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AXIOMS FOR GROUNDED TRUTH

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Abstract. We axiomatize Leitgeb's (2005) theory of truth and show that this theory proves all arithmetical sentences of the system of ramified analysis up to ϵ_0. We also give alternative axiomatizations of Kripke's (1975) theory of truth (Strong Kleene and supervaluational version) and show that they are at least as strong as the Kripke-Feferman system KF and Cantini's VF, respectively.

§1. Introduction. Since the mid-1970s there has been a lot of research into developing formal theories of type-free truth, both semantic and axiomatic.[1] One concept that has attracted particular attention is that of grounding—the idea that a determinate truth-value is only assigned to those sentences that are determined by the truth-free fragment of the language.[2] The present paper introduces three new axiomatic theories of grounded truth. There are a variety of reasons for which axiomatic theories of truth are useful. Firstly, while semantic theories of truth usually rely on resources that go beyond the means of the object language, the axiomatic approach does not.[3] Secondly, they provide a basis for comparing different notions of truth. Thirdly, axiomatic theories of truth admit interesting reductions of set existence assumptions to axioms of truth.[4]

The theories found in the present paper are formulated in the language of Peano arithmetic together with unary predicates $T(x)$ and $G(x)$, meaning 'x is true' and 'x is grounded', respectively. Each of the proposed systems will share the same truth-theoretic axioms, viz. the T-schema and the compositional truth axioms *restricted* to members of G. The idea is that instead of choosing between equally plausible but jointly inconsistent truth axioms we take all of them but restrict them in a uniform way. We will provide a list of grounding axioms that can roughly be divided into base and closure axioms. Considering different subsets of these grounding axioms yields a family of theories of grounded truth of differing proof-theoretic strength.[5] In this article we focus only on three of the possible systems.

These three systems are intended as axiomatizations of the following *semantic* theories of truth: (1) Kripke's minimal fixed point (in its closed off version) based on the Strong Kleene evaluation schema[6], which is usually axiomatized by KF.[7] (2) Kripke's minimal

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[1] For an overview of semantic approaches see Field (2008). For axiomatic approaches consult Halbach (2011).

[2] See Herzberger (1970), Kripke (1975), Yablo (1982), and Leitgeb (2005).

[3] In a series of recent papers it has been argued that this constitutes a rationale for prefering axiomatic over semantic theories of truth. See for example Halbach & Horsten (2005).

[4] See Halbach (2000).

[5] Roughly, two theories over Peano arithmetic as base system are said to be proof-theoretically equivalent if they prove the same arithmetical statements.

[6] Kripke (1975).

[7] Feferman (1991).

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fixed point based on supervaluations (in Cantini's version), which is usually axiomatized by VF.[8] (3) Leitgeb's minimal fixed point based on his notion of semantic dependence[9], which has not been axiomatized so far. By axiomatizing these semantic theories in our modular system we hope to advance understanding of differences and similarities between them.

This paper is organized as follows. In order to motivate our axioms we briefly review the semantic theories mentioned above (section 2). In section 3 we lay down our list of axioms. In section 4 we prove the consistency of our axiom systems. Their proof-theoretic strength is assessed in section 5. The first main result is that our axiom system for Leitgeb's theory is at least as strong as $\mathrm{RA}_{<\epsilon_0}$; our second main result is that nothing is lost in proof-theoretic strength if we pass from KF or VF to their alternatives presented here. We conclude with some final observations and say a few words as to how our systems differ from Feferman's axioms for determinateness and truth.[10]

Some remarks on terminology. Our object language will be the language $\mathcal{L}_T$ of Peano arithmetic enlarged by the unary predicate T. $\mathcal{L}_{PA}$ denotes the language without the T-predicate. By PA we understand the usual axioms of Peano arithmetic, while PAT consists of the axioms of Peano arithmetic with induction for all sentences in the language $\mathcal{L}_T$. The models considered have the form $(\mathbb{N}, S)$, where $\mathbb{N}$ is the standard model of PA and $S \subseteq \omega$ interprets the truth predicate T. We will identify the sentences of $\mathcal{L}_T$ with their numerical codes $g(\phi)$ relative to some (unspecified) Gödel numbering g. If ϕ is a sentence, then $\ulcorner \phi \urcorner$ stands for the Gödel numeral of ϕ, that is, $\ulcorner \phi \urcorner^{\mathbb{N}} = g(\phi)$. As is well known, most syntactic operations on the expressions of the language $\mathcal{L}_T$ are primitive recursive (p. r.) and hence can be represented in $\mathcal{L}_T$. We assume that our language contains function symbols for all p. r. functions. We will use the dot notation to denote these functions; for example, $\dot{\neg}$ represents that function that maps the code of a sentence to the code of its negation; $\dot{\wedge}$ represents that function that sends the codes of two formulae to the code of their conjunction, etc. For details see Halbach (2011). In section 3 we will expand the language $\mathcal{L}_T$ by the unary predicate G. Models will then have the form $(\mathbb{N}, X, Y)$, where $X \subseteq \omega$ interprets G and $Y \subseteq \omega$ interprets T.

§2. Semantic theories of grounded truth.

