beta \leqslant 1 - \varepsilon$. These three inequalities will be shown in Claims 3.17, 3.18 and 3.19, respectively. We will first compute bounds for ξ^* which we will require to bound the above functions. We start by proving two technical results, where we obtain bounds on $k\ell - \xi^*$ and $e_{k,\ell} \cdot \xi^*$, respectively. One of them (Claim 3.8) was also used in the proof of (3.11).

Claim 3.13. *Let $k \geqslant 3, \ell \geqslant 2$ and let ξ^* satisfy (2.4). Then $\xi^* > k\ell - 0.36$. Moreover, $k\ell - \xi^* < 0.19$ for $k = 3, \ell \geqslant 4$ and $k \geqslant 4, \ell \geqslant 2$.*

Proof. Recall that

$$k\ell = \frac{\xi^* Q(\xi^*, \ell)}{Q(\xi^*, \ell + 1)}.$$

By definition we have

$$\frac{k\ell}{\xi^*} = \frac{Q(\xi^*, \ell)}{Q(\xi^*, \ell + 1)} = 1 + \frac{\mathbb{P}(\mathrm{Po}(\xi^*) = \ell)}{Q(\xi^*, \ell + 1)} = 1 + \frac{1}{\sum_{i \geqslant 1} \frac{(\xi^*)^i}{(\ell+1)\dots(\ell+i)}}. \tag{3.17}$$

Let

$$\mathcal{S} := \sum_{i \geqslant 1} \frac{(\xi^*)^i}{(\ell + 1)\dots(\ell + i)} \quad \text{and} \quad \mathcal{S}_i := \frac{(\xi^*)^i}{(\ell + 1)\dots(\ell + i)}.$$

Substituting

$$\xi^* = \frac{k\ell}{1 + 1/\mathcal{S}},$$

we obtain

$$\mathcal{S}_i = \frac{\left(\frac{1}{1+1/\mathcal{S}}\right)^i}{\left(\frac{1}{k}+\frac{1}{k\ell}\right)\dots\left(\frac{1}{k}+\frac{i}{k\ell}\right)}. \tag{3.18}$$

By (3.18) we have

$$\mathcal{S} > \mathcal{S}_1 = \frac{k\ell \cdot \frac{\mathcal{S}}{\mathcal{S}+1}}{\ell+1} \implies \mathcal{S} > \frac{k\ell}{\ell+1} - 1 \geqslant 1. \tag{3.19}$$

So

$$\xi^* = \frac{k\ell}{1+1/\mathcal{S}} > \frac{k\ell}{2}$$

and thus $\xi^* \geqslant 3\ell/2$. Therefore we obtain

$$\mathcal{S} > \frac{k\ell/2}{\ell+1} + \frac{(k\ell/2)^2}{(\ell+1)(\ell+2)} + \frac{(k\ell/2)^3}{(\ell+1)(\ell+2)(\ell+3)}.$$

The right-hand side is increasing in k and ℓ. Therefore, substituting $k = 3$ and $\ell = 2$ we obtain $\mathcal{S} > 2.2$, implying that

$$\xi^* > (11/16)k\ell \geqslant (33/16)\ell. \tag{3.20}$$

In order to improve the bound upon $k\ell - \xi^*$, we use the fact that $k\ell - \xi^* = \xi^*/\mathcal{S}$, and show that $\mathcal{S}/\xi^* > 1$. We obtain

$$\frac{\mathcal{S}}{\xi^*} = \sum_{i\geqslant 1}\frac{(\xi^*)^{i-1}}{(\ell+1)\dots(\ell+i)} = \frac{1}{\ell+1}\left(\sum_{i\leqslant\ell}\frac{(\xi^*)^{i-1}}{(\ell+2)\dots(\ell+i)} + \sum_{i\geqslant\ell+1}\frac{(\xi^*)^{i-1}}{(\ell+2)\dots(\ell+i)}\right)$$

$$\overset{(3.20)}{>} \frac{1}{\ell+1}\left(\ell + \sum_{i\geqslant\ell+1}\frac{(2\ell)^{i-1}}{(\ell+2)\dots(\ell+i)}\right).$$

For $\ell \geqslant 3$ observe that the term for $i = \ell+1$ is

$$\frac{(2\ell)^{i-1}}{(\ell+2)(\ell+3)\dots(2\ell+1)} > \frac{2\ell \cdot 2\ell}{(2\ell-1)(2\ell+1)} > 1.$$

For $\ell = 2$ we have

$$\sum_{i\geqslant\ell+1}\frac{(2\ell)^{i-1}}{(\ell+2)\dots(\ell+i)} > \sum_{i=3}^{5}\frac{4^3}{(2+i)(2+i-1)\dots 5} > 1.$$

By (3.17), we have

$$k\ell - \xi^* = \frac{1}{\sum_{i\geqslant 1}\frac{(\xi^*)^{i-1}}{(\ell+1)\dots(\ell+i)}},$$

and so

$$\frac{1}{k\ell - \xi^*} > \sum_{i\geqslant\ell+1}\frac{(\xi^*)^{i-1}}{(\ell+1)\dots(\ell+i)} > \sum_{i\geqslant\ell+1}\frac{(k\ell-1)^{i-1}}{(\ell+1)\dots(\ell+i)}.$$

