Sender-Receiver Games with Cooperation

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Abstract

We consider generalized sender-receiver games in which the sender also has a decision to make, but this decision does not directly affect the receiver. We introduce specific perfect Bayesian equilibria, in which the players agree on a joint decision after that a message has been sent ("talk and cooperate equilibrium," TCE). We establish that a TCE exists provided that the receiver has a "uniform punishment decision" (UPD) against the sender.

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1 Introduction

Starting with Crawford and Sobel (1982), strategic information transmission has been investigated in various directions. In the standard model, an informed agent sends a costless message to a decision maker. The utility of both individuals depends on the information of the sender and the decision of the receiver.

In this paper, we consider generalized sender-receiver games in which there is a single informed player (as usual) but the sender and the receiver have simultaneous decisions to make. We minimally depart from standard sender-receiver games: we still assume that the sender's action does not directly affect the receiver's utility. This is a restrictive assumption, but it is likely to hold in many contexts where the informed player is himself a decision maker. Suppose for instance that the receiver is an investor while the sender is a financial expert who can make investments on his own, as an insider (see, e.g., Benabou and Laroque (1992) and Morgan and Stocken (2003)). The financial expert's investments convey information but are typically negligible with respect to the investor's profit.

In our generalized sender-receiver game, the informed player first sends a message to the uninformed one. Then the receiver proposes a joint decision to the sender and finally, the latter can reject the proposal. In case of rejection, the players make their decisions independently of each other. If the informed player accepts to cooperate, he only reveals information through his message to the uninformed one.

We ask whether our generalized sender-receiver game has a perfect Bayesian equilibrium (PBE) in which information transmission is followed by agreement of the sender. To formulate the question in a precise way, we assume that the informed player has finitely many types and that both players have finitely many actions. We define a TCE ("talk and cooperate equilibrium") as a PBE of the sender-receiver game in which the informed player always accepts the receiver's proposal on equilibrium path, namely, talks[1] and then cooperates. A TCE may very well be nonrevealing, but then, no information ever transpires from the joint decision that is actually made.

We first propose a characterization of TCE in terms of three properties: incentive compatibility (IC), individual rationality for the informed

[1] The sender's message is *cheap* talk in the sense that it is does not have a direct effect on the players' utility.

player (IR) and optimality for the uninformed one (Opt). (IC) says that, for every type, the sender is indifferent between any two messages that he sends with positive probability given his type. This property is familiar in sender-receiver games in which the sender can randomize over finitely many messages (see, e.g., Aumann and Hart (2003)). The sender's individual rationality condition (IR) is formulated at the posterior stage, namely, after the sender has sent his message. (Opt) says that the receiver proposes a joint decision that maximizes his own posterior expected utility given the sender's message.

A consequence of (IC) is that the interim expected utility of the sender, given his type, coincides with his posterior expected utility, given his type and any message that is possibly sent by this type. Hence the sender's individual rationality condition (IR) can as well be formulated at the interim stage. With this reformulation of our characterization, our model is related to a particular case of the principal-agent problems considered in Bester and Strausz (2001). When no decision of the principal is contractible, Bester and Strausz (2001)'s model reduces to a standard sender-receiver game with *exogenous* interim participation constraints for the sender (see (6), p. 1082). In our model, the sender's interim reservation utility emerges from the decisions that he could make without cooperating with the receiver. In other words, our (IR) condition, when it is formulated at the interim stage, can be interpreted as an *endogenous* participation constraint for each type of the sender.

Once a characterization of TCE is available, the next question is whether the existence of a TCE can be guaranteed under meaningful assumptions on the players' utility functions. A key to existence turns out to be that the receiver has a "uniform punishment decision" (UPD). Such a decision enables the receiver to credibly punish the sender as if the latter's type were common knowledge.

We show on an example that existence of a TCE may fail if no UPD is available. We nonetheless prove that, if the receiver has a UPD, then a TCE exists, without any further assumption. For this, we rely on results of Simon, Spież and Toruńczyk (1995) and Renault (2000). We provide two examples in which a TCE cannot be constructed in a straightforward way, i.e., in which there is no nonrevealing TCE and no completely revealing TCE. These examples show that UPD is satisfied if the receiver's optimal choice, when he considers all sender's types equally likely, consists of a status quo decision that keeps the sender at his individually rational level, whatever his type.

The paper is organized as follows: the basic Bayesian game, without information transmission, is described in Section 2. The solution concept, "talk and cooperate" equilibrium (TCE) is defined in Section 3.1 and characterized in Section 3.2. Our existence result is stated in Section 3.3. The next section is devoted to examples. Section 5 investigates the extension of our results to the case where the receiver cares about the sender's decision. This more general case allows us to make further connections with the literature.

2 Game without information transmission

2.1 Basic Bayesian game

We start with a two-person Bayesian game, which describes the players' information, decisions and utility functions. This game will be extended later in order to account for information transmission and cooperation. It stands for the default game to be played when no agreement is reached.

Let K be a finite set. At a virtual initial stage of the game, an element k of K is chosen according to a prior probability distribution $p \in \Delta(K)$;[2] only player 1 is informed of k, which will be referred to as player 1's type. Unless specified otherwise, we assume that $p^\ell > 0$ for every $\ell \in K$. Player 1 and player 2 choose simultaneously an action in respective finite sets A_1 and A_2. If the pair of actions $a \in A = A_1 \times A_2$ is chosen, player 1's (von Neumann-Morgenstern) utility is $U^k(a_1, a_2)$ and player 2's (von Neumann-Morgenstern) utility is $V^k(a_2)$. We denote this basic Bayesian game as $B(p)$.[3]

Player 1 is an expert with private information and will become the sender. Player 2 is a decision maker who cares for player 1's information. As in standard sender-receiver games, both players are affected by player 2's decisions. The novelty is that player 1 also has a decision to make. However, the game retains a fundamental feature of standard sender-receiver games, namely, player 2 is not directly affected by the expert's action.

As a benchmark, we can consider the (Bayesian) Nash equilibria of the game $B(p)$ described above, in which there is no communication. In such an equilibrium, player 2 chooses an action that maximizes his expected utility

[2] All along the paper, given a finite set E, we denote as $\Delta(E)$ the set of probability distributions over E.

[3] It is convenient to keep track of the parameter p, which can be interpreted as player 2's prior belief, because p will typically be updated.

at the prior p and player 1 best replies as a function of his type. A difference with standard sender-receiver games is that after $B(p)$ has been played, some of player 1's information may be revealed through his decisions.