In theories of grounded truth, a determinate truth-value is assigned only to those sentences 'whose truth value is *grounded* in atomic facts from the base language L, that is can be determined from such facts by evaluation according to the rules of truth for the connectives and quantifiers, and where statements of the form $T(\overline{A})$ are evaluated to be true (false) only when A itself has already been verified (falsified).'[11] Technically, the extension of the truth predicate is obtained by a fixed point construction. We briefly review the constructions.

2.1. Kripke. This is a general recipe for constructing interesting extensions for the truth predicate by recursion along the ordinals. Fix some monotone evaluation schema V. (Often, this requires that we do not only have an extension but also an anti-extension for the truth predicate at hand. We assume throughout that the anti-extension of T relative to some model $(\mathbb{N}, S)$ consists of the negations of sentences in S.) An evaluation schema is monotone if for all X, Y with $X \subseteq Y$ and all sentences ϕ we have:

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[8] Cantini (1990).
[9] Leitgeb (2005).
[10] Feferman (2008).
[11] Feferman (1991, pp. 18–19).

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$$\text{(i) } (\mathbb{N}, X) \models_V \phi \Rightarrow (\mathbb{N}, Y) \models_V \phi$$
$$\text{(ii) } (\mathbb{N}, X) \not\models_V \phi \Rightarrow (\mathbb{N}, Y) \not\models_V \phi$$

Then the construction is as follows. Let $S_0 = \emptyset$. Given S_α, let $S_{\alpha+1}$ consist of all the sentences that come out true relative to $(\mathbb{N}, S_\alpha)$ under the given evaluation schema. At limit points we just take unions. For cardinality reasons, there must be some fixed point $S_\beta = S_{\beta+1}$. We say that a sentence is *grounded* (relative to the evaluation schema) if and only if the sentence or its negation is in the fixed point.

In the present paper we will focus on two monotone evaluation schemes: the Strong Kleene schema and the supervaluational schema (in Cantini's version). We assume that the reader is familiar with the Strong Kleene schema.[12] The successor rule for the supervaluational schema can be written as follows:

$$S_{\alpha+1} := \{\phi \mid \text{for all consistent } X \supseteq S_\alpha : \ (\mathbb{N}, X) \models \phi\}$$

Here, X is consistent if there is no sentence ϕ such that both $\phi, \neg\phi \in X$, and $\models$ refers to the classical evaluation schema.

Now let K be the fixed point obtained by using the Strong Kleene schema and let C be the fixed point obtained by the supervaluational schema.[13] For the rest of the paper, we will consider $(\mathbb{N}, K)$ and $(\mathbb{N}, C)$ as classical models, that is in what follows $\models$ will always refer to the classical evaluation schema.

One significant difference between $(\mathbb{N}, K)$ and $(\mathbb{N}, C)$ is that the latter satisfies the global reflection principle for Peano arithmetic with full induction for the language $\mathcal{L}_T$ while the former does not, that is

$$(\mathbb{N}, C) \models \forall x \left(Sent_{\mathcal{L}_T}(x) \wedge Prov_{PAT}(x) \rightarrow Tx\right),$$

where $Prov_{PAT}(x)$ defines the set of sentences that are provable from Peano arithmetic with full induction for the language $\mathcal{L}_T$. For example we have $(\mathbb{N}, C) \models T\ulcorner\lambda \vee \neg\lambda\urcorner$ but $(\mathbb{N}, K) \not\models T\ulcorner\lambda \vee \neg\lambda\urcorner$ (where λ is the Liar sentence).

2.2. Leitgeb. Leitgeb's approach is novel insofar as it clearly seperates the concept of grounding from that of truth. Accordingly, we are confronted not with a single recursion that gives us the set of grounded truths, but rather the set of grounded sentences is given by a first recursion and the set of grounded truths is extracted by a second recursion. The central notion for the construction of the set of grounded sentences is the notion of *dependence* or determination:

ϕ depends on Φ iff for all Ψ_1, Ψ_2: if $\Psi_1 \cap \Phi = \Psi_2 \cap \Phi$, then $\phi^{\Psi_1} = \phi^{\Psi_2}$.

Here, ϕ^{Ψ_i} denotes the truth value of ϕ in the model $(\mathbb{N}, \Psi_i)$.

Thus, a sentence ϕ is determined by a collection of sentences Φ iff for all Ψ_1, Ψ_2: if Ψ_1, Ψ_2 agree on Φ, then they agree on the truth-value of ϕ.

By transfinite recursion define G_α, L_α for $\alpha \in ON$ by:

(i) $G_0 = \emptyset$
(ii) $G_{\alpha+1} = \{\phi | \phi \text{ depends on } G_\alpha\}$
(iii) $G_\beta = \bigcup_{\alpha<\beta} G_\alpha$, when β is a limit ordinal.
(iv) $L_0 = \emptyset$

[12] Cf. Halbach (2011).

[13] In what follows, I will also refer to K as Kripke's hierarchy and to C as Cantini's hierarchy, eventhough both fixed points can be traced back to the work of Kripke.

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(v) $L_{\alpha+1} = \{\phi \in G_{\alpha+1} | (\mathbb{N}, L_\alpha) \models \phi\}$
(vi) $L_\beta = \bigcup_{\alpha<\beta} L_\alpha$, when β is a limit ordinal.

