Mathematical Proceedings of the Cambridge Philosophical Society

Vol. 156 | May 2014 | Part 3

Math. Proc. Camb. Phil. Soc. (2014), **156**, 383–407
doi:10.1017/S0305004113000728
First published online 15 January 2014

Counting imaginary quadratic points via universal torsors, II

By ULRICH DERENTHAL† and CHRISTOPHER FREI‡

Mathematisches Institut, Ludwig-Maximilians-Universität München
Theresienstr. 39, 80333 *München, Germany*
e-mail: ulrich.derenthal@math.lmu.de, christopher.frei@math.lmu.de

(*Received* 12 *April* 2013)

Abstract

We prove Manin's conjecture for four singular quartic del Pezzo surfaces over imaginary quadratic number fields, using the universal torsor method.

1. *Introduction*

Let K be a number field, S a del Pezzo surface defined over K with only **ADE**-singularities, U the open subset obtained by removing the lines from S, and H a height function on S coming from an anticanonical embedding. If $S(K)$ is Zariski dense in S then generalizations (e.g. [**BT98b**]) of Manin's conjecture [**FMT89**, **BM90**] predict an asymptotic formula, as $B \to \infty$, for the quantity

$$N_{U,H}(B) := |\{\mathbf{x} \in U(K) \mid H(\mathbf{x}) \leqslant B\}|,$$

namely

$$N_{U,H}(B) = c_{S,H} B(\log B)^{\rho-1}(1 + o(1)),$$

where ρ is the rank of the Picard group of a minimal desingularization of S and $c_{S,H}$ is a positive real number.

Much progress has been made in recent years in proving Manin's conjecture for specific del Pezzo surfaces over $\mathbb{Q}$ via the *universal torsor method*. In [**DF13**], the authors extended

† Supported by grant DE 1646/2-1 of the Deutsche Forschungsgemeinschaft and by the Hausdorff Research Institute for Mathematics in Bonn which he would like to thank for the hospitality.
‡ Partially supported by a research fellowship of the Alexander von Humboldt Foundation.

this method to imaginary quadratic fields in the case of a quartic del Pezzo surface of type $\mathbf{A}_3$ with five lines.

In this paper, we continue this investigation by proving Manin's conjecture over imaginary quadratic fields for quartic del Pezzo surfaces of types $\mathbf{A}_3 + \mathbf{A}_1$, $\mathbf{A}_4$, $\mathbf{D}_4$, and $\mathbf{D}_5$.

For more information about Manin's conjecture and the universal torsor method, we refer to the introductory section of [**DF13**] and the references mentioned therein.

1·1. *Results*

Let K be an imaginary quadratic field. We define the anticanonically embedded quartic del Pezzo surfaces $S_i \subset \mathbb{P}^4_K$ over K by the following equations:

$$S_0: \quad x_0x_1 - x_2x_3 = x_0x_3 + x_1x_3 + x_2x_4 = 0 \quad \text{of type } \mathbf{A}_3 \text{ (5 lines)}, \tag{1·1}$$

$$S_1: \quad x_0x_3 - x_2x_4 = x_0x_1 + x_1x_3 + x_2^2 = 0 \quad \text{of type } \mathbf{A}_3 + \mathbf{A}_1, \tag{1·2}$$

$$S_2: \quad x_0x_1 - x_2x_3 = x_0x_4 + x_1x_2 + x_3^2 = 0 \quad \text{of type } \mathbf{A}_4, \tag{1·3}$$

$$S_3: \quad x_0x_3 - x_1x_4 = x_0x_1 + x_1x_3 + x_2^2 = 0 \quad \text{of type } \mathbf{D}_4, \tag{1·4}$$

$$S_4: \quad x_0x_1 - x_2^2 = x_3^2 + x_0x_4 + x_1x_2 = 0 \quad \text{of type } \mathbf{D}_5. \tag{1·5}$$

All of them are split over K, hence rational over K, and therefore, their rational points over K are Zariski dense. The Weil height on $\mathbb{P}^4_K(K)$ is defined by

$$H(x_0 : \cdots : x_4) := \frac{\max\{\|x_0\|_\infty, \ldots, \|x_4\|_\infty\}}{\mathfrak{N}(x_0\mathcal{O}_K + \cdots + x_4\mathcal{O}_K)}, \tag{1·6}$$

where $\mathcal{O}_K$ is the ring of integers in K, $\|\cdot\|_\infty := |\cdot|^2$ is the square of the usual complex absolute value, and $\mathfrak{N}\mathfrak{a}$ is the absolute norm of a fractional ideal $\mathfrak{a}$.

For S_0, Manin's conjecture was proved over arbitrary imaginary quadratic fields in [**DF13**]. In this article, we prove Manin's conjecture for $S_1, \ldots, S_4$ over imaginary quadratic fields:

THEOREM 1. *Let K be an imaginary quadratic field. For $i \in \{1, \ldots, 4\}$, let U_i be the complement of the lines in the del Pezzo surface $S_i \subset \mathbb{P}^4_K$ defined by* (1·2)–(1·5). *Then there are positive real constants $c_{S_i,H}$ such that, for $B \geqslant 3$, we have*

$$N_{U_i,H}(B) = c_{S_i,H} B(\log B)^5 + O(B(\log B)^4 \log\log B).$$

Since these quartic del Pezzo surfaces are split, their minimal desingularizations $\widetilde{S}_i$ have Picard groups of rank 6, hence Theorem 1 agrees with Manin's conjecture. The leading constants are of the shape

$$c_{S_i,H} := \alpha(\widetilde{S}_i) \cdot \frac{(2\pi)^6 h_K^6}{\Delta_K^4 \omega_K^6} \cdot \theta_0 \cdot \omega_\infty(\widetilde{S}_i)$$

with a rational number $\alpha(\widetilde{S}_i)$ defined in [**Pey95**, définition 2·4], [**BT95**, definition 2·4·6] (see [**DF13**, section 8]), h_K the class number, Δ_K the discriminant and ω_K the number of units in the ring of integers of K, the Euler product

$$\theta_0 := \prod_{\mathfrak{p}} \left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}}\right)^6 \left(1 + \frac{6}{\mathfrak{N}\mathfrak{p}} + \frac{1}{\mathfrak{N}\mathfrak{p}^2}\right), \tag{1·7}$$

and a complex integral $\omega_\infty(\widetilde{S}_i)$. We give $\alpha(\widetilde{S}_i)$ and $\omega_\infty(\widetilde{S}_i)$ explicitly in the proof of each case.

We note that Manin's conjecture for S_4 is implied by [**CLT02**] over arbitrary number fields, since S_4 is an equivariant compactification of $\mathbb{G}_a^2$. On the other hand, $S_0, \ldots, S_3$ are neither toric nor equivariant compactifications of $\mathbb{G}_a^2$ [**DL10**], so that [**BT98a**, **CLT02**] do not apply. Finally, S_1 and S_3 (but not S_0, S_2, S_4) are equivariant compactifications of some semidirect products $\mathbb{G}_a \rtimes \mathbb{G}_m$ [**DL12**], so similar methods as in [**BT98a**, **CLT02**] may apply to them, but this has been worked out only over $\mathbb{Q}$ and with further restrictions in [**TT12**].

Over $\mathbb{Q}$, Manin's conjecture was proved for $S_0, \ldots, S_4$ with main terms of the shape $BP(\log B)$ for suitable polynomials P of degree 5, and with error terms of the form $O(B(\log B)^4 \log\log B)$ for S_0 [**DF13**], $O(B(\log B)^4(\log\log B)^2)$ for S_1 [**Der09**], $O(B(\log B)^{4+5/7})$ for S_2 [**BD09**] and $O(B(\log B)^3)$ for S_3 [**DT07**]. For S_4, a power-saving error term $O(B^{11/12+\epsilon})$ was achieved in [**BB07**]. The error terms for S_1 and S_2 could easily be improved to $O(B(\log B)^4 \log\log B)$.

1·2. *Methods*

The general strategy in our proofs of Theorem 1 for $S_1, \ldots, S_4$ is the one proposed in [**DF13**].

In the first step, the rational points $S_i(K)$ are parameterized by integral points on universal torsors over S_i, satisfying certain *height conditions* and *coprimality conditions*, following the strategy from [**DF13**, section 4]. Since the Cox rings of all minimal desingularizations $\widetilde{S}_i$ have only one relation [**Der06**], the universal torsors are open subsets of hypersurfaces in $\mathbb{A}_K^9$, with coordinates $(\eta_1, \ldots, \eta_9)$ and one relation, the *torsor equation*.

In the second step, we approximate the number of these integral points on universal torsors subject to height and coprimality conditions by an integral. In all cases η_9 appears linearly in the torsor equation, so it is uniquely defined by $\eta_1, \ldots, \eta_8$. We first count pairs (η_8, η_9) for given $(\eta_1, \ldots, \eta_7)$ using the method from [**DF13**, section 5] and then sum the result over another variable using the results from [**DF13**, section 6]. The summations over the remaining variables are handled in all cases by a direct application of the results of [**DF13**, section 7].

In the third step, we show that the integrals from the second step satisfy the asymptotic formulas from Theorem 1. Here, the shape of the effective cone of $\widetilde{S}_i$ is crucial; after all, the volume of its dual intersected with a certain hyperplane appears as $\alpha(\widetilde{S}_i)$ in Peyre's refinement [**Pey95**] of Manin's conjecture.

Though the proofs for $S_0, \ldots, S_4$ have many features in common, each case has its own difficulties.

In the case of S_0, the first step is mostly covered by our general results from [**DF13**], whereas the second step requires dichotomies with different orders of summation according to the relative size of the variables.

The first step in the case of S_1 is mostly covered by the general results as well, but the second summation in the second step requires additional effort in order to obtain sufficiently good error terms.

In the case of S_2, parts of the first step need to be treated individually, and the second summation in the second step is more complicated, since η_8 does not appear linearly in the torsor equation. Additionally, the second summation requires a dichotomy similarly as in the case of S_0, in order to handle the error terms.

The case of S_3 is probably the most simple one. Parts of the first step need to be treated individually, but the summations in the second step go through without additional tricks, so it just remains to bound the error terms.

Finally, in the case of S_4, parts of the first step need to be treated individually and the second summation in the second step is slightly more complicated, since η_8 does not appear linearly in the torsor equation.

1·3. *Notation*

Throughout this article, we use the notation introduced in [**DF13**, section 1·4]. In particular, $\mathcal{C}$ denotes a fixed system of integral representatives for the ideal classes of the ring of integers $\mathcal{O}_K$. Moreover, $\mathfrak{p}$ always denotes a nonzero prime ideal of $\mathcal{O}_K$, and products indexed by $\mathfrak{p}$ are understood to run over all such prime ideals. We say that $x \in K$ is *defined* (resp. *invertible*) modulo an ideal $\mathfrak{a}$ of $\mathcal{O}_K$, if $v_\mathfrak{p}(x) \geqslant 0$ (resp. $v_\mathfrak{p}(x) = 0$) for all $\mathfrak{p} \mid \mathfrak{a}$, where $v_\mathfrak{p}$ is the usual $\mathfrak{p}$-adic valuation. For x, y defined modulo $\mathfrak{a}$, we write $x \equiv_\mathfrak{a} y$ if $v_\mathfrak{p}(x-y) \geqslant v_\mathfrak{p}(\mathfrak{a})$ for all $\mathfrak{p} \mid \mathfrak{a}$.