We denote as "decisions" possibly randomized actions in the basic Bayesian game, i.e., elements of $\Delta(A_i)$, $i = 1, 2$, and $\Delta(A)$. We still write U^k and V^k for the linear extensions of the utility functions over randomized actions. "Strategies" will appear in Section 3, when the game $B(p)$ will be extended by allowing information transmission. In the next subsections, we define two further basic notions in $B(p)$.

2.2 Optimal decisions for the uninformed player

Let g be the mapping that associates player 2's maximal expected utility to every $q \in \Delta(K)$:

$$g(q) = \max_{a_2 \in A_2} \sum_{k \in K} q^k V^k(a_2).$$

The mapping g is convex as a maximum of linear functions. Let

$$\begin{aligned} \mathcal{R}(q) &= \left\{ \tau \in \Delta(A_2) : \sum_{k \in K} q^k V^k(\tau) = g(q) \right\} \qquad (1) \\ &= Conv \left\{ a_2 \in A_2 : \sum_{k \in K} q^k V^k(a_2) = g(q) \right\} \end{aligned}$$

The set $\mathcal{R}(q)$ is convex and compact.

We will also consider joint decisions $\alpha \in \Delta(A)$ that are optimal for player 2 given his belief q, namely,

$$\begin{aligned} F(q) &= \left\{ \alpha \in \Delta(A) : \sum_{k \in K} q^k V^k(\alpha) = g(q) \right\} \qquad (2) \\ &= \left\{ \alpha \in \Delta(A) : marg_{A_2}(\alpha) \in \mathcal{R}(q) \right\} \end{aligned}$$

where $marg_{A_2}(\alpha)$ denotes the marginal distribution of α over A_2. Finally, let

$$\mathcal{R} = \cup_{q \in \Delta(K)} \mathcal{R}(q). \qquad (3)$$

The set $\mathcal{R}$ is the set of "rationalizable" decisions of player 2, i.e., the decisions that are optimal for some belief of player 2. $\mathcal{R}$ is compact, but not necessarily convex. We assume in the sequel that $A_2 \subset \mathcal{R}$, namely, that actions that cannot be rationalized have been eliminated from A_2; even then, $\mathcal{R}$ can be strictly included in $\Delta(A_2)$ (see Section 4).

2.3 Individual rationality for the informed player

We consider two possible notions of individually rational utility vectors for the informed player. The relevance of these definitions will become fully clear in the next section. Recall that $\mathcal{R}$ is the set of rationalizable decisions of player 2, defined by (3).

Definition 1 *An interim utility vector $u = (u^k)_{k \in K}$ is individually rational for player 1 iff player 2 has a rationalizable decision that prevents player 1 from getting more than u^k when his type is k, namely,*

$$\exists \tau \in \mathcal{R}\ \forall k \in K\ \forall a_1 \in A_1 : U^k(a_1, \tau) \leq u^k. \tag{4}$$

Given player 1's type k, let us define

$$\underline{m}^k = \min_{\tau \in \mathcal{R}} \max_{a_1 \in A_1} U^k(a_1, \tau). \tag{5}$$

Definition 2 *An interim utility vector $u = (u^k)_{k \in K}$ is type by type individually rational for player 1 iff $u^k \geq \underline{m}^k$ for every $k \in K$.*

It is clear that individual rationality, as defined by (4), implies type by type individual rationality. But in general the reverse is not true (see Example 1 in Section 4). This motivates the following definition

Definition 3 *Let $\tau \in \Delta(A_2)$ be a decision of player 2; τ is a uniform punishment decision (UPD) iff $\tau \in \mathcal{R}$ and*

$$\forall k \in K\ \forall a_1 \in A_1 : U^k(a_1, \tau) \leq \underline{m}^k$$

With these definitions the following result is immediate:

Lemma 4 *Let $u = (u^k)_{k \in K}$ be an interim utility vector for the informed player. If the uninformed player has a UPD, u is individually rational iff u is type by type individually rational.*

Observe that all the notions defined in this subsection are independent of the prior belief p as far as $p^\ell > 0$ for every $\ell \in K$. Note also that, in Definition 1, player 2's decision τ is already "uniform," in the sense that τ prevents player 1 from getting more than u^k simultaneously for all types k and actions a_1. However, in this definition u^k can be fairly high. The additional property of a UPD τ is that for every k, τ maintains player at his *minmax* level $\underline{m}^k$. The examples of Section 4 will illustrate that UPD is easily fulfilled if the optimal action of player 2 when he views all types of player 1 as equally likely, is a status quo that cannot give player 1 more than his reservation utility $\underline{m}^k$, whatever his type k and action.

3 Game with information transmission

3.1 "Talk and cooperate" equilibrium (TCE)

We now extend the basic game $B(p)$ into a sender-receiver game by allowing the informed player to voluntarily send a costless message to the uninformed one. In the extended game, cooperation of the informed player is explicitly required. We first consider a scenario in which the informed player can conclude an agreement with the uninformed one after having sent a message. Let S be some given, finite set of messages, such that $\mid S \mid \geq \mid K \mid$.[4]

Sender-receiver game $G_{post}(p)$, with cooperation agreement at the posterior stage:

- A type k is chosen in K according to p, only player 1 is informed of k.
- Player 1 sends a message $s \in S$ to player 2.
- Player 2 proposes a joint decision $\alpha \in \Delta(A)$ to player 1.
- If player 1 accepts player 2's proposal, the joint decision α is enforced, player 1 gets $U^k(\alpha)$ and player 2 gets $V^k(\alpha)$.
- If player 1 rejects player 2's proposal, player 1 chooses an action $a_1 \in A_1$, player 2 simultaneously chooses an action $a_2 \in A_2$, player 1 gets $U^k(a_1, a_2)$ and player 2 gets $V^k(a_2)$.

[4]For a precise analysis of the number of messages, see Bester and Strausz (2001), Hart (1985) and Heumann (2015).

In the previous description, it is understood that player 1 can make use of a lottery in $\Delta(S)$ to select the message that he sends. If player 2 proposes a joint decision α that is randomized, it is understood that the players can rely on a third party to perform the lottery α on their behalf. This is meant to reflect the players' cooperation possibilities.

The equilibria of the basic Bayesian game $B(p)$ can be recovered as equilibria of $G_{post}(p)$ in which player 1 rejects any proposal of player 2 after having sent an uninformative message. We are rather interested in perfect Bayesian equilibria of $G_{post}(p)$ in which player 1 always accepts player 2's proposal, so that both players cooperatively use any information revealed by player 1. This motivates the following definition.

