Trading under Market Impact

-Crossing Networks Interacting with Dealer Markets-

Jana Bielagk (Humboldt University Berlin)
Ulrich Horst (Humboldt University Berlin)
Santiago Moreno-Bromberg (University of Zurich)

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Trading under market impact*
-crossing networks interacting with dealer markets-

Jana Bielagk[1], Ulrich Horst[2]

Department of Mathematics, Humboldt-University Berlin, Unter den Linden 6, 10099 Berlin, Germany.

Santiago Moreno–Bromberg[3]

Department of Banking and Finance, University of Zurich, Andreasstrasse 15, 8050 Zurich, Switzerland

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Abstract

We use a model with agency frictions to analyze the structure of a *dealer market* that faces competition from a *crossing network*. Traders are privately informed about their *types* (e.g. their portfolios), which is something the dealer must take into account when engaging his counterparties. Instead of participating in the dealer market, the traders may take their business to a crossing network. We show that the presence of such a network results in more trader types being serviced by the dealer and that, under certain conditions, the book's *spread* shrinks. We allow for the pricing on the dealer market to determine the structure of the crossing network and show that the same conditions that lead to a reduction of the spread imply the existence of an equilibrium book/crossing network pair.

Keywords: Asymmetric information; crossing networks; dealer markets; non-linear pricing; principal-agent games.

JEL classification: D42; D53; G12; G14.

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[1]Email: `bielagk@math.hu-berlin.de`
[2]Email: `horst@math.hu-berlin.de`
[3]Email: `santiago.moreno@bf.uzh.ch`

1 Introduction

In traditional financial markets, liquidating large positions (relative to the available liquidity) leads to an unfavorable price impact. In response to this problem, alternative trading venues such as crossing networks (CNs for short) have been established (e.g. Liquidnet and POSIT, among others). The most prominent feature of these venues is that trading occurs at a price taken from a reference market (a stock market or a dealer market), but the execution of orders is uncertain. This leads to the question of how the prices and traded volumes in the dealer market (DM for short) are affected by the emergence of the CN.[1]

We model the price-setting market using a contract-theoretic framework with private information, where the principal represents a monopolistic DM run by a profit-maximizing dealer. On the other hand, the privately-informed agents correspond to traders who choose between engaging the dealer and their so-called outside option. In our case, the latter corresponds to either abstaining from trading or trading in a CN, depending on which offers the highest (expected) utility. The price in the CN depends in a pre-specified manner on the price schedule offered by the dealer (e.g. the midpoint of the bid-ask spread of the DM) and, simultaneously, it determines the traders' outside option, thereby influencing the dealer's optimal strategy.[2] This feedback loop leads to a fixed point problem, which we analyze step by step.

First, we formulate the optimization problems of the dealer and the traders. The latter are parameterized by their types (e.g. their inventory positions). The dealer does not know each trader's type, but only the distribution of types; hence, he faces a screening problem. For a given execution price in the CN, the dealer's objective is to devise a pricing schedule so as to maximize his expected profits from trading, roughly defined as the gains from trading certain positions net of the associated costs. We show that this problem has a unique solution on the set of traders participating in the DM.

Second, we analyze the qualitative influence of the CN on the existing market. We prove that the set of *reserved types*, i.e. those who trade nothing in exchange for nothing, shrinks after the introduction of the CN. This means that the presence of the CN results in more traders earning positive rents. Furthermore, we see that, in the particular case of uniformly distributed types, the spread of the DM narrows and the indirect utility, i.e. the highest attainable utility a trader can obtain from engaging the dealer, increases in the presence of the CN. In the sequel we also distinguish between *excluded types* (those traders who favor the CN over the DM) and *fully serviced types* (those traders who earn strictly positive rents from participating in the DM). Analyzing the *market segmentation* brought about by the presence of a CN is one of our main foci.

In order to have a benchmark, we first analyze the problem without a CN in the spirit of Biais et al. (2000). Adding a non-trivial outside option complicates computations significantly, due to the discontinuities of the model. These discontinuities, closely related to the market-segmentation issue, arise because small changes in the structure of the DM may result in a large amount of traders switching from being fully serviced to being excluded (or vice versa). This in turn results in considerable changes of the dealer's expected utility. We overcome this difficulty by using an "accounting trick" following Jullien (2003): instead of excluding traders with whom trading is too costly, we assume the dealer has access to

[1]The European MiFID 2, approved in 2014 and to be implemented by January 3, 2018, introduced the new category "organized trading facility (OTF)". Among others, the OTF regime captures broker CNs. Given the increased regulatory scrutiny to which CNs are subject, analyzing their effect on primary markets is a current, relevant issue.

[2]Put differently, this work is based on a particular setting, in which the competitor to the monopoly is not modeled, and instead adopts an automatic quotation system inspired by some existing financial markets.

a second, fictitious technology that allows him to offer trades at their expected costs, giving zero profits to the dealer. In this case, there are no longer excluded types, as previously excluded traders become fully serviced, albeit under the alternative technology. We show that on the set of types who are serviced using the "original technology", the resulting optimal pricing schedule coincides with the solution to the original price-schedule design problem. The caveat is that we have to keep track of the points where the dealer switches between using one technology or the other. We illustrate the results by means of several examples with and without a CN. The segmentation of the market can become quite complex, as is demonstrated in particular in Example 3.10.

Having understood the dealer's problem, we show that, under certain conditions, there exists an *equilibrium price schedule*. By this we mean the following: given an exogenous sell/buy price pair (π_-, π_+) in the CN, the dealer optimally chooses his pricing schedule. If we then specify the mechanism via which prices in the DM determine those in the CN, we in general have that the prices emerging from the dealer's choice do not coincide with the original (π_-, π_+). An equilibrium price schedule is such that the optimal reaction of the dealer to the prices (π_-, π_+) in the CN results in a price schedule that induces (π_-, π_+) via the price-generation mechanism. Our study of such a feedback loop is novel and it is a crucial component in our analysis of the interactions between DMs and CNs, which is typically not unidirectional. As an application we consider a problem of optimal portfolio liquidation where traders can chose between a DM and a CN (in this particular case a dark pool). We obtain the existence of an equilibrium price and discuss the lack of the uniqueness thereof.

Related literature

Market impact. The analysis of optimal trading under market impact has received considerable attention from the mathematical-finance community. Starting with the contribution of Almgren and Chriss (2001), the existence of optimal trading strategies under illiquidity has been established by many authors, including Forsyth et al. (2012), Gatheral and Schied (2011), Kratz and Schöneborn (2015) and Schied et al. (2010), just to name a few. This literature typically assumes that block trading takes place under some (exogenous) pricing schedule, which describes the liquidity available for trading at different price levels. Horst and Naujokat (2014) and Kratz and Schöneborn (2015) were the first to also allow orders to be simultaneously submitted both to a DM and a CN. However, neither allows for an impact of off-exchange trading on the dynamics of the associated DM, which is precisely the feedback effect that we focus on[3]

Market segmentation. The literature on the impact of market segmentation and, more specifically, the impact of alternative trading venues on existing markets, has grown significantly in the last two decades, see for instance Gomber et al. (2013), Degryse et al. (2005) and Oriol (2012) for references to both theoretical and empirical papers. For instance, Fagart (1996) and Pouyet et al. (2008) study models of dealers competing for traders with private information about their types, with focus on the precise information structure in the market. The former analyzes a model with two equal dealers and a trader who may have one out of two possible types. The latter allows more than two identical dealers and an unspecified type space. Contracts and the resulting competitive equilibria are called efficient if, among other requirements, the dealers obtain zero profit. Our setting differs significantly from the aforementioned works by having only one profit-maximizing dealer who competes with a CN whose price-schedule adjusts mechanically to the dealer's price. In other words, whereas they have true Nash

[3] Zhu (2014), Buti et al. (2016) and Ye (2016) are examples of recent literature that also analyzes this impact.

Equilibria[4], we have only a one-sided optimization. A priori, it is not clear whether equilibrium prices exist in our setting.

A common approach (that we do not follow) in the theoretical literature is to assume that the market participants trade only a single unit of the stock. For instance, in their seminal work, Hendershott and Mendelson (2002) derive conditions for the viability of the alternative trading institutions in a modeling framework where a random number of informed and liquidity traders, each buying or selling a single unit, chooses between a DM and a CN. In their model, dealers receive multiple single-unit orders and cannot distinguish between the informed and the liquidity orders. Degryse et al. (2009) and Daniëls et al. (2013) also address competition between DMs and CNs.

Empirical evidence. Besides the cited theoretical works, there exists a growing empirical literature on the effects of market segmentation. For instance, Battalio (1997) empirically observes a positive effect of new OTC[5] trading (and hence increased market segmentation) on NYSE-listed securities and a tightening spread. Gresse (2006) finds that risk-sharing benefits from CNs dominate fragmentation costs and cream-skimming[6]; if dealers are allowed to trade in the CN, then they can offer better prices. Næs and Ødegaard (2006) analyze different costs associated to trading in CN such as direct trading costs (e.g. originating from the spread), adverse selection costs and opportunity costs from delayed trading. They find that implicit costs are larger than explicit trading costs. Nimalendran and Ray (2014) present evidence of informed traders in CNs, suggesting that information and price discovery happen in CN due to concurrent trading. Degryse et al. (2015) empirically supports the theory of *cream-skimming* and finds a negative impact of dark trading on the related lit market. Buti et al. (2011), provide empirical evidence that high CN activity is associated with narrower spreads, but no causality is concluded. Apergis and Voliotis (2015) finds empirical evidence for negative spill-over effects of dark trading. Foley and Putniņš (2016) show that two-sided dark pools (i.e., dark limit order books) are beneficial, whereas the impact of one-sided dark pools (where crossing occurs e.g. at the midpoint of the bid-ask spread) is not clear and has an adverse-selection effect. We can sum up the empirical results with a statement from Section 7.3 in Gomber et al. (2013): "It is also possible that all types of dark pool trading activity may not have a uniform impact on the markets, given the different types of market structure that are clubbed in its definition."

2 The model and an existence result

We consider a quote-driven market for an asset, in which a risk-neutral *dealer* engages a group of privately-informed *traders*[7]. The dealer market (DM for short) is described by a pricing schedule $T : \mathbb{R} \to \mathbb{R}$, where q units of the asset are offered to be traded, on a take-it-or-leave-it basis, for the amount $T(q)$. For $q \in \mathbb{R}$, we refer to the pair $\big(q, T(q)\big)$ as a *contract*. Following Biais et al. (2000), we assume that $T(0) = 0$ and that T is absolutely continuous; thus, we may write

$$T(q) = \int_0^q t(s)ds, \quad q \geq 0,$$

[4] Glosten (1994) and Parlour and Seppi (2003) also use equilibrium models so as to analyze the impact of alternative trading venues on DMs and trading behavior.

[5] OTC (over-the-counter) - trading that occurs away from traditional exchange markets

[6] Cream-skimming refers to the effect that informed dealers tend to prefer the DM, whereas uninformed traders move to the CN, leading to a higher (adverse selection) risk for the dealer who faces the better-informed traders.

[7] Our dealer is called the PRINCIPAL in the contract-theory jargon, whereas the traders are commonly referred to as AGENTS.

and analogously for negative values of q. Here $t(s)$ is the marginal price at which the s-th unit is traded. As we shall see below, pricing schedules are, in general, not differentiable at zero. Hence, for a particular schedule T the *spread* is

$$\mathcal{S}(T) := |T'(0_+) - T'(0_-)| = |t(0_+) - t(0_-)|,$$

where $t(0_-)$ and $t(0_+)$ are the *best-bid* and *best-ask* prices, respectively. We denote by $C : \mathbb{R} \to \mathbb{R}$ the dealer's inventory or risk costs associated with a position q (e.g. the impact costs of unwinding a portfolio of size q in a limit-order book). We assume that the mapping $q \mapsto C(q)$ is strictly convex, coercive[8] and it satisfies $C(0) = 0$.

The traders' idiosyncratic characteristics are indexed by $\theta \in \Theta := [\underline{\theta}, \overline{\theta}]$. So as to have buyers and sellers, we assume that zero belongs to the interior of Θ. Saying that a trader's *type* is θ means that if he trades q shares for $T(q)$ dollars his utility is $u(\theta, q) - T(q)$, where

$$u(\theta, q) := \theta\psi_1(q) + \psi_2(q)$$

and $\psi_1, \psi_2 : \mathbb{R} \to \mathbb{R}$ are smooth functions that satisfy $\psi_1(0) = \psi_2(0) = 0$, ψ_1 is strictly increasing and $C(q) - \psi_2(q) \geq 0$ holds for all $q \in \mathbb{R}$.