Both sequences are monotone and reach a fixed-point. We denote these by G and L, respectively. G is the set of grounded sentences, while L contains those that are true. If we let L^- be the set of sentences whose negation is in L, then we get the equation $G = L \cup L^-$. G has some nice closure properties:

PROPOSITION 2.1. *1. All arithmetical sentences are grounded.*
2. *All theorems of PAT (and their negations) are grounded.*
3. *ϕ is grounded iff $T\ulcorner\phi\urcorner$ is grounded.*
4. *ϕ is grounded iff $\neg\phi$ is grounded.*
5. *If ϕ, ψ are grounded, then $\phi \wedge \psi$ is grounded.*
6. *If ϕ, ψ are grounded, then $\phi \vee \psi$ is grounded.*
7. *If, for all n, $\phi(\overline{n})$ is grounded, then $\forall x \phi$ is grounded.*[14]

In fact, the sentences mentioned under (1) and (2) come in at the first stage of the hierarchy; (3)–(7) provide closure conditions.

It is easy to show that $L \subseteq C$. In fact, the inclusion is proper.[15] For example, the sentence $T\ulcorner 1 = 1\urcorner \vee \lambda$, where λ is the Liar, enters Cantini's construction at stage 2, but it is not in G, since it depends on λ. However, since we know that $1 = 1$ is true, we can also determine the truth value of $T\ulcorner 1 = 1\urcorner \vee \lambda$. As another example, we may take the sentence $T\ulcorner 1 \neq 1\urcorner \wedge \lambda$, which is easily seen to be false (since $1 \neq 1$ is false). This sentence is not in G, but it is again a part of C. It follows from the first example that L (in contrast to C) invalidates the claim that Modus Ponens preserves truth:

$$(\mathbb{N}, L) \not\models \forall x \forall y\, (T\,(x \dot{\to} y) \to (Tx \to Ty))$$

However, by a little modification we can make L equal to C.[16] This can be done by introducing the notion of *conditional* dependence, and by further restricting the quantifiers in the definition to *consistent* supersets.[17]

We say that ϕ c-depends$_\Sigma$ on Φ iff for all consistent $\Psi_1, \Psi_2 \supseteq \Sigma$: if $\Psi_1 \cap \Phi = \Psi_2 \cap \Phi$, then $\phi^{\Psi_1} = \phi^{\Psi_2}$. Now by transfinite recursion define G'_α, L'_α for $\alpha \in ON$ by:

1. $G'_0 = \emptyset$
2. $G'_{\alpha+1} = \{\phi | \phi \text{ c-depends}_{L'_\alpha} \text{ on } G'_\alpha\}$
3. $G'_\beta = \bigcup_{\alpha<\beta} G'_\alpha$, when β is a limit ordinal.
4. $L'_0 = \emptyset$
5. $L'_{\alpha+1} = \{\phi \in G'_{\alpha+1} | \left(\mathbb{N}, L'_\alpha\right) \models \phi\}$
6. $L'_\beta = \bigcup_{\alpha<\beta} L'_\alpha$, when β is a limit ordinal.

It can then be shown that $L'_\alpha = C_\alpha$ for all $\alpha \in ON$. Indeed, we have $G'_\alpha = C_\alpha \cup C^-_\alpha$, and the above construction shows how we can extract the grounding hierarchy corresponding to C in a recursive way.

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[14] These facts were established in Leitgeb (2005).
[15] loc. cit.
[16] Cf. Vugt & Bonnay (2009) or Meadows (2011).
[17] Conditionality takes care of the first example, consistency takes care of the second example.

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The following picture emerges: a grounding hierarchy starts with some initial set of grounded sentences—for example, the arithmetical sentences, or, if you work within a classical framework, the arithmetical sentences plus all arithmetical truths and falsehoods in the language $\mathcal{L}_T$. Their truth-value is determined by the world (the base model) and the chosen evaluation schema. At the next stage of the hierarchy, all sentences are added that depend on the initial set, and so on. Of course, the extension of 'depends on' varies with the evaluation schema. The Strong Kleene evaluation schema yields a different extension than the supervaluational schema. In the next section we will see that all members of such a grounding hierarchy satisfy the compositional truth axioms plus the T-schema. The systems will only differ with respect to the extension of the grounding predicate G.

§3. The axioms. The novelty of Leitgeb's approach is the clear seperation of the concepts of grounding and truth. Accordingly, in axiomatizing his theory we introduce two unary predicates $T(x)$ and $G(x)$, meaning 'x is true' and 'x is grounded', respectively, and give seperate axioms for both predicates. Leitgeb begins his paper by raising the following question: 'What kinds of sentences with truth predicate may be inserted plausibly and consistently into the T-scheme? We state an answer in terms of dependence: those sentences which depend directly or indirectly on nonsemantic state of affairs (only).' It seems, then, that what Leitgeb wants to get is a restricted version of the T-schema, viz.

$$G(\ulcorner \phi \urcorner) \rightarrow (T(\ulcorner \phi \urcorner) \leftrightarrow \phi)$$