2. *The quartic del Pezzo surface of type* $\mathbf{A}_3 + \mathbf{A}_1$

2·1. *Passage to a universal torsor*

Up to a permutation of the indices, we use the notation of [**Der06**].

For any given $\mathbf{C} = (C_0, \ldots, C_5) \in \mathcal{C}^6$, we define $u_\mathbf{C} := \mathfrak{N}(C_0^3 C_1^{-1} \cdots C_5^{-1})$ and

$$\mathcal{O}_1 := C_5 \qquad \mathcal{O}_2 := C_4 \qquad \mathcal{O}_3 := C_0 C_1^{-1} C_4^{-1} C_5^{-1}$$
$$\mathcal{O}_4 := C_1 C_2^{-1} \qquad \mathcal{O}_5 := C_3 \qquad \mathcal{O}_6 := C_2 C_3^{-1}$$
$$\mathcal{O}_7 := C_0 C_1^{-1} C_2^{-1} C_3^{-1} \qquad \mathcal{O}_8 := C_0 C_4^{-1} \qquad \mathcal{O}_9 := C_0 C_5^{-1}.$$

Let

$$\mathcal{O}_{j*} := \begin{cases} \mathcal{O}_j^{\neq 0}, & j \in \{1, \ldots, 7\}, \\ \mathcal{O}_j, & j \in \{8, 9\}. \end{cases}$$

For $\eta_j \in \mathcal{O}_j$, let

$$I_j := \eta_j \mathcal{O}_j^{-1}.$$

For $B \geqslant 0$, let $\mathcal{R}(B)$ be the set of all $(\eta_1, \ldots, \eta_8) \in \mathbb{C}^8$ with $\eta_1 \neq 0$ and

$$\|\eta_2\eta_3\eta_4\eta_5\eta_6\eta_7\eta_8\|_\infty \leqslant B, \tag{2·1}$$
$$\left\|\eta_1^2\eta_2^2\eta_3^3\eta_4^2\eta_6\right\|_\infty \leqslant B, \tag{2·2}$$
$$\left\|\eta_1\eta_2\eta_3^2\eta_4^2\eta_5^2\eta_6^2\eta_7\right\|_\infty \leqslant B, \tag{2·3}$$
$$\left\|\eta_3\eta_4\eta_5\eta_6\eta_7(\eta_4\eta_5^3\eta_6^2\eta_7 + \eta_2\eta_8)\right\|_\infty \leqslant B, \tag{2·4}$$
$$\left\|\frac{\eta_2\eta_7\eta_8^2 + \eta_4\eta_5^3\eta_6^2\eta_7^2\eta_8}{\eta_1}\right\|_\infty \leqslant B. \tag{2·5}$$

We observe for future reference that (2·1) and (2·4) imply the condition

$$\left\|\eta_3\eta_4^2\eta_5^4\eta_6^3\eta_7^2\right\|_\infty \leqslant 4B. \tag{2·6}$$

Let $M_\mathbf{C}(B)$ be the set of all

$$(\eta_1, \ldots, \eta_9) \in \mathcal{O}_{1*} \times \cdots \times \mathcal{O}_{9*}$$

that satisfy the *height conditions*

$$(\eta_1, \ldots, \eta_8) \in \mathcal{R}(u_\mathbf{C} B),$$

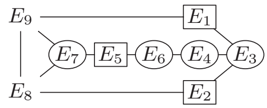

Fig. 1. Configuration of curves on $\widetilde{S}_1$.

the *torsor equation*

$$\eta_4\eta_5^3\eta_6^2\eta_7 + \eta_2\eta_8 + \eta_1\eta_9 = 0 \tag{2·7}$$

and the *coprimality conditions*

$$I_j + I_k = \mathcal{O}_K \text{ for all distinct nonadjacent vertices } E_j, E_k \text{ in Figure 1.} \tag{2·8}$$

LEMMA 2. *We have*

$$N_{U_1,H}(B) = \frac{1}{\omega_K^6}\sum_{\mathbf{C}\in\mathcal{C}^6}|M_{\mathbf{C}}(B)|.$$

Proof. We observe that the statement of our lemma is a specialization of [**DF13**, claim 4·1]. We prove it using the strategy from [**DF13**, section 4] based on the construction of the minimal desingularization $\pi : \widetilde{S}_1 \to S_1$ by the following sequence of blow-ups: Starting with the curves $E_8^{(0)} := \{y_0 = 0\}$, $E_3^{(0)} := \{y_1 = 0\}$, $E_9^{(0)} := \{y_2 = 0\}$, $E_7^{(0)} := \{-y_0 - y_2 = 0\}$ in $\mathbb{P}_K^2$, we:

(i) blow up $E_3^{(0)} \cap E_7^{(0)}$, giving $E_4^{(1)}$;
(ii) blow up $E_4^{(1)} \cap E_7^{(1)}$, giving $E_6^{(2)}$;
(iii) blow up $E_6^{(2)} \cap E_7^{(2)}$, giving $E_5^{(3)}$;
(iv) blow up $E_3^{(3)} \cap E_8^{(3)}$, giving $E_2^{(4)}$;
(v) blow up $E_3^{(4)} \cap E_9^{(4)}$, giving $E_1^{(5)}$.

With the inverse $\pi \circ \rho^{-1} : \mathbb{P}_K^2 \dashrightarrow S_1$ of the projection $\phi = \rho \circ \pi^{-1} : S_1 \dashrightarrow \mathbb{P}_K^2$, $(x_0 : \cdots : x_4) \mapsto (x_0 : x_2 : x_3)$ given by

$$\psi((y_0 : y_1 : y_2)) = (y_0y_1(y_0+y_2) : -y_1^3 : y_1^2(y_0+y_2) : y_1y_2(y_0+y_2) : y_0y_2(y_0+y_2)) \tag{2·9}$$

and the map Ψ from [**DF13**, claim 4·2] sending $(\eta_1, \ldots, \eta_9)$ to

$$(\eta_2\eta_3\eta_4\eta_5\eta_6\eta_7\eta_8, -\eta_1^2\eta_2^2\eta_3^3\eta_4^2\eta_6, \eta_1\eta_2\eta_3^2\eta_4^2\eta_5^2\eta_6^2\eta_7, \eta_1\eta_3\eta_4\eta_5\eta_6\eta_7\eta_9, \eta_7\eta_8\eta_9),$$

we can proceed exactly as in the proof of [**DF13**, lemma 9·1].

2·2. *Summations*

2·2·1. *The first summation over η_8 with dependent η_9*

LEMMA 3. *Write $\boldsymbol{\eta}' := (\eta_1, \ldots, \eta_7)$ and $\mathbf{I}' := (I_1, \ldots, I_7)$. For $B > 0$, $\mathbf{C} \in \mathcal{C}^6$, we have*

$$|M_{\mathbf{C}}(B)| = \frac{2}{\sqrt{|\Delta_K|}}\sum_{\boldsymbol{\eta}'\in\mathcal{O}_{1*}\times\cdots\times\mathcal{O}_{7*}}\theta_8(\mathbf{I}')V_8(\mathfrak{N}I_1, \ldots, \mathfrak{N}I_7; B) + O_{\mathbf{C}}(B(\log B)^2),$$

where

$$V_8(t_1, \ldots, t_7; B) := \frac{1}{t_1}\int_{h(\sqrt{t_1},\ldots,\sqrt{t_7},\eta_8;B)\leqslant 1} \mathrm{d}\eta_8,$$

with a complex variable η_8, and where

$$\theta_8(\mathbf{I}') := \prod_{\mathfrak{p}}\theta_{8,\mathfrak{p}}(J_{\mathfrak{p}}(\mathbf{I}')),$$

with $J_{\mathfrak{p}}(\mathbf{I}') := \{j \in \{1, \dots, 7\} \ : \ \mathfrak{p} \mid I_j\}$ *and*

$$\theta_{8,\mathfrak{p}}(J) := \begin{cases} 1 & \text{if } J = \emptyset, \{1\}, \{2\}, \{7\}, \\ 1 - \frac{1}{\mathfrak{N}\mathfrak{p}} & \text{if } J = \{4\},\{5\},\{6\},\{1,3\},\{2,3\},\{3,4\},\{4,6\},\{5,6\},\{5,7\}, \\ 1 - \frac{2}{\mathfrak{N}\mathfrak{p}} & \text{if } J = \{3\}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By [**DF13**, lemma 3·2], the set $\mathcal{R}(\boldsymbol{\eta}', u_{\mathbf{C}}B)$ of all $\eta_8 \in \mathbb{C}$ with $(\eta_1, \dots, \eta_8) \in \mathcal{R}(u_{\mathbf{C}}B)$ has class m, with an absolute constant m. Moreover, by [**DF13**, lemma 3·4, (1)] applied to (2·5), this set is contained in the union of at most 2 balls of radius

$$R(\boldsymbol{\eta}'; u_{\mathbf{C}}B) := \left(u_{\mathbf{C}}B \left\|\eta_1\eta_2^{-1}\eta_7^{-1}\right\|_\infty\right)^{1/4} \ll_{\mathbf{C}} \left(B\mathfrak{N}\left(I_1 I_2^{-1} I_7^{-1}\right)\right)^{1/4}.$$

We apply [**DF13**, proposition 5·3] with $(A_1, A_2, A_3, A_0) := (4, 6, 5, 7)$, $(B_1, B_0) := (2, 8)$, $(C_1, C_0) := (1, 9)$, $D := 3$, and $u_{\mathbf{C}}B$ instead of B. (Moreover, we choose Π_1 and Π_2 as in [**DF13**, remark 5·2].)

Similarly as in [**DF13**, lemma 9·2], we see that the resulting main term is the one given in the lemma. The error term from [**DF13**, proposition 5·3] is

$$\ll \sum_{\boldsymbol{\eta}',\ (2\cdot10)} 2^{\omega_K(I_3)+\omega_K(I_3I_4I_5I_6)} \left(\frac{R(\boldsymbol{\eta}'; u_{\mathbf{C}}B)}{\mathfrak{N}(I_1)^{1/2}} + 1\right),$$

where, using (2·3) and the definitions of $u_{\mathbf{C}}$ and the $\mathcal{O}_j$, the sum runs over all $\boldsymbol{\eta}'$ with

$$\mathfrak{N}\left(I_1 I_2 I_3^2 I_4^2 I_5^2 I_6^2 I_7\right) \leqslant B. \tag{2·10}$$

Since $|\mathcal{O}_K^\times| < \infty$, we can sum over the I_j instead of the η_j, which then run over all nonzero ideals of $\mathcal{O}_K$ with (2·10), so the error term is bounded by

$$\ll_{\mathbf{C}} \sum_{\mathbf{I}',\ (2\cdot10)} 2^{\omega_K(I_3)+\omega_K(I_3I_4I_5I_6)} \left(\frac{B^{1/4}}{\mathfrak{N}I_1^{1/4}\mathfrak{N}I_2^{1/4}\mathfrak{N}I_7^{1/4}} + 1\right)$$

$$\ll \sum_{\substack{I_1,\dots,I_6 \\ \mathfrak{N}I_j \leqslant B}} \left(\frac{2^{\omega_K(I_3)+\omega_K(I_3I_4I_5I_6)}B}{\mathfrak{N}I_1\mathfrak{N}I_2\mathfrak{N}I_3^{3/2}\mathfrak{N}I_4^{3/2}\mathfrak{N}I_5^{3/2}\mathfrak{N}I_6^{3/2}} + \frac{2^{\omega_K(I_3)+\omega_K(I_3I_4I_5I_6)}B}{\mathfrak{N}I_1\mathfrak{N}I_2\mathfrak{N}I_3^2\mathfrak{N}I_4^2\mathfrak{N}I_5^2\mathfrak{N}I_6^2}\right)$$

$$\ll B(\log B)^2.$$

2·2·2. *The second summation over* η_7.