Besides participating in the DM or abstaining from trading (which we discuss below), each trader has the possibility to submit an order to a *crossing network* (CN for short). The latter is an alternative trading venue where trades take place at fixed bid/ask prices $\pi := (\pi_-, \pi_+)$, but where execution is not guaranteed.[9] For a specific π, the quantity $\mathrm{w}(\theta; \pi) \geq 0$ represents the expected utility of the θ-type investor who decides to trade in the CN. Following Daniëls et al. (2013) and Hendershott and Mendelson (2002) we focus on the case where a trader chooses **exclusively** between abstaining from trading or doing it either in the DM or the CN, i.e., we do not allow for simultaneous participation in the DM and the CN. Initially, we take π as given. Later, we analyze the case where it is endogenously determined through the interaction between the DM and the CN via the feedback of the spread in the former into the pricing in the latter. It is key to our analysis that the dealer is able to match the utilities that traders enjoy in the CN, even if this comes at a loss. As we show below, this requires $\mathrm{w}(\cdot; \pi)$ to be a convex function. Finally, we assume there is a fixed cost of entry $\kappa > 0$ to the CN. This captures the idea that "small" traders do not benefit from off-market participation. More precisely, we work under the following assumption:

Assumption 2.1. *The traders' utility* $\mathrm{w}(\cdot; \pi)$ *from participating in the CN satisfies*

$$\mathrm{w}(\cdot; \pi) = \widetilde{\mathrm{w}}(\cdot; \pi) - \kappa,$$

where $\widetilde{\mathrm{w}}(\cdot; \pi)$ *is a convex function that satisfies* $\widetilde{\mathrm{w}}(0; \pi) = 0$ *and* $\kappa > 0$ *is the fixed cost of accessing the CN.*

The traders' third option is to abstain from trading altogether. Given that $u(\theta, 0) = 0$ and $C(0) = 0$, from a modelling perspective we may equate this situation with the trade of the $(0, 0)$ contract in the DM. In other words, abstaining from trading is equivalent to trading "nothing for nothing" in the DM. Clearly, this only comes into consideration for traders who do not benefit from trading in the CN, i.e.

[8] By coercive we mean that $\lim_{|q|\to\infty} C(q) = \infty$.

[9] In other words, the crossing network presents traders with possibly better prices at the cost of an uncertain execution. CN trading often benefits traders who intend to unwind large positions, which might result in a price impact.

those whose types are such that $\mathrm{w}(\theta;\pi) < 0$. In the sequel we refer to $u_0(\cdot;\pi) := \max\{\mathrm{w}(\cdot;\pi), 0\}$ as the traders' outside option(s).[10]

Trading in the DM is anonymous: the dealer is unable to determine a trader's type before he engages the latter. The only ex-ante information the dealer has is the distribution of the individual types over Θ, which is described by a density $f : \Theta \to \mathbb{R}_+$. Below we specify the traders' and the dealer's optimization problems and analyze the impact of the CN on the DM, especially on its spread.

2.1 The traders' problem

Until further notice we consider π to be fixed. Given the pricing schedule T, the problem of a trader of type θ is to determine,

$$q_m(\theta) := \operatorname*{argmax}_{q \in \mathbb{R}} \Big\{ u(\theta, q) - T(q) \Big\}$$

and then choose, for $q_m \in q_m(\theta)$, between his *indirect-utility* $v(\theta) := u\big(\theta, q_m\big) - T\big(q_m\big)$ from trading in the DM and his outside option $u_0(\theta;\pi)$. As the supremum of affine functions, the indirect utility function is convex (thus the need for u_0 to be convex if we wish to analyze a situation where the dealer can match the CN).

The choice of a pricing schedule T induces a segmentation of the type space. We say that a trader of type θ *participates* in the DM if $v(\theta) \geq u_0(\theta;\pi)$, assuming that ties are broken in the dealer's favor. Conversely, we say that a trader of type θ *is excluded* from trading in the DM if $v(\theta) < u_0(\theta;\pi)$. For a given schedule T, we denote the set of excluded types by $\Theta_e(T;\pi)$. We say that a trader of type θ is *fully serviced* if he earns strictly positive profits from interacting with the dealer.

2.2 The dealer's problem

In a monopolistic setting like ours, contracts that are not chosen do not play a role in the game's outcome.[11] This implies that there is no loss of generality in assuming that the DM is described by *books* of the form $\big\{\big(q(\theta), \tau(\theta)\big), \theta \in \Theta\big\}$, where $\tau : \Theta \to \mathbb{R}$, $\tau = T \circ q$, is an absolutely continuous function. Studying the structure of the DM through contracts that are indexed by θ significantly simplifies the analysis of the dealer's decisions. We then write $\Theta_e(q, \tau;\pi)$ instead of $\Theta_e(T;\pi)$ for the set of excluded types.

At the onset, a trader of type θ could misrepresent his type by choosing a contract $\big(q(\widetilde{\theta}), \tau(\widetilde{\theta})\big)$, with $\widetilde{\theta} \neq \theta$. The dealer strives to avoid this situation because he wants to exploit the information contained in the density of types. This requires that he offers *incentive-compatible* books, i.e. those that satisfy

$$\max_{\widetilde{\theta} \in \Theta} \big\{ u\big(\theta, q(\widetilde{\theta})\big) - \tau(\widetilde{\theta}) \big\} = u\big(\theta, q(\theta)\big) - \tau(\theta).$$

In the presence of an incentive-compatible book, the contract that yields a trader of type θ his indirect utility is precisely the one the dealer has designed for him.

The dealer's objective is to maximize his expected income from engaging the traders. Taking into

[10] Clearly, from the dealer's perspective, once a trader of type θ has chosen his outside option, it is irrelevant whether or not $\mathrm{w}(\cdot;\pi) < 0$. However, we show below that, in general, traders with better outside options get better deals in the DM.

[11] The *Revelation Principle* (see, e.g. Myerson (1991)) states that, when studying outcomes in hidden-information games such as ours, there is no loss of generality in focusing on direct-revelation mechanisms, i.e. those mechanisms where the set of types indexes the contracts. Furthermore, from the *Taxation Principle* (see e.g. Rochet (1985)) there is also no loss of generality in writing $\tau(\theta)$ instead of $T(q(\theta))$.

account the impact of the CN on the traders' optimal actions, his problem is to devise (q^*, τ^*) so as to solve

$$\mathcal{P}(\pi) := \begin{cases} \sup_{(q,\tau)} \int_{\Theta_e^c(q,\tau;\pi)} \Big(\tau(\theta) - C\big(q(\theta)\big)\Big) f(\theta) d\theta, \\ \text{subject to incentive compatibility} \\ (q(\theta), \tau(\theta)) \in \operatorname{argmax}_{\widetilde{\theta}\in\Theta} \big\{u\big(\theta, q(\widetilde{\theta})\big) - \tau(\widetilde{\theta})\big\}, \quad \forall \theta \in \Theta \\ \text{and } \tau \text{ absolutely continuous.} \end{cases}$$

From the *Envelope Theorem*, if a contract $\big\{(q(\theta), \tau(\theta)), \theta \in \Theta\big\}$ is incentive compatible, then $\psi_1(q(\theta))$ belongs to the subdifferential $\partial v(\theta)$. Given that for almost all $\theta \in \Theta$ it holds that $\partial v(\theta) = v'(\theta)$ and ψ_1 is strictly increasing, we have that for almost all $\theta \in \Theta$

$$q(\theta) = \psi_1^{-1}\big(v'(\theta)\big). \tag{1}$$

Therefore, starting from a convex indirect-utility function we can recover, for almost all types, the quantities in the incentive-compatible book that generated it. Furthermore, the indirect utility function may be written as

$$\begin{aligned} v(\theta) &= \theta \psi_1\big(\psi_1^{-1}\big(v'(\theta)\big)\big) + \psi_2\big(\psi_1^{-1}\big(v'(\theta)\big)\big) - \tau(\theta) \\ &= \theta\, v'(\theta) + (\psi_2 \circ \psi_1^{-1})\big(v'(\theta)\big) - \tau(\theta). \end{aligned} \tag{2}$$

It follows from Eqs. (1) and (2) that the traders' indirect utility function contains all the information about the quantities and the pricing schedule, which allows us to write $\Theta_e^c(v;\pi)$ instead of $\Theta_e^c(q,\tau;\pi)$. In particular, introducing the functions

$$\widetilde{K}(q) := C\big(\psi_1^{-1}(q)\big) - \psi_2\big(\psi_1^{-1}(q)\big) \quad \text{and} \quad i(\theta, v, q) := \theta \cdot q - v - \widetilde{K}(q)$$

and denoting by $\mathcal{C}$ the set of all real-valued convex functions over Θ, we can restate the dealer's problem as

$$\mathcal{P}(\pi) = \sup_{v\in\mathcal{C}} \int_{\Theta_e^c(v;\pi)} i\big(\theta, v(\theta), v'(\theta)\big) f(\theta) d\theta.$$

We prove in Theorem 2.4 below that, under suitable assumptions, Problem $\mathcal{P}(\pi)$ admits a solution. The latter is, in fact, quasi-unique in the sense that on the set of participating types the solution is indeed unique. However, traders are excluded by offering them any incentive-compatible, indirect-utility function that lies below u_0. In other words, there is no uniqueness on the set of excluded types. From the traders' point of view there is no ambiguity: they either trade in the dealer market or they take their outside option. The non-uniqueness is also a non-issue for the dealer because it it only appears in subdomains of the type space that he does not access. With this in mind, in the sequel we denote by $v(\cdot;\pi)$ "the" solution to Problem $\mathcal{P}(\pi)$, which requires the following technical assumption:

Assumption 2.2. *The functions ψ_1, ψ_2 and C are such that $\widetilde{K}$ is strictly convex, coercive, continuously differentiable and it satisfies $\widetilde{K}'(0) = 0$.*

For any $v \in \mathcal{C}$, we refer to $\Theta_0(v) := \big\{\theta \in \Theta \,|\, v(\theta) = 0\big\}$ as the set of *reserved traders*. Determining this set is essential to our analysis, as it is precisely at the *boundary types* where $t(0_-)$ and $t(0_+)$ are determined. We prove in Lemma B.1 that, by virtue of Assumption 2.1, these limits are always well defined. We prove in Proposition A.2 that there is no loss of generality in assuming that any admissible

$v \in \mathcal{C}$ satisfies $v(0) = 0$; thus, $\Theta_0(v) \neq \emptyset$. For simplicity, given π we write

$$\Theta_0(\pi) := \Theta_0(v(\cdot; \pi)).$$

Remark 2.3. *A well defined spread requires $\Theta_0(\pi)$ to be a proper interval $[\underline{\theta}_0(\pi), \overline{\theta}_0(\pi)]$. We show below that this follows from Assumption 2.1. There must also be an $\epsilon > 0$ such that $(\underline{\theta}_0(\pi) - \epsilon, \underline{\theta}_0(\pi))$ and $(\overline{\theta}_0(\pi), \overline{\theta}_0(\pi) + \epsilon)$ belong to the set of fully-serviced traders. The existence of such an ϵ is proved in Lemma B.1. Economically, this conditions means that the CN is not beneficial for low-type traders. There are several instances where the proofs of our results concern conditions on points to the left of $\underline{\theta}_0(\pi)$ or to the right of $\overline{\theta}_0(\pi)$ that are analogous. So as to streamline the presentation, whenever we find ourselves in one of these "either-or" situations, we deal only with the positive case.*

Our first main result pertains the existence of a solution to the dealer's problem. The corresponding proof is presented in Appendix A.

Theorem 2.4. *Problem $\mathcal{P}(\pi)$ admits a solution, which is unique on the set of participating types.*

Remark 2.5. *If the dealer can profitably match all traders' outside option, then the quasi-uniqueness of a solution to Problem $\mathcal{P}(\pi)$ is in fact uniqueness and it follows directly from Assumption 2.2. Indeed, in that case $\big(i(\theta, v(\theta), v'(\theta))\big)_+ = \big(i(\theta, v(\theta), v'(\theta))\big)$ and problem $\widetilde{\mathcal{P}}(\pi)$ reduces to maximizing a strictly concave, coercive functional over a convex set that is closed with respect to uniform convergence. In the general case, we construct the quasi-unique solution in Section 3.2. Assumption 2.2 remains crucial, as it guarantees that the maximization problems through which we define the optimal quantities have unique maximizers.*

In the following section we analyze the effect of the CN on the spread and the set of participating traders. This requires us to compare scenarios with and without the presence of a CN. In the sequel we use the subindices "m" and "o" to distinguish structures or quantities with and without a CN, respectively.

3 The impact of a crossing network

In this section we look at the impact that a CN has on the spread, on participation and on the traders' welfare. In order to do so, we provide a characterization of the solution to Problem $\mathcal{P}(\pi)$.