Definition 5 *A "talk and cooperate" equilibrium (TCE) for $B(p)$ is a perfect Bayesian equilibrium of $G_{post}(p)$ in which player 1 accepts player 2's proposal on equilibrium path.*

Let us formalize the previous definition further. A strategy for player 1 (the sender) in $G_{post}(p)$ is fully described by a signaling mapping $\mu : K \to \Delta(S)$ together with a mapping $\sigma : K \times S \times \Delta(A) \to \{y, n\} \times \Delta(A_1)$ indicating whether player 1 accepts player 2's proposal ("y") or not ("n") and in the latter case, which decision player 1 chooses. For every $k \in K$ and $s \in S$, we write $\mu(s \mid k)$ for $\mu(s)(k)$. Let S_μ be the support of the probability over S induced by p and μ. Recalling that $p^k > 0$ for every $k \in K$,

$$S_\mu = \{s \in S \mid \exists k \in K : \mu(s \mid k) > 0\}.$$

A strategy of player 2 (the receiver) is fully described by a proposal mapping $\chi : S \to \Delta(A)$ together with a mapping $\tau : S \to \Delta(A_2)$ indicating which decision player 2 chooses if player 1 rejects his proposal. A TCE consists of a pair of strategies $((\mu, \sigma), (\chi, \tau))$ forming a perfect Bayesian equilibrium (PBE) of $G_{post}(p)$ and such that σ prescribes "y" on every $(k, s, \chi(s))$, $k \in K$, $s \in S_\mu$.

3.2 Characterization of TCE

Our first result is a tractable characterization of TCE. To state it, we first need to make precise what incentive compatibility means in our framework. Given a signaling mapping $\mu : K \to \Delta(S)$ and a restricted proposal mapping $\chi_\mu : S_\mu \to \Delta(A)$ (for instance, χ_μ is the restriction to S_μ of a proposal mapping χ defined all over S), we say that μ is *incentive compatible* given χ_μ iff $\forall k \in K, \forall s, s' \in S_\mu$

$$\begin{aligned} \mu(s \mid k) > 0 \text{ and } \mu(s' \mid k) > 0 &\Rightarrow U^k(\chi_\mu(s)) = U^k(\chi_\mu(s')) \\ \mu(s \mid k) > 0 \text{ and } \mu(s' \mid k) = 0 &\Rightarrow U^k(\chi_\mu(s)) \geq U^k(\chi_\mu(s')) \end{aligned} \tag{6}$$

These conditions are appropriate because player 1 sends his message directly, by himself.[5] If he randomizes over messages s and $s' \in S$, he must be indifferent between s and s'. As a consequence, player 1's *interim utility vector* from μ and χ_μ, which we denote as $U(\chi_\mu) = (U^k(\chi_\mu))_{k \in K}$, satisfies

$$U^k(\chi_\mu) = U^k(\chi_\mu(s)) \ \forall k \in K, \forall s \in S_\mu : \mu(s \mid k) > 0. \tag{7}$$

Finally, for every $s \in S_\mu$, player 2 can compute a posterior probability distribution $p_s(\mu)$ over K by Bayes formula.

We can now state our characterization.

Proposition 6 *Let $\mu : K \to \Delta(S)$ be a signaling mapping for the sender and $\chi_\mu : S_\mu \to \Delta(A)$ be a restricted proposal mapping for the receiver. The mappings μ and χ_μ are part of a TCE for the Bayesian game $B(p)$ iff the following conditions hold:*

(IC) *μ is incentive compatible given χ_μ (i.e., (6)).*

(IR) *the sender's interim utility vector from μ and χ_μ (namely, (7)) is individually rational (i.e., (4)).*

(Opt) *for every message $s \in S_\mu$, $\chi_\mu(s) \in F(p_s(\mu))$ (with F defined by (2)), where $p_s(\mu)$ denotes the receiver's posterior belief computed from p and μ.*

Proof

Let $((\mu, \sigma), (\chi, \tau))$ be a TCE. Let us show that μ and the restriction χ_μ of χ to S_μ satisfy (IC), (IR) and (Opt).

If player 1 of type k sends message $s \in S_\mu$ and accepts $\chi_\mu(s)$, he gets $U^k(\chi_\mu(s))$. Let us set for every $s' \in S_\mu$, $z^k(s') = \max_{a_1 \in A_1} U^k(a_1, \tau(s'))$.

[5] Similar incentive compatibility conditions appear in Aumann and Hart (2003) and Hart (1985).

Player 1's equilibrium conditions imply that, for every type k: $\forall s \in S_\mu$ such that $\mu(s \mid k) > 0$, $\forall s' \in S_\mu$ (even such that $\mu(s' \mid k) = 0$): $U^k(\chi_\mu(s)) \geq U^k(\chi_\mu(s'))$ and $U^k(\chi_\mu(s)) \geq z^k(s')$. Indeed, player 1 can consider any message $s' \in S_\mu$, and, having sent s', can accept or reject $\chi_\mu(s')$; in the latter case, player 1 cannot do better than best replying to $\tau(s')$. Since $((\mu, \sigma), (\chi, \tau))$ forms a PBE by definition, if player 1 rejects player 2's proposal after having sent s', player 2 forms some belief over K and makes an optimal decision, namely, $\tau(s') \in \mathcal{R}$. Player 1's equilibrium conditions thus imply (IC) together with, for every $k \in K$ and $s \in S_\mu$ such that $\mu(s \mid k) > 0$,

$$U^k(\chi_\mu) = U^k(\chi_\mu(s)) \geq z^k(s') \quad \text{for every } s' \in S_\mu.$$

In particular, (IR) is satisfied.

For player 2, the Nash equilibrium conditions (i.e., on equilibrium path) prescribe that for every message $s \in S_\mu$, his proposal $\chi_\mu(s)$ be optimal given his posterior belief $p_s(\mu)$, namely, (Opt).

Conversely, assume that the mappings μ and χ_μ satisfy (IC), (IR) and (Opt). We will complete them into a TCE. By (IR), the interim utility vector $U(\chi_\mu)$ associated with μ and χ_μ by (7) is individually rational for player 1. Hence, there exist some belief q and some $\tau^* \in \mathcal{R}(q)$ such that (4) holds. Let us define player 1's strategy in $G_{post}(p)$ as follows, given his type k,

- choose a message $s \in S$ according to $\mu(\cdot \mid k)$
- accept α (i.e., choose "y") iff $U^k(\alpha) \geq U^k(\chi_\mu)$
- in case of rejection of player 2's proposal, play a best reply to τ^*, i.e., maximize $U^k(\cdot, \tau^*)$.