LEMMA 4. *Write* $\boldsymbol{\eta}'' := (\eta_1, \dots, \eta_6)$. *For* $B \geqslant 3$, $\mathbf{C} \in \mathcal{C}^6$, *we have*

$$|M_{\mathbf{C}}(B)| = \left(\frac{2}{\sqrt{|\Delta_K|}}\right)^2 \sum_{\boldsymbol{\eta}'' \in \mathcal{O}_{1*}\times\cdots\times\mathcal{O}_{6*}} \mathcal{A}(\theta_8(\mathbf{I}'), I_7) V_7(\mathfrak{N}I_1, \dots, \mathfrak{N}I_6; B) + O_{\mathbf{C}}(B(\log B)^4 \log\log B).$$

Here, $\mathcal{A}(\theta_8(\mathbf{I}'), I_7)$ *is as in* [**DF13**, (2·1)] *and, for* $t_1, \dots, t_6 \geqslant 1$,

$$V_7(t_1, \dots, t_6; B) := \frac{\pi}{t_1} \int_{\substack{(\sqrt{t_1},\dots,\sqrt{t_7},\eta_8)\in\mathcal{R}(B) \\ t_7 \geqslant 1}} \mathrm{d}t_7\, \mathrm{d}\eta_8,$$

with a real variable t_7 *and a complex variable* η_8.

Proof. Following the strategy described in [**DF13**, section 6] in the case $b_0 = 1$, we write

$$|M_{\mathbf{C}}(B)| = \frac{2}{\sqrt{|\Delta_K|}} \sum_{\eta'' \in \mathcal{O}_{1*} \times \cdots \times \mathcal{O}_{6*}} \sum_{\eta_7 \in \mathcal{O}_{7*}} \vartheta(I_7) g(\mathfrak{N} I_7) + O_{\mathbf{C}}(B(\log B)^2), \tag{2·11}$$

where $\vartheta(\mathfrak{a}) := \theta_8(I_1, \ldots, I_6, \mathfrak{a})$ and $g(t) := V_8(\mathfrak{N} I_1, \ldots, \mathfrak{N} I_6, t; B)$. The conditions (2·2) and (2·6) imply that $g(t) = 0$ unless

$$\mathfrak{N} I_1^2 \mathfrak{N} I_2^2 \mathfrak{N} I_3^3 \mathfrak{N} I_4^2 \mathfrak{N} I_6 \leqslant B \quad \text{and} \quad t \leqslant t_2 := \left(\frac{4B}{\mathfrak{N} I_3 \mathfrak{N} I_4^2 \mathfrak{N} I_5^4 \mathfrak{N} I_6^3} \right)^{1/2}. \tag{2·12}$$

Moreover, applying [**DF13**, lemma 3·4 (2)] to (2·5), we see that

$$\begin{aligned} g(t) &\ll \frac{1}{\mathfrak{N} I_1} \cdot \left(\frac{\mathfrak{N} I_1 B}{\mathfrak{N} I_2 t} \right)^{1/2} \\ &= \frac{B}{\mathfrak{N} I_1 \cdots \mathfrak{N} I_6 t} \left(\frac{B}{\mathfrak{N} I_1^2 \mathfrak{N} I_2^2 \mathfrak{N} I_3^3 \mathfrak{N} I_4^2 \mathfrak{N} I_6} \right)^{-1/4} \left(\frac{B}{\mathfrak{N} I_3 \mathfrak{N} I_4^2 \mathfrak{N} I_5^4 \mathfrak{N} I_6^3 t^2} \right)^{-1/4}. \end{aligned}$$

In particular, we always have $g(t) \ll B/(\mathfrak{N} I_1 \cdots \mathfrak{N} I_6 t)$.

By [**DF13**, lemma 5·4, lemma 2·2], ϑ satisfies the condition [**DF13**, (6·1)] with $C = 0$ and $c_\vartheta = 2^{\omega(I_1 \cdots I_4 I_6)}$.

Let $t_1 := (\log B)^{14}$. A straightforward application of [**DF13**, proposition 6·1] would not yield sufficiently good error terms, so, using a strategy as in the proof of [**DF13**, proposition 7·2], we split the sum over η_7 into the two cases $\mathfrak{N} I_7 \leqslant t_1$ and $\mathfrak{N} I_7 > t_1$.

Let us start with the second case. We may assume that $t_2 \geqslant t_1$. Using [**DF13**, proposition 6·1] with the upper bound $g(t) \ll B/(\mathfrak{N} I_1 \cdots \mathfrak{N} I_6 t)$, we see that

$$\sum_{\substack{\eta_7 \in \mathcal{O}_{7*} \\ \mathfrak{N} I_7 > t_1}} \vartheta(I_7) g(\mathfrak{N} I_7) = \frac{2\pi}{\sqrt{|\Delta_K|}} \mathcal{A}(\vartheta(\mathfrak{a}), \mathfrak{a}, \mathcal{O}_K) \int_{t \geqslant t_1} g(t) \, \mathrm{d}t + O\left(\frac{2^{\omega_K(I_1 \cdots I_4 I_6)} B}{\mathfrak{N} I_1 \cdots \mathfrak{N} I_6} t_1^{-1/2} \right).$$

When summing the error term over the remaining variables, we may sum over all $\mathbf{I}''$ with $\mathfrak{N} I_j \leqslant B$, so the error term is

$$\ll t_1^{-1/2} \sum_{\mathbf{I}''} \frac{2^{\omega_K(I_1 \cdots I_4 I_6)} B}{\mathfrak{N} I_1 \cdots \mathfrak{N} I_6} \ll (\log B)^{-7} B (\log B)^{11} = B(\log B)^4.$$

Now let us consider the sum over all η_7 with $\mathfrak{N} I_7 \leqslant t_1$. Since $0 \leqslant \vartheta(I_7) \leqslant 1$, we obtain an upper bound

$$\begin{aligned}
&\sum_{\eta'' \in \mathcal{O}_{1*}\times\cdots\times\mathcal{O}_{6*}} \sum_{\substack{\eta_7\in\mathcal{O}_{7*}\\ \mathfrak{N}I_7\leqslant t_1}} \vartheta(I_7)g(\mathfrak{N}I_7)\\
&\ll \sum_{\substack{\mathbf{I}'',I_7\\ (2\cdot 12)\text{ with } t=\mathfrak{N}I_7\\ \mathfrak{N}I_7\leqslant t_1}} \frac{B}{\mathfrak{N}I_1\cdots\mathfrak{N}I_6\mathfrak{N}I_7}\left(\frac{B}{\mathfrak{N}I_1^2\mathfrak{N}I_2^2\mathfrak{N}I_3^3\mathfrak{N}I_4^2\mathfrak{N}I_6}\right)^{-\frac14}\left(\frac{B}{\mathfrak{N}I_3\mathfrak{N}I_4^2\mathfrak{N}I_5^4\mathfrak{N}I_6^3\mathfrak{N}I_7^2}\right)^{-\frac14}\\
&\ll \sum_{\substack{I_2,\dots,I_7\\ (2\cdot 12)\text{ with } t=\mathfrak{N}I_7\\ \mathfrak{N}I_7\leqslant t_1}} \frac{B}{\mathfrak{N}I_2\cdots\mathfrak{N}I_7}\left(\frac{B}{\mathfrak{N}I_3\mathfrak{N}I_4^2\mathfrak{N}I_5^4\mathfrak{N}I_6^3\mathfrak{N}I_7^2}\right)^{-\frac14}\\
&\ll \sum_{\substack{I_2,I_3,I_4,I_6,I_7\\ \mathfrak{N}I_j\leqslant B,\ \mathfrak{N}I_7\leqslant t_1}} \frac{B}{\mathfrak{N}I_2\mathfrak{N}I_3\mathfrak{N}I_4\mathfrak{N}I_6\mathfrak{N}I_7} \ll B(\log B)^4\log t_1 \ll B(\log B)^4\log\log B.
\end{aligned}$$

Our proof is finished once we see that

$$\sum_{\eta''\in\mathcal{O}_{1*}\times\cdots\times\mathcal{O}_{6*}} \mathcal{A}(\vartheta(\mathfrak{a}),\mathfrak{a})\int_1^{t_1} g(t)\,\mathrm{d}t \ll B(\log B)^4\log\log B.$$

This follows from an analogous computation as above with the integral over t instead of the sum over I_7, and using that $0\leqslant\mathcal{A}(\vartheta(\mathfrak{a}),\mathfrak{a})\leqslant 1$.

LEMMA 5. *If* $\mathbf{I}''$ *runs over all six-tuples* $(I_1,\dots,I_6)$ *of nonzero ideals of* $\mathcal{O}_K$ *then we have*

$$N_{U_1,H}(B) = \left(\frac{2}{\sqrt{|\Delta_K|}}\right)^2 \sum_{\mathbf{I}''} \mathcal{A}(\theta_8(\mathbf{I}'',I_7),I_7)V_7(\mathfrak{N}I_1,\dots,\mathfrak{N}I_6;B) + O(B(\log B)^4\log\log B).$$

Proof. This is entirely analogous to [**DF13**, lemma 9·4].

2·2·3. *The remaining summations*

LEMMA 6. *We have*

$$N_{U_1,H}(B) = \left(\frac{2}{\sqrt{|\Delta_K|}}\right)^8\left(\frac{h_K}{\omega_K}\right)^6\theta_0V_0(B) + O(B(\log B)^4\log\log B),$$

where θ_0 *is as in* (1·7) *and*

$$V_0(B) := \int_{\substack{(\eta_1,\dots,\eta_8)\in\mathcal{R}(B)\\ \|\eta_1\|_\infty,\dots,\|\eta_7\|_\infty\geqslant 1}} \frac{1}{\|\eta_1\|_\infty}\,\mathrm{d}\eta_1\cdots\mathrm{d}\eta_8,$$

with complex variables $\eta_1,\dots,\eta_8$.

Proof. By [**DF13**, lemma 3·4, (6)], applied to (2·5), we have

$$V_7(t_1,\dots,t_6;B) \ll \frac{B^{2/3}}{t_1^{1/3}t_2^{1/3}t_4^{1/3}t_5t_6^{2/3}} = \frac{B}{t_1\cdots t_6}\left(\frac{B}{t_1^2t_2^2t_3^3t_4^2t_6}\right)^{-1/3}.$$

We apply [**DF13**, proposition 7·3] with $r=5$ and use polar coordinates, similarly to [**DF13**, lemma 9·5, lemma 9·9].