3.1 A benchmark without a CN

We first analyze the benchmark case where the traders do not have access to a CN, i.e. all traders' outside option is zero. The corresponding dealer's problem is denoted by $\mathcal{P}_o$ whose solution we characterize using a Lagrange-multiplier approach. To this end, let us introduce the following definition:

$$I[v] := \int_\Theta i\big(\theta, v(\theta), v'(\theta)\big) f(\theta) d\theta.$$

Let $BV_+(\Theta)$ be the space of non-negative functions of bounded variation $\gamma : \Theta \to \mathbb{R}_+$, which we place in duality with $C(\Theta, \mathbb{R})$, the space of real-valued, continuous functions on Θ, via the standard pairing

$$\langle v, \gamma \rangle := \int_\Theta v(\theta) d\gamma(\theta)$$

for $v \in C(\Theta, \mathbb{R})$, where $d\gamma$ is the distributional derivative of γ. It follows from Pontryagin's Maximum Principle and the fact that f is a probability density function that there is no loss of generality in assuming that γ is absolutely continuous and that $\gamma(\overline{\theta}) = 1$. The Lagrangian for the dealer's problem is

$$\mathcal{L}(v, \gamma) := I[v] + \langle v, \gamma \rangle, \quad v \in \mathcal{C},$$

with corresponding Karush-Kuhn-Tucker conditions

$$\langle v, \gamma \rangle = 0 \quad \text{and} \quad d\gamma(\theta) = 0 \Rightarrow v(\theta) > 0. \tag{3}$$

Regularity properties of the solutions to variational problems subject to convexity constraints were studied in Carlier and Lachand-Robert (2001). Their methodology can be directly adapted to prove the following result, which formalizes the *vox populi* saying "quality does not jump".

Proposition 3.1. *If $v \in \mathcal{C}$ is a stationary point of $\mathcal{L}(v, \gamma)$, then $v \in C^1(\Theta)$.*

The fact that, at the optimum, the mapping $\theta \mapsto v'(\theta)$ is continuous, implies that q is also a continuous function of the types. This is extremely useful, specially in the presence of a CN. If we integrate by parts, then $\mathcal{L}(v, \gamma)$ can be transformed into

$$\Sigma(q, \gamma) := \int_{\Theta} \left(\left(\theta + \frac{F(\theta) - \gamma(\theta)}{f(\theta)} \right) \psi_1\big(q(\theta)\big) - \widetilde{C}\big(q(\theta)\big) \right) f(\theta) d\theta,$$

where $q(\theta) = \psi_1^{-1}\big(v'(\theta)\big)$, as described above, and $\widetilde{C}(q) := C(q) - \psi_2(q)$. The idea is to maximize the mapping

$$q \mapsto \sigma(\theta, q, \Gamma) := \left(\theta + \frac{F(\theta) - \Gamma}{f(\theta)} \right) \psi_1(q) - \widetilde{C}(q)$$

pointwise, for a given Γ (we use Γ whenever we are dealing with an arbitrary but fixed value of γ). From Assumption 2.2 it follows that we can write down the unique maximizer as

$$l(\theta, \Gamma) := K^{-1}\left(\frac{F(\theta) + \theta\, f(\theta) - \Gamma}{f(\theta)} \right),$$

where $K(q) := \widetilde{C}'(q)/\psi_1'(q)$. For each $\theta \in \Theta$ and $\Gamma \in [0, 1]$, the quantity $l(\theta, \Gamma)$ is a candidate for the optimal $q(\theta)$ and convexity (or incentive compatibility) is verified if the mapping $\theta \mapsto l(\theta, \Gamma)$ is increasing. The crux is then to determine the Lagrange multiplier γ. In the sequel we denote $\Theta_o := \Theta_0(v_o^*)$, where v_o^* solves Problem $\mathcal{P}_o$. In other words, if $\theta \in \Theta_o$, then $q(\theta) = T(\theta) = v(\theta) = 0$.

From Lemma A.2 we have that, unless $v(\underline{\theta}) = 0$, the quantity $q(\underline{\theta}) < 0$ and the complementary-slackness condition imply that $\gamma(\theta) = 0$ for $\theta \in [\underline{\theta}, \widetilde{\theta})$ for some $\widetilde{\theta} > \underline{\theta}$. The left endpoint $\underline{\theta}_0$ of Θ_o is then determined by solving the equation

$$K^{-1}\left(\theta + \frac{F(\theta)}{f(\theta)} \right) = 0.$$

Furthermore, as v must be convex, once $v(\hat{\theta}) > 0$ then $v(\theta) > 0$ for all $\theta > \hat{\theta}$. This implies that the right endpoint $\overline{\theta}_0$ of Θ_o is determined by solving the equation

$$K^{-1}\left(\theta - \frac{1 - F(\theta)}{f(\theta)} \right) = 0.$$

The quantities $F(\theta)/f(\theta)$ and $(1 - F(\theta))/f(\theta)$ are know as the *Hazard rates*, and sufficient conditions for the mapping $\theta \mapsto l(\theta, \Gamma)$ to be non-decreasing are

$$\frac{d}{d\theta}\left(\frac{F(\theta)}{f(\theta)}\right) \geq 0 \geq \frac{d}{d\theta}\Big(\frac{1-F(\theta)}{f(\theta)}\Big),$$

see, e.g. Biais et al. (2000) for a discussion on this condition.

Let us assume that we have determined Θ_o. What remains is then to connect the participation constraint with the spread. Differentiating Eq. (2) and noting that $v'(\theta) = \psi_1(q(\theta))$ we have that

$$\tau'(\theta) = q'(\theta)\big(\theta\psi_1'(q(\theta)) + \psi_2'(q(\theta))\big).$$

Observe that $\tau'(\underline{\theta}_0)$ and $\tau'(\overline{\theta}_0)$ are in fact $T'(0_-)$ and $T'(0_+)$ as, by construction, $q(\underline{\theta}_0) = q(\overline{\theta}_0) = 0$. If we define $\phi_1 := \psi_1'(0)$ and $\phi_2 := \psi_2'(0)$, then we have that the spread is given by the expressions

$$t(0_-) = q'(\underline{\theta}_0-)\big(\underline{\theta}_0\phi_1 + \phi_2\big) \quad \text{and} \quad t(0_+) = q'(\overline{\theta}_0+)\big(\overline{\theta}_0\phi_1 + \phi_2\big). \tag{4}$$

Our objective in Section 3.2 is to compare these values to those obtained in the presence of a crossing network.

Before we proceed we present two examples so as to illustrate the use of the methodology described hitherto. The first revisits Mussa and Rosen (1978). The second is slightly more advanced. We use it below to illustrate the complex structure of optimal pricing schedules and utilities in the presence of CNs.

Example 3.2. *Let us assume that $\Theta = [-r, r]$ for some $r > 0$, that types are uniformly distributed and that*

$$u(\theta, q) = \theta q.$$

We also set $C(q) = 0.5\, q^2$. By direct computation we find that $\underline{\theta}_0 = -\frac{r}{2}$ and $\overline{\theta}_0 = \frac{r}{2}$. Given that a trader of type $\theta \in \Theta_o$ is brought down to reservation utility and hence trades $q(\theta) = 0$, the expression

$$q(\theta) = \theta + \frac{F(\theta) - \gamma(\theta)}{f(\theta)} = 2\theta + r - 2r\gamma(\theta)$$

implies that the Lagrange multiplier is

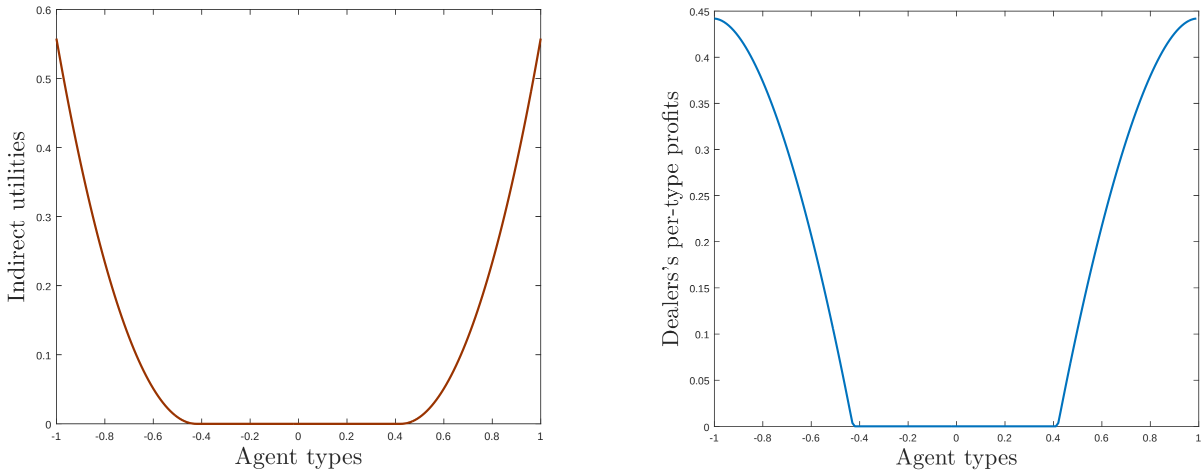
$$\gamma(\theta) = \begin{cases} 0 & ,\theta < \underline{\theta}_0 \\ \frac{1}{2} + \frac{\theta}{r} & ,\theta \in \Theta_o \\ 1 & ,\theta > \overline{\theta}_0 \end{cases} .$$

In particular, $q'(\underline{\theta}_0-) = q'(\overline{\theta}_0+) = 2$ and hence $t(0_-) = -r$ and $t(0_+) = r$. Thus, the spread increases linearly in the highest/lowest type.

Example 3.3. *Let us assume that the distribution of types over $\Theta = [-1, 1]$ is given by $f(\theta) = (2\theta+3)/4$ for $\theta \in [-1, 0)$ and $f(\theta) = (3-2\theta)/4$ for $\theta \in [0, 1]$; that $C(q) = 0.5\, q^2$ and that $u(\theta, q) - \tau = \theta \cdot q + 0.25\, q^2 - \tau$. It is straightforward to show that the conditions on the Hazard rates are satisfied and that*

$$K^{-1}\Big(\theta + \frac{F(\theta)}{f(\theta)}\Big) = 2\Big[\frac{3\theta^2 + 6\theta + 2}{2\theta + 3}\Big] \quad \textit{and} \quad K^{-1}\Big(\theta - \frac{1-F(\theta)}{f(\theta)}\Big) = 2\Big[\frac{3\theta^2 - 6\theta + 2}{2\theta - 3}\Big].$$

Furthermore, $\Theta_o \approx \big[-0.423, 0.423\big]$*. For the spread, we have that* $t(0_-) = q'(\underline{\theta}_0)\underline{\theta}_0 \approx -1.359$ *and* $t(0_+) = q'(\overline{\theta}_0)\overline{\theta}_0 \approx 1.359$*. In order to obtain* v *we integrate* q *(*$\psi_1(q) = q$*) and take into account that* $v \equiv 0$ *over* Θ_o*. We plot graph*$\{v_o\}$ *in Figure 1, as well as the per-type profits of the dealer.*



(a) Indirect Utilities (b) Profits

Figure 1: An example without a crossing network

3.2 Introducing a crossing network

We now analyze the dealer's problem when there is a CN that gives a trader of type θ the expected utility $u_0(\theta;\pi)$. Recall that the dealer's problem is

$$\mathcal{P}(\pi) = \sup_{v\in\mathcal{C}} \int_\Theta \Big(\theta\, v'(t) - v(t) - \widetilde{K}\big(v'(\theta)\big)\Big)\mathbb{1}_{\{\Theta_e^c(v)\}}(\theta) f(\theta) d\theta.$$

Dealing with the presence of the zero-one indicator function $\mathbb{1}_{\{\Theta_e^c\}}$ is quite cumbersome (see, e.g. Horst and Moreno-Bromberg (2011)) because its domain of definition may change with different book choices. In contrast to the setting studied in Horst and Moreno-Bromberg (2011), however, here the CN is passive. This lack of non-cooperative-games component allows for an alternative way to proceed, which, as mentioned in Section 2, has as a key requirement that, disregarding negative expected unwinding costs, the dealer is able match the CN. As a consequence of Assumption 2.1 and the structure of u we can show this is always possible. More specifically

Proposition 3.4. *There exists an incentive compatible book* $\big\{(q_c(\theta), \tau_c(\theta)), \theta \in \Theta\big\}$ *such that for almost all* $\theta \in \Theta$ *it holds that* $u\big(\theta, q_c(\theta)\big) - \tau_c(\theta) = u_0(\theta;\pi)$.

Observe that the incentive compatible book $\big\{(q_c(\theta), \tau_c(\theta)), \theta \in \Theta\big\}$ that replicates $u_0(\cdot;\pi)$ does not say anything about

$$\tau_c(\theta) - C(q_c(\theta)),$$

which may be negative. In other words, matching the CN for all types may result in type-wise losses. With Proposition 3.4 in hand, we may make use of the following *accounting trick*, which was introduced in Jullien (2003): let us assume that the dealer had access to a fictitious market such that the unwinding costs from trading in it, denoted in the sequel by C_c, satisfy $C_c(q(\theta)) = \tau(\theta)$. In this way, we may again assume that the dealer trades with all market participants, but now his costs of unwinding are given by the function $\mathbb{C} : \mathbb{R} \to \mathbb{R}$ defined as

$$\mathbb{C}(q) := \min\big\{C(q), C_c(q)\big\}, \quad q \in \mathbb{R}.$$

In terms of incentives, nothing is distorted by introducing the cost function $\mathbb{C}$, but we must identify the points where there is switching from using C to using C_c and vice versa. These switching points determine the market's segmentation.