To define the proposal mapping χ of player 2, let $a_1 \in A_1$ be an arbitrary action of player 1 and set

$$\chi(s) = \begin{array}{ll} \chi_\mu(s) & \text{if } s \in S_\mu \\ (a_1, \tau^*) & \text{if } s \notin S_\mu. \end{array}$$

Finally, if player 1 rejects the proposal, player 2 chooses $\tau(s) = \tau^*$, for every $s \in S$. It can be checked that the strategy profile $((\mu, \sigma), (\chi, \tau))$ forms a PBE of $G_{post}(p)$. ∎

In the sender-receiver game $G_{post}(p)$, the receiver makes a proposal after having received a message of the sender who then agrees or not. However, (7) shows that given incentive compatibility, the sender's interim utility vector does not depend on the message that he sends. This motivates an alternate possible scenario, in which the sender is asked to cooperate before sending his message, namely right after having learnt his type. More precisely, let us consider the following game $G_{int}(p)$:

Sender-receiver game $G_{int}(p)$, with cooperation agreement at the interim stage:

- A type k is chosen in K according to p, only player 1 is informed of k.
- Player 1 cooperates or not. In the latter case, player 1 chooses an action $a_1 \in A_1$, player 2 simultaneously chooses an action $a_2 \in A_2$, player 1 gets $U^k(a_1, a_2)$ and player 2 gets $V^k(a_2)$.
- If player 1 cooperates, he sends a message $s \in S$ to player 2 and then
- player 2 chooses a joint decision α in A, player 1 gets $U^k(\alpha)$ and player 2 gets $V^k(\alpha)$.

In the game $G_{int}(p)$, as in Bester and Strausz (2001), player 1 faces interim participation constraints and if player 1 participates, he delegates the joint decision to player 2, who acts as a principal. However, in our framework, the utility that player 1 gets if he does not cooperate is not the effect of an exogenous outside option; it rather depends on player 2's reaction in the default Bayesian game. By proceeding as above, one can establish the following analog of Proposition 6:

Proposition 7 *Let $\mu : K \to \Delta(S)$ be a signaling mapping for the sender and $\chi_\mu : S_\mu \to \Delta(A)$ be a restricted proposal mapping for the receiver. The mappings μ and χ_μ are part of a PBE in $G_{int}(p)$ in which the sender participates whatever his type iff (IC), (IR) and (Opt) hold.*

The previous result says in particular that the incentive compatibility constraints (IC) are identical, whether the sender is asked to cooperate at the interim stage or at the posterior stage. This property holds because the sender fully controls the message that he sends (see Remark 2 below).

A direct consequence of Proposition 7 is that a TCE of $B(p)$ can be defined equivalently as a PBE of $G_{int}(p)$ in which the sender cooperates whatever his type. In other words, in our framework, "talk and cooperate" turns out to be equivalent to "cooperate and talk."[6]

3.3 Existence of TCE

If player 1 cannot make a private use of his information, a nonrevealing TCE trivially exists. This happens in standard sender-receiver games or if the expert's preferences are independent of the state, as in Chakraborty and Harbaugh (2010).[7] The next example illustrates that existence of a TCE may fail in more general environments.

Example 1: Player 1 has two types ($\mid K \mid = 2$), we now denote as $p \in [0,1]$ the probability of type 1. Both players have two actions ($A_1 = \{T, B\}$, $A_2 = \{L, R\}$). The utility functions are:

$$(U^1, V^1)(\cdot) = \begin{array}{ccc} & L & R \\ T & 1,4 & 4,0 \\ B & 0,4 & 0,0 \end{array} \qquad (U^2, V^2)(\cdot) = \begin{array}{ccc} & L & R \\ T & 0,0 & 0,4 \\ B & 4,0 & 1,4 \end{array}$$

Player 1 knows the state, player 2 wants to match the state (by playing L when player 1 is of type 1 and R when player 1 is of type 2); player 1 also wants to match the state but if he does, he is happier if player 2 does not.

The following properties are immediate: all decisions of player 2 are rationalizable (given his belief p that player 1's type is $k = 1$, he plays R if $p \leq \frac{1}{2}$, L if $p \geq \frac{1}{2}$ and any mixed decision if $p = \frac{1}{2}$), the minmax level of player 1 of type k is $\underline{m}^k = 1$, $k = 1, 2$.

Assume that some signal s is sent with positive probability by both types of player 1. Any joint decision $\chi(s) \in \Delta(A)$ that player 2 can propose upon receiving s is type independent and thus satisfies $U^1(\chi(s)) + U^2(\chi(s)) \leq 4$. If player 1 rejects player 2's proposal, player 1 can then match the state (i.e., play $\sigma(1) = T$, $\sigma(2) = B$) while player 2 will play some $\tau(s) \in \Delta(\{L, R\})$, which will depend on his belief over k after having received s and observed player 1's rejection. For this σ and *every* $\tau(s)$, $U^1(\sigma, \tau(s)) + U^2(\sigma, \tau(s)) = 5$.

[6] Schlag and Vida (2013) illustrate the importance of the timing of talk and commitment in a different context.

[7] In such a nonrevealing equilibrium, player 2 proposes a decision maximizing his own expected utility and maintains this decision in case of rejection by player 1.

Hence one of player 1's types will reject $\chi(s)$ and a TCE cannot involve any such signal s.

Assume now that player 1 fully reveals his type. Let $\chi(k)$ be player 2's proposal when player 1 claims to be of type k. Optimality for player 2 then leads to $\chi(1) = L$, $\chi(2) = R$, so that player 1's interim utility vector is at best $(1, 1)$. Again, by rejecting the proposal and matching the state, player 1 can guarantee a higher payoff to at least one of his types. ■

In the previous example, player 2 has no UPD (recall Definition 3) and Lemma 4 does not hold: individual rationality for player 1 (Definition 1) amounts to $u^1 + u^2 = 5$ and is more demanding than type by type individual rationality (Definition 2), i.e., $u^1 \geq 1$ and $u^2 \geq 1$.[8] Having a UPD turns out to be a sufficient condition for the existence of a TCE, at every prior p, whatever the players' utility functions.