2·3. *Proof of Theorem* 1 *for* S_1

Let $\alpha(\widetilde{S}_1) := \frac{1}{8640}$ and

$$\omega_\infty(\widetilde{S}_1) := \frac{12}{\pi}\int_{\|z_0z_1(z_0+z_2)\|_\infty,\,\|z_1^3\|_\infty,\,\|z_1^2(z_0+z_2)\|_\infty,\,\|z_1z_2(z_0+z_2)\|_\infty,\,\|z_0z_2(z_0+z_2)\|_\infty\leqslant 1} \mathrm{d}z_0\,\mathrm{d}z_1\,\mathrm{d}z_2.$$

We will use the conditions

$$\left\|\eta_1^2\eta_2^2\eta_4^2\eta_6\right\|_\infty \leqslant B, \tag{2·13}$$

$$\left\|\eta_1^2\eta_2^2\eta_4^2\eta_6\right\|_\infty \leqslant B \text{ and } \left\|\eta_1^{-1}\eta_2^{-1}\eta_4^2\eta_5^6\eta_6^4\right\|_\infty \leqslant B. \tag{2·14}$$

LEMMA 7. *Let* $\mathcal{R}(B)$ *be as in* (2·1)–(2·5) *and define*

$$V_0'(B) := \int_{\substack{(\eta_1,\ldots,\eta_8)\in\mathcal{R}(B)\\ \|\eta_1\|_\infty,\|\eta_2\|_\infty,\|\eta_4\|_\infty,\|\eta_5\|_\infty,\|\eta_6\|_\infty\geqslant 1\\ (2\cdot 14)}} \frac{1}{\|\eta_1\|_\infty}\,\mathrm{d}\eta_1\cdots\mathrm{d}\eta_8,$$

with complex variables $\eta_1, \ldots, \eta_8$. *Then*

$$\pi^6\alpha(\widetilde{S}_1)\omega_\infty(\widetilde{S}_1)B(\log B)^5 = 4V_0'(B). \tag{2·15}$$

Proof. We use the following substitutions on $\omega_\infty(\widetilde{S}_1)$: Let $\eta_1, \eta_2, \eta_4, \eta_5, \eta_6 \in \mathbb{C}\setminus\{0\}$ and $B > 0$. Let η_3, η_7, η_8 be complex variables. With $l := (B\left\|\eta_1\eta_2\eta_4\eta_5^3\eta_6^2\right\|_\infty)^{1/2}$, we apply the coordinate transformation $z_0 = l^{-1/3}\eta_2\cdot\eta_8$, $z_1 = l^{-1/3}\eta_1\eta_2\eta_4\eta_5\eta_6\cdot\eta_3$, $z_2 = l^{-1/3}(-\eta_2\cdot\eta_8 - \eta_4\eta_5^3\eta_6^2\cdot\eta_7)$, of Jacobi determinant

$$\frac{\|\eta_1\eta_2\eta_4\eta_5\eta_6\|_\infty}{B}\frac{1}{\|\eta_1\|_\infty}, \tag{2·16}$$

and obtain

$$\omega_\infty(\widetilde{S}_1) = \frac{12}{\pi}\frac{\|\eta_1\eta_2\eta_4\eta_5\eta_6\|_\infty}{B}\int_{(\eta_1,\ldots,\eta_8)\in\mathcal{R}(B)}\frac{1}{\|\eta_1\|_\infty}\,\mathrm{d}\eta_3\,\mathrm{d}\eta_7\,\mathrm{d}\eta_8. \tag{2·17}$$

The negative curves $[E_1], \ldots, [E_7]$ generate the effective cone of $\widetilde{S}_1$. We have $[-K_{\widetilde{S}_1}] = [2E_1 + 2E_2 + 3E_3 + 2E_4 + E_6]$ and $[E_7] = [E_1 + E_2 + E_3 - 2E_5 - E_6]$. Hence, [**DF13**, lemma 8·1] (with the roles of η_3 and η_6 exchanged) gives

$$\alpha(\widetilde{S}_1)(\log B)^5 = \frac{1}{3\pi^5}\int_{\substack{\|\eta_1\|_\infty,\|\eta_2\|_\infty,\|\eta_4\|_\infty,\|\eta_5\|_\infty,\|\eta_6\|_\infty\geqslant 1\\ (2\cdot 14)}}\frac{\mathrm{d}\eta_1\,\mathrm{d}\eta_2\,\mathrm{d}\eta_4\,\mathrm{d}\eta_5\,\mathrm{d}\eta_6}{\|\eta_1\eta_2\eta_4\eta_5\eta_6\|_\infty}. \tag{2·18}$$

The lemma follows by substituting (2·17) and (2·18) in (2·15).

To finish our proof, we compare $V_0(B)$ from Lemma 6 with $V_0'(B)$ defined in Lemma 7. Let

$$\mathcal{D}_0(B) := \{(\eta_1,\ldots,\eta_8)\in\mathcal{R}(B) \mid \|\eta_1\|_\infty,\ldots,\|\eta_7\|_\infty \geqslant 1\},$$
$$\mathcal{D}_1(B) := \{(\eta_1,\ldots,\eta_8)\in\mathcal{R}(B) \mid \|\eta_1\|_\infty,\ldots,\|\eta_7\|_\infty \geqslant 1, (2\cdot 13)\},$$
$$\mathcal{D}_2(B) := \{(\eta_1,\ldots,\eta_8)\in\mathcal{R}(B) \mid \|\eta_1\|_\infty,\ldots,\|\eta_7\|_\infty \geqslant 1, (2\cdot 14)\},$$
$$\mathcal{D}_3(B) := \{(\eta_1,\ldots,\eta_8)\in\mathcal{R}(B) \mid \|\eta_1\|_\infty,\ldots,\|\eta_6\|_\infty \geqslant 1, (2\cdot 14)\},$$
$$\mathcal{D}_4(B) := \{(\eta_1,\ldots,\eta_8)\in\mathcal{R}(B) \mid \|\eta_1\|_\infty,\|\eta_2\|_\infty,\|\eta_4\|_\infty,\|\eta_5\|_\infty,\|\eta_6\|_\infty \geqslant 1, (2\cdot 14)\}.$$

Moreover, let

$$V_i(B) := \int_{\mathcal{D}_i(B)}\frac{\mathrm{d}\eta_1\cdots\mathrm{d}\eta_8}{\|\eta_1\|_\infty}.$$

Then $V_0(B)$ is as in Lemma 6 and $V_4(B) = V_0'(B)$. We show that, for $1 \leqslant i \leqslant 4$, $V_i(B) - V_{i-1}(B) = O(B(\log B)^4)$. This holds for $i = 1$, since, by (2·2) and $\|\eta_3\|_\infty \geqslant 1$, we have $\mathcal{D}_1(B) = \mathcal{D}_0(B)$.

Moreover, using [**DF13**, lemma 3·4 (2)] and (2·5) to bound the integral over η_8, we have

$$V_2(B) - V_1(B) \ll \int_{\substack{1 \leqslant \|\eta_1\|_\infty, \dots, \|\eta_7\|_\infty \leqslant B \\ \|\eta_1^{-1}\eta_2^{-1}\eta_4^2\eta_5^6\eta_6^4\|_\infty > B \\ (2\cdot 6)}} \frac{B^{1/2}}{\|\eta_1\eta_2\eta_7\|_\infty^{1/2}} \,\mathrm{d}\eta_1 \cdots \mathrm{d}\eta_7 \ll B(\log B)^4.$$

Moreover,

$$V_3(B) - V_2(B) \ll \int_{\substack{\|\eta_1\|_\infty, \dots, \|\eta_6\|_\infty \geqslant 1 \\ \|\eta_7\|_\infty < 1,\ (2\cdot 2),\ (2\cdot 14)}} \frac{B^{1/2}}{\|\eta_1\eta_2\eta_7\|_\infty^{1/2}} \,\mathrm{d}\eta_1 \cdots \mathrm{d}\eta_7 \ll B(\log B)^4.$$

Finally, using [**DF13**, lemma 3·4 (4)] and (2·5) to bound the integral over η_7, η_8, we have

$$V_4(B) - V_3(B) \ll \int_{\substack{\|\eta_1\|_\infty, \|\eta_2\|_\infty, \|\eta_4\|_\infty, \|\eta_5\|_\infty, \|\eta_6\|_\infty \geqslant 1 \\ \|\eta_3\|_\infty < 1,\ (2\cdot 13)}} \frac{B^{2/3}}{\|\eta_1\eta_2\eta_4\eta_5^3\eta_6^2\|_\infty^{1/3}} \,\mathrm{d}\eta_1 \cdots \mathrm{d}\eta_6 \ll B(\log B)^4.$$

Using Lemma 6 and Lemma 7, this shows Theorem 1 for S_1.

3. *The quartic del Pezzo surface of type* $\mathbf{A}_4$

3·1. *Passage to a universal torsor*

We use the notation of [**Der06**], except that we swap η_8 and η_9.

For any given $\mathbf{C} = (C_0, \dots, C_5) \in \mathcal{C}^6$, we define $u_{\mathbf{C}} := \mathfrak{N}(C_0^3 C_1^{-1} \cdots C_5^{-1})$ and

$$\begin{aligned}
&\mathcal{O}_1 := C_3C_4^{-1} && \mathcal{O}_2 := C_4C_5^{-1} && \mathcal{O}_3 := C_0C_1^{-1}C_3^{-1}C_4^{-1}\\
&\mathcal{O}_4 := C_1C_2^{-1} && \mathcal{O}_5 := C_5 && \mathcal{O}_6 := C_2\\
&\mathcal{O}_7 := C_0C_1^{-1}C_2^{-1} && \mathcal{O}_8 := C_0C_3^{-1} && \mathcal{O}_9 := C_0^2C_3^{-1}C_4^{-1}C_5^{-1}.
\end{aligned}$$

Let

$$\mathcal{O}_{j*} := \begin{cases} \mathcal{O}_j^{\neq 0}, & j \in \{1, \dots, 7\},\\ \mathcal{O}_j, & j \in \{8, 9\}.\end{cases}$$

For $\eta_j \in \mathcal{O}_j$, let

$$I_j := \eta_j \mathcal{O}_j^{-1}.$$

For $B \geqslant 0$, let $\mathcal{R}(B)$ be the set of all $(\eta_1, \dots, \eta_8) \in \mathbb{C}^8$ with $\eta_5 \neq 0$ and

$$\left\|\eta_1^2\eta_2^4\eta_3^3\eta_4^2\eta_5^3\eta_6\right\|_\infty \leqslant B, \tag{3·1}$$

$$\|\eta_1\eta_2\eta_3\eta_4\eta_6\eta_7\eta_8\|_\infty \leqslant B, \tag{3·2}$$

$$\left\|\eta_1^2\eta_2^3\eta_3^2\eta_4\eta_5^2\eta_8\right\|_\infty \leqslant B, \tag{3·3}$$

$$\left\|\eta_1\eta_2^2\eta_3^2\eta_4^2\eta_5\eta_6^2\eta_7\right\|_\infty \leqslant B, \tag{3·4}$$

$$\left\|\frac{\eta_1\eta_7\eta_8^2 + \eta_3\eta_4^2\eta_6^3\eta_7^2}{\eta_5}\right\|_\infty \leqslant B \tag{3·5}$$

and let $M_{\mathbf{C}}(B)$ be the set of all

$$(\eta_1, \dots, \eta_9) \in \mathcal{O}_{1*} \times \cdots \times \mathcal{O}_{9*}$$

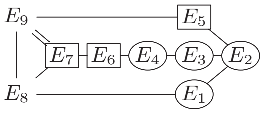

Fig. 2. Configuration of curves on $\widetilde{S}_2$.