In addition, we also add the compositional axioms for T (relative to G). The idea here is to take *all the naïve principles of truth and restrict them in a uniform way*. This leaves us with the task of specifying the axioms of grounding. We start with an axiom that occupies a peculiar status:

$$\text{G0 } \forall x \left(G(x) \leftrightarrow \left(Tx \vee T \dot{\neg} x\right)\right)$$

This axiom says that the grounded sentences are exactly those that are true or have a true negation. This is exactly the definition of grounding that Kripke gave: truth comes first, and grounding is derived. Thus the predicate G is eliminable. However, on Leitgeb's approach the order is reversed: grounding is conceptually prior to truth. First the grounding hierarchy is defined, then the truth hierarchy is defined based on the grounding hierarchy. In fact, as far as Leitgeb's theory is concerned, G0 (though sound) might even be dropped – and there might be a good reason for doing so: as we will see below, G0 together with the T-schema (T7 below) implies (T-Out). Now (T-Out) proves the Liar sentence. Thus theories which contain G0 and T7 prove untrue sentences. This has led to criticism by Field.[18] And we will see below that our axioms for Leitgeb's theory—even if we drop G0—are still sufficient to prove all arithmetical statements of ramified analysis.[19,20]

[18] See Field (2008).

[19] The same can be done for Kripke's theory without loss of proof-theoretic strength (but note that this is contrary to the spirit of his construction). For Cantini's theory, however, one needs G0 in order to get the strength of ID_1. See section 5 below.

[20] One might require that if G is added as a primitive predicate symbol, then the axioms for G and T should be expanded so as to cover sentences that contain the predicate G. I agree. Leitgeb's construction and our axioms can be modified in order to meet that requirement. However, in order to keep our exposition as transparent as possible, we refrain from doing so in this paper.

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3.1. Axioms of Grounding. We start with the base and closure axioms, Axioms G1–G7. These mirror (1)–(7) of Proposition 2.1 stated above for Leitgeb's theory.

Base Axioms.

G1 $\forall x \left(Sent_{\mathcal{L}_{PA}}(x) \rightarrow G(x)\right)$
G2 $\forall x \left(Prov_{PAT}(x) \vee Prov_{PAT}\left(\dot{\neg} x\right) \rightarrow G(x)\right)$

Closure Axioms.

G3 $\forall x \left(G\left(\dot{T} x\right) \leftrightarrow G(x)\right)$
G4 $\forall x \left(G(x) \leftrightarrow G\left(\dot{\neg} x\right)\right)$
G5 $\forall x \forall y \left(G(x) \wedge G(y) \rightarrow G\left(x \dot{\vee} y\right)\right)$
G6 $\forall x \forall y \left(G(x) \wedge G(y) \rightarrow G\left(x \dot{\wedge} y\right)\right)$
G7 $\forall x \forall v \left(\forall t G(x(t/v)) \rightarrow G\left(\dot{\forall} v x\right)\right)$

Here and below, v ranges over the set of (codes of) variables, t ranges over the set of (codes of) closed terms, and x, y range over the (codes of the) formulae of $\mathcal{L}_T$.

Jump Axiom.

G8 $\forall x (\phi(x) \rightarrow G(x)) \rightarrow \forall x (Rel(x, \ulcorner \phi \urcorner) \rightarrow G(x))$, for $\phi(x) \in \mathcal{L}_{PA}$,

where $Rel(x, \ulcorner \phi \urcorner)$ says that x is a sentence in which every occurence of a subformula Tt is relativized to ϕ, that is every occurrence of a subformula Tt of x occurs in the context $\phi(t) \wedge Tt$.[21]

Axioms for Conditional Dependence.

G9 $\forall x \forall y \left(T(x) \rightarrow G\left(x \dot{\vee} y\right)\right)$
G10 $\forall x \forall y \left(T\left(\dot{\neg} x\right) \rightarrow G\left(x \dot{\wedge} y\right)\right)$
G11 $\forall x \forall y \left(T\left(x \dot{\vee} y\right) \wedge T\left(\dot{\neg} x\right) \rightarrow G(y)\right)$
G12 $\forall x \forall y \left(T\left(x \dot{\wedge} y\right) \rightarrow G(x) \wedge G(y)\right)$

3.2. Axioms of Truth. We take the compositional truth axioms (relativized to G) for the full language (including the truth-predicate) plus the uniform Tarskian biconditionals (again, relativized to G). In the first and second axiom, the variables s, t range over the set of closed terms, while s° denotes the value of s. In the last axiom, $\underset{.}{t}$ indicates that t is substituted for the free variable in ϕ.