Proposition 8 *Assume that the uninformed player has a uniform punishment decision (UPD). Then for every $p \in \Delta(K)$, the Bayesian game $B(p)$ has a talk and cooperate (TCE) equilibrium.*

The proof of this proposition relies on results of Simon et al. (1995) and Renault (2000) (see Appendix). As a by-product, we get that there exists a TCE in which the sender's interim utility vector $U(\chi_\mu)$ satisfies

$$U^k(\chi_\mu) \geq \inf_{r \in \Delta(K)} \max_{x \in F(r)} U^k(x) \quad \text{for every } k \in K. \tag{8}$$

In the previous inequalities, the lower bound on $U^k(\chi_\mu)$ is computed by allowing for a maximum over $F(r)$, which reflects the players' possible cooperation. We deduce that $U(\chi_\mu)$ is type by type individually rational:

$$\inf_{r \in \Delta(K)} \max_{x \in F(r)} U^k(x) \geq \inf_{r \in \Delta(K)} \min_{\tau \in \mathcal{R}(r)} \max_{a_1 \in A_1} \sum_k q^k U^k(a_1, \tau) = \underline{m}^k \quad \text{for every } k \in K.$$

Hence, by lemma 4, $U(\chi_\mu)$ is individually rational.[9]

[8] Observe that incentive compatibility is not an issue in the example. The problem only rests on individual rationality. If player 1 were forced to cooperate, $\chi(1) = (T, L)$, $\chi(2) = (B, R)$ would be achievable in the sense that it is incentive compatible for player 1 and posterior optimal for player 2. Note also that even if the definition of TCE is weakened by replacing PBE by Nash equilibrium, the previous example still has no TCE.

[9] The lower bound in (8) is strictly higher than $\underline{m}^k$ as soon as the uninformed player is indifferent between some of his actions.

Remark 1: We have noted above that individual rationality for the informed player, namely (IR), can be interpreted as an interim participation constraint. If the uninformed player has a UPD, by Lemma 4, (IR) can be formulated type by type and thus takes the same form as in Bester and Strausz (2001), with $\underline{m}^k$ as type k's (endogenous) reservation utility. More precisely, when no decision of the principal is contractible, Bester and Strausz (2001)'s model amounts to considering exogenous reservation utility levels u_0^k, $k \in K$, for player 1 and reducing A_1 to a singleton. Bester and Strausz (2001)'s equilibrium conditions are similar to the above ones (namely, (IC), (IR) with respect to u_0^k, $k \in K$, and (Opt)). They establish a version of the revelation principle but do not address the problem of existence of an equilibrium. Sufficient conditions can be derived by using the same tools as for Proposition 8 (see Salamanca (2017)).

Remark 2: Even assuming that player 2 has a UPD, the existence of a TCE is made difficult by the incentive compatibility conditions ((IC), namely (6)), which require player 1 to be indifferent between the various messages that he sends with positive probability. Introducing a mediator in the game $G_{int}(p)$ could be helpful by transforming (IC) into linear inequalities. If a mediator is added to the game $G_{post}(p)$, veto-incentive compatibility (see Forges (1999)), should likely be considered.

4 Examples of (partially revealing) TCE

In both examples below, as in Example 1, the informed player has two possible types ($\mid K \mid = 2$) and two possible actions ($A_1 = \{T, B\}$); we denote as $p \in [0, 1]$ the probability of type 1. Unlike in Example 1, the uninformed player has a UPD. We will show that there is no nonrevealing TCE and no completely revealing TCE. As expected from Proposition 8, there exists a partially revealing TCE and indeed we construct one. Example 2 is just meant to indicate the pattern of the equilibria that can be constructed.

Example 2: Player 2 has three actions ($A_2 = \{L, C, R\}$) and the utility functions are:

$$(U^1, V^1)(\cdot) = \begin{array}{cccc} & L & C & R \\ T & 1,4 & 1,3 & 1,0 \\ B & 0,4 & 0,3 & 4,0 \end{array} \qquad (U^2, V^2)(\cdot) = \begin{array}{cccc} & L & C & R \\ T & 4,0 & 0,3 & 0,4 \\ B & 1,0 & 1,3 & 1,4 \end{array}$$

As in Example 1, player 1 knows the state and player 2 wants to match the state (by playing L when player 1 is of type 1 and by playing R when player 1 is of type 2). Player 2 has an additional decision C, which can be interpreted as "status quo". All decisions of player 2 are rationalizable: R is optimal if $0 \leq p \leq \frac{1}{4}$, C is optimal if $\frac{1}{4} \leq p \leq \frac{3}{4}$ and L is optimal if $\frac{3}{4} \leq p \leq 1$.[10] For player 1, not matching the state (i.e., playing T when type 1 and R when type 2) is safe (in the sense that it guarantees a utility that does not not depend on player 2's decision) while matching the state (i.e., playing B when type 1 and T when type 2) is risky (in the sense that it only gives a high utility in case of differentiation with respect to player 2's decision). We check that $\underline{m}^1 = \underline{m}^2 = 1$ and that C is a UPD of player 2.

Let us take $p = \frac{1}{2}$. There is no nonrevealing TCE because optimality for player 2 implies he should play C, but then player 1 cannot achieve individual rationality without revealing his type. There is no completely revealing TCE either; optimality for player 2 leads to L if type 1 is reported and R if type 2 is reported. By individual rationality for player 1, the joint decision must be (T, L) if type 1 is reported and (B, R) if type 2 is reported, but this is not incentive compatible.

However, a partially revealing TCE is easily constructed. At this TCE, player 1 sends signals m and s so as to reach the posteriors $p_m = \frac{1}{4}$ and $p_s = \frac{3}{4}$. Let $\chi(m) = (B; \frac{3}{4}C, \frac{1}{4}R)$ and $\chi(s) = (T; \frac{1}{4}L, \frac{3}{4}C)$. For player 1, $U(\chi(m)) = U(\chi(s))$. For player 2, any mixture of C and R (resp., of L and C) is optimal at $p_m = \frac{1}{4}$ (resp., $p_s = \frac{3}{4}$).

The interim utility vector of player 1 at the TCE is $(1, 1)$ and the expected utility of player 2 is 3. Both players just get the same expected utility as in the Nash equilibrium of the Bayesian game $B(\frac{1}{2})$. In the latter game, player 1's action completely reveals his type but player 2 has to make his decision simultaneously, so that he cannot make any use from player 1's revealed information. In the TCE, player 1 reveals some information before a joint decision is made; no further information is revealed. ■

In the next example, we construct a TCE in which both players get strictly more than their individually rational level.