Let

$$S_i(k,\ell) = \frac{(k\ell - 1)^{i-1}}{(\ell + 1)\ldots(\ell + i)}.$$

Clearly $S_i(k,\ell)$ is increasing with respect to k. Taking the derivative with respect to ℓ, we obtain that

$$\frac{\partial}{\partial \ell} S_i(k,\ell) = S_i(k,\ell)\left(\frac{k(i-1)}{k\ell - 1} - \sum_{j=1}^{i}\frac{1}{\ell + j}\right)$$

$$> S_i(k,\ell)\left(\frac{i-1}{\ell} - \sum_{j=1}^{i}\frac{1}{\ell + j}\right) = \frac{S_i(k,\ell)}{\ell}\left(\sum_{j=1}^{i}\frac{j}{\ell + j} - 1\right)$$

$$> \frac{S_i(k,\ell)}{\ell}\left(\frac{i-1}{\ell + i - 1} + \frac{i}{\ell + i} - 1\right) \overset{i \geqslant \ell + 1}{>} \frac{S_i(k,\ell)}{\ell}\left(\frac{1}{2} + \frac{\ell + 1}{2\ell + 1} - 1\right) > 0.$$

Therefore, for all $i \geqslant \ell + 1$, $S_i(k,\ell)$ increases with respect to ℓ. We have

$$\frac{1}{\sum_{i \geqslant \ell + 1} S_i(3,3)} < \frac{1}{\sum_{i=4}^{12} S_i(3,3)} < 0.34,$$

$$\frac{1}{\sum_{i \geqslant 4} S_i(3,4)} < \frac{1}{\sum_{i=4}^{14} S_i(3,4)} < 0.15,$$

and

$$\frac{1}{\sum_{i \geqslant 3} S_i(4,2)} < \frac{1}{\sum_{i=3}^{13} S_i(4,2)} < 0.19.$$

For the case $(k,\ell) = (3,2)$ we compute ξ^* using its definition which gives us $k\ell - \xi^* < 0.36$ for this case. □

We are now ready to bound $e_{k,\ell} \cdot \xi^*$.

Proof of Claim 3.8. We write

$$\frac{xQ(x,\ell)}{Q(x,\ell+1)} = \frac{x(Q(x,\ell+1) + \mathbb{P}(\text{Po}(x) = \ell))}{Q(x,\ell+1)} = x + \frac{\frac{1}{\ell!}}{\frac{1}{(\ell+1)!} + \frac{x}{(\ell+2)!} + \frac{x^2}{(\ell+3)!} + \cdots}.$$

By definition,

$$e_{k,\ell} \cdot k\ell = 1 - \frac{1}{\ell!}\frac{\frac{1}{(\ell+2)!} + \frac{2\xi^*}{(\ell+3)!} + \frac{3\xi^{*2}}{(\ell+4)!} + \cdots}{\left(\frac{1}{(\ell+1)!} + \frac{\xi^*}{(\ell+2)!} + \frac{\xi^{*2}}{(\ell+3)!} + \cdots\right)^2}$$

$$= 1 - (k\ell - \xi^*) \cdot \frac{\frac{1}{(\ell+2)!} + \frac{2\xi^*}{(\ell+3)!} + \frac{3\xi^{*2}}{(\ell+4)!} + \cdots}{\frac{1}{(\ell+1)!} + \frac{\xi^*}{(\ell+2)!} + \frac{\xi^{*2}}{(\ell+3)!} + \cdots}$$

$$= 1 - (k\ell - \xi^*) \cdot \left(1 - \frac{\frac{\ell+1}{(\ell+2)!} + \frac{(\ell+1)\xi^*}{(\ell+3)!} + \frac{(\ell+1)\xi^{*2}}{(\ell+4)!} + \cdots}{\frac{1}{(\ell+1)!} + \frac{\xi^*}{(\ell+2)!} + \frac{\xi^{*2}}{(\ell+3)!} + \cdots}\right)$$

$$= 1 - (k\ell - \xi^*) \cdot \left(1 - \frac{\ell+1}{\xi^*} \cdot \left(1 - \frac{\frac{1}{(\ell+1)!}}{1(\ell+1)! + \frac{\xi^*}{(\ell+2)!} + \frac{\xi^{*2}}{(\ell+3)!} + \cdots}\right)\right)$$

$$= 1 - (k\ell - \xi^*)\left(1 - \frac{\ell+1}{\xi^*} + \frac{k\ell - \xi^*}{\xi^*}\right)$$

$$= 1 - (k\ell - \xi^*)\left(-\frac{\ell+1}{\xi^*} + \frac{k\ell}{\xi^*}\right).$$