T1 $\forall s \forall t \left(T(s \dot{=} t) \leftrightarrow s^\circ = t^\circ\right)$
T2 $\forall t \left(G(t^\circ) \rightarrow \left(T \dot{T} t \leftrightarrow T t^\circ\right)\right)$
T3 $\forall x \left(G(x) \rightarrow \left(T \dot{\neg} x \leftrightarrow \neg T x\right)\right)$
T4 $\forall x \forall y \left(G(x) \wedge G(y) \rightarrow \left(T\left(x \dot{\wedge} y\right) \leftrightarrow Tx \wedge Ty\right)\right)$
T5 $\forall x \forall y \left(G(x) \wedge G(y) \rightarrow \left(T\left(x \dot{\vee} y\right) \leftrightarrow Tx \vee Ty\right)\right)$
T6 $\forall x \forall v \left(G\left(\dot{\forall} v x\right) \rightarrow \left(T\left(\dot{\forall} v x\right) \leftrightarrow \forall t T(x(t/v))\right)\right)$
T7 $\forall t \left(G\left(\ulcorner \phi\left(\underset{.}{t}\right) \urcorner\right) \rightarrow \left(T\left(\ulcorner \phi\left(\underset{.}{t}\right) \urcorner\right) \leftrightarrow \phi(t^\circ)\right)\right)$

[21] It would be more natural just to relativize (in the usual sense) all quantifiers to ϕ; however, the present formulation is more convenient for the proof of Proposition 5.3 below. Given an axiom to the effect that G is closed under arithmetical equivalence (as considered in remark 6 below) both formulations would amount to the same.

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3.3. Remarks. 1. G2 is sound with respect to $(\mathbb{N}, C)$ and $(\mathbb{N}, L)$, but it is not sound with respect to $(\mathbb{N}, K)$, since, for example, the disjunction $\lambda \vee \neg\lambda$ is not grounded according to Kripke (where λ is again the Liar sentence).

2. Axioms G3–G7 give closure conditions. This is why most of them have the form of a conditional rather than a biconditional. The result of adding the right-to-left direction of G5–G7 is not sound (with respect to L and C). For example, $\lambda \vee \neg\lambda$ (where λ is the Liar sentence) is grounded according to Leitgeb and Cantini, but none of its disjuncts is. In the case of Kripke, one might add axioms such as:

- $\forall x \forall y \left(G\left(x \dot{\vee} y\right) \rightarrow G(x) \vee G(y)\right)$

If one intends to axiomatize Kripke's construction with the Weak Kleene schema, then one might add stronger axioms like:

- $\forall x \forall y \left(G\left(x \dot{\vee} y\right) \leftrightarrow G(x) \wedge G(y)\right)$

As far as I can see, this adds nothing to the proof-theoretic strength of the system (as long as G8 is assumed).

3. The idea underlying jump axiom G8 is that a sentence that attributes truth or falsity to some subset of G is itself grounded. In the statement of the axiom, ϕ must be arithmetical (i.e. T-free), since otherwise an inconsistency would occur.

4. As pointed out above, axioms G9–G12 are not satisfied by Leitgeb's hierarchy; rather, they apply only to the constructions of Kripke and Cantini.

5. Notice that replacing $G(x) \wedge G(y)$ by $G\left(x \dot{\vee} y\right)$ in the antecedent of T5 would be unsound with respect to L and C. For example, while $\lambda \vee \neg\lambda$ is grounded on both Leitgeb's and Cantini's approach, none of its disjuncts is grounded. For similar reasons, the antecedent of T4 can not be replaced by $G\left(x \dot{\wedge} y\right)$. For this reason, T7 can not be derived from T1–T6. However, given the Strong or Weak Kleene schema, such a replacement is indeed sensible. In that event, T7 will be derivable.

6. One might consider the idea of adding a further axiom that states that G is closed under PAT-equivalence:

- $\forall x \forall y \left(Prov_{PAT}\left(x \dot{\leftrightarrow} y\right) \wedge G(x) \rightarrow G(y)\right)$

(This is indeed true for L and C, but not for K.) This would give a more unified picture, but would not add anything to the proof-theoretic strength.

§4. Models. Here we present a fairly simple model of $\mathrm{G0} + \mathrm{G1} + \mathrm{G3}-\mathrm{G8} + \mathrm{T1}-\mathrm{T7}$. (Notice that models now have the form $(\mathbb{N}, X, Y)$, where $X \subseteq \omega$ interprets G and $Y \subseteq \omega$ interprets T.) Let $\Sigma_0 := \mathcal{L}_{PA}$, and let $\Sigma_{\alpha+1} \subseteq Fm_{\mathcal{L}_T}$ be the smallest superset of Σ_α such that (i) whenever $\phi, \psi \in \Sigma_\alpha$, then $\phi \wedge \psi, \phi \vee \psi, \neg\phi, T\ulcorner\phi\urcorner \in \Sigma_{\alpha+1}$, (ii) whenever $\phi(t) \in \Sigma_\alpha$ for all t, then $\forall x \phi \in \Sigma_{\alpha+1}$, (iii) whenever ψ is a T-free formula that (elementarily) defines a subset of Σ_α and ϕ is relativized (in the above sense) to ψ, then $\phi \in \Sigma_{\alpha+1}$. At limit points, we take unions. Let Σ be the fixed-point of this hierarchy. This will serve as our grounding hierarchy. The set of grounded truths is extracted as usual: let $\Gamma_0 := \emptyset$, put $\Gamma_{\alpha+1} := \{\phi \in \Sigma_{\alpha+1} | (\mathbb{N}, \Gamma_\alpha) \models \phi\}$, and take unions at limit points. Let Γ be the fixed-point of this hierarchy. Then $(\mathbb{N}, \Sigma, \Gamma) \models \mathrm{G0} + \mathrm{G1} + \mathrm{G3}-\mathrm{G8} + \mathrm{T1}-\mathrm{T7}$, as is easily verified.