that satisfy the *height conditions*

$$(\eta_1, \ldots, \eta_8) \in \mathcal{R}(u_{\mathbf{C}} B),$$

the *torsor equation*

$$\eta_3\eta_4^2\eta_6^3\eta_7 + \eta_1\eta_8^2 + \eta_5\eta_9 = 0 \tag{3·6}$$

and the *coprimality conditions*

$$I_j + I_k = \mathcal{O}_K \text{ for all distinct nonadjacent vertices } E_j, E_k \text{ in Figure 2.} \tag{3·7}$$

LEMMA 8. *We have*

$$N_{U_2,H}(B) = \frac{1}{\omega_K^6} \sum_{\mathbf{C}\in\mathcal{C}^6} |M_{\mathbf{C}}(B)|.$$

Proof. This is a specialization of [**DF13**, claim 4·1] and we prove it using the strategy from [**DF13**, section 4] with the data supplied in [**Der06**]. Starting with the curves $E_3^{(0)} := \{y_0 = 0\}$, $E_8^{(0)} := \{y_1 = 0\}$, $E_7^{(0)} := \{y_2 = 0\}$, $E_9^{(0)} := \{-y_0y_2 - y_1^2 = 0\}$ in $\mathbb{P}_K^2$, we prove [**DF13**, claim 4·2] for the following sequence of blow-ups:

(i) blow up $E_3^{(0)} \cap E_8^{(0)} \cap E_9^{(0)}$, giving $E_1^{(1)}$;
(ii) blow up $E_1^{(1)} \cap E_3^{(1)} \cap E_9^{(1)}$, giving $E_2^{(2)}$;
(iii) blow up $E_2^{(2)} \cap E_9^{(2)}$, giving $E_5^{(3)}$;
(iv) blow up $E_3^{(3)} \cap E_7^{(3)}$, giving $E_4^{(4)}$;
(v) blow up $E_4^{(4)} \cap E_7^{(4)}$, giving $E_6^{(5)}$.

The inverse $\pi \circ \rho^{-1} : \mathbb{P}_K^2 \dashrightarrow S_2$ of the projection $\phi = \rho \circ \pi^{-1} : S_2 \dashrightarrow \mathbb{P}_K^2$, $(x_0 : \cdots : x_4) \mapsto (x_0 : x_2 : x_3)$ is given by

$$(y_0 : y_1 : y_2) \longmapsto \big(y_0^3 : y_0y_1y_2 : y_0^2y_1 : y_0^2y_2 : -y_2\big(y_1^2 + y_0y_2\big)\big), \tag{3·8}$$

and the map Ψ appearing in [**DF13**, claim 4·2] sends $(\eta_1, \ldots, \eta_9)$ to

$$\big(\eta_1^2\eta_2^4\eta_3^3\eta_4^2\eta_5^3\eta_6,\ \eta_1\eta_2\eta_3\eta_4\eta_6\eta_7\eta_8,\ \eta_1^2\eta_2^3\eta_3^2\eta_4\eta_5^2\eta_8,\ \eta_1\eta_2^2\eta_3^2\eta_4^2\eta_5\eta_6^2\eta_7,\ \eta_7\eta_9\big).$$

As in the proof of [**DF13**, lemma 9·1], we see that the hypotheses of [**DF13**, lemma 4·3] are satisfied, so [**DF13**, claim 4·2] holds in our situation for $i = 0$.

Note that [**DF13**, lemma 4·4] applies in steps (3), (4), (5) of the above chain of blow-ups. In steps (1), (2), we are in the situation of [**DF13**, remark 4·5], so that we must derive some coprimality conditions using the torsor equation. We use the notation of [**DF13**, lemma 4·4, remark 4·5].

For (1), we start with the parameterization provided by [**DF13**, lemma 4·3], consisting of $(\eta_3', \eta_7', \eta_8', \eta_9')$ satisfying certain coprimality conditions and other conditions. Since $\eta_3' \neq 0$, there is a unique $C_1 \in \mathcal{C}$ such that $[I_3' + I_8' + I_9'] = [C_1^{-1}]$. We choose $\eta_1'' \in C_1$ such that $I_1'' = I_3' + I_8' + I_9'$; this is unique up to multiplication by $\mathcal{O}_K^\times$. We define $\eta_3'' := \eta_3'/\eta_1''$, $\eta_8'' := \eta_8'/\eta_1''$, $\eta_9'' := \eta_9'/\eta_1''$ and $\eta_7'' := \eta_7'$. To show that $(\eta_1'', \eta_3'', \eta_7'', \eta_8'', \eta_9'')$ lies in the set described in [**DF13**, claim 4·2] for $i = 1$, everything is provided by the proof of [**DF13**, lemma 4·4] except the coprimality conditions involving $\eta''_1, \eta''_3, \eta''_8, \eta''_9$. Considering the configuration of $E_1^{(1)}, E_3^{(1)}, E_8^{(1)}, E_9^{(1)}$, these are $I''_3 + I''_8 = \mathcal{O}_K$ (which holds because $I''_3 + I''_8 + I''_9 = \mathcal{O}_K$ by construction and because of the relation $\eta''_3\eta''_7 + \eta''_1\eta''^2_8 + \eta''_9 = 0$) and $I''_1 + I''_8 + I''_9 = \mathcal{O}_K$ (which holds because otherwise the relation would give non-triviality of $I''_1 + I''_8 + I''_9 + I''_3 I''_7$ contradicting the previous condition or the condition $I''_1 + I''_7 = \mathcal{O}_K$ provided by the proof of [**DF13**, lemma 4·4]).

For (2), we replace $''$ by $'$ in the result of the previous step. We choose $C_2 \in \mathcal{C}$ such that $[I'_1 + I'_3 + I'_9] = [C_2^{-1}]$ and $\eta''_2 \in C_4 = \mathcal{O}''_2$ such that $I''_2 = I'_1 + I'_3 + I'_9$. It remains to check the pairwise coprimality of I''_1, I''_3, I''_9. By construction, $I''_1 + I''_3 + I''_9 = \mathcal{O}_K$; considering the torsor equation $\eta''_3\eta''_7 + \eta''_1\eta''^2_8 + \eta''_9 = 0$ shows $I''_1 + I''_3 = \mathcal{O}_K$ directly, $I''_1 + I''_9 = \mathcal{O}_K$ using $I''_1 + I''_7 = \mathcal{O}_K$, and $I''_3 + I''_9 = \mathcal{O}_K$ using $I''_3 + I''_8 = \mathcal{O}_K$.

Since steps (3), (4), (5) are covered by [**DF13**, lemma 4·4], this shows [**DF13**, claim 4·2]. We deduce [**DF13**, claim 4·1] in the same way as in [**DF13**, lemma 9·1].

3·2. *Summations*

3·2·1. *The first summation over η_8 with dependent η_9*

Let $\boldsymbol{\eta}' := (\eta_1, \ldots, \eta_7)$ and $\mathbf{I}' := (I_1, \ldots, I_7)$. Let $\theta_0(\mathbf{I}') := \prod_{\mathfrak{p}} \theta_{0,\mathfrak{p}}(J_{\mathfrak{p}}(\mathbf{I}'))$, where $J_{\mathfrak{p}}(\mathbf{I}') := \{j \in \{1, \ldots, 7\} : \mathfrak{p} \mid I_j\}$ and

$$\theta_{0,\mathfrak{p}}(J) := \begin{cases} 1 & \text{if } J = \emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \\ & \text{or } J = \{1,2\}, \{2,3\}, \{2,5\}, \{3,4\}, \{4,6\}, \{6,7\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\theta_0(\mathbf{I}') = 1$ if and only if $I_1, \ldots, I_7$ satisfy the coprimality conditions from (3·7), and $\theta_0(\mathbf{I}') = 0$ otherwise.

We apply [**DF13**, proposition 5·3] with $(A_1, A_2, A_3, A_0) := (3, 4, 6, 7)$, $(B_1, B_0) := (1, 8)$, $(C_1, C_0) := (5, 9)$, and $D := 2$. For given η_2, η_5, we write

$$\eta_3\eta_4^2\eta_6^3\eta_7 = \eta_{A_0}^{a_0}\Pi(\boldsymbol{\eta}_A) = \Pi_1\Pi_2^2,$$

where Π_1, Π_2 are chosen as follows: Let $\mathfrak{A} = \mathfrak{A}(\eta_2, \eta_5)$ be a prime ideal not dividing $I_2 I_5$ such that $\mathfrak{A}\mathcal{O}_6^{-1}\mathcal{O}_8 = \mathfrak{A}C_0C_2^{-1}C_3^{-1}$ is a principal fractional ideal $t\mathcal{O}_K$, for a suitable $t = t(\eta_2, \eta_5) \in K^\times$. Then we define $\Pi_2 = \Pi_2(\eta_2, \eta_5) := \eta_6 t$ and $\Pi_1 := \Pi_1(\eta_2, \eta_5) := \eta_3\eta_4^2\eta_6\eta_7 t^{-2}$.

LEMMA 9. *We have*

$$|M_{\mathbf{C}}(B)| = \frac{2}{\sqrt{|\Delta_K|}} \sum_{\boldsymbol{\eta}' \in \mathcal{O}_{1*} \times \cdots \times \mathcal{O}_{7*}} \theta_8(\boldsymbol{\eta}', \mathbf{C}) V_8(\mathfrak{N} I_1, \ldots, \mathfrak{N} I_7; B) + O_{\mathbf{C}}(B(\log B)^3),$$

where

$$V_8(t_1, \ldots, t_7; B) := \frac{1}{t_5} \int_{(\sqrt{t_1}, \ldots, \sqrt{t_7}, \eta_8) \in \mathcal{R}(B)} \mathrm{d}\eta_8.$$

Moreover,

$$\theta_8(\boldsymbol{\eta}', \mathbf{C}) := \sum_{\substack{\mathfrak{k}_{\mathfrak{c}} \mid I_2 \\ \mathfrak{k}_{\mathfrak{c}} + I_1 I_3 = \mathcal{O}_K}} \frac{\mu_K(\mathfrak{k}_{\mathfrak{c}})}{\mathfrak{N}\mathfrak{k}_{\mathfrak{c}}} \tilde{\theta}_8(\mathbf{I}', \mathfrak{k}_{\mathfrak{c}}) \sum_{\substack{\rho \mod \mathfrak{k}_{\mathfrak{c}} I_5 \\ \rho\mathcal{O}_K + \mathfrak{k}_{\mathfrak{c}} I_5 = \mathcal{O}_K \\ \rho^2 \equiv_{\mathfrak{k}_{\mathfrak{c}} I_5} \eta_6\eta_7 A}} 1,$$

with

$$\tilde{\theta}_8(\mathbf{I}', \mathfrak{k}_{\mathfrak{c}}) := \theta_0(\mathbf{I}') \frac{\phi_K^*(I_2 I_3 I_4 I_6)}{\phi_K^*(I_2 + \mathfrak{k}_{\mathfrak{c}} I_5)}.$$

Here, $A := -\eta_3\eta_4^2/(t(\eta_2, \eta_5)^2\eta_1)$ *and* $\eta_6\eta_7 A$ *is invertible modulo* $\mathfrak{k}_{\mathfrak{c}} I_5$ *whenever* $\theta_0(\mathbf{I}') \neq 0$.