If we define, for any traded quantity q, the function $\widetilde{\mathbb{C}}(q) := \mathbb{C}\big(q\big) - \psi_2\big(q\big)$, then we may re-use the machinery from Section 3.1 with minor modifications;[12] namely, denoting by $\mathbb{I}$ the dealer's utility corresponding to the cost function $\mathbb{C}$, we may write the Lagrangian of the dealer's problem as

$$\mathbb{L}(v, \gamma) := \mathbb{I}[v] + \langle v - u_0(\cdot;\pi), \gamma\rangle,$$

with the corresponding complementary-slackness conditions. From here on, we may proceed as in Section 3.1 in order to find the quantities that the dealer chooses to offer. Strictly speaking we should find the pointwise maximizer in q of the expression

$$\Big(\theta + \frac{F(\theta) - \Gamma}{f(\theta)}\Big)\psi_1(q) - \mathbb{K}(q), \tag{5}$$

where $\mathbb{K}(q) := \widetilde{\mathbb{C}}(q) - \psi_2(q)$. This may fortunately be avoided, given that whenever $\mathbb{C}(q) = C_c(q)$, the participation constraint binds and $q(\theta) = q_c(\theta)$. Next, we study the connection between the solution to the fictitious problem $\mathbb{P}(\pi)$ and that to $\mathcal{P}(\pi)$.

Whenever the participation constraint does not bind, the dealer selects the quantity to be chosen via the pointwise maximization of the mapping $q \mapsto \sigma(\theta, q, \Gamma)$. What makes the current problem trickier than the case without a CN is that now we must pay more attention to the evolution of the multiplier γ. If we compare $l(\theta, 0)$ and $l(\theta, 1)$ to $q_c(\theta)$ we may pinpoint the set where the participation constraint may bind. Observe that $\big\{l(\theta, 1), \theta \in \Theta\big\}$ and $\big\{l(\theta, 0), \theta \in \Theta\big\}$ are the sets of the lowest and highest quantities the dealer may offer in an individually-rational way. Hence, as long as $l(\theta, 1) \le q_c(\theta) \le l(\theta, 0)$, there is the possibility of *profitable matching*.

There might be instances where the participation constraint is binding for some type $\theta \in \Theta$, i.e. $\big(q(\theta), \tau(\theta)\big) = \big(q_c(\theta), \tau_c(\theta)\big)$, and $\tau_c(\theta) - C\big(q_c(\theta)\big) < 0$. In such cases $\mathbb{C}\big(q_c(\theta)\big) = C_c\big(q_c(\theta)\big)$ and $\theta \in \Theta_e(v)$ for the corresponding indirect utility function, and we say there is *exclusion*.

Remark 3.5. *It is at this point that the quasi-uniqueness mentioned in Remark 2.5 can be addressed. The dealer's problem $\mathbb{P}(\pi)$ using the cost function $\mathbb{C}$ results in the condition*

$$\big(i(\theta, v(\theta), v'(\theta))\big)_+ = \big(i(\theta, v(\theta), v'(\theta))\big)$$

being trivially satisfied. As a consequence, problem $\mathbb{P}(\pi)$ admits a unique solution. The latter coincides, by construction, with the solution to $\mathcal{P}(\pi)$ whenever $\mathbb{C}(q(\theta)) = C(q(\theta))$. The caveat is that the solution to problem $\mathbb{P}(\pi)$ is blind towards what is offered to excluded types, as their outside option is costlessly matched (they are effectively reserved). Constructing incentive compatible contracts for the excluded types is, thanks to the convexity of the indirect utility function, relatively simple. For instance if an interval of

[12] Observe that Assumption 2.1 and Proposition 3.4 imply that $\widetilde{\mathbb{C}}$ satisfies Assumption 2.2.

types (θ_1, θ_2) were excluded (but θ_1 and θ_2 participated) one could consider any two supporting lines to graph$\{v(\cdot;\pi)\}$ at $(\theta_1, v(\theta_1;\pi))$ and $(\theta_2, v(\theta_2;\pi))$. From the resulting indirect-utility function on (θ_1,θ_2) one could extract the corresponding quantities and prices. The resulting global convexity of the indirect-utility function offered by the dealer would imply that all incentives would remain unchanged. Whether the dealer would suffer losses from the contracts offered to types on (θ_1,θ_2) would be irrelevant, given the corresponding traders would not participate.

As mentioned above, here it is not necessary to determine $\gamma(\theta)$ in order to do likewise with $q(\theta)$. On the other hand, however, if we interpret γ as the shadow cost of satisfying the participation constraint, we may wish to identify the multiplier so as to have a measure of the impact of the CN on the dealer's profits. The following result, which deals with points where there is switching between matching and fully servicing, extends Proposition 3.1.

Proposition 3.6. *For $\pi \in \mathbb{R}^2$ given, let $\widetilde{\theta} \in \Theta$ be such that there exists $\epsilon > 0$ such that $v(\theta;\pi) = u_0(\theta;\pi)$ on $(\widetilde{\theta}-\epsilon, \widetilde{\theta}]$ and $v(\theta;\pi) > u_0(\theta;\pi)$ on $(\widetilde{\theta}, \widetilde{\theta}+\epsilon]$. Furthermore, assume that*

$$\int_{\widetilde{\theta}-\epsilon}^{\widetilde{\theta}} \big(\tau(\theta) - C(q(\theta))\big) f(\theta) d\theta > 0,$$

where $\big\{\big(q(\theta), \tau(\theta)\big), \theta \in \Theta\big\}$ implements $v(\cdot;\pi)$. In other words, there is profitable matching on $(\widetilde{\theta}-\epsilon, \widetilde{\theta}]$ and the dealer fully services types on $(\widetilde{\theta}, \widetilde{\theta}+\epsilon]$. Then $\partial v(\widetilde{\theta};\pi)$ is a singleton. The result also holds if the order of the matching and full-servicing intervals is switched.

The rationale behind Proposition 3.6 is that, as long as the dealer is able to match the traders' outside option without incurring in a loss, it is possible to normalize the latter to zero and directly apply Proposition 3.1. This is, naturally, not the case when matching u_0 results in losses. We put Proposition 3.6 to work in Example 3.10.

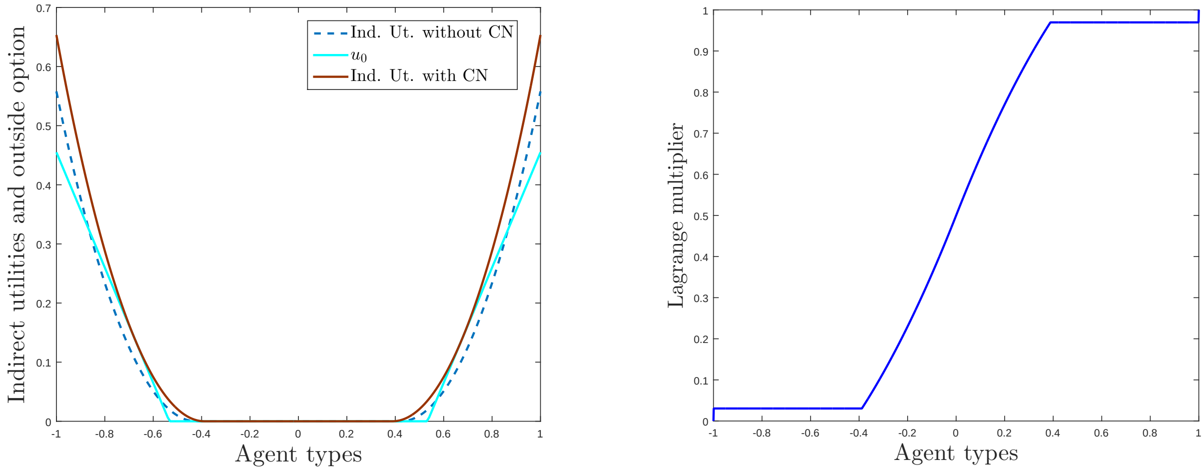
Before moving on, we present below a modification to Example 3.3 that shows how even traders without access to a non-trivial outside option benefit from the presence of the CN and that the optimal Lagrange multiplier need not be continuous.

Example 3.7. *Let f, Θ, C and u be as in Example 3.3 and assume that the CN offers the traders the following expected profits:*

$$u_0(\theta;(3.2,3.2)) = \begin{cases} -0.975\theta - 0.52, & \text{if } \theta \le -\frac{8}{15}; \\ 0.975\theta - 0.52, & \text{if } \theta \ge \frac{8}{15}; \\ \text{convex and negative for } \theta \in (-\frac{8}{15}, \frac{8}{15}). \end{cases}$$

Matching this outside option would require the dealer to offer the contracts $(\pm 0.975, 0.52)$. This is profitable, hence the indirect utility never lies below u_0. To illustrate this, we have plotted the indirect-utility function in Figure 2(a). It strictly dominates the one plotted in Figure 1(a) for all types who earn positive profits. The smooth pasting condition ($l(\theta,\gamma(\theta)) = q_c(\theta)$ where v touches u_0, i.e. in ± 0.675) determines the optimal Lagrange multiplier, namely $\gamma(-1) = 0$ and $\gamma \equiv 0.030$ on $(-1,-0.389]$. For positive types we obtain symmetrically $\gamma(1) = 1$ and $\gamma \equiv 0.970$ on $[0.389, 1)$. The new spread, given by $\big(t(0_-), t(0_+)\big) = (-1.282, 1.282)$, is strictly smaller than in the case without a CN.

The following theorem, the second of our main results, analyzes the impact of the CN on the DM and the traders' welfare.



(a) Indirect Utilities (b) Lagrange Multiplier

Figure 2: An example without exclusion

Theorem 3.8. *For a given price $\pi = (\pi_-, \pi_+)$ let $\mathcal{S}_m$ and $\mathcal{S}_o$ be the spreads with and without the presence of the crossing network and v_o and $v(\cdot;\pi)$ the corresponding indirect-utility functions, respectively. In the presence of the crossing network*

1. *less types are reserved, i.e. $\Theta_0(v_o) \supseteq \Theta_0(\pi)$. Furthermore, the inclusion is strict if there exists $\theta \in \Theta$ such that $u_0(\theta;\pi) > v_o(\theta)$;*
2. *if the types are uniformly distributed ($f \equiv (\overline{\theta} - \underline{\theta})^{-1}$) the spread narrows, i.e. $\mathcal{S}_o \geq \mathcal{S}_m$;*
3. *the type-wise welfare increases, i.e. $v_o(\theta) \leq v(\theta;\pi)$ for all $\theta \in \Theta$.*

We finalize this section with two examples that showcase the results obtained thus far. Example 3.9 showcases that, in the simple case where the outside option is such that the dealer (only) excludes all high-enough (in absolute value) types, then the results of Theorem 3.8 follow trivially.

Example 3.9. *Let us revisit Example 3.2 with an extremely steep outside option that warrants exclusion, namely, for $r_0 < r$ let*

$$u_0(\theta) = \begin{cases} \infty, & \text{if } \theta \in [-r, -r_0) \bigcup (r_0, r]; \\ 0, & \text{otherwise.} \end{cases}$$

Recall that, for a given value Γ of the Lagrange multiplier, the corresponding quantity is

$$q(\theta;\Gamma) := 2\theta + r - 2r\Gamma.$$

In Example 3.2 the participation constraint does not bind for high types. In particular, $\gamma \equiv 0$ on $[-r, \underline{\theta}_0)$ and to find the left-hand endpoint of the reserved set we set $\Gamma = 0$ and solve $2\theta + r = 0$. In the current setting, the participation constraint binds for $\theta < -r_0$ and the multiplier is constant on $(-r_0, \underline{\theta}_0(\Gamma))$, where

$$\underline{\theta}_0(\Gamma) := -\frac{r}{2}\big[1 - 2\Gamma\big].$$

By construction, the choice of Γ bears no weight on the trader types that are serviced to the left of $\theta = -r_0$, but only on how many additional low types benefit from the presence of the outside option. By integrating

$q(\theta;\Gamma)$ and noting that the corresponding indirect-utility function $v(\cdot;\Gamma)$ must satisfy $v(\underline{\theta}_0(\Gamma);\Gamma)=0$, we have, for $\theta \in [-r_0, \underline{\theta}_0(\Gamma)]$

$$v(\theta;\Gamma) = \theta^2 + \theta r\big[1-2\Gamma\big] + \frac{r^2}{4}\big[1-2\Gamma\big]^2.$$

Given that the indirect-utility function also satisfies $v(\theta;\Gamma) = \theta q(\theta;\Gamma) - \tau(\theta;\Gamma)$, we have that the dealer market on $[-r_0, \underline{\theta}_0(\Gamma)]$ is described by the quantity-price pairs $\big(q(\theta;\Gamma), \theta^2 - \frac{r^2}{4}\big[1-2\Gamma\big]^2\big)$. As a consequence, the per-type profit is

$$\Pi(\theta;\Gamma) := -\theta^2 - \frac{3}{4}r^2\big[1-2\Gamma\big]^2 - 2\theta r\big[1-2\Gamma\big],$$

where the third term on the right-hand side is positive and dominates the first two. Finally, we have that each choice of Γ results in the dealer obtaining the aggregate profits from negative types

$$P(\Gamma) := \frac{1}{2r}\int_{-r_0}^{\underline{\theta}_0(\Gamma)} \Pi(\theta;\Gamma)d\theta.$$

The mapping $\Gamma \mapsto P(\Gamma)$ is strictly concave and the first-order conditions yield that it is maximized at $\Gamma = (r-r_0)/(2r)$. As a result $\underline{\theta}_0(\Gamma) = -r_0/2$ and $v(\theta;\Gamma) = \theta^2 + r_0\theta + r_0^2/4$, which correspond to the boundary of the reserved set and the indirect-utility function for negative trader types in the problem without a CN on $[-r_0, r_0]$.