Thus,

$$e_{k,\ell} = \frac{1}{k\ell} - \frac{k\ell - \xi^*}{k\ell}\left(-\frac{\ell+1}{\xi^*} + \frac{k\ell}{\xi^*}\right)$$

$$= \frac{1}{k\ell} + \frac{\ell+1}{\xi^*} - \frac{\ell+1}{k\ell} - \frac{k\ell - \xi^*}{\xi^*}$$

$$= \frac{1}{\xi^*}\left(1 - (k\ell - \xi^*) + \frac{k\ell - \xi^*}{k}\right).$$

By Claim 3.13 we have $k\ell - \xi^* > 0.36$ for $k = 3$ and $\ell \geqslant 2$. Plugging this value into the above equation, we obtain that for $k = 3$, $e_{k,\ell} > 0.77/\xi^*$. For other values of k and ℓ we use

$$e_{k,\ell} \cdot \xi^* > 1 - (k\ell - \xi^*).$$

which by the second part of Claim 3.13 is at least 0.81. □

Claim 3.14. *For every $t \geqslant 1$, the function $x \mapsto xQ(x, t-1)/Q(x, t)$ is increasing for $x > 0$.*

Proof. Set

$$g_t(x) := \frac{1}{(t-1)!} \cdot \frac{1}{\frac{1}{t!} + \frac{x}{(t+1)!} + \frac{x^2}{(t+2)!} + \cdots}.$$

Then

$$\frac{xQ(x, t-1)}{Q(x, t)} = \frac{x(Q(x, t) + \mathbb{P}(\text{Po}(x) = t-1))}{Q(x, t)} = x + g_t(x).$$

To see the claim it thus suffices to show that

$$-g_t'(x) < 1.$$

But

$$-g_t'(x) = \frac{1}{(t-1)!} \frac{\frac{1}{(t+1)!} + \frac{2x}{(t+2)!} + \frac{3x^2}{(t+3)!} + \cdots}{\left(\frac{1}{t!} + \frac{x}{(t+1)!} + \frac{x^2}{(t+2)!} + \cdots\right)^2}.$$

We therefore need to prove that

$$\frac{1}{(t-1)!}\left(\frac{1}{(t+1)!} + \frac{2x}{(t+2)!} + \frac{3x^2}{(t+3)!} + \cdots\right) < \left(\frac{1}{t!} + \frac{x}{(t+1)!} + \frac{x^2}{(t+2)!} + \cdots\right)^2.$$
$$(3.21)$$

We compare the coefficients on both sides one by one. Note that

$$\frac{1}{(t-1)!(t+1)!} < \frac{1}{t!^2}$$

holds as it is equivalent to $\binom{2t}{t-1} < \binom{2t}{t}$. Moreover,

$$\frac{2}{(t-1)!(t+2)!} < \frac{2}{t!(t+1)!} \Leftrightarrow t < t+2.$$

Next, the coefficient of x^s for $s \geqslant 2$ on the right-hand side is

$$\begin{cases} 2 \sum\limits_{i=0}^{\lfloor(s-1)/2\rfloor} \dfrac{1}{(t+i)!(t+s-i)!} + \dfrac{1}{(t+\lceil(s-1)/2\rceil)!^2} & \text{if } s \text{ is even,} \\ 2 \sum\limits_{i=0}^{\lfloor(s-1)/2\rfloor} \dfrac{1}{(t+i)!(t+s-i)!} & \text{if } s \text{ is odd.} \end{cases}$$

Note that in any case we have less than $s+1$ summands. So it suffices to show that each one of them is larger than $1/((t-1)!(t+s+1)!)$. This is indeed sufficient as the coefficient of x^s on the left-hand side is equal to $(s+1)/((t-1)!(t+s+1)!)$. But this is the case, as for any $0 \leqslant i \leqslant s$ we have

$$\frac{1}{(t-1)!(t+s+1)!} < \frac{1}{(t+i)!(t+s-i)!} \Leftrightarrow \prod_{j=0}^{i}(t+j) < \prod_{j=0}^{i}(t+s-j+1).$$

This now concludes the proof of the claim. $\square$

We immediately obtain the following.

Corollary 3.15. *Let $k \geqslant 3, \ell \geqslant 2$ and ξ^* satisfies (2.4). Then*

$$\xi^* \frac{Q(\xi^*,\ell)}{Q(\xi^*,\ell+1)}$$

is increasing with respect to ξ^*.

Claim 3.16. *For any $k \geqslant 3$ and $\ell \geqslant 2$ we have $\xi^* < k\ell$ and*

$$\xi^* > k\ell - \frac{e^{-k\ell}(k\ell) \cdot (k\ell - 0.36)^\ell}{\ell!} \left(1 - \exp\left(\frac{-(k\ell - \ell + 0.64)^2}{2k\ell - 0.72}\right)\right)^{-1}.$$

Proof. We have

$$k \cdot \ell = \xi^* \cdot \frac{Q(\xi^*,\ell)}{Q(\xi^*,\ell+1)}.$$

As

$$\frac{Q(\xi^*,\ell)}{Q(\xi^*,\ell+1)} > 1$$

for all ξ^* and ℓ, we deduce that $\xi^* < k\ell$. By Claim 3.13 we know that for all $k \geqslant 3$ and $\ell \geqslant 2$, $\xi^* > k\ell - 0.36$. In order to improve upon the above bound, note first that

$$\xi^* = k\ell \cdot \frac{Q(\xi^*, \ell+1)}{Q(\xi^*, \ell)} = k\ell - k\ell \frac{\mathbb{P}(\text{Po}(\xi^*) = \ell)}{Q(\xi^*, \ell)} \geqslant k\ell - k\ell \frac{\mathbb{P}(\text{Po}(k\ell - 0.36) = \ell)}{Q(k\ell - 0.36, \ell)}. \quad (3.22)$$