Example 3: Player 2 has five actions ($A_2 = \{LL, L, C, R, RR\}$) and the utility functions are:

[10] However no mixture of decisions L and R is rationalizable, i.e., the set $\mathcal{R}$ is strictly included in $\Delta(A_2)$.

$$
(U^1,V^1)(\cdot) \quad = \quad \begin{array}{cccccc} & LL & L & C & R & RR \\ T & 6,10 & 10,9 & 0,7 & 4,4 & 3,0 \\ B & 1,10 & 2,9 & -1,7 & 5,4 & 7,0 \end{array}
$$

$$
(U^2,V^2)(\cdot) \quad = \quad \begin{array}{cccccc} & LL & L & C & R & RR \\ T & 7,0 & 5,4 & -1,7 & 2,9 & 1,10 \\ B & 3,0 & 4,4 & 0,7 & 10,9 & 6,10 \end{array}
$$

The utilities associated with action T for type 1 and action B for type 2 are as in Forges (1990), where the informed player has no decision to make. The preferences associated with the other action (i.e., B for type 1 and T for type 2) are widely opposed to the ones of player 2. As in the two previous examples, all decisions of player 2 are rationalizable (RR is optimal for $0 \leq p \leq \frac{1}{5}$, R is optimal for $\frac{1}{5} \leq p \leq \frac{2}{5}$, C is optimal for $\frac{2}{5} \leq p \leq \frac{3}{5}$, L is optimal for $\frac{3}{5} \leq p \leq \frac{4}{5}$ and LL is optimal for $\frac{4}{5} \leq p \leq 1$). Decision C is a UPD for player 2, the minmax level of player 1 is $\underline{m}^k = 0$, $k = 1,2$. The benchmark Nash equilibrium of the Bayesian game $B(\frac{1}{2})$ gives $(0,0)$ as interim utility vector to player 1 and 7 as expected utility to player 2.

There is no nonrevealing TCE at $p = \frac{1}{2}$. Indeed, individual rationality for player 1 means that both types k must get a utility of at least 0 and optimality for player 2 requires that he takes action C. These conditions cannot be met at any $x \in \Delta(A)$.

There is no completely revealing TCE at any $p \in (0,1)$. To check this, assume there is such a TCE, $\chi(k) \in \Delta(A)$, $k = 1,2$. Optimality for player 2 implies that $\chi(1) = ((\alpha_{1,}1-\alpha_1), LL)$ and $\chi(2) = ((\alpha_{2,}1-\alpha_2), RR)$, with α_k denoting the probability of T when k is reported. Individual rationality for player 1 is satisfied since both types of player 1 get more than 0 at $\chi(k)$, $k = 1,2$. The incentive compatibility conditions are

$$
\begin{aligned}
6\alpha_1 + (1-\alpha_1) &\geq 3\alpha_2 + 7(1-\alpha_2), \\
\alpha_2 + 6(1-\alpha_2) &\geq 7\alpha_1 + 3(1-\alpha_1).
\end{aligned}
$$

They imply $\alpha_1 \geq \alpha_2 + 3$, which is impossible.

Here is a partially revealing TCE at $p = \frac{1}{2}$: player 1 sends messages m and s so as to reach the posteriors $p_m = \frac{1}{5}$ and $p_s = \frac{4}{5}$. If player 2 receives message m, he proposes the joint decision $\chi(m) = (B; \frac{1}{6}R, \frac{5}{6}RR)$ while if he receives message s, he proposes the joint decision $\chi(s) = (T; \frac{5}{6}LL, \frac{1}{6}L)$. Both types of player 1 are indifferent between sending m or s and accepting player

2's proposal; the corresponding interim utility vector is $(\frac{20}{3}, \frac{20}{3})$. By rejecting player 2's proposal, player 1 cannot hope for more than 0, whatever his type. Player 2's expected utility at this TCE is 8. $\blacksquare$

5 Extension to more general utility functions

The sender-receiver game $G_{post}(p)$ studied in this paper can be equivalently described as follows: (i) the informed player sends a message to the uninformed one and then, as a function of the message, (ii) the players conclude an agreement which comes into effect if both players approve it. The advantage of this description is that it allows to extend the basic Bayesian game $B(p)$ into a game with possible cooperation, regardless of the restrictive assumption on the uninformed player's payoff function. If both players have a decision to make, we do not assume anymore that player 2 makes a proposal to player 1 but rather that both players must approve a joint agreement that is proposed to them. As a starting point, we just ask whether there exist proposals that are accepted by both players without trying to justify how these proposals are generated.

In Forges, Horst and Salomon (2016), we proceed as described in the previous paragraph. Under a weaker assumption[11], we establish an existence result for a solution concept that is similar to TCE in that it is achieved by talk and approval, but differs from TCE in that it is implemented in Nash equilibrium rather than in PBE. More precisely, we assume that if a proposed agreement is rejected by at least one of the players, there is no restriction on the actions that can be taken in the (possibly updated) Bayesian game.[12]

The following example illustrates the difficulty of establishing the existence of a TCE, which as a PBE involves credible punishments, if player 2's utility depends on player 1's action, even if it does not depend on player 1's type.

Example 4: We modify player 2's utility function in Example 1 by assuming

[11] Player 2 has a uniform punishment strategy that need not be rationalizable.

[12] Forges (2013), Peters and Szentes (2012), Peters (2015) allow for early commitment to punishment, while Celik and Peters (2011, 2016) do not.

$V^1 = V^2 = V$, as follows:

$$(U^1, V)(\cdot) = \begin{array}{c|cc} & L & R \\ T & 1,4 & 4,0 \\ B & 0,0 & 0,4 \end{array} \qquad (U^2, V)(\cdot) = \begin{array}{c|cc} & L & R \\ T & 0,4 & 0,0 \\ B & 4,0 & 1,4 \end{array}$$

Consider the following scenario: player 1 reports his type, 1 or 2, and according to player 1's message, the agreement $\chi(1) = (T; \frac{1}{2}L, \frac{1}{2}R)$ or the agreement $\chi(2) = (B; \frac{1}{2}L, \frac{1}{2}R)$ is submitted to the players' approval. If player 1 can threaten player 2 to make the decision $(\frac{1}{2}T, \frac{1}{2}B)$ if player 2 does not approve the agreement, i.e., to punish player 2 at his minmax level 2, then the previous scenario induces a Nash equilibrium.