Example 3.10. *We stay with the basic setup of Examples 3.3 and 3.7, but now assume that $u_0(\theta;\pi) = \left(\frac{1-\pi_+}{3}\theta^{6/5} - 0.001\right)_+$ for $\theta \geq 0$ and $u_0(\theta;\pi) \equiv 0$ otherwise. For any type θ such that $u_0(\theta) > 0$ it holds that*

$$\big(q_c(\theta), \tau_c(\theta)\big) = \Big(\frac{2}{5}(1-\pi_+)\theta^{1/5}, \frac{2}{5}(1-\pi_+)\theta^{6/5} + \frac{1}{25}(1-\pi_+)^2\theta^{2/5} - \big(\frac{1}{3}(1-\pi_+)\theta^{6/5} - 0.001\big)_+\Big).$$

We assume $\pi = (0, 1/2)$. The first thing to notice is that the dealer's per-type profit for offering $(q_c(\theta), \tau_c(\theta))$, i.e. $\tau_c(\theta) - C(q_c(\theta)) = \theta^{6/5}/30 - \theta^{2/5}/100 + 0.001$, is negative for types $\theta \in (0.0035, 0.1667)$. On the other hand, the inequality $u_0(\theta;1/2) \geq 0$ only holds for $\theta \geq 0.014$. Combining both arguments we see that $\Theta_e(\pi) \subset (0.014, 0.1667)$. Next we observe that the inequality

$$l(\theta,1) = K^{-1}\Big(\theta - \frac{1-F(\theta)}{f(\theta)}\Big) \geq \frac{\sqrt[5]{\theta}}{5}$$

holds for all $\theta \in [0.4761, 1]$. As a consequence, we have that profitable matching may occur on the interval $(0.1667, 0.4761)$, over which $q(\theta) = q_c(\theta)$ and $\mathbb{C}\big(q(\theta)\big) = C\big(q(\theta)\big)$. Furthermore, Proposition 3.6 implies that the corresponding indirect utility function is differentiable at $\theta = 0.4761$. In order to obtain $v(\theta;\pi)$ for $\theta \in [0.4761, 1]$, we integrate $l(\cdot, 1)$ and determine the corresponding integration constant c by equating

$$2\int_0^{0.4761}\left(\frac{3\theta^2 - 6\theta + 2}{2\theta - 3}\right)d\theta + c = \frac{1}{6}(0.4761)^{6/5} - 0.001.$$

We know from the example without a CN that $\gamma(t) = 0$ for $\theta \in [-1, -0.423)$. On $[-0.423, 0)$ the multiplier must satisfy

$$K^{-1}\Big(\theta - \frac{\gamma(\theta) - F(\theta)}{f(\theta)}\Big) = 0,$$

which results in $\gamma(\theta) = (3\theta^2 + 6\theta + 2)/4$ on the said interval. What remains to be determined is $\overline{\theta}_0$ and $\gamma(\overline{\theta}_0)$. To this end, we define the family of functions $v(\cdot;\Gamma)$ such that $v'(\theta;\Gamma) = l(\theta,\Gamma)$ whenever this quantity is positive and $v(\theta;\Gamma) = 0$ for $\theta \in [0, \theta(\Gamma)]$, where $\theta(\Gamma)$ is the solution to the equation $l(\theta,\Gamma) = 0$.

As $\gamma(0) = 0.5$*, we have that* $\Gamma > 0.5$.[13] *In fact,* $\Gamma = \gamma(\overline{\theta}_0) = 0.5105$, $\overline{\theta}_0 = 0.007$ *and the intersection of* $v(\cdot;\Gamma)$ *and* $u_0(\cdot;1/2)$ *occurs at* $\theta = 0.0159$.

Summarizing, the types on $[-1,-0.423) \cup (0.007, 0.0159] \cup (0.1667, 1]$ *are fully serviced, those on* $[-0.423, 0.007]$ *are reserved and the ones that lie on* $(0.0159, 0.1667)$ *are excluded. The left-hand side of the spread is the same as in the example without a CN, whereas the right-hand side is* $t(0_+) = 0.0281$. *This is significantly smaller than in Example 3.3.*

Determining $\gamma(\theta)$ *on* $(0, 0.007]$ *is relatively simple, as we again must solve* $l(\theta,\gamma(\theta)) = 0$*, which results in* $\gamma(\theta) = (-3\theta^2 + 6\theta + 2)/4$. *Finally, in order to determine* γ *on* $\Theta_e(\pi)$ *we must rewrite the virtual surplus using* $\mathbb{C}(q(\theta)) = \tau_c(\theta)$*, which results in*

$$\mathbb{C}(q) = (5^5/6)q^6 - (1/4)q^2 + 0.001.$$

The pointwise maximization of the resulting virtual surplus must equal $q_c(\theta) = \sqrt[5]{\theta}/5$. *After some lengthy arithmetic that we choose to spare the reader from, we obtain*

$$\gamma(\theta) = F(\theta) - f(\theta)\left[5^5 q_c(\theta)^5 - \theta\right] = F(\theta) \quad \textit{for } \theta \in \Theta_e(\pi).$$

Finally, in the profitable-matching region we solve $l(\theta,\gamma(\theta)) = \sqrt[5]{\theta}/5$ *so as to find the multiplier, which yields*

$$\gamma(\theta) = F(\theta) - f(\theta)\left[\frac{1}{10}\theta^{1/5} - \theta\right] \quad \textit{for } \theta \in [0.1667, 0.4761).$$

or

$$\gamma(\theta) = \frac{1}{10}\theta^{1/5} \cdot \frac{2\theta - 3}{4} - \frac{3\theta^2 - 6\theta - 2}{4} \quad \textit{for } \theta \in [0.1667, 0.4761).$$

Observe that, in contrast with Example 3.7, here $\gamma(\theta) = 1$ *for types that are strictly smaller than one. This means that the rightmost types do not profit from the introduction of the CN via changes in the quantities they are offered, but rather from changes in the corresponding prices. Intuitively speaking this has to do with how steep the outside option is for large types and, as a consequence, whether or not it is matched over a non-trivial interval.*

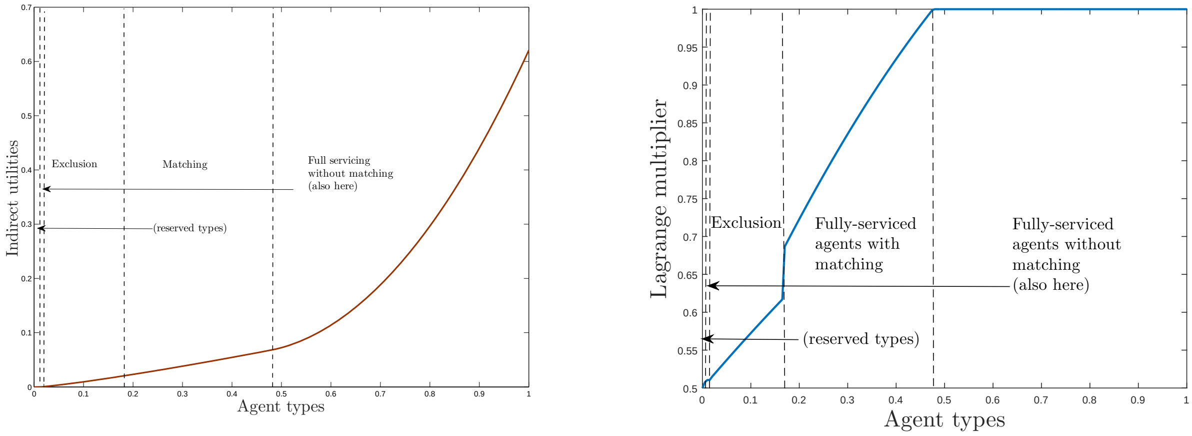
We present in Figure 3(a) the indirect utilities for positive types (the ones for negative ones being the same as in Figure 1(a)). The values of γ *have been plotted in Figure 3(b). In Figure 4 we provide a magnification around small values of* θ *so as to highlight the switching between reservation, full servicing and exclusion. Observe the jump of the Lagrange multiplier at the boundary between fully-serviced and excluded types (Figure 4(b)) and between excluded and matched ones (Figure 3(b)).*

We revisit this example in the upcoming section, where we look into the existence of equilibrium prices in the CN.

4 An equilibrium price in the crossing network

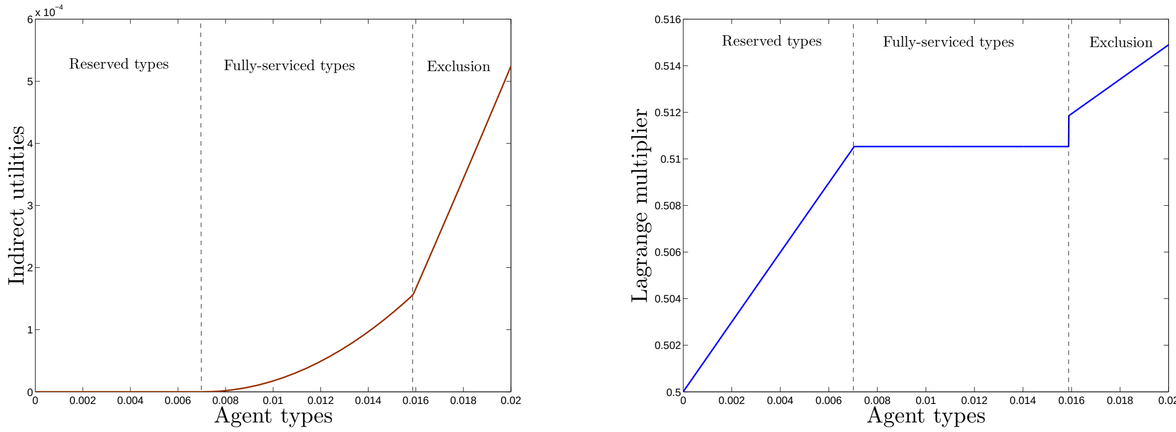
Motivated by the fact that prices in CNs are obtained from those in a primary venue, it is natural to assume that pricing in the DM has an impact on the pricing schedule π. For example, trading in the CN could take place at the best-bid and best-ask prices of the primary market. We analyze such an example, within a portfolio-liquidation framework, in Section 5.

[13] Pasting when passing from servicing to excluding need not be smooth.



(a) Indirect Utilities (b) Lagrange Multiplier

Figure 3: An example with exclusion



(a) Indirect Utilities (b) Lagrange Multiplier

Figure 4: An example with exclusion (magnified)

The pecuniary interaction between the DM and the CN, however, is not unidirectional: the dealer anticipates the effect that his choice of book structure has on the CN. Our main focus is the impact of the CN on the spread in the DM. Specifically, if we denote by $t(0;\pi) := \big(t(0_-;\pi), t(0_+;\pi)\big)$ the best bid-ask prices in the DM for a given CN price schedule π, then we call π^* an *equilibrium price* if $\pi^* = t(0;\pi^*)$. In this section we analyze the existence of an equilibrium price π^*.

We make the following natural assumption on the impact of π on the traders' outside option.

Assumption 4.1. *Let $\pi_1 \leq \pi_2$, where "$\leq$" is the lexicographic order in $\mathbb{R}^2$, then for all $\theta \in \Theta$ it holds that $u_0(\theta;\pi_1) \geq u_0(\theta;\pi_2)$. Furthermore, we assume that there exists $\big(\underline{\pi}_-, \overline{\pi}_+\big) \in \mathbb{R}^2$ such that $u_0(\cdot;\pi) \leq 0$ for all (π_-, π_+) such that $\pi_- \leq \underline{\pi}_-$ and $\overline{\pi}_+ \leq \pi_+$.*

Observe that, from Assumption 4.1, there is no loss of generality in assuming that π^* belongs to some closed and bounded subset of $\mathbb{R}^2$, which we denote by Π. As a consequence we have that $t(0;\cdot) : \Pi \to \Pi$.