Proof. It is clear that $\theta_8(\boldsymbol{\eta}', \mathbf{C}) = \theta_1(\boldsymbol{\eta}')$ from [**DF13**, proposition 5·3], and a simple argument as in the proof of [**DF13**, lemma 9·2] shows that $V_8(\mathfrak{N}I_1, \dots, \mathfrak{N}I_7; B) = V_1(\boldsymbol{\eta}', u_{\mathbf{C}}B)$. Hence, the main term is correct and it remains to bound the error term arising from [**DF13**, proposition 5·3].

Similarly as in [**DF13**, lemma 9·2], we see that the set $\mathcal{R}(\boldsymbol{\eta}', B)$ of all η_8 with $(\eta_1, \dots, \eta_8) \in \mathcal{R}(u_{\mathbf{C}}B)$ is of bounded class and (using [**DF13**, lemma 3·5, (1)] on (3·5)) contained in two balls of radius $R(\boldsymbol{\eta}'; u_{\mathbf{C}}B) \ll_{\mathbf{C}} (B\mathfrak{N}I_5\mathfrak{N}I_1^{-1}\mathfrak{N}I_7^{-1})^{1/4}$.

The error term is

$$\ll \sum_{\boldsymbol{\eta}', (3\cdot 9)} 2^{\omega_K(I_2)+\omega_K(I_2I_3I_4I_6)+\omega_K(I_2I_5)} \left(\frac{R(\boldsymbol{\eta}'; u_{\mathbf{C}}B)}{\mathfrak{N}(I_5)^{1/2}} + 1\right),$$

where, using (3·4), the sum runs over all $\boldsymbol{\eta}' \in \mathcal{O}_{1*} \times \cdots \times \mathcal{O}_{7*}$ with

$$\mathfrak{N}(I_1 I_2^2 I_3^2 I_4^2 I_5 I_6^2 I_7) \leqslant B. \tag{3·9}$$

Since $|\mathcal{O}_K^\times| < \infty$, we can sum over the I_j instead of the η_j, which then run over all nonzero ideals of $\mathcal{O}_K$ with (3·9), and obtain

$$\begin{aligned}
&\ll_{\mathbf{C}} \sum_{\mathbf{I}', (3\cdot 9)} 2^{\omega_K(I_2)+\omega_K(I_2I_3I_4I_6)+\omega_K(I_2I_5)} \left(\frac{B^{1/4}}{\mathfrak{N}I_1^{1/4}\mathfrak{N}I_5^{1/4}\mathfrak{N}I_7^{1/4}} + 1\right)\\
&\ll \sum_{I_1,\dots,I_6} \left(\frac{2^{\omega_K(I_2)+\omega_K(I_2I_3I_4I_6)+\omega_K(I_2I_5)} B}{\mathfrak{N}I_1\mathfrak{N}I_2^{3/2}\mathfrak{N}I_3^{3/2}\mathfrak{N}I_4^{3/2}\mathfrak{N}I_5\mathfrak{N}I_6^{3/2}} + \frac{2^{\omega_K(I_2)+\omega_K(I_2I_3I_4I_6)+\omega_K(I_2I_5)} B}{\mathfrak{N}I_1\mathfrak{N}I_2^{2}\mathfrak{N}I_3^{2}\mathfrak{N}I_4^{2}\mathfrak{N}I_5\mathfrak{N}I_6^{2}}\right)\\
&\ll B(\log B)^3.
\end{aligned}$$

For the further summations, we define

$$\theta_8'(\mathbf{I}') := \sum_{\substack{\mathfrak{k}_{\mathfrak{c}} \mid I_2\\ \mathfrak{k}_{\mathfrak{c}} + I_1 I_3 = \mathcal{O}_K}} \frac{\mu_K(\mathfrak{k}_{\mathfrak{c}})}{\mathfrak{N}\mathfrak{k}_{\mathfrak{c}}} \tilde{\theta}_8(\mathbf{I}', \mathfrak{k}_{\mathfrak{c}})$$

and distinguish between two cases: Similarly to [**BD09**], let $M_{\mathbf{C}}^{(86)}(B)$ be the main term in Lemma 9 with the additional condition $\mathfrak{N}I_6 > \mathfrak{N}I_7$ on the $\boldsymbol{\eta}'$, and let $M_{\mathbf{C}}^{(87)}(B)$ be the main term with the additional condition $\mathfrak{N}I_6 \leqslant \mathfrak{N}I_7$. Moreover, we define

$$N_{86}(B) := \frac{1}{\omega_K^6} \sum_{\mathbf{C} \in \mathcal{C}^6} M_{\mathbf{C}}^{(86)}(B)$$

and $N_{87}(B)$ analogously, so

$$N_{U_2,H}(B) = N_{86}(B) + N_{87}(B) + O(B(\log B)^3). \tag{3·10}$$

3·2·2. *The second summation over η_6 in $M_{\mathbf{C}}^{(86)}(B)$*

LEMMA 10. *Write $\boldsymbol{\eta}'' := (\eta_1, \dots, \eta_5, \eta_7)$ and $\mathcal{O}'' := \mathcal{O}_{1*} \times \cdots \times \mathcal{O}_{5*} \times \mathcal{O}_{7*}$. We have*

$$M_{\mathbf{C}}^{(86)}(B) = \left(\frac{2}{\sqrt{|\Delta_K|}}\right)^2 \sum_{\boldsymbol{\eta}'' \in \mathcal{O}''} \mathcal{A}(\theta_8'(\mathbf{I}'), I_6) V_{86}(\mathfrak{N} I_1, \dots, \mathfrak{N} I_5, \mathfrak{N} I_7; B) + O_{\mathbf{C}}(B (\log B)^4),$$

where, for $t_1, \dots, t_5, t_7 \geqslant 1$,

$$V_{86}(t_1, \dots, t_5, t_7; B) := \frac{\pi}{t_5} \int_{\substack{(\sqrt{t_1}, \dots, \sqrt{t_7}, \eta_8) \in \mathcal{R}(B) \\ t_6 > t_7}} \mathrm{d}t_6 \, \mathrm{d}\eta_8,$$

with a real variable t_6 and a complex variable η_8.

Proof. We follow the strategy described in [**DF13**, section 6] in the case $b_0 \geqslant 2$. We write

$$M_{\mathbf{C}}^{(86)}(B) = \frac{2}{\sqrt{|\Delta_K|}} \sum_{\boldsymbol{\eta}'' \in \mathcal{O}''} \sum_{\substack{\mathfrak{k}_{\mathfrak{c}} \mid I_2 \\ \mathfrak{k}_{\mathfrak{c}} + I_1 I_3 = \mathcal{O}_K}} \frac{\mu(\mathfrak{k}_{\mathfrak{c}})}{\mathfrak{N}\mathfrak{k}_{\mathfrak{c}}} \Sigma,$$

where

$$\Sigma := \sum_{\substack{\eta_6 \in \mathcal{O}_{6*} \\ \mathfrak{N} I_6 > \mathfrak{N} I_7}} \vartheta(I_6) \sum_{\substack{\rho \mod \mathfrak{k}_{\mathfrak{c}} I_5 \\ \rho \mathcal{O}_K + \mathfrak{k}_{\mathfrak{c}} I_5 = \mathcal{O}_K \\ \rho^2 \equiv_{\mathfrak{k}_{\mathfrak{c}} I_5} \eta_6 \eta_7 A}} g(\mathfrak{N} I_6),$$

with $\vartheta(I_6) := \tilde{\theta}_8(\mathbf{I}', \mathfrak{k}_{\mathfrak{c}})$ and $g(t) := V_8(\mathfrak{N} I_1, \dots, \mathfrak{N} I_5, t, \mathfrak{N} I_7; B)$.

By [**DF13**, lemma 5·5, lemma 2·2], the function ϑ satisfies [**DF13**, (6·1)] with $C{:=}0$, $c_\vartheta{:=}2^{\omega_K(I_1 I_2 I_3 I_5)}$. By (3·4), we have $g(t) = 0$ whenever $t > t_2 := B^{1/2}/(\mathfrak{N} I_1^{1/2} \mathfrak{N} I_2 \mathfrak{N} I_3 \mathfrak{N} I_4 \mathfrak{N} I_5^{1/2} \mathfrak{N} I_7^{1/2})$, and, by Lemma [**DF13**, lemma 3·5, (2)] applied to (3·5), we have $g(t) \ll B^{1/2}/(\mathfrak{N} I_1^{1/2} \mathfrak{N} I_5^{1/2} \mathfrak{N} I_7^{1/2})$. Using [**DF13**, proposition 6·1], we obtain

$$\Sigma = \frac{2\pi}{\sqrt{|\Delta_K|}} \phi_K^*(\mathfrak{k}_{\mathfrak{c}} I_5) \mathcal{A}(\vartheta(\mathfrak{a}), \mathfrak{a}, \mathfrak{k}_{\mathfrak{c}} I_5) \int_{t \geqslant \mathfrak{N} I_7} g(t) \, \mathrm{d}t + O\left(\frac{2^{\omega_K(I_1 I_2 I_3 I_5)} B^{1/2}}{\mathfrak{N} I_1^{1/2} \mathfrak{N} I_5^{1/2} \mathfrak{N} I_7^{1/2}} \left(\frac{B^{1/4} \mathfrak{N}\mathfrak{k}_{\mathfrak{c}}^{1/2} \mathfrak{N} I_5^{1/4}}{\mathfrak{N} I_1^{1/4} \mathfrak{N} I_2^{1/2} \mathfrak{N} I_3^{1/2} \mathfrak{N} I_4^{1/2} \mathfrak{N} I_7^{1/4}} + \mathfrak{N}(\mathfrak{k}_{\mathfrak{c}} I_5) \log B\right)\right).$$

Using [**DF13**, lemma 6·3] we see that the main term in the lemma is correct.

For the error term, we may sum over $\mathfrak{k}_{\mathfrak{c}}$ and over the ideals I_j instead of the η_j, since $|\mathcal{O}_K^\times| < \infty$. By (3·1) and our condition $\mathfrak{N} I_6 > \mathfrak{N} I_7$ it suffices to sum over $\mathfrak{k}_{\mathfrak{c}}$ and all $(I_1, \dots, I_5, I_7)$ satisfying

$$\mathfrak{N} I_1^2 \mathfrak{N} I_2^4 \mathfrak{N} I_3^3 \mathfrak{N} I_4^2 \mathfrak{N} I_5^3 \mathfrak{N} I_7 \leqslant B. \tag{3·11}$$

Thus, the total error is bounded by

$$\sum_{\substack{I_1,\ldots,I_5,I_7\\(3\cdot 11)}} \left(\frac{2^{\omega_K(I_2)+\omega_K(I_1I_2I_3I_5)} B^{3/4}}{\mathfrak{N}I_1^{3/4}\mathfrak{N}I_2^{1/2}\mathfrak{N}I_3^{1/2}\mathfrak{N}I_4^{1/2}\mathfrak{N}I_5^{1/4}\mathfrak{N}I_7^{3/4}} + \frac{2^{\omega_K(I_2)+\omega_K(I_1I_2I_3I_5)} B^{1/2}\log B}{\mathfrak{N}I_1^{1/2}\mathfrak{N}I_5^{-1/2}\mathfrak{N}I_7^{1/2}} \right)$$

$$\ll \sum_{\substack{I_1,\ldots,I_5\\ \mathfrak{N}I_j \leqslant B}} \left(\frac{2^{\omega_K(I_2)+\omega_K(I_1I_2I_3I_5)} B}{\mathfrak{N}I_1^{5/4}\mathfrak{N}I_2^{3/2}\mathfrak{N}I_3^{5/4}\mathfrak{N}I_4\mathfrak{N}I_5} + \frac{2^{\omega_K(I_2)+\omega_K(I_1I_2I_3I_5)} B\log B}{\mathfrak{N}I_1^{3/2}\mathfrak{N}I_2^{2}\mathfrak{N}I_3^{3/2}\mathfrak{N}I_4\mathfrak{N}I_5} \right)$$

$$\ll B(\log B)^4.$$

LEMMA 11. *If* $\mathbf{I}''$ *runs over all six-tuples* $(I_1, \ldots, I_5, I_7)$ *of nonzero ideals of* $\mathcal{O}_K$ *then we have*

$$N_{86}(B) = \left(\frac{2}{\sqrt{|\Delta_K|}}\right)^2 \sum_{\mathbf{I}''} \mathcal{A}(\theta_8'(\mathbf{I}'), I_6) V_{86}(\mathfrak{N}I_1, \ldots, \mathfrak{N}I_5, \mathfrak{N}I_7; B) + O(B(\log B)^4).$$

Proof. This is analogous to [**DF13**, lemma 9·4].