One might also start with $\Sigma_0 := \mathcal{L}_{PA} \cup \{\phi | \mathrm{PAT} \vdash \phi\} \cup \{\phi | \mathrm{PAT} \vdash \neg\phi\}$, thus obtaining a model of $\mathrm{G0}-\mathrm{G8} + \mathrm{T1}-\mathrm{T7}$.

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It is easily seen (see remarks in section 3) that Leitgeb's fixed point satisfies all axioms above except those for conditional dependence.[22] Kripke's fixed point satisfies all axioms except G2, which states that all theorems of PAT are grounded. Finally, Cantini's hierarchy satisfies all of the axioms presented above:[23]

1. $(\mathbb{N}, L \cup L^-, L) \models \mathrm{LG} := \mathrm{PAT} + \mathrm{G1}-\mathrm{G8} + \mathrm{T1}-\mathrm{T7}$
2. $(\mathbb{N}, K \cup K^-, K) \models \mathrm{KG} := \mathrm{PAT} + \mathrm{G0}-\mathrm{G1} + \mathrm{G3}-\mathrm{G12} + \mathrm{T1}-\mathrm{T7}$
3. $(N, C \cup C^-, C) \models \mathrm{CG} := \mathrm{PAT} + \mathrm{G0}-\mathrm{G12} + \mathrm{T1}-\mathrm{T7}$

Thus, the constructions of Leitgeb and Cantini start with the same initial set of grounded sentences—the arithmetical sentences plus all theorems of PAT. But Cantini's hierarchy grows more in width—it satisfies the axioms for conditional dependence, whereas Leitgeb's does not. This is why L does not satisfy the claim that Modus Ponens preserves truth. Kripke's hierarchy, on the other hand, satisfies the same growth axioms, but it starts with a smaller initial set—it only satisfies G1 but not G2; this is why Kripke's hierarchy does not satisfy the global reflection principle for PAT.

§5. Proof-theoretic strength. $(\mathbb{N}, K)$ is usually axiomatized by KF, which has the same proof-theoretic strength as the system of Ramified Analysis up to ϵ_0, $\mathrm{RA}_{<\epsilon_0}$, while $(\mathbb{N}, C)$ is usually axiomatized by the system VF, which has the same proof-theoretic strength as the system ID_1 of elementary inductive definitions.[24] What can be said about the proof-theoretic strength of the three systems LG, KG and CG?

PROPOSITION 5.1. *Assume PAT as background theory. Then:*

1. *G1 + T7 proves* $T\ulcorner\phi\urcorner \leftrightarrow \phi$, *for all* $\phi \in \mathcal{L}_{PA}$.
2. *G0 + T7 proves (T-Out)* $\forall t\,(T(\ulcorner\phi(\dot{t})\urcorner) \rightarrow \phi(t^\circ))$, *for all* $\phi \in \mathcal{L}_T$.
3. *G0 + T3 proves (Cons)* $\forall x\,(T\dot{\neg}x \rightarrow \neg Tx)$.
4. *G0 + T3 proves (DN)* $\forall x\,(T\dot{\neg}\dot{\neg}x \leftrightarrow Tx)$.
5. *G0 + T6 proves (U-Inf)* $\forall x\forall v\,(\forall t T(x(t/v)) \rightarrow T(\dot{\forall}vx))$.
6. *G0 + T6 proves* $\forall x\forall v\,(T(\dot{\forall}vx) \rightarrow \forall t T(x(t/v)))$.
7. *G0 + G4 + G11 + T3 + T5 proves (MP)* $\forall x\forall y\,(T(x\dot{\rightarrow}y) \rightarrow (Tx \rightarrow Ty))$.
8. *G0 + G12 + T5 proves (Conj)* $\forall x\forall y\,(T(x\dot{\wedge}y) \leftrightarrow Tx \wedge Ty)$.
9. *G0 + G3 + T2 proves (T-SYM)* $\forall t\,(T\dot{T}t \leftrightarrow Tt^\circ)$.
10. *G0 + G4 + G9 + G11 + T3 + T5 + T7 proves* $T\ulcorner\phi\urcorner \vee T\ulcorner\neg\phi\urcorner \rightarrow ((\phi \rightarrow T\ulcorner\psi\urcorner) \leftrightarrow (T\ulcorner\phi \rightarrow \psi\urcorner))$.
11. $PAT \vdash \phi \Rightarrow G2 + T7 \vdash T\ulcorner\phi\urcorner$.

Proof. Straightforward. For example, in order to prove (7), assume that Tx and $T(x\dot{\rightarrow}y)$, that is $T(\dot{\neg}x\dot{\vee}y)$ holds. Tx and (4) imply $T\dot{\neg}\dot{\neg}x$. Applying G11 yields Gy. By G4 we also get that $G\dot{\neg}x$. Thus by assumption and T5 we have $T\dot{\neg}x \vee Ty$. But from $T\dot{\neg}\dot{\neg}x$ and T3 we get $\neg T\dot{\neg}x$. Thus Ty must hold, as desired. □

[22] However, for the reasons given above, we omit G0 from the Leitgeb system.

[23] In (1)–(3), we assume that induction is now expanded to all sentences of the language including the predicates G and T.

[24] See Feferman (1991) for KF and Cantini (1990) for VF.