We are now ready to state of our third main result.

Theorem 4.2. *If types are uniformly distributed, then the mapping $\pi \mapsto t(0;\pi)$ has a fixed point.*

Summarizing, we have that the dealer can correctly anticipate the movements in prices in the CN when he designs the optimal pricing schedule for the DM. Furthermore, the presence of the CN is beneficial in terms of liquidity, market participation and the traders' welfare.

Remark 4.3. *The requirement of uniformly distributed types can be relaxed to the extent that if f and K are such that Conditions* (B1) *are satisfied, then the required monotonicity properties still apply. Unfortunately, these conditions cannot be verified ex-ante because they include the end points of the set of reserved traders.*

Example 4.4. *Let us go back to Example 3.10 (with exclusion), but introduce the feedback loop between the DM and the CN through the iteration $\pi_{i+1} = t(0;\pi_i)$. We initialize the recursion by setting $\pi_0 = (0, 1/2)$ and $\kappa = 0.001$, which are the parameters in the aforementioned example.*

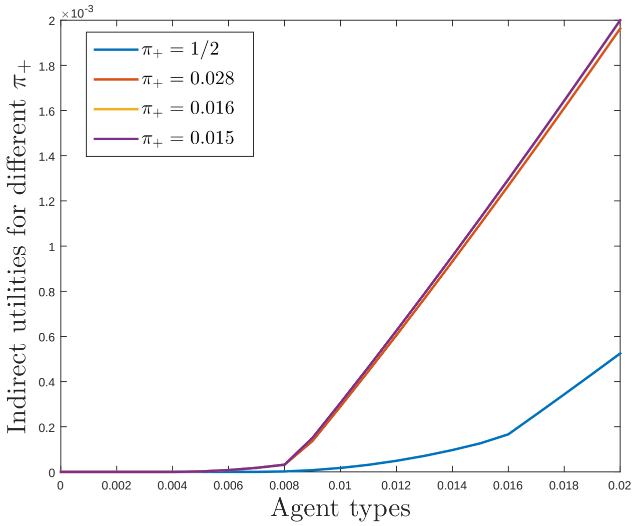


Figure 5: The indirect-utility functions corresponding to the iteration $\pi_{i+1} = t(0;\pi_i)$.

We observe a very swift convergence. Indeed, it takes only four iterations to reach $\|v(\cdot;\pi_i) - v(\cdot;\pi_{i+1})\|_\infty \leq 10^{-5}$ and the indirect-utility functions in the third and fourth iteration are almost indistinguishable. The equilibrium price is $\pi^* = (0, 0.015)$. We present in Figure 5 the plots of the first four iterates. It is evident that each iteration results in a smaller set of reserved traders and in a higher indirect utility for all types. The spreads, the right endpoints of the reserved regions, the Lagrange multipliers at the right endpoint of the reserved regions and the exclusion regions are provided in Table 1. It is interesting to observe that, as the spread decreases to its equilibrium level, the number of trader types that are reserved decreases and the sets of excluded types grow (in terms of inclusions). This last fact obeys the fact that, when the traders have a more attractive outside option, it is harder for the dealer to match it profitably.

5 Portfolio liquidation and dark-pool trading

In this section we present an application of our methodology to portfolio liquidation. We assume that the market participants' aim is to liquidate their current holdings on some traded asset. The sizes of the

Table 1: The numbers of the feedback loop

π_+	Θ_o	Γ	$\Theta_e(\pi_+)$
1/2	[-0.423,0.0070]	0.5105	[0.0159, 0.1667]
0.0281	[-0.423,0.0040]	0.5061	[0.0083, 0.4872]
0.0161	[-0.423,0.0040]	0.5060	[0.0082, 0.4954]
0.0158	[-0.423,0.0040]	0.5060	[0.0082, 0.4955]

traders' portfolios are heterogeneous and saying that a trader's type is θ means that he holds θ shares of the asset prior to trading. We set $\Theta = [-1, 1]$ and $f \equiv 1/2$. If a trader of type θ trades q shares for τ dollars, his utility is

$$\hat{u}(\theta, q) - \tau := -\alpha(\theta - q)^2 - \tau,$$

where $0 < \alpha$ denotes the traders' (homogeneous) sensitivity towards inventory holdings. Notice that $-\alpha\theta^2$ is the type-dependent reservation utility of a trader of type θ. If we "normalize" the said utility to zero, we may write

$$u(\theta, q) - \tau = 2\alpha\theta q - \alpha q^2 - \tau.$$

In this example the crossing network takes the form of a *dark pool* (DP for short). Choosing to trade in the latter entails two kinds of costs for the traders: On the one hand, there is a direct, fixed cost $\kappa > 0$ of engaging in dark-pool trading. On the other hand, execution in the DP is not guaranteed. We denote by $p \in [0, 1]$ the probability that an order is executed where we assume for simplicity that the probability of order execution is independent of the order size. Pricing in the DP is linear. Namely, for a given execution price π, the utility that a trader of type θ extracts from submitting an order of q shares to be traded in the DP is

$$p\big[(2\theta\alpha - \pi)q - \alpha q^2\big] - \kappa,$$

where again we have normalized reservation utilities to zero. The problem of optimal submission to the DP for a θ-type trader is

$$\max_q \Big\{p\big[(2\theta\alpha - \pi)q - \alpha q^2\big]\Big\},$$

which yields the optimal submission level

$$q_d(\theta) := \theta - \frac{\pi}{2\alpha}.$$

We obtain that opting for the DP results in a trader of type θ enjoying the expected utility

$$u_0(\theta; \pi) = \alpha p \left(\theta - \frac{\pi}{2\alpha}\right)^2 - \kappa.$$

We assume that $p\pi^2 < 4\alpha\kappa$ so as to keep the DP unattractive for small types.

We assume that the dealer's costs/profits of unwinding a portfolio of size q are $C(q) = \epsilon\, q + \beta q^2$ where $\beta > 0$ and ϵ is non-negative. Observe that, as $u_0(\cdot; \pi)$ does not satisfy Assumption 2.1, some restrictions must be imposed on the problem's parameters so as to still have Lemma C.1. Namely, it must hold that

$$\pi < 2\sqrt{\frac{\alpha\kappa}{p}}. \tag{6}$$

Condition (6) imposes a hard upper bound on possible equilibrium DP prices. It should be noted that Assumption 4.1 is not satisfied by $u_0(\cdot;\pi)$, which, together with the way in which we shall define the pricing feedback loop from the DM to the DP, implies that our equilibrium result does not apply "as is" to the current setting.

5.1 The dealer market without a dark pool

In the absence of a DP, the dealer's optimal choices of quantities are, for negative types

$$l(\theta,0)=\frac{\alpha}{\alpha+\beta}(2\theta+1)-\frac{\epsilon}{2(\alpha+\beta)}$$

and for positive types

$$l(\theta,1)=\frac{\alpha}{\alpha+\beta}(2\theta-1)-\frac{\epsilon}{2(\alpha+\beta)},$$

where the boundary of Θ_0 is given by

$$\underline{\theta}_0=\frac{1}{2}\Big(\frac{\epsilon}{2\alpha}-1\Big)\quad\text{and}\quad\overline{\theta}_0=\frac{1}{2}\Big(\frac{\epsilon}{2\alpha}+1\Big).$$

In order to guarantee that $\Theta_0\subset[-1,1]$ the condition $\epsilon<2\alpha$ must be imposed on the corresponding parameters. From the relation $v'(\theta)=\psi_1\big(q(\theta)\big)$ we have that the indirect-utility function is

$$v(\theta)=\begin{cases}\frac{2\alpha^2}{\alpha+\beta}\theta^2+\frac{\alpha}{\alpha+\beta}\big(2\alpha-\epsilon\big)\theta+c_1, & \theta\le\underline{\theta}_0;\\ \frac{2\alpha^2}{\alpha+\beta}\theta^2-\frac{\alpha}{\alpha+\beta}\big(2\alpha+\epsilon\big)\theta+c_2, & \theta\ge\overline{\theta}_0,\end{cases}$$

where

$$c_1=\frac{2\alpha^2}{4(\alpha+\beta)}\Big(\frac{\epsilon}{2\alpha}+1\Big)^2+\frac{\alpha(2\alpha+\epsilon)}{2(\alpha+\beta)}\Big(\frac{\epsilon}{2\alpha}+1\Big)$$

and

$$c_2=\frac{2\alpha^2}{4(\alpha+\beta)}\Big(\frac{\epsilon}{2\alpha}-1\Big)^2-\frac{\alpha(2\alpha+\epsilon)}{2(\alpha+\beta)}\Big(\frac{\epsilon}{2\alpha}-1\Big).$$

When it comes to the spread, observe that $q'\equiv\frac{2\alpha}{\alpha+\beta}$, $\psi_1\equiv 2\alpha$ and $\psi_2\equiv 0$, which yields

$$[t(0_-),t(0_+)]=\frac{4\alpha^2}{\alpha+\beta}[\underline{\theta}_0,\overline{\theta}_0].$$

Below we analyze how the spread changes with the introduction of the DP.

5.2 The impact of a dark pool

We first take an exogenous execution price π and determine, for each $\theta\in\Theta$, what is the quantity-price pair $\big(q_c(\theta;\pi),\tau_c(\theta;\pi)\big)$ that the dealer must offer so as to match a DP with execution price π. Using the relation $q_c(\theta;\pi)=u_0'(\theta;\pi)$ we obtain

$$\begin{aligned}&q_c(\theta;\pi)=2\alpha p\left(\theta-\frac{\pi}{2\alpha}\right)\ \text{and}\\ &\tau_c(\theta;\pi)=\kappa+4\alpha^2p(\theta-\alpha p)\left(\theta-\frac{\pi}{2\alpha}\right)-\alpha p\left(\theta-\frac{\pi}{2\alpha}\right)^2.\end{aligned}\tag{7}$$

From the Envelope Theorem and the structure of $u(\theta,q)$ we have that the traders' indirect utility function satisfies

$$\frac{v'(\theta)}{2\alpha} = l\big(\theta, \gamma(\theta)\big). \tag{8}$$

In order to determine the spread in the presence of the DP we must determine $\underline{\theta}_{0,m}$ and $\overline{\theta}_{0,m}$ together with $\gamma\big(\underline{\theta}_{0,m}\big)$ and $\gamma\big(\overline{\theta}_{0,m}\big)$. For an arbitrary $\Gamma \in [0,1]$ we have

$$l(\theta,\Gamma) = \frac{\alpha}{\alpha+\beta}\big[2\theta + 1 - 2\Gamma\big] - \frac{\epsilon}{2(\alpha+\beta)}.$$

Indexed by Γ, the candidates for $\underline{\theta}_{0,m}$ are then given by

$$\underline{\theta}_{0,m}(\Gamma) = \frac{1}{2}\Big(\frac{\epsilon}{2\alpha} + 2\Gamma - 1\Big).$$

As it must hold that $\underline{\theta}_{0,m}(\Gamma) \le 0$, then $\Gamma \le 0.5(1-\epsilon/2\alpha)$. Integrating Expression (8) we have that, on the interval $[\widetilde{\theta}_m(\Gamma), \underline{\theta}_{0,m}(\Gamma)]$, the traders' indirect utility is given by

$$v(\theta;\Gamma) = \frac{2\alpha^2}{\alpha+\beta}\theta^2 + 2\alpha\Big[\frac{\alpha}{\alpha+\beta}(1-2\Gamma) - \frac{\epsilon}{2(\alpha+\beta)}\Big]\theta + c_{1,m}, \tag{9}$$

where $\widetilde{\theta}_m(\Gamma)$ is the first intersection to the left of $\underline{\theta}_{0,m}(\Gamma)$ of $v(\cdot;\Gamma)$ and $u_0(\cdot;\pi)$ and $c_{1,m}$ is determined by the equation

$$v\big(\underline{\theta}_{0,m}(\Gamma);\Gamma\big) = 0.$$

Unless the inequality $\Gamma \le 0.5(1-\epsilon/2\alpha)$ is tight, in which case the types below $\widetilde{\theta}_m(\Gamma)$ are excluded, Proposition 3.6 implies that Γ must be chosen so as to satisfy the smooth-pasting condition $u_0'\big(\widetilde{\theta}_m(\Gamma);\pi\big) = v'\big(\widetilde{\theta}_m(\Gamma);\pi\big)$, which is equivalent to

$$\widetilde{\theta}_m(\Gamma) = \Big[\frac{2\alpha}{\alpha+\beta} - p\Big]^{-1}\Big[\frac{\epsilon}{2(\alpha+\beta)} - \frac{\alpha}{\alpha+\beta}(1-2\Gamma) - \frac{p\pi}{2\alpha}\Big].$$

Observe that, besides the requirement $\Gamma \ge 0.5(1-\epsilon/2\alpha)$, the strategy to determine $\overline{\theta}_{0,m}$ is exactly the same as for $\underline{\theta}_{0,m}$. Summarizing, from Eq. (9) we observe that, if Γ_- and Γ_+ correspond to the optimal choices for the negative and positive endpoints of $\Theta_0(\pi)$, then

$$q'\big(\underline{\theta}_{0,m}(\Gamma_-)\big) = \frac{1}{2\alpha}v''\big(\underline{\theta}_{0,m}(\Gamma_-);\Gamma_-\big) = \frac{1}{2\alpha}v''\big(\overline{\theta}_{0,m}(\Gamma_+);\Gamma_+\big) = q'\big(\overline{\theta}_{0,m}(\Gamma_+)\big) = \frac{2\alpha}{\alpha+\beta}.$$

The spread is then

$$[t_m(0_-), t_m(0_+)] = \frac{4\alpha^2}{\alpha+\beta}[\underline{\theta}_{0,m}(\Gamma_-), \overline{\theta}_{0,m}(\Gamma_+)] \subset \frac{4\alpha^2}{\alpha+\beta}[\underline{\theta}_0, \overline{\theta}_0],$$

i.e. the presence of a dark pool strictly narrows the spread in the dealer's market.