Let X be a Poisson random variable with parameter $\mu = k\ell - 0.36$. Thus, $Q(k\ell - 0.36, \ell) = 1 - \mathbb{P}(X \leqslant \ell - 1)$. We define $\delta = 1 - (\ell - 1)/\mu$. Now, for any $t < 0$ we have

$$\mathbb{P}(X \leqslant \ell - 1) = \mathbb{P}(X \leqslant (1-\delta)\mu) = \mathbb{P}(e^{tX} \geqslant e^{t(1-\delta)\mu})$$
$$\leqslant \frac{\mathbb{E}(e^{tX})}{e^{t(1-\delta)\mu}} = \frac{\exp(-\mu + \mu \cdot e^t)}{\exp(t(1-\delta)\mu)}.$$

Setting $t = \log(\ell - 1) - \log(\mu)$, we have

$$\mathbb{P}(X \leqslant \ell - 1) < \left(\frac{e^{-\delta}}{(1-\delta)^{(1-\delta)}} \right)^{\mu} < \exp\left(\frac{-(\mu - \ell + 1)^2}{2\mu} \right). \quad (3.23)$$

The combination of (3.22) and (3.23) leads us to the stated lower bound. $\qquad \square$

In what follows we use the definition

$$t(k, \ell) := \left(1 - \frac{0.36}{k\ell} \right)^{\ell} \left(1 - \exp\left(\frac{-(k\ell - \ell + 0.64)^2}{2k\ell - 0.72} \right) \right)^{-1}.$$

We are now ready to deduce the inequalities in (3.16), starting with a bound on $h(\beta)$.

Claim 3.17. *For any $k \geqslant 3$ and $\ell \geqslant 2$ there exists $C_1 > 0$ such that, for any $0 < \varepsilon < 1$ and any $0.6 \leqslant \beta \leqslant 1 - \varepsilon$, we have $h(\beta) \leqslant -C_1\varepsilon$.*

Proof. By Claim 3.16, we have

$$k\ell - t(k, \ell) \cdot \frac{e^{-k\ell}(k\ell)^{\ell+1}}{\ell!} < \xi^* < k\ell.$$

Using these bounds for ξ^* we obtain

$$e^{h(\beta)} < \beta^{-(\ell+1)\beta} \left(\frac{(2^k - 1)(\ell+1)}{k\ell - (\ell+1)(1-\beta)} \right)^{\ell(1-\beta)}$$
$$\times \left(\frac{t(k,\ell) \cdot \frac{e^{-k\ell}(k\ell)^{\ell+1}}{\ell!}}{k\ell - (\ell + k\ell + 1)(1-\beta)} \right)^{1-\beta} \left(1 - \frac{(\ell+1)(1-\beta)}{k\ell} \right)^{k\ell}$$
$$= \left(\frac{2^k - 1}{e^k \cdot \beta^{\beta/(1-\beta)}} \right)^{\ell(1-\beta)} \left(1 - \frac{(\ell+1)(1-\beta)}{k\ell} \right)^{-\ell(1-\beta)}$$
$$\times \left(1 - \frac{(\ell + k\ell + 1)(1-\beta)}{k\ell} \right)^{-(1-\beta)}$$
$$\times \left(\frac{(\ell+1)^{\ell} \cdot t(k,\ell)}{\beta^{\beta/(1-\beta)}\ell!} \right)^{1-\beta} \left(1 - \frac{(\ell+1)(1-\beta)}{k\ell} \right)^{k\ell}. \quad (3.24)$$

Using the inequality

$$(1-x)^{-1} \leqslant \exp\left(x + \frac{x^2}{1.4}\right)$$

for $x \leqslant 0.4$ we can deduce

$$\beta^{-\beta/(1-\beta)} = (1-(1-\beta))^{-\beta/(1-\beta)} \leqslant e^{\beta+(1-\beta)\beta/1.4}. \tag{3.25}$$

Also,

$$\left(1 - \frac{(\ell+1)(1-\beta)}{k\ell}\right)^{-1} \leqslant \exp\left\{\frac{(\ell+1)(1-\beta)}{k\ell} + \frac{(\ell+1)^2(1-\beta)^2}{1.4(k\ell)^2}\right\},$$

$$\left(1 - \frac{(1+\ell+k\ell)(1-\beta)}{k\ell}\right)^{-1/\ell} \leqslant \exp\left\{\frac{(1-\beta)(1+\ell+k\ell)}{k\ell^2} + \frac{(1-\beta)^2(1+\ell+k\ell)^2}{k^2\ell^3}\right\},$$

$$\left(1 - \frac{(\ell+1)(1-\beta)}{k\ell}\right)^{k\ell} < \exp\left(-(\ell+1)(1-\beta) - \frac{(\ell+1)^2(1-\beta)^2}{2k\ell}\right).$$