3·2·3. *The remaining summations for* $N_{86}(B)$

LEMMA 12. *We have*

$$N_{86}(B) = \pi^6 \left(\frac{2}{\sqrt{|\Delta_K|}}\right)^8 \left(\frac{h_K}{\omega_K}\right)^6 \theta_0 V_{860}(B) + O(B(\log B)^4 \log\log B),$$

where θ_0 *is as in* (1·7) *and*

$$V_{860}(B) := \int_{t_1,\ldots,t_5,t_7 \geqslant 1} V_{86}(t_1, \ldots, t_5, t_7; B)\, \mathrm{d}t_1 \cdots \mathrm{d}t_5\, \mathrm{d}t_7,$$

with real variables $t_1, \ldots, t_5, t_7$.

Proof. By [**DF13**, lemma 3·5 (5)], applied to (3·5), we have, for $t_7 \geqslant 1$,

$$V_{86}(t_1, \ldots, t_5, t_7; B) \ll \frac{B}{t_1\cdots t_5 t_7}\left(\frac{B}{t_1^3t_2^6t_3^4t_4^2t_5^5}\right)^{-1/6}.$$

Furthermore, using (3·1) to bound t_6 and (3·3) to bound $\|\eta_8\|_\infty$, we see that

$$V_{86}(t_1, \ldots, t_5, t_7; B) \ll \frac{1}{t_5}\left(\frac{B}{t_1^2t_2^4t_3^3t_4^2t_5^3}\right)\left(\frac{B}{t_1^2t_2^3t_3^2t_4t_5^2}\right) = \frac{B}{t_1\cdots t_5t_7}\left(\frac{B}{t_1^3t_2^6t_3^4t_4^2t_5^5}\right).$$

We apply [**DF13**, proposition 7·3] with $r = 5$.

3·2·4. *The second summation over* η_7 *in* $M_{\mathbf{C}}^{(87)}(B)$

LEMMA 13. *Write* $\boldsymbol{\eta}'' := (\eta_1, \ldots, \eta_6)$. *We have*

$$M_{\mathbf{C}}^{(87)}(B) = \left(\frac{2}{\sqrt{|\Delta_K|}}\right)^2 \sum_{\boldsymbol{\eta}'' \in \mathcal{O}_{1*}\times\cdots\times\mathcal{O}_{6*}} \mathcal{A}(\theta_8'(\mathbf{I}'), I_7) V_{87}(\mathfrak{N}I_1, \ldots, \mathfrak{N}I_6; B) + O_{\mathbf{C}}(B(\log B)^4),$$

where, for $t_1, \ldots, t_6 \geqslant 1$,

$$V_{87}(t_1, \ldots, t_6; B) := \frac{\pi}{t_5} \int_{\substack{(\sqrt{t_1},\ldots,\sqrt{t_7},\eta_8)\in\mathcal{R}(B)\\ t_7\geqslant t_6}} \mathrm{d}t_7\, \mathrm{d}\eta_8,$$

with a real variable t_7 *and a complex variable* η_8.

Proof. Again, we apply the strategy described in [**DF13**, section 6] in the case $b_0 \geqslant 2$. However, this time we must examine the arithmetic function more carefully, since a straightforward application as in Lemma 10 would not yield sufficiently good error terms. We write

$$M_{\mathbf{C}}^{(87)}(B) = \frac{2}{\sqrt{|\Delta_K|}} \sum_{\boldsymbol{\eta}'' \in \mathcal{O}_{1*} \times \cdots \times \mathcal{O}_{6*}} \sum_{\substack{\mathfrak{k}_\mathfrak{c} \mid I_2 \\ \mathfrak{k}_\mathfrak{c} + I_1 I_3 = \mathcal{O}_K}} \frac{\mu_K(\mathfrak{k}_\mathfrak{c})}{\mathfrak{N}\mathfrak{k}_\mathfrak{c}} \Sigma, \tag{3·12}$$

where

$$\Sigma := \sum_{\substack{\eta_7 \in \mathcal{O}_{7*} \\ \mathfrak{N}I_7 \geqslant \mathfrak{N}I_6}} \vartheta(I_7) \sum_{\substack{\rho \mod \mathfrak{k}_\mathfrak{c} I_5 \\ \rho\mathcal{O}_K + \mathfrak{k}_\mathfrak{c} I_5 = \mathcal{O}_K \\ \rho^2 \equiv_{\mathfrak{k}_\mathfrak{c} I_5} \eta_6 \eta_7 A}} g(\mathfrak{N}I_7), \tag{3·13}$$

with $\vartheta(I_7) := \tilde{\theta}_8(\mathbf{I}', \mathfrak{k}_\mathfrak{c})$ and $g(t) := V_8(\mathfrak{N}I_1, \ldots, \mathfrak{N}I_6, t; B)$.

The key observation is that, as in [**BD09**], we can replace $\vartheta(I_7)$ by the function

$$\vartheta'(I_7) := \theta_0'(\mathbf{I}') \frac{\phi_K^*(I_2 I_3 I_4 I_6)}{\phi_K^*(I_2 + \mathfrak{k}_\mathfrak{c} I_5)},$$

where θ_0' encodes all coprimality conditions that are encoded by θ_0, except for allowing $I_5 + I_7 \neq \mathcal{O}_K$. For the representation $\theta_0' = \prod_\mathfrak{p} \theta_{0,\mathfrak{p}}'(J_\mathfrak{p}(\mathbf{I}'))$ as a product of local factors, this amounts to

$$\theta_{0,\mathfrak{p}}'(J) := \begin{cases} 1 & \text{if } \theta_{0,\mathfrak{p}}(J) = 1 \text{ or } J = \{5, 7\}, \\ 0 & \text{otherwise.} \end{cases}$$

Replacing ϑ by ϑ' in (3·13) does not change Σ for any $\boldsymbol{\eta}'' \in \mathcal{O}_{1*} \times \cdots \times \mathcal{O}_{6*}$ and $\mathfrak{k}_\mathfrak{c}$ as in (3·12), since the sum over ρ is zero whenever $I_5 + I_7 \neq \mathcal{O}_K$. Indeed, we know from Lemma 9 that $\eta_6\eta_7 A$ is invertible modulo $\mathfrak{k}_\mathfrak{c} I_5$ whenever $\mathfrak{k}_\mathfrak{c}$ is as in (3·12) and $\theta_0(\mathbf{I}') \neq 0$. This implies that $v_\mathfrak{p}(\eta_6 A \mathcal{O}_7) = 0$ for any fixed $\boldsymbol{\eta}''$, $\mathfrak{k}_\mathfrak{c}$ as in (3·12) with $\Sigma \neq 0$ and any $\mathfrak{p} \mid \mathfrak{k}_\mathfrak{c} I_5$. Therefore, if $\mathfrak{p} \mid I_5 + I_7$ then the second and third condition under the sum over ρ in (3·13) contradict each other.

Since $\vartheta'(I_7) = \vartheta(I_7)$ whenever $I_5 + I_7 = \mathcal{O}_K$, we have $\mathcal{A}(\vartheta'(\mathfrak{a}), \mathfrak{a}, \mathfrak{k}_\mathfrak{c} I_5) = \mathcal{A}(\vartheta(\mathfrak{a}), \mathfrak{a}, \mathfrak{k}_\mathfrak{c} I_5))$.

Moreover, we obtain immediately from the definition that $\vartheta' \in \Theta(I_1 I_2 I_3 I_4, 1, 1, 1)$ (see [**DF13**, definition 2·1]). Hence, by [**DF13**, lemma 2·2], the function ϑ' satisfies [**DF13**, (6·1)] with $c_\theta := 2^{\omega_K(I_1 I_2 I_3 I_4)}$, $C := 0$.

By (3·4), $g(t) = 0$ whenever $t > t_2 := B/(\mathfrak{N}I_1 \mathfrak{N}I_2^2 \mathfrak{N}I_3^2 \mathfrak{N}I_4^2 \mathfrak{N}I_5 \mathfrak{N}I_6^2)$, and, by [**DF13**, lemma 3·5 (2)] applied to (3·5), $g(t) \ll B^{1/2}/(\mathfrak{N}I_1^{1/2} \mathfrak{N}I_5^{1/2}) \cdot t^{-1/2}$. With [**DF13**, proposition 6·1], we obtain

$$\Sigma = \frac{2\pi}{\sqrt{|\Delta_K|}} \phi_K^*(\mathfrak{k}_\mathfrak{c} I_5) \mathcal{A}(\vartheta(\mathfrak{a}), \mathfrak{a}, \mathfrak{k}_\mathfrak{c} I_5) \int_{t \geqslant \mathfrak{N}I_6} g(t)\, \mathrm{d}t$$
$$+ O\left(\frac{2^{\omega_K(I_1 I_2 I_3 I_4)} B^{1/2}}{\mathfrak{N}I_1^{1/2} \mathfrak{N}I_5^{1/2}} \left(\sqrt{\mathfrak{N}(\mathfrak{k}_\mathfrak{c} I_5)} \log B + \frac{\mathfrak{N}\mathfrak{k}_\mathfrak{c} I_5}{\mathfrak{N}I_6^{1/2}} \log(\mathfrak{N}I_6 + 2) \right) \right).$$

As in Lemma 10, the main term in the lemma is correct, and for the error term we may sum over the ideals $\mathfrak{k}_\mathfrak{c}$ and I_j instead of the η_j. By (3·1), (3·4), and our condition $\mathfrak{N}I_7 \geqslant \mathfrak{N}I_6$, it suffices to sum over $\mathfrak{k}_\mathfrak{c}$ and the $(I_1, \ldots, I_6)$ satisfying (3·1) and

$$\mathfrak{N}I_1^3 \mathfrak{N}I_2^6 \mathfrak{N}I_3^5 \mathfrak{N}I_4^4 \mathfrak{N}I_5^4 \mathfrak{N}I_6^4 \leqslant B^2 \tag{3·14}$$

Thus, the total error is bounded by

$$\sum_{\substack{I_1,\dots,I_6\\(3\cdot 14)}} \left(\frac{2^{\omega_K(I_2)+\omega_K(I_1I_2I_3I_4)}B^{1/2}\log B}{\mathfrak{N}I_1^{1/2}} + \frac{2^{\omega_K(I_2)+\omega_K(I_1I_2I_3I_4)}\mathfrak{N}I_5^{1/2}B^{1/2}\log B}{\mathfrak{N}I_1^{1/2}\mathfrak{N}I_6^{1/2}}\right)$$
$$\ll \sum_{\substack{I_1,\dots,I_5\\ \mathfrak{N}I_j\leqslant B}} \frac{2^{\omega_K(I_2)+\omega_K(I_1I_2I_3I_4)}B\log B}{\mathfrak{N}I_1^{5/4}\mathfrak{N}I_2^{3/2}\mathfrak{N}I_3^{5/4}\mathfrak{N}I_4\mathfrak{N}I_5} + \sum_{\substack{I_1,\dots,I_4,I_6\\ \mathfrak{N}I_j\leqslant B}} \frac{2^{\omega_K(I_2)+\omega_K(I_1I_2I_3I_4)}B\log B}{\mathfrak{N}I_1^{3/2}\mathfrak{N}I_2^{2}\mathfrak{N}I_3^{3/2}\mathfrak{N}I_4\mathfrak{N}I_6}$$
$$\ll B(\log B)^4.$$

LEMMA 14. *If* $\mathbf{I}''$ *runs over all six-tuples* $(I_1,\dots,I_6)$ *of nonzero ideals of* $\mathcal{O}_K$ *then we have*

$$N_{87}(B) = \left(\frac{2}{\sqrt{|\Delta_K|}}\right)^2 \sum_{\mathbf{I}''} \mathcal{A}(\theta_8'(\mathbf{I}'), I_7) V_{87}(\mathfrak{N}I_1,\dots,\mathfrak{N}I_6; B) + O(B(\log B)^4).$$

Proof. This is analogous to [**DF13**, lemma 9·4].