5.3 An equilibrium price

A standard (but not unique) way in which dark-pool prices are generated is by computing the average of some publicly available best-bid and best-ask prices. In the case of the US, this is usually the mid-quote of the National Best Bid and Offer (NBBO). Borrowing from this idea we define the price-iteration in the DP as follows:

$$\pi_{i+1} = \frac{1}{2}\big(t_i(0_+) - t_i(0_-)\big), \quad i \in \mathbb{N},$$

where $\{t_i(0_-), t_i(0_+)\}$ are the best bid and ask prices in the DM in the presence of a DP with execution price π_i. We know from the previous section that the sequence $\{\pi_i, i \in \mathbb{N}\} \subset ((4\alpha^2)/(\alpha+\beta))[\underline{\theta}_0, \overline{\theta}_0]$; hence, by the Bolzano-Weierstrass Theorem it has at least one convergent subsequence. The limit of each of the said subsequences is an equilibrium price. The (possible) non-uniqueness of these prices is due to the fact that by virtue of its definition, the sequence of dark-pool prices need not be monotonic. The problem of non-uniqueness of equilibria in models of competing DMs and CNs has been observed before. We refer to Daniëls et al. (2013) for a detailed discussion.

6 Conclusions

We have presented a hidden-information model to study the structure of the limit-order book of a dealer who provides liquidity to traders of unknown preferences. Furthermore, we have established a link between the traders' indirect-utility function and the bid-ask spread in the DM. Making use of the aforementioned link, we have studied how the presence of a type-dependent outside option impacts the spread of the DM, as well as the set of trader types who participate in the DM and their welfare. In particular, we have shown, in a portfolio-liquidation setting, that the presence of a dark pool results in a shrinkage of the spread in the DM. Finally, we have established that, under certain conditions, the feedback loop introduced by the impact that the spread has on the structure of the outside option leads to an equilibrium price.

Appendix

Appendix A Existence of a solution to Problem $\mathcal{P}(\pi)$

In this appendix we prove the existence of a solution to the dealer's problem in the presence of a CN. Some of the arguments are somewhat standard, but we give them for completeness. The first important result that we require is that the dealer's optimal choices lead to him never losing money on types that participate.

Proposition A.1. *If $(q^*, \tau^*) : \Theta \to \mathbb{R}^2$ is an optimal allocation, then for all participating types it holds that $\tau^*(\theta) - C\big(q^*(\theta)\big) \geq 0$.*

Proof. Assume the contrary, i.e. that the set

$$\widetilde{\Theta} := \big\{\theta \,|\, v(\theta;\pi) \geq u_0(\theta;\pi), \tau^*(\theta) < C\big(q^*(\theta)\big)\big\},$$

where $v(\theta;\pi) = u\big(\theta, q^*(\theta)\big) - \tau^*(\theta)$ has positive measure. Define a new pricing schedule via

$$\widetilde{\tau}(\theta) := \max\big\{\tau^*(\theta), C\big(q^*(\theta)\big)\big\}.$$

The incentives for types in $\widetilde{\Theta}^c$ do not change because their prices remain unchanged, whereas prices for others have increased. Profits corresponding to trading with types in $\widetilde{\Theta}$ increase to zero. As a consequence the dealer's welfare strictly increases, which violates the optimality of (q^*, τ^*). $\square$

A consequence of Proposition A.1 is that, together with Assumption 2.2, it allows us to restrict the admissible set of the dealer's problem to a compact one. We prove this in several steps.

Lemma A.2. *If $v : \Theta \to \mathbb{R}$ is a non-negative, convex function that solves $\mathcal{P}$, then $v(0) = 0$.*

PROOF. Assume that $v \in \mathcal{C}$ solves $\mathcal{P}$ and $v(0) > 0$. This implies that $\psi_2\big(q(0)\big) - \tau(0) \geq 0$. Given that, from Assumption 2.1, a trader of type $\theta = 0$ has no access to a profitable outside option, then he participates. From Proposition A.1 it must then hold that $\tau(0) \geq C\big(q(0)\big)$ which in turn implies that $\psi_2\big(q(0)\big) \geq C\big(q(0)\big)$. This relation, however, can only hold for $q(0) = 0$, which implies that $\tau(0) = v(0) = 0$. $\square$

Lemma A.3. *If $v \in \mathcal{C}$ solves $\mathcal{P}$, then $|\partial v| \leq \overline{q}$.*

PROOF. From Assumption 2.2 and the compactness of Θ we have that the mapping $q \mapsto i(\theta, v, q)$ tends to $-\infty$ as $|q| \to \infty$ uniformly on Θ for $v \geq 0$. From Proposition A.1 $i\big(\theta, v(\theta), v'(\theta)\big)$ must be non-negative for all participating types, which concludes the proof. $\square$

As $\overline{q}$ could depend on π, we define

$$\mathcal{A}(\pi) := \{v \in \mathcal{C} \,|\, v \geq 0, v(0) = 0, |\partial v| \leq \overline{q}\}$$

as new admissibility set for problem $\mathcal{P}(\pi)$. The previous results show that if we replace $\mathcal{C}$ by $\mathcal{A}(\pi)$ in the definition of $\mathcal{P}(\pi)$, the solution to the problem does not change.

Corollary A.4. *The admissible set $\mathcal{A} \subset \mathcal{C}$ of Problem $\mathcal{P}$ is uniformly bounded and uniformly equicontinuous.*

PROOF. From Lemmas A.2 and A.3, a uniform bound for all $v \in \mathcal{A}$ is given by $\max_{\theta\in\Theta} \big\{u_0(\theta;\pi)\big\} + \overline{q}\|\Theta\|$. Lemma A.3 guarantees that for any $v \in \mathcal{A}$ it holds that $|\partial v| \leq \overline{q}$. In other words, $\mathcal{A}$ is composed of convex functions whose subdifferentials are uniformly bounded, hence $\mathcal{A}$ is uniformly equicontinuous. $\square$

Notice that, when it comes to determining quantities and prices for trader types who do participate, Proposition A.1 results in the dealer having to solve the problem

$$\widetilde{\mathcal{P}}(\pi) := \begin{cases} \sup_{v\in\mathcal{A}} \int_\Theta \big(i\big(\theta, v(\theta), v'(\theta)\big)\big)_+ f(\theta)d\theta \\ \text{s.t. } v(\theta) \geq u_0(\theta;\pi) \text{ for all } \theta \in \Theta. \end{cases}$$

The last auxiliary result that we need is the following proposition, whose proof is a direct consequence of Fatou's Lemma, together with Lemmas A.2 and A.3.

Proposition A.5. *The mapping*

$$v \mapsto \int_\Theta \big(i(\theta, v(\theta), v'(\theta))\big)_+ f(\theta)d\theta$$

is upper semi-continuous in $\mathcal{A}$ with respect to uniform convergence.

We are now ready to prove our first main result:

Proof of Theorem 2.4: Assume that $\mathcal{A} \bigcap \big\{v \in \mathcal{C} | v(\cdot) \geq u_0(\cdot;\pi)\big\}$ is non-empty and consider a maximizing sequence $\big\{\widetilde{v}_n\big\}_{n\in\mathbb{N}}$ of Problem $\widetilde{\mathcal{P}}(\pi)$. From Corollary A.4 we have that, passing to a subsequence if necessary, there exists $\widetilde{v} \in \mathcal{A}$ such that $\widetilde{v}_n \to \widetilde{v}$ uniformly. A direct application of Proposition A.5 yields that $\widetilde{v}$ is a solution to $\widetilde{\mathcal{P}}(\pi)$. To finalize the proof we must construct from $\widetilde{v}$ a solution to Problem $\mathcal{P}(\pi)$. To this end, let us define the sets

$$\Theta_- := \big\{\theta \in \Theta | i\big(\theta, \widetilde{v}(\theta), \widetilde{v}'(\theta)\big) < 0\big\} \quad \text{and} \quad \Theta_+ := \Theta_-^c.$$

It is well known that if a sequence of convex functions converges uniformly (to a convex function), then there is also uniform convergence of the derivatives wherever they exist, which is almost everywhere. This fact, together with the continuity of the mappings $\theta \mapsto \widetilde{v}(\theta)$ and $(\theta, v, q) \mapsto i(\theta, v, q)$, implies that

Θ_- is the union of a disjoint set of open intervals:

$$\Theta_- = \bigcup_{i=1}^{\infty}(a_i, b_i).$$

Define, for each $i \geq 1$,

$$\widetilde{v}_{a,i} := \inf\left\{q | q \in \partial\widetilde{v}(a_i)\right\} \quad \text{and} \quad \widetilde{v}_{b,i} := \sup\left\{q | q \in \partial\widetilde{v}(b_i)\right\}$$

and consider the support lines to graph$\{\widetilde{v}\}$ at a_i and b_i given by

$$l_i(\theta) = \widetilde{v}(a_i) + \widetilde{v}_{a,i}(\theta - a_i) \quad \text{and} \quad L_i(\theta) = \widetilde{v}(b_i) + \widetilde{v}_{b,i}(\theta - b_i),$$

respectively. Let $c_i \in (a_i, b_i)$ be, for each $i \geq 1$, the unique solution to the equation $l_i(\theta) = L_i(\theta)$ and define on $(a_i, b_i) =: \Theta_i$

$$v_i^*(\theta) := \begin{cases} l_i(\theta) & \theta \leq c_i; \\ L_i(\theta) & \theta > c_i. \end{cases}$$

Finally define

$$v^*(\theta) := \begin{cases} \widetilde{v}(\theta) & \theta \in \Theta_+; \\ v_i^*(\theta) & \theta \in \Theta_i, i \in \mathbb{N}, \end{cases}$$

then v^* is a solution to Problem $\mathcal{P}(\pi)$ and $\Theta_e(v^*) = \Theta_-$, which concludes the proof. □

Appendix B The impact of a CN on the DM

In order to prove Theorem 3.8, we require a result that guarantees that our notion of the spread is well defined in the presence of a CN. This could be loosely summarized by saying that the first (in terms of moving away from $\theta = 0$) types to earn positive utility trade in the DM.

Lemma B.1. *There exists $\epsilon = \epsilon(\pi)$ such that the types that belong to*

$$(\underline{\theta}_0(\pi) - \epsilon, \underline{\theta}_0(\pi)) \cup (\overline{\theta}_0(\pi), \overline{\theta}_0(\pi) + \epsilon)$$

are fully serviced.

Proof. Let us denote by $\hat{\theta}$ the positive solution to the equation $u_0(\theta; \pi) = 0$. If there exists $\eta > 0$ such that types on $(\hat{\theta}, \hat{\theta} + \eta)$ can be matched profitably, then the result follows either because $\overline{\theta}_0(\pi) < \hat{\theta}$ or because $\overline{\theta}_0(\pi) = \hat{\theta}$ and the types on $(\hat{\theta}, \hat{\theta} + \epsilon)$, for some $0 < \epsilon \leq \eta$, are fully serviced. Let us now assume that such an η does not exist, we claim then that $\overline{\theta}_0(\pi) < \hat{\theta}$ must hold. Proceeding by the way of contradiction, let us assume that $\overline{\theta}_0(\pi) = \hat{\theta}$ (which is equivalent to $\overline{\theta}_0(\pi) \geq \hat{\theta}$) and that there exists $\delta > 0$ such that $(\hat{\theta}, \hat{\theta} + \delta) \subset \Theta_e(\pi)$. This configuration can be improved upon as follows: let $a > 0$ be such that $\hat{\theta} - a > 0$. By construction $l(\hat{\theta} - a, \gamma(\hat{\theta} - a)) = 0$. Let us fix $\gamma(\theta) \equiv \gamma(\hat{\theta} - a) =: \Gamma(a)$ for $\theta \in (\hat{\theta} - a, \theta_a)$, where θ_a the solution to $v_a(\theta) = u_0(\theta; \pi)$ on $(\hat{\theta} - a, \overline{\theta}]$ if it exists or $\theta_a = \overline{\theta}$ otherwise, given that we denote by v_a the indirect-utility function corresponding to $\Gamma(a)$. In particular $\theta_a > \hat{\theta}$ and $l(\theta, \Gamma(a)) > 0$ for $\theta \in (\hat{\theta} - a, \theta_a)$.