By Stirling's formula and (3.25) we have

$$\frac{(\ell+1)^\ell}{\ell! \cdot \beta^{\beta/(1-\beta)}} < \frac{(1+1/\ell)^\ell \exp(\ell)}{\sqrt{2\pi\ell}} \exp\left(\beta + \frac{\beta(1-\beta)}{1.4}\right).$$

Now combining the last two terms in (3.24), we obtain

$$\left(\frac{(\ell+1)^\ell \cdot t(k,\ell)}{\beta^{\beta/(1-\beta)}\ell!}\right)^{1-\beta}\left(1 - \frac{(\ell+1)(1-\beta)}{k\ell}\right)^{k\ell}$$

$$< \left(\frac{(1+1/\ell)^\ell \cdot t(k,\ell)}{\sqrt{2\pi\ell}}\right)^{1-\beta}$$

$$\times \exp\left(\beta(1-\beta) + \frac{\beta(1-\beta)^2}{1.4} - (1-\beta) - \frac{(\ell+1)^2(1-\beta)^2}{2k\ell}\right)$$

$$= \left(\frac{(1+1/\ell)^\ell \cdot t(k,\ell)}{\sqrt{2\pi\ell}}\right)^{1-\beta}$$

$$\times \exp\left(\beta(1-\beta) + \frac{\beta(1-\beta)^2}{1.4} - (1-\beta) - \left(1+\frac{1}{\ell}\right)\frac{(\ell+1)(1-\beta)^2}{2k}\right).$$

Also, recall that

$$t(k,\ell) = \left(1 - \frac{0.36}{k\ell}\right)^\ell \left(1 - \exp\left(\frac{-(k\ell-\ell+0.64)^2}{2k\ell-0.72}\right)\right)^{-1}.$$

Substituting these bounds in (3.24), we obtain

$$e^{h(\beta)} < \left(\left(\frac{2^k-1}{\exp(k-\Delta_{k,\ell,\beta})}\right)^\ell \cdot \frac{(1+1/\ell)^\ell \exp\left(\beta + \frac{\beta(1-\beta)}{1.4} - 1\right)}{\sqrt{2\pi\ell} \cdot \left(1 - \exp\left(\frac{-(k\ell-\ell+0.64)^2}{2k\ell-0.72}\right)\right)}\right)^{1-\beta}, \tag{3.26}$$

where

$$\Delta_{k,\ell,\beta} := \beta + \frac{(1-\beta)\beta}{1.4} + \frac{(\ell+1)(1-\beta)}{k\ell} + \frac{(\ell+1)^2(1-\beta)^2}{1.4(k\ell)^2} + \frac{(1-\beta)(1+k\ell+\ell)}{k\ell^2}$$
$$+ \frac{(1-\beta)^2(1+k\ell+\ell)^2}{k^2\ell^3} - \left(1+\frac{1}{\ell}\right)\frac{(\ell+1)(1-\beta)}{2k\ell}$$
$$= \beta + \frac{(1-\beta)\beta}{1.4} + \frac{(\ell+1)(1-\beta)}{2k\ell} + \frac{(\ell+1)^2(1-\beta)^2}{1.4(k\ell)^2} + \frac{(1-\beta)(1+k\ell+\ell)}{k\ell^2}$$
$$+ \frac{(1-\beta)^2(1+k\ell+\ell)^2}{k^2\ell^3} - \frac{1}{\ell}\frac{(\ell+1)(1-\beta)}{2k\ell}$$
$$= \beta + \frac{(1-\beta)\beta}{1.4} + \frac{1-\beta}{2k}\left(1+\frac{1}{\ell}\right) + \frac{(1-\beta)^2}{1.4\,k^2}\left(1+\frac{1}{\ell}\right)^2 + \frac{1-\beta}{\ell}\left(\frac{1}{2k\ell}+1+\frac{1}{2k}\right)$$
$$+ \frac{(1-\beta)^2}{\ell}\left(\frac{1}{k\ell}+1+\frac{1}{k}\right)^2.$$

We note that each term in $\Delta_{k,\ell,\beta}$ is decreasing in k and ℓ. The partial derivative of $\Delta_{k,\ell,\beta}$ with respect to β is given by

$$\Delta'_{k,\ell,\beta} := \frac{\partial \Delta_{k,\ell,\beta}}{\partial \beta} = \frac{12}{7} - \frac{10}{7}\beta - \frac{1}{2k}\left(1+\frac{1}{\ell}\right) - \frac{1-\beta}{0.7k^2}\left(1+\frac{1}{\ell}\right)^2 - \frac{1}{\ell}\left(\frac{1}{2k\ell}+1+\frac{1}{2k}\right)$$
$$- \frac{2(1-\beta)}{\ell}\left(\frac{1}{k\ell}+1+\frac{1}{k}\right)^2.$$