Suppose however that, in case of rejection of the proposal, the players play the Bayesian game, with possibly updated beliefs. Then, in a PBE, player 1 must match the state, namely, choose T when $k = 1$ and B when $k = 2$. This implies that, in this example, there is no way to associate a joint decision with player 1's message that would be consistent with a PBE in which player 1 sends a message, both players approve the joint decision and play the Bayesian game if one of them rejects the decision.

To be fully precise, we can proceed as in Example 1 for player 1. If, in a PBE, this player rejects an agreement after having sent a message, he must then match the state (i.e., play $\sigma(1) = T$, $\sigma(2) = B$). Player 2's best response, depending on his belief over k, can be any $\tau \in \Delta(\{L, R\})$. By matching the state, player 1 gets an interim utility vector (u^1, u^2) satisfying $u^1 + u^2 = 5$ for every τ. Hence, as in Example 1, player 1 rejects any agreement $\chi(s)$ that follows a message s that is sent with positive probability by both of his types. Assume thus that player 1 reveals his type k at the message stage. If player 2 now rejects a proposal $\chi(k)$, the default game is played, with player 2's belief $p^k_k = 1$. Hence, to avoid rejection by player 2, (T, L) must be played if $k = 1$ and (B, R) must be played if $k = 2$. But this gives a utility of 1 to each type of player 1 and is again be rejected by this player. ■

This example is not a counter-example to an extension of our existence result (Proposition 8) when the uninformed player's utility depends on the informed player's *action*, because UPD does not hold in the game above. Yet, in games like the previous one, in which the uninformed player's utility does not depend on the informed player's *type*, there always exist a Nash equilibrium with talk (by the informed player) and cooperation (of both players), *even if UPD does not hold* (see Forges, Horst and Salomon (2016), Section 3.2.2). The example thus suggests that, as soon as the uninformed player's utility depends on the informed player's action, the techniques that we used so far do not suffice to establish the existence of a TCE.

We go on by modifying Example 1 in a way that allows for more cooperation opportunities. Then a TCE is easily constructed.

Example 5:

$$(U^1, V^1)(\cdot) = \begin{array}{ccc} & L & R \\ T & 1,1 & 4,0 \\ B & 0,4 & 0,0 \end{array} \qquad (U^2, V^2)(\cdot) = \begin{array}{ccc} & L & R \\ T & 0,0 & 0,4 \\ B & 4,0 & 1,1 \end{array}$$

The interpretation is now that player 1 knows the state of nature, both players like to match the state and like it even more if the other does not match the state.[13]

Let us start by assuming that both players know that the state is $k = 1$. Then, the joint decision

$$\chi = \begin{array}{ccc} & L & R \\ T & 0 & \frac{1}{2} \\ B & \frac{1}{2} & 0 \end{array}$$

can be implemented as a subgame perfect equilibrium of the extended game in which both players get to approve χ. Indeed, if they both accept χ, χ is enforced and each of them gets 2. If one of the players rejects χ, the default game is played, subgame perfectness calls for (T, L), with utility 1 for each player.

Let us go back to the game in which only player 1 knows the state, while player 2 believes that $k = 1$ with probability p. The previous (nonrevealing) decision is not individually rational for player 1. Indeed, we can proceed as examples 1 and 4. If both players accept χ, player 1's types both get 2. If player 1 unilaterally rejects χ, player 2 forms a belief q over the state and makes some decision $(\tau, 1-\tau) \in \Delta(\{L, R\})$. By matching the state, namely, by playing T if $k = 1$ and B if $k = 2$, player 1 is sure to get an interim utility

[13] In this example, for each type k of player 1, the set of feasible payoffs - $(U^k(\alpha), V^k(\alpha))$, $\alpha \in \Delta(A)$ - that give each player more than his Nash equilibrium payoff - equal to 1 here - has a nonempty interior. This property did not hold in Examples 1 and 4.

vector (u^1, u^2) such that $u^1 + u^2 = 5$, hence, whatever τ, one of player 1's types profits from rejection.

Let us next assume that player 1 sends the message m or s to player 2 and that as a function of the message, the following agreements are submitted to the players' approval:

$$\chi(m) = \begin{array}{ccc} & L & R \\ L & 0 & \frac{3}{4} \\ R & \frac{1}{4} & 0 \end{array} \qquad \chi(s) = \begin{array}{ccc} & L & R \\ L & 0 & \frac{1}{4} \\ R & \frac{3}{4} & 0 \end{array}.$$

Then we can construct a PBE implementing these agreements. Player 1 reveals his type (i.e., sends m when $k = 1$ and s when $k = 2$) and at equilibrium, both players approve the agreement associated with player 1's message. If one of them rejects it, player 1 plays T if $k = 1$ and B if $k = 2$. If player 1 rejects, player 2 believes that each type is equally likely and chooses L or R with probability $\frac{1}{2}$. If he himself is the only one to reject, player 2 chooses L (resp., R) if m (resp., s) has been sent.

If both players accept the agreement, both types of player 1 get 3 and player 2 gets 1. Player 1 cannot profit from lying and accepting. Given player 2's reaction, if player 1 rejects, each of his types can get an expected utility of at most 2.5. Given player 1's reaction, if player 2 rejects, his expected utility cannot exceed 1. Furthermore, off equilibrium path, every player plays optimally given the strategy of the other, with player 2 believing that player 1's types are equally likely in case of rejection by the latter.

As a benchmark, the expected payoffs of the Nash equilibrium of the Bayesian game with prior probability p, without information transmission are

$$\begin{array}{ll} \min\{p + 4(1-p), 4p + (1-p)\} \leq 2.5 & \text{for player 1} \\ \max\{p, 1-p\} \leq 1 & \text{for player 2.} \end{array}$$

Hence the TCE proposed above is Pareto-improving with respect to this benchmark. ■

6 Appendix: proof of Proposition 8

To establish Proposition 8, we rely on a lemma, which follows from a theorem of Simon et al. (1995) (see Renault (2000), Simon (2002), Simon et al. (2008)).

Lemma *Let us fix a finite set K, a compact, convex set X, linear functions $U^k : X \to \mathbb{R}$, $k \in K$, a non-empty convex valued, upper-hemi-continuous correspondence $F : \Delta(K) \to X$ and a lower-semi-continuous function $\varphi : \Delta(K) \to \mathbb{R}$ satisfying the following*
Assumption A: $\forall q, r \in \Delta(K)\ \exists x$ *such that* $x \in F(r)$ *and* $q \cdot U(x) \geq \varphi(q)$.