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PROPOSITION 5.2. *PAT + G0−G4 + G9 + G11 + T1−T7 is at least as strong as* ID_1.

Proof. Cantini (1990) has shown that properties (1), (2), (5), (7), (10) and (11) of Proposition 5.1 suffice to establish the desired result. □

We will now show that that the systems LG and KG are at least as strong as Ramified Analysis $RA_{<\epsilon_0}$. In order to do so, we show that both LG and KG are able to define[25] all truth-predicates of the system of Ramified Truth $RT_{<\epsilon_0}$.[26] The language of $RT_{<\epsilon_0}$ is that of PA plus predicates T_α for all $\alpha < \epsilon_0$. The sublanguages $\mathcal{L}_\alpha$ of $\mathcal{L}_T$ are defined by recursion over the ordinals up to ϵ_0. $\mathcal{L}_0$ is just $\mathcal{L}_{PA}$. For $0 < \alpha < \epsilon_0$, ϕ is a formula of the language $\mathcal{L}_\alpha$ if ϕ is relativized (in the sense explained above) to the formula $Sent_{\mathcal{L}_\beta}(x)$ for some $\beta < \alpha$. Using a primitive recursive function h which substitutes every occurrence $T_\alpha t$ in a formula of the language of $RT_{<\epsilon_0}$ by $Tt \wedge Sent_{\mathcal{L}_\alpha}(t)$ (and is otherwise structure-preserving)[27], we can translate every formula of $RT_{<\epsilon_0}$ into one of $\mathcal{L}_\alpha$ (for some α).

PROPOSITION 5.3. *PAT + G1 + G3−G8* $\vdash \forall\alpha \prec \beta \forall x.\, Sent_{\mathcal{L}_\alpha}(x) \rightarrow G(x)$, *for all* $\beta < \epsilon_0$.

That is, the translation of every sentence of the Tarski-Hierarchy up to but excluding ϵ_0 is provably grounded.

Proof. By transfinite induction up to ϵ_0. The case $\alpha = 0$ is covered by G1. Since all truth predicates in a formula of $\mathcal{L}_{\alpha+1}$ are always relativized to some language with lower index, the induction step follows from jump axiom G8 (plus G3-G7). □

In what follows, we write $\phi_\alpha(t)$ for the formula $Tt \wedge Sent_{\mathcal{L}_\alpha}(t)$.

PROPOSITION 5.4. *PAT + G1 + G3−G8 + T1−T7 proves (i)-(vii) for all* $\alpha < \epsilon_0$:

(i) $\forall s \forall t\, (\phi_\alpha(s \dot{=} t) \leftrightarrow s^\circ = t^\circ)$

(ii) $\forall x \left(Sent_{\mathcal{L}_\alpha}(x) \rightarrow \left(\phi_\alpha\left(\dot{\neg} x\right) \leftrightarrow \neg\phi_\alpha(x)\right)\right)$

(iii) $\forall x \forall y \left(Sent_{\mathcal{L}_\alpha}\left(x \dot{\wedge} y\right) \rightarrow \left(\phi_\alpha\left(x \dot{\wedge} y\right) \leftrightarrow \phi_\alpha(x) \wedge \phi_\alpha(y)\right)\right)$

(iv) $\forall x \forall y \left(Sent_{\mathcal{L}_\alpha}\left(x \dot{\vee} y\right) \rightarrow \left(\phi_\alpha\left(x \dot{\vee} y\right) \leftrightarrow \phi_\alpha(x) \vee \phi_\alpha(y)\right)\right)$

(v) $\forall x \forall v \left(Sent_{\mathcal{L}_\alpha}\left(\dot{\forall} v x\right) \rightarrow \left(\phi_\alpha\left(\dot{\forall} v x\right) \leftrightarrow \forall t.\phi_\alpha(x(t/v))\right)\right)$

(vi) $\forall t \left(Sent_{\mathcal{L}_\alpha}(t^\circ) \rightarrow \left(\phi_\alpha\left(\dot{\phi_\beta}\left(\underset{\cdot}{t}\right)\right) \leftrightarrow \phi_\beta(t^\circ)\right)\right)$, *for* $\beta < \alpha$

(vii) $\forall t \forall \beta \prec \alpha \left(Sent_{\mathcal{L}_\alpha}(t^\circ) \rightarrow \left(\phi_\alpha\left(\dot{\phi_\beta}(t)\right) \leftrightarrow \phi_\alpha(t)\right)\right)$

Proof. (i) is just a restriction of T1. (ii) follows from T3 and the fact that every sentence of $Sent_{\mathcal{L}_\alpha}$ is grounded by Proposition 5.3. For (iii), just observe that if $x \wedge y$ is a sentence of $\mathcal{L}_\alpha$, then both x and y are also sentences of $\mathcal{L}_\alpha$. Thus, the claim follows from proposition 5.3 and T4. (iv) and (v) are proved in the same manner. For (vi), use T7 and proposition 5.3 For (vii), use T2 and proposition 5.3 □

PROPOSITION 5.5. *PAT + G1 + G3−G8 + T1−T7 defines all truth-predicates of* $RT_{<\epsilon_0}$.

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[25] In the sense of Fujimoto (2010).