Observe that each term (with the sign) in

$$\frac{\partial \Delta_{k,\ell,\beta}}{\partial \beta}$$

is increasing with k and ℓ. Let

$$p(k,\ell,\beta) := \left(\frac{2^k-1}{\exp(k-\Delta_{k,\ell,\beta})}\right) \quad \text{and} \quad g(k,\ell) := \frac{\exp(1)}{\sqrt{2\pi\ell}\cdot\left(1-\exp\left(\frac{-(k\ell-\ell+0.64)^2}{2k\ell-0.72}\right)\right)}.$$

One can check that

$$e^{h(\beta)} < ((p(k,\ell,\beta))^\ell g(k,\ell))^{1-\beta}.$$

We start with the case $k \geqslant 4$. First note that $\Delta'_{4,2,\beta} = -519/448 + (297/448)\beta$, which is negative for all $\beta < 1$. Also, as $(2^k-1)\cdot\exp(-k)$ is decreasing in k and $\Delta_{k,\ell,\beta}$ is decreasing in k and ℓ, we infer that for $k \geqslant 4, \ell \geqslant 2$, thus the maximum value of $p(k,\ell,\beta)$ is $p(4,2,0.6)$. Direct substitution yields $p(4,2,0.6) = 15e^{-3237/1120} < 0.84$. We note that the partial derivative of

$$-\frac{(k\ell-\ell+0.64)^2}{2k\ell-0.72}$$

with respect to ℓ is

$$
\begin{aligned}
&-\frac{2(k\ell - \ell + 0.64)}{2k\ell - 0.72}\left(k - 1 - \frac{(k\ell - \ell + 0.64)k}{2k\ell - 0.72}\right) \\
&< -\frac{2(k\ell - \ell + 0.64)}{2k\ell - 0.72}\left(k - 1 - \frac{(k\ell - \ell + 0.64)k}{2k\ell}\right) \\
&= -\frac{2(k\ell - \ell + 0.64)}{2k\ell - 0.72}\left(k - 1 - \frac{\ell(k - 1) + 0.64}{2\ell}\right) \\
&< -\frac{2(k\ell - \ell + 0.64)}{2k\ell - 0.72}\left(\frac{k - 1}{2} - \frac{0.64}{2\ell}\right) < 0
\end{aligned}
\tag{3.27}
$$

and with respect to k it is

$$
\begin{aligned}
&-\frac{2\ell(k\ell - \ell + 0.64)}{2k\ell - 0.72}\left(1 - \frac{k\ell - \ell + 0.64}{2k\ell - 0.72}\right) \\
&= -\frac{2\ell(k\ell - \ell + 0.64)}{2k\ell - 0.72}\left(1 - \frac{k\ell - \ell + 0.64}{2k\ell}\right) \\
&\leqslant -\frac{2\ell(k\ell - \ell + 0.64)}{2k\ell - 0.72}\left(\frac{1}{2} + \frac{1}{2k} - \frac{0.64}{2k}\right) < 0.
\end{aligned}
\tag{3.28}
$$

We can now conclude that

$$
\sqrt{2\pi\ell}\left(1 - e^{-(k\ell - \ell + 0.64)^2/2k\ell - 0.72}\right)
$$

is increasing in k and ℓ and therefore $g(k, \ell)$ is decreasing in k and ℓ. Direct substitution yields that $g(3, 2) < 0.91$, which completes the proof for $k \geqslant 4, \ell \geqslant 2$.

For the case $k = 3$, first note that

$$
\Delta'_{3,5,\beta} = 229/875 - (52/125)\beta,
$$

which implies that $\Delta_{3,5,\beta}$ is maximized at $\beta = \beta_{\max} = 229/364$. Therefore, for $\ell \geqslant 5$, $p(3, \ell, \beta)$ is maximized at $p(3, 5, \beta_{\max})$. Numerical computations show that

$$
p(3, 5, \beta_{\max}) = 7e^{-50231/25480} < 0.98.
$$

For the cases $\ell \leqslant 4$, first note that

$$
\Delta'_{3,4,\beta} = -1/21 - 17\beta/96 \overset{\beta > 0}{<} 0.
$$

Now let

$$
m(k, \ell, \beta) := p(k, \ell, \beta)^\ell g(k, \ell).
$$

Recall that $\Delta'_{k,\ell,\beta}$ is increasing in k and ℓ. Also, $\Delta_{3,4,\beta}$ is decreasing in β. We can therefore conclude that for all $\beta \geqslant 0.6$ and $\ell \leqslant 4$, $m(3, \ell, \beta) \leqslant m(3, \ell, 0.6)$. One can check that $m(3, 3, 0.6) < 0.93$ and $m(3, 4, 0.6) < 0.62$. The case $\ell = 2$ is more tedious. We substitute

$k = 3, \ell = 2$ in (3.26). Then

$$e^{(h(\beta))/(1-\beta)} < \left(\frac{7}{\exp\left(3 - \Delta_{3,2,\beta}\right)} \right)^2 \cdot \frac{(1 + 1/2)^2 \exp\left(\beta + \frac{\beta(1-\beta)}{1.4} - 1\right)}{\sqrt{4\pi} \cdot \left(1 - \exp\left(\frac{-(4.64)^2}{11.28}\right)\right)}$$

$$< \left(\frac{7}{\exp\left(3 - \Delta_{3,2,\beta} - \frac{\beta}{2} - \frac{\beta(1-\beta)}{2.8}\right)} \right)^2 \cdot \frac{2.25 \cdot \exp(-1)}{\sqrt{4\pi} \cdot \left(1 - \exp\left(\frac{-(4.64)^2}{11.28}\right)\right)} \qquad (3.29)$$