Then there exist a finite set S and for every $p \in \Delta(K)$ (such that $p^k > 0$ for every $k \in K$*) mappings $\mu : K \to \Delta(S)$ and $\chi : S \to X$ such that $S_\mu = S$, (IC) and (Opt) hold. Furthermore, the interim utility vector $U(\chi_\mu)$ satisfies*

$$q \cdot U(\chi_\mu) \geq \varphi(q) \text{ for every } q \in \Delta(K) \tag{9}$$

To get some intuition on how we are going to make use of the previous lemma, let us first show that, for an appropriate definition of the mapping φ, (9) says that $U(\chi_\mu)$ is individually rational for the informed player. Let us take $\varphi = f_1$, with f_1 defined by

$$f_1(q) = \min_{\tau \in \mathcal{R}} \sum_k q^k \max_{a_1 \in A_1} U^k(a_1, \tau) \quad \text{for every } q \in \Delta(K).$$

The mapping f_1 corresponds to the minmax level of player 1 in the Bayesian game that is defined as $B(q)$ but in which player 2's decisions are restricted to $\mathcal{R}$. It is easily checked that, if $u = (u^k)_{k \in K}$ is individually rational according to Definition 1, then

$$q \cdot u \geq f_1(q) \text{ for every } q \in \Delta(K).$$

When the set $\mathcal{R}$ is convex, the converse also holds. However $\mathcal{R}$ is not necessarily convex and even when it is the case, f_1 and F (defined by (2)) may not satisfy Assumption A.[14]

Instead of f_1, let us consider the following mapping

$$f(q) = \min_{\tau \in \mathcal{R}} \max_{a_1 \in A_1} \sum_k q^k U^k(a_1, \tau) \quad \text{for every } q \in \Delta(K). \tag{10}$$

The mapping f corresponds to the minmax level of player 1 in an auxiliary game, again defined as $B(q)$, in which player 1 is not informed (and player

[14] This can be illustrated on Example 1, in which, denoting the probability of type 1 as $q \in [0,1]$, $f_1(q) = \min\{4 - 3q, 3q + 1\}$.

2's decisions are still restricted to $\mathcal{R}$). We will show below that f and F (defined by (2)) do satisfy Assumption A.

The lemma will then provide us with a TCE such that (9) holds with $\varphi = f$. This does not yet guarantee that player 1's utility vector $U(\chi_\mu)$ at the TCE is individually rational, since $f_1 \geq f$. However the mappings f_1 and f coincide at the extreme points of $\Delta(K)$ and thanks to UPD, type by type individual rationality is sufficient for individual rationality (by Lemma 4).

To get some further intuition on the lemma, consider the following dramatic strengthening of Assumption A:

$$\forall r \in \Delta(K)\ \exists x \text{ such that } x \in F(r) \text{ and } \forall q \in \Delta(K)\ q \cdot U(x) \geq \varphi(q).$$

Given our comments on (9), the previous condition says that for every prior probability r, there exists a decision x that is optimal for the uninformed player and such that the utility vector $U(x)$ is individually rational for the informed player (admitting that individual rationality is captured by the mapping φ). This implies that, for every prior r, there exists a nonrevealing TCE.

Assumption A, as it appears in the statement of the lemma, just says that if player 1 and player 2 hold respective, possibly different, priors, q and r, there is a joint decision that is optimal for player 2 at his prior r and gives player 1, at his prior q, an expected utility of at least $\varphi(q)$. The assumption is likely to be satisfied if $\varphi(q)$ is the minmax level of player 1 when he cannot make use his information, i.e., in our framework, if $\varphi = f$.

Proof of Proposition 8

Let us consider the set $X = \Delta(A)$, the extension to X of the utility functions U^k of the informed player, the correspondence F defined by (2) and the following mapping φ:

$$\varphi(q) = \inf_{r \in \Delta(K)} \max_{x \in F(r)} q \cdot U(x). \tag{11}$$

The assumptions of the previous lemma are then satisfied. Indeed, φ is constructed so as to satisfy Assumption A and is easily shown to be Lipschitz of constant $M = \max_{k,x} \mid U^k(x) \mid$. To check the latter property, fix any $q \in \Delta(K)$ and $\varepsilon > 0$. By the definition of inf max,

$$\exists r^* = r^*(q, \varepsilon) \text{ s.t. } \forall x \in F(r^*)\ \sum_k q^k U^k(x) \leq \varphi(q) + \varepsilon$$

$$\forall r \; \exists x^* = x^*(q,r) \in F(r) \; \sum_k q^k U^k(x^*) \geq \varphi(q).$$

Let us take q_1, q_2 and $\varepsilon > 0$. Consider $r_2^* = r^*(q_2, \varepsilon)$ and $x_1^* = x^*(q_1, r_2^*) \in F(r_2^*)$:

$$\begin{aligned} \varphi(q_1) - \varphi(q_2) &\leq \sum_k q_1^k U^k(x_1^*) - \sum_k q_2^k U^k(x_1^*) + \varepsilon \\ &\leq \mid \sum_k U^k(x_1^*)(q_1^k - q_2^k) + \varepsilon \mid \leq M \parallel q_1 - q_2 \parallel + \varepsilon \end{aligned}$$

Let then $\varepsilon \to 0$ and do the same by exchanging q_1 and q_2.

By the lemma, there exist a signaling mapping μ and a proposal mapping χ such that $S_\mu = S$ (and thus $\chi_\mu = \chi$), satisfying (IC) and (Opt). To show that μ and χ define a TCE, there remains to show that $U(\chi)$ satisfies (IR).

The definitions of F, φ and f (see (2), (11) and (10)) imply that, for every $q \in \Delta(K)$,

$$\begin{aligned} f(q) &\leq \inf_{r \in \Delta(K)} \min_{\tau \in \mathcal{R}(r)} \max_{a_1 \in A_1} \sum_k q^k U^k(a_1, \tau) \\ &\leq \inf_{r \in \Delta(K)} \max_{\tau \in \mathcal{R}(r)} \max_{a_1 \in A_1} \sum_k q^k U^k(a_1, \tau) \leq \varphi(q). \end{aligned}$$

By (9), $q \cdot U(\chi) \geq f(q)$ for every $q \in \Delta(K)$. In particular, by taking q as the k^{th} extreme point of $\Delta(K)$, we get that $U^k(\chi) \geq \underline{m}^k$. Hence, $U(\chi)$ is type by type individually rational. Since the uninformed player has a UPD, by Lemma 4, $U(\chi)$ is also individually rational. ■

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