We now have that types $\theta \in (\hat{\theta} - a, \theta_a)$ are fully serviced. By Assumption 2.1, $v_a'(\hat{\theta} - a) = 0 < u_0'(\hat{\theta}; \pi)$; therefore, there exists $a_1 > 0$ such that for all $a \leq a_1$ it holds that $\theta_a < \hat{\theta} + \delta$. If we could show that there exists $a \leq a_1$ such that the dealer could offer types in $(\hat{\theta} - a, \theta_a)$ the quantities $q_a(\theta) = l(\theta, \Gamma(a))$ at a profit, we would contradict the optimality of $\overline{\theta}_0(\pi)$ and the proof would be finalized, as incentives above θ_a would not be distorted and the dealer's profits would strictly increase. In order to do so, observe that the dealer's typewise profit when offering $q_a(\theta)$ is

$$P(\theta) := \theta\psi_1(q_a(\theta)) + \psi_2(q_a(\theta)) - v_a(\theta) - C\big(q_a(\theta)\big).$$

In particular, $P(\hat\theta - a) = 0$ and

$$\begin{aligned} P'(\hat\theta - a) &= \psi_1(q_a(\hat\theta - a)) + (\hat\theta - a)\psi_1'(q_a(\hat\theta - a))q_a'(\hat\theta - a) + v_a'(\hat\theta - a) \\ &\quad - \widetilde{C}'\big(q_a(\hat\theta - a)\big)q_a'(\hat\theta - a) \\ &= \psi_1(0) + (\hat\theta - a)\psi_1'(0)q_a'(\hat\theta - a) + v_a'(\hat\theta - a) - \widetilde{C}'\big(0\big)q_a'(\hat\theta - a) \\ &= (\hat\theta - a)\psi_1'(0)q_a'(\hat\theta - a). \end{aligned}$$

The step from the second to the third equality follows, because by construction $v_a'(\hat\theta - a) = 0$; by assumption $\psi_1(0) = 0$ and, from Assumption 2.2, $\widetilde{C}'\big(0\big) = 0$. Furthermore, given that ψ_1 is strictly increasing and $q_a'(\hat\theta - a) > 0$, then $P'(\hat\theta - a) > 0$. Therefore, there exists $b > 0$ such that $P(\theta) > 0$ if $\theta \in (\hat\theta - a, \hat\theta - a + b)$. As a consequence, if $a < a_1$ is small enough, then $P(\theta) > 0$ for $\theta \in (\hat\theta - a, \theta_a)$, as required. $\square$

We are now ready to prove our second main result:

Proof of Theorem 3.8: (1) Observe that if π is such that $\big(\underline{\theta}_0(\pi), \overline{\theta}_0(\pi)\big) = \Theta_0(\pi) \subset \Theta_o$, then the result follows immediately from Lemma B.1. If we revert the inclusion, two situations are possible, as the addition of the CN-constraint to Problem $\mathcal{P}_o$ may or may not bind for some types. The latter case being trivial, let us look at the case where there is a point $\theta_a > \overline{\theta}_0$ on which it holds that $v_o(\theta_a) = u_0(\theta_b; \pi)$ and such that $v_o(\theta) > u_0(\theta; \pi)$ for $\theta < \theta_a$ and vice versa for $\theta > \theta_a$. The Lagrange multiplier γ_m is active on $(\theta_a, \overline{\theta}]$, which implies that $\gamma_m(\theta_a) < 1$. We know from Jullien (2003), p. 9, that for all θ such that $l(\theta, \Gamma) > 0$, the latter is decreasing in Γ. As a consequence, the root of the equation

$$K^{-1}\Big(\theta + \frac{F(\theta) - \gamma_m(\theta_a)}{f(\theta)}\Big) = 0$$

is strictly smaller than that of $l(\theta, 1) = 0$, which yields the desired result.

(2) Let us denote by $t_o(0_-)$ and $t_o(0_+)$ the best bid and ask prices without the presence of a CN and by $t_m(0_-)$ and $t_m(0_+)$ the corresponding marginal prices with one; thus,

$$t_o(0_-) = q_o'(\underline{\theta}_{0,o-})\big(\underline{\theta}_{0,o}\phi_1 + \phi_2\big) \text{ and } t_o(0_+) = q_o'(\overline{\theta}_{0,o+})\big(\overline{\theta}_{0,o}\phi_1 + \phi_2\big)$$

and

$$t_m(0_-) = q_m'(\underline{\theta}_{0,m-})\big(\underline{\theta}_{0,m}\phi_1 + \phi_2\big) \text{ and } t_m(0_+) = q_m'(\overline{\theta}_{0,m+})\big(\overline{\theta}_{0,m}\phi_1 + \phi_2\big).$$

From Part (1) we know that $\underline{\theta}_{0,o} \le \underline{\theta}_{0,m}$ (both negative) and $\overline{\theta}_{0,m} \le \overline{\theta}_{0,o}$ (both positive) and, given that ϕ_1 and ϕ_2 do not depend on the presence of the CN, all we have left to do is show that

$$q_m'(\underline{\theta}_{0,m-}) \le q_o'(\underline{\theta}_{0,o-}) \text{ and } q_m'(\underline{\theta}_{0,m+}) \le q_o'(\underline{\theta}_{0,o+}).$$

Using the well-known relation $(f^{-1})'(a) = 1/f'(f^{-1}(a))$ we have that

$$\begin{aligned} q_m'(\underline{\theta}_{0,m-}) &= \frac{1}{K'\Big(K^{-1}\big(\underline{\theta}_{0,m} - \frac{\gamma(\underline{\theta}_{0,m-}) - F(\underline{\theta}_{0,m})}{f(\underline{\theta}_{0,m})}\big)\Big)} \frac{d}{d\theta}\big(\theta - \frac{\gamma(\theta) - F(\theta)}{f(\theta)}\big)\Big|_{\theta = \underline{\theta}_{0,m-}} \\ &= \frac{1}{K'\big(q_m(\underline{\theta}_{0,m})\big)} \frac{d}{d\theta}\big(\theta - \frac{\gamma(\theta) - F(\theta)}{f(\theta)}\big)\Big|_{\theta = \underline{\theta}_{0,m-}} \\ &= \frac{1}{K'(0)}\Big(1 - \frac{d}{d\theta}\big(\frac{\gamma(\underline{\theta}_{0,m-}) - F(\theta)}{f(\theta)}\big)\Big)\Big|_{\theta = \underline{\theta}_{0,m}}, \end{aligned}$$

where we have used the fact that γ is constant on $(\underline{\theta}_{0,m} - \delta, \underline{\theta}_{0,m})$ for some $\delta > 0$. We may proceed analogously for the other three quantities. We have to show that

$$\begin{aligned} \frac{1}{K'(0)}\frac{d}{d\theta}\Big(\frac{\gamma(\underline{\theta}_{0,m-})-F(\theta)}{f(\theta)}\Big)\Big|_{\theta=\underline{\theta}_{0,m}} &\geq \frac{1}{K'(0)}\frac{d}{d\theta}\Big(\frac{-F(\theta)}{f(\theta)}\Big)\Big|_{\theta=\underline{\theta}_{0,o}} \\ \frac{1}{K'(0)}\frac{d}{d\theta}\Big(\frac{\gamma(\overline{\theta}_{0,m+})-F(\theta)}{f(\theta)}\Big)\Big|_{\theta=\overline{\theta}_{0,m}} &\geq \frac{1}{K'(0)}\frac{d}{d\theta}\Big(\frac{1-F(\theta)}{f(\theta)}\Big)\Big|_{\theta=\overline{\theta}_{0,o}}, \end{aligned} \tag{B1}$$

which hold with equality under the assumption that $f \equiv (\overline{\theta}-\underline{\theta})^{-1}$.

(3) If follows from Part (1) that, if θ participates in the presence of the CN, then $q_o(\theta) \leq q_m(\theta)$. Assume now that the inequality $v_o(\theta) > v(\theta;\pi)$ holds for all θ in a non-empty interval (θ_1,θ_2) and $v_o(\theta_1) = v(\theta_1;\pi)$ and $v_o(\theta_2) = v(\theta_2;\pi)$. By the convexity of v_o and $v(\cdot;\pi)$, this would imply the existence of $\theta_3 \in (\theta_1,\theta_2)$ such that $v_o'(\theta) > v'(\theta;\pi)$ holds almost surely in (θ_1,θ_3). However $v_o'(\theta) = \psi_1(q_o(\theta))$, $v'(\theta;\pi) = \psi_1(q_m(\theta))$ and ψ_1 is strictly increasing; hence, this would imply that $q_o(\theta) > q_m(\theta)$ for almost all $\theta \in (\theta_1,\theta_3)$, which is a contradiction. □

Appendix C The existence of an equilibrium price.

The restriction of possible equilibrium prices to Π, together with Assumptions 2.1 and 4.1, yields the next result.

Lemma C.1. *There exists a non-empty interval $[\epsilon_1,\epsilon_2] \subset \Theta$ such that*

1. $0 \in (\epsilon_1,\epsilon_2)$;
2. $u_0(\theta;\pi) = 0$ *for all* $\theta \in [\epsilon_1,\epsilon_2]$ *and all* $\pi \in \Pi$.

In the sequel we make use of the results obtained in Section 3.2 to show that the mapping $\pi \mapsto t(0;\pi)$ has the required monotonicity properties so as to use the following result (see, e.g. Aliprantis and Border (2007)):

Theorem C.2. *(Tarski's Fixed Point Theorem) Let $(X,\leq)$ be a non-empty, complete lattice. If $f: X \to X$ is order preserving, then the set of fixed points of f is also a non-empty, complete lattice.*

Proof of Theorem 4.2. Lemmas B.1 and C.1 guarantee that we have a well-defined spread; thus, we may decompose the analysis of the mapping $\pi \mapsto t(0;\pi)$ into that of the mappings $\pi_- \mapsto t(0_-;\pi_-)$ and $\pi_+ \mapsto t(0_+;\pi_+)$. In other words, for a given price π, the dealer's optimal response to $u_0(\cdot;\pi)$ is, modulo a normalization of γ, equivalent to the combination of his actions towards negative and positive types separately. We shall concentrate on the existence of a fixed point of the mapping $\pi_+ \mapsto t(0_+;\pi_+)$.

From Assumption 4.1 we have that if $\pi_{1+} < \pi_{2+}$, then $u_0(\theta;\pi_{1+}) > u_0(\theta;\pi_{2+})$ for all $\theta > 0$. If for $i = 1,2$ it holds that $u_0(\theta;\pi_{i+}) < v_o(\theta)$ for all $\theta > 0$, then $v(\theta;\pi_{1+}) = v(\theta;\pi_{2+})$ on the same domain and $t(0_+;\pi_{1+}) = t(0_+;\pi_{2+})$. Next assume that $u_0(\theta;\pi_{i+}) \geq v_o(\theta)$ on a subset Θ_i of $(0,\overline{\theta}]$, for $i = 1,2$. Given that $u_0(\theta;\pi_{1+}) > u_0(\theta;\pi_{2+})$ for all $\theta > 0$, then $\overline{\theta}(\pi_1) < \overline{\theta}(\pi_2)$ and the first point $\widetilde{\theta}_1$ such that $v(\theta;\pi_{1+}) = u_0(\theta;\pi_{1+})$ holds satisfies $\widetilde{\theta}_1 < \widetilde{\theta}_2$, where the latter is the analogous to $\widetilde{\theta}_1$ in the presence of $u_0(\theta;\pi_{2+})$. The existence of $\widetilde{\theta}_1$ and $\widetilde{\theta}_2$ is guaranteed by the fact that in both cases the indirect-utility functions intersect the corresponding outside options. Arguing as in the proof of Theorem 3.8, Part (2), this also implies that $\overline{\theta}_0(\pi_1) < \overline{\theta}_0(\pi_2)$; hence $t(0_+;\pi_{1+}) < t(0_+;\pi_{2+})$. In other words, the mapping $\pi_+ \mapsto t(0_+;\pi_+)$ is order-preserving and, using Tarski's Fixed Point Theorem, we may conclude it has a fixed point. □

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