Now we check that the partial derivative of

$$\Delta_{3,2,\beta} + \frac{\beta}{2} + \frac{\beta(1-\beta)}{2.8}$$

with respect to β is

$$\Delta'_{3,2,\beta} + \frac{1}{2} - \frac{\beta}{2.8} = -\frac{99}{56} + \frac{33}{28}\beta + \frac{1}{2} - \frac{\beta}{2.8} = -\frac{51}{56} + \frac{13}{28}\beta$$

$$\overset{\beta \leqslant 1}{\leqslant} -\frac{25}{28},$$

which implies that the right-hand side is decreasing with respect to β for $\beta \leqslant 1$. We complete the proof by calculating the above expression for $\beta = 0.6$, which gives $e^{h(\beta)} < (0.91)^{1-\beta}$. $\qquad \square$

The following claim, which bounds $f(\beta, \beta)$, also imposes an upper bound on ε.

Claim 3.18. *For any $k \geqslant 3$ and $\ell \geqslant 2$ there exists $C_2 > 0$ such that the following holds. For any $\varepsilon < 1/e$, if $0.6 < \beta \leqslant 1 - \varepsilon$ we have*

$$f(\beta, \beta) < -C_2 \varepsilon.$$

Proof. By Lemma 2.8, it follows that substituting $q = \beta$ in

$$\frac{k\ell(1-q)}{1-\beta}$$

we have

$$I_{\xi^*}\left(\frac{k\ell(1-\beta)}{1-\beta} \right) = 0.$$

So,

$$f(\beta, \beta) = -(k\ell - \ell - 1)H(\beta) + \ell(1-\beta)\log(2^k - 1).$$

Note that for any $k \geqslant 3$ and $\ell \geqslant 2$ this function is convex with respect to β, as $-H(\beta)$ is convex and the linear term that is added preserves its convexity. Note that $-H(1 - \varepsilon) < -\varepsilon \log(1/\varepsilon)$, whereby it follows that there exists a constant $C_2 = C_2(k, \ell) > 0$ such that for any $0 < \varepsilon < 1/e$ we have

$$f(1 - \varepsilon, 1 - \varepsilon) < -C_2 \varepsilon \log(1/\varepsilon) < -C_2 \varepsilon.$$

Since $H(0.6) > 0.6$, we have

$$f(0.6, 0.6) < -0.6(k\ell - \ell - 1) + 0.4\ell \log(2^k - 1).$$

The derivative of this function with respect to k is

$$-0.6\ell + \ell \cdot 0.4 \frac{2^k \log 2}{2^k - 1}.$$

A simple calculation shows that the second summand is less than 0.32ℓ for all $k \geqslant 3$. The derivative with respect to ℓ is $-0.6k + 0.6 + 0.4 \log(2^k - 1)$ which is again a decreasing function in k and less than -0.42 at $k = 3$. So, we may set $k = 3$ and $\ell = 2$, thus obtaining $f(0.6, 0.6) < -1.8 + 0.8 \log 7 < -0.24$. The above analysis along with the convexity of $f(\beta, \beta)$ implies the claimed statement. $\qquad \square$

Claim 3.19. *For all $k \geqslant 3$ and $\ell \geqslant 2$, there is a $C_3 > 0$ such that, for all ε and for all $\beta \leqslant 1 - \varepsilon$,*

$$f(\beta, 1 - (\ell + 1)(1 - \beta)/k\ell) \leqslant -C_3 \varepsilon.$$

Proof. Substituting $1 - (\ell + 1)(1 - \beta)/k\ell$ for q into the formula of f, we obtain

$$f\left(\beta, 1 - \frac{(\ell + 1)(1 - \beta)}{k\ell}\right) = (\ell + 1)H(\beta) + \ell(1 - \beta)\log(2^k - 1)$$
$$- k\ell H\left(\frac{k\ell - (\ell + 1)(1 - \beta)}{k\ell}\right) - (1 - \beta)I_{\xi^*}(\ell + 1).$$

Note that for $\beta = 1$ the expression is equal to 0. To deduce the bound we are aiming for, we will show that in fact $f(\beta, 1 - (\ell + 1)(1 - \beta)/k\ell)$ is an increasing function with respect to β. That is, we will show that its first derivative with respect to β is positive for any $\beta \leqslant 1$. Finally, Taylor's theorem based around $\beta = 1$ implies the claim.