[26] $RA_{<\epsilon_0}$ and $RT_{<\epsilon_0}$ prove the same arithmetical sentences. Cf. Halbach (2011, p. 129). An explicit formulation of $RT_{<\epsilon_0}$ is also found there.

[27] For an explicit definition of h see Halbach (2011, p. 223).

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Thus, KG is at least as strong as KF, and CG is as least as strong as VF. Hence, nothing is lost when we pass from KF to KG or from VF to CG. The system LG for Leitgeb's theory proves all arithmetical sentences of KF. Notice that Kripke's construction over Weak Kleene logic (closed off version) also satisfies PAT + G1 + G3−G8 + T1−T7.

§6. Comparison.

We conclude with some final remarks.

1. All systems containing G0 prove the consistency axiom (Cons). It can be consistently added to KF, but is usually not a part of it, because it differs in character from the other axioms of KF.[28] Here we have it as a consequence. Furthermore, (T-Out) and (MP) are also consequences of our theories, but they are again no part of KF in its usual setting, eventhough they can be consistently added; in fact, they are consequences of KF + (Cons).[29]

2. The systems LG and CG prove the weak T-rule (property (11) of Proposition 5.1). As far as I can see, it is not possible to derive the stronger global reflection principle for PAT in these systems. In CG, it is possible to prove that all axioms of predicate logic are true, that all axioms of PA are true and that the usual inference rules of predicate logic are truth-preserving. However, the most straightforward proof that all instances of induction containing the truth predicate are true requires that the truth predicate is complete, that is for all x, either Tx or $T\dot{\neg}x$. And this is not a theorem of CG.[30] Of course, the reflection principle can consistently be added to both LG and CG (it can also consistently be added to KG, but the principle is not sound w.r.t. $(\mathbb{N}, K)$). I refrained from doing so because it differs in character from the other axioms. Notice that CG plus global reflection for PAT proves axioms V1–V7 of VF. CG alone proves V1–V2 and V4–V7. Some authors complain that the axioms of VF seem somewhat unrelated and lack a common denominator. I hope the present axiomatization shows that it is possible to reformulate VF in a way that makes it look more natural.

3. The systems presented here bear some resemblance to Feferman's (2008) theory DT of determinate and meaningful truth. Feferman's idea, which derives from Bertrand Russell, is that any predicate has a domain of significance, and it makes no sense to apply a predicate outside its domain. For his theory of truth, he therefore uses two predicates, D and T, where the laws for T—the compositional axioms—apply only to members of D, and D is strongly compositional, that is if ϕ is a compound sentence, then all its subformulae must also be in D. As Feferman says, D consists of the determinate *and* meaningful sentences. And whereas it is possible that a sentence has a determinate truth value without its components having one (witness $\lambda \vee \neg\lambda$), it is not possible for a sentence to be meaningful without its components being meaningful. In short, DT has axioms similar to our axioms for the Weak Kleene fixed points, as indicated in section 3.3. Indeed, Feferman even utilizes Kripke's construction using the Weak Kleene evaluation schema (plus rules for his conditional) in order to provide a model for his theory. DT might therefore be considered as part of the family of theories introduced in the present paper.

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[28] Cf. the discussion in Feferman (1991, pp. 19–20).
[29] Cf. Halbach (2011, p. 214).
[30] Notice that the completeness axiom is inconsistent with (T-Out).

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BIBLIOGRAPHY

Cantini, A. (1990). A theory of formal truth arithmetically equivalent to ID_1. *Journal of Symbolic Logic*, **55**, 244–259.

Feferman, S. (1991). Reflecting on incompleteness. *Journal of Symbolic Logic*, **56**, 1–49.

Feferman, S. (2008). Axioms for determinateness and truth. *Review of Symbolic Logic*, **1**, 204–217.

Field, H. (2008). *Saving Truth from Paradox*. New York, NY: Oxford University Press.

Fujimoto, K. (2010). Relative truth definability of axiomatic truth theories. *Bulletin of Symbolic Logic*, **16**, 305–344.

Halbach, V. (2000). Truth and reduction. *Erkenntnis*, **53**, 97–126.

Halbach, V. (2011). *Axiomatic Theories of Truth*. Cambridge, UK: Cambridge University Press.

Halbach, V., & Horsten, L. (2005). The deflationst's axioms for truth. In Beall, J. C., and Armour-Garb, B., editors. *Deflationism and Paradox*. New York, NY: Oxford University Press, pp. 203–217.

Herzberger, H. (1970). Paradoxes of grounding in semantics. *Journal of Philosophy*, **67**, 145–167.

Kripke, S. (1975). Outline of a theory of truth. *Journal of Philosophy*, **72**, 690–716.

Leitgeb, H. (2005). What truth depends on. *Journal of Philosophical Logic*, **34**, 155–192.

Meadows, T. (2011). Truth, dependence, and supervaluation: Living with the ghost. *Journal of Philosophical Logic* (forthcoming).

Vugt, F., & Bonnay, D. (2009). What makes a sentence be about the world? Towards a unified account of groundedness. Unpublished manuscript.

Yablo, S. (1982). Grounding, dependence, and paradox. *Journal of Philosophical Logic*, **11**, 117–137.

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