We get

$$\frac{\partial f\left(\beta, 1 - \frac{(\ell+1)(1-\beta)}{k\ell}\right)}{\partial \beta} = (\ell + 1)\log\left(\frac{1 - \beta}{\beta}\right) - \ell \log(2^k - 1)$$
$$- (\ell + 1)\log\left(\frac{(\ell + 1)(1 - \beta)}{k\ell - (\ell + 1)(1 - \beta)}\right) + I_{\xi^*}(\ell + 1).$$

Substituting for $I_{\xi^*}(\ell + 1)$ the value given in Lemma 2.8 and since

$$e^{\xi^*} Q(\xi^*, \ell + 1) = \xi^{*\ell+1}/\ell!(k\ell - \xi^*),$$

we obtain for $\beta < 1$

$$\frac{\partial f\left(\beta, 1 - \frac{(\ell+1)(1-\beta)}{k\ell}\right)}{\partial \beta} = \log\left(\left(\frac{k\ell - (\ell + 1)(1 - \beta)}{(\ell + 1)\beta}\right)^{\ell+1} (2^k - 1)^{-\ell} \cdot \frac{\ell + 1}{k\ell - \xi^*}\right).$$

We will show that the fraction inside the logarithm is greater than 1. Note first that

$$\frac{k\ell - (\ell + 1)(1 - \beta)}{(\ell + 1)\beta} = \frac{1}{\beta}\left(\frac{k\ell - (\ell + 1)}{\ell + 1}\right) + 1 = \frac{1}{\beta}\left(\frac{(k-1)\ell - 1}{\ell + 1}\right) + 1$$

is decreasing with respect to β, so we obtain a lower bound by setting $\beta = 1$. Substituting $\beta = 1$ we obtain

$$\frac{\partial f\left(\beta, 1 - \frac{(\ell+1)(1-\beta)}{k\ell}\right)}{\partial \beta} > \log\left(\left(\frac{k\ell}{\ell+1}\right)^{\ell+1}(2^k-1)^{-\ell} \cdot \frac{\ell+1}{k\ell-\xi^*}\right).$$

By Claim 3.16, for all $k \geqslant 3$ and $\ell \geqslant 2$ we have

$$k\ell - \xi^* \leqslant \frac{e^{-k\ell}(k\ell)^{\ell+1}}{\ell!(1 - e^{-(k\ell-\ell+0.64)^2/2k\ell-0.72})},$$

which yields

$$
\begin{aligned}
\left(\frac{k\ell}{\ell+1}\right)^{\ell+1}(2^k-1)^{-\ell} \cdot \frac{(\ell+1)}{k\ell-\xi} \quad &\geqslant \quad \frac{e^{k\ell}\ell!(1 - e^{-(k\ell-\ell+0.64)^2/2k\ell-0.72})}{(2^k-1)^\ell(\ell+1)^\ell} \\
&= \quad \frac{e^{k\ell}\ell!(1 - e^{-(k\ell-\ell+0.64)^2/2k\ell-0.72})}{\ell^\ell(2^k-1)^\ell(1+1/\ell)^\ell} \quad\quad (3.30) \\
&\overset{1+x\leqslant e^x}{>} \quad \frac{\ell!}{e \cdot \ell^\ell} \cdot \frac{e^{k\ell}(1 - e^{-(k\ell-\ell+0.64)^2/2k\ell-0.72})}{(2^k-1)^\ell}.
\end{aligned}
$$

Using the bounds $\ell! \geqslant \sqrt{2\pi\ell}(\ell/e)^\ell$ and $1 + x \leqslant e^x$, we can further bound the right-hand side of (3.30) as follows:

$$\frac{\ell!}{e \cdot \ell^\ell} \cdot \frac{e^{k\ell}(1 - e^{-(k\ell-\ell+0.64)^2/2k\ell-0.72})}{(2^k-1)^\ell} \geqslant \frac{\sqrt{2\pi\ell}}{e^{\ell+1}} \cdot \frac{e^{k\ell}(1 - e^{-(k\ell-\ell+0.64)^2/2k\ell-0.72})}{(2^k-1)^\ell}. \quad (3.31)$$

We note that the partial derivative of

$$-\frac{(k\ell-\ell+0.64)^2}{2k\ell-0.72}$$

with respect to k and ℓ is less than 0 (see (3.27) and (3.28)). We can therefore conclude that

$$\sqrt{2\pi\ell}(1 - e^{-(k\ell-\ell+0.64)^2/2k\ell-0.72})$$

is increasing in k and ℓ. Also the first derivative of the function $e^k/(2^k-1)$ with respect to k is $e^k(2^k(1-\log 2)-1)/(2^k-1)^2$, which is positive for any $k \geqslant 3$. Moreover, the first derivative of the function $e^{k\ell-\ell-1}/(2^k-1)^\ell$ with respect to ℓ is $e^{k\ell-\ell-1}(2^k-1)^{-\ell}(k-\log(2^k-1)-1)$, which is positive for any $k \geqslant 3$ and $\ell \geqslant 2$. So we infer that the right-hand side of the above inequality is increasing in both k and ℓ. Plugging the values $(k,\ell) = (3,2)$ into (3.31), we obtain that the right-hand side is greater than 1.2. The above arguments establish the fact that the derivative of $f(\beta, 1 - (\ell+1)(1-\beta)/k\ell)$ with respect to β is positive, for all $k \geqslant 3$ and $\ell \geqslant 2$. □

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