3·2·5. *The remaining summations for* $N_{87}(B)$

LEMMA 15. *We have*

$$N_{87}(B) = \pi^6 \left(\frac{2}{\sqrt{|\Delta_K|}}\right)^8 \left(\frac{h_K}{\omega_K}\right)^6 \theta_0 V_{870}(B) + O(B(\log B)^4 \log\log B),$$

where θ_0 *is given in* (1·7) *and*

$$V_{870}(B) := \int_{t_1,\dots,t_6\geqslant 1} V_{87}(t_1,\dots,t_6; B)\,\mathrm{d}t_1\cdots\mathrm{d}t_6,$$

with real variables $t_1,\dots,t_6$.

Proof. By [**DF13**, lemma 3·5 (6)], applied to (3·5), we have

$$V_{87}(t_1,\dots,t_6; B) \ll \frac{1}{t_5}\cdot\frac{B^{3/4}t_5^{3/4}}{t_1^{1/2}t_3^{1/4}t_4^{1/2}t_6^{3/4}} = \frac{B}{t_1\cdots t_6}\left(\frac{B}{t_1^2t_2^4t_3^3t_4^2t_5^3t_6}\right)^{-1/4}.$$

Furthermore, using (3·3) and (3·4) to bound $\|\eta_8\|_\infty$ and t_7, respectively, we see that

$$V_{87}(t_1,\dots,t_6; B) \ll \frac{1}{t_5}\left(\frac{B}{t_1^2t_2^3t_3^2t_4t_5^2}\right)\left(\frac{B}{t_1t_2^2t_3^2t_4^2t_5t_6^2}\right) = \frac{B}{t_1\cdots t_6}\cdot\left(\frac{B}{t_1^2t_2^4t_3^3t_4^2t_5^3t_6}\right).$$

We apply [**DF13**, proposition 7·3] with $r = 5$.

3·2·6. *Combining the summations*

LEMMA 16. *We have*

$$N_{U_2,H}(B) = \left(\frac{2}{\sqrt{|\Delta_K|}}\right)^8 \left(\frac{h_K}{\omega_K}\right)^6 \theta_0 V_0(B) + O(B(\log B)^4 \log\log B),$$

where θ_0 *is given in* (1·7) *and*

$$V_0(B) := \int_{\substack{(\eta_1,\dots,\eta_8)\in\mathcal{R}(B)\\ \|\eta_1\|_\infty,\dots,\|\eta_7\|_\infty\geqslant 1}} \frac{1}{\|\eta_5\|_\infty}\,\mathrm{d}\eta_1\cdots\mathrm{d}\eta_8,$$

with complex variables $\eta_1,\dots,\eta_8$.

Proof. This follows from (3·10), Lemma 12 and Lemma 15, using polar coordinates, similarly to [**DF13**, lemma 9·9].

3·3. *Proof of Theorem* 1 *for* S_2

Let $\alpha(\widetilde{S}_2) := \frac{1}{21600}$ and

$$\omega_\infty(\widetilde{S}_2) := \frac{12}{\pi} \int_{\|z_0^3\|_\infty, \|z_0z_2z_3\|_\infty, \|z_0^2z_2\|_\infty, \|z_0^2z_3\|_\infty, \|z_3(z_2^2+z_0z_3)\|_\infty \leqslant 1} \mathrm{d}z_0 \, \mathrm{d}z_1 \, \mathrm{d}z_2.$$

We use the conditions

$$\left\|\eta_1^2\eta_2^4\eta_4^2\eta_5^3\eta_6\right\|_\infty \leqslant B, \tag{3·15}$$

$$\left\|\eta_1^2\eta_2^4\eta_4^2\eta_5^3\eta_6\right\|_\infty \leqslant B \text{ and } \left\|\eta_1^{-1}\eta_2^{-2}\eta_4^2\eta_5^{-3}\eta_6^4\right\|_\infty \leqslant B. \tag{3·16}$$

LEMMA 17. *Let* $\mathcal{R}(B)$ *be as in* (3·1)–(3·5). *Define*

$$V_0'(B) := \int_{\substack{(\eta_1,\dots,\eta_8)\in\mathcal{R}(B) \\ \|\eta_1\|_\infty, \|\eta_2\|_\infty, \|\eta_4\|_\infty, \|\eta_5\|_\infty, \|\eta_6\|_\infty \geqslant 1 \\ (3·16)}} \frac{1}{\|\eta_5\|_\infty} \, \mathrm{d}\eta_1 \cdots \mathrm{d}\eta_8,$$

where $\eta_1, \dots, \eta_8$ *are complex variables. Then*

$$\pi^6\alpha(\widetilde{S}_2)\omega_\infty(\widetilde{S}_2)B(\log B)^5 = 4V_0'(B). \tag{3·17}$$

Proof. The proof is analogous to the proof of Lemma 7. Let $\eta_1, \eta_2, \eta_4, \eta_5, \eta_6 \in \mathbb{C}$, $B > 0$, and let $l := (B\left\|\eta_1\eta_2^2\eta_4\eta_5^3\eta_6^2\right\|_\infty)^{1/2}$. Let η_3, η_7, η_8 be complex variables. Applying the coordinate transformation $z_0 = l^{-1/3}\eta_1\eta_2^2\eta_4\eta_5^2\eta_6 \cdot \eta_3$, $z_2 = l^{-1/3}\eta_1\eta_2\eta_5 \cdot \eta_8$, $z_3 = l^{-1/3}\eta_4\eta_6^2 \cdot \eta_7$ to $\omega_\infty(\widetilde{S}_2)$, we obtain

$$\omega_\infty(\widetilde{S}_2) = \frac{12}{\pi} \frac{\|\eta_1\eta_2\eta_4\eta_5\eta_6\|_\infty}{B} \int_{(\eta_1,\dots,\eta_8)\in\mathcal{R}(B)} \frac{1}{\|\eta_5\|_\infty} \, \mathrm{d}\eta_3 \, \mathrm{d}\eta_7 \, \mathrm{d}\eta_8. \tag{3·18}$$

The negative curves $[E_1], \dots, [E_7]$ generate the effective cone of $\widetilde{S}_1$. Because of $[-K_{\widetilde{S}_1}] = [2E_1 + 4E_2 + 3E_3 + 2E_4 + 3E_5 + E_6]$ and $[E_7] = [E_1 + 2E_2 + E_3 + 2E_5 - E_6]$, [**DF13**, lemma 8·1] implies

$$\alpha(\widetilde{S}_2)(\log B)^5 = \frac{1}{3\pi^5} \int_{\substack{\|\eta_1\|_\infty, \|\eta_2\|_\infty, \|\eta_4\|_\infty, \|\eta_5\|_\infty, \|\eta_6\|_\infty \geqslant 1 \\ (3·16)}} \frac{\mathrm{d}\eta_1 \, \mathrm{d}\eta_2 \, \mathrm{d}\eta_4 \, \mathrm{d}\eta_5 \, \mathrm{d}\eta_6}{\|\eta_1\eta_2\eta_4\eta_5\eta_6\|_\infty}. \tag{3·19}$$

The lemma follows by substituting (3·18) and (3·19) in (3·17).

To finish our proof, we compare $V_0(B)$ from Lemma 16 with $V_0'(B)$ defined in Lemma 17. Let

$$\mathcal{D}_0(B) := \{(\eta_1, \dots, \eta_8) \in \mathcal{R}(B) \mid \|\eta_1\|_\infty, \dots, \|\eta_7\|_\infty \geqslant 1\},$$

$$\mathcal{D}_1(B) := \{(\eta_1, \dots, \eta_8) \in \mathcal{R}(B) \mid \|\eta_1\|_\infty, \dots, \|\eta_7\|_\infty \geqslant 1, (3·15)\},$$

$$\mathcal{D}_2(B) := \{(\eta_1, \dots, \eta_8) \in \mathcal{R}(B) \mid \|\eta_1\|_\infty, \dots, \|\eta_7\|_\infty \geqslant 1, (3·16)\},$$

$$\mathcal{D}_3(B) := \{(\eta_1, \dots, \eta_8) \in \mathcal{R}(B) \mid \|\eta_1\|_\infty, \dots, \|\eta_6\|_\infty \geqslant 1, (3·16)\},$$

$$\mathcal{D}_4(B) := \{(\eta_1, \dots, \eta_8) \in \mathcal{R}(B) \mid \|\eta_1\|_\infty, \|\eta_2\|_\infty, \|\eta_4\|_\infty, \|\eta_5\|_\infty, \|\eta_6\|_\infty \geqslant 1, (3·16)\}.$$

Moreover, let

$$V_i(B) := \int_{\mathcal{D}_i(B)} \frac{\mathrm{d}\eta_1 \cdots \mathrm{d}\eta_8}{\|\eta_5\|_\infty}.$$

Then clearly $V_0(B)$ is as in Lemma 16 and $V_4(B) = V_0'(B)$. We show that, for $1 \leqslant i \leqslant 4$, $V_i(B) - V_{i-1}(B) = O(B(\log B)^4)$. This holds for $i = 1$, since $R_1 = R_0$. Moreover, using [**DF13**, lemma 3.5, (4)] and (3·5) to bound the integral over η_7 and η_8, we have

$$V_2(B) - V_1(B) \ll \int_{\substack{\|\eta_1\|_\infty,\dots,\|\eta_6\|_\infty \geqslant 1 \\ \|\eta_1\eta_2^2\eta_3^2\eta_4^2\eta_5\eta_6^2\|_\infty \leqslant B \\ \|\eta_1^{-1}\eta_2^{-2}\eta_4^2\eta_5^{-3}\eta_6^4\|_\infty > B}} \frac{B^{3/4}}{\|\eta_1^2\eta_3\eta_4^2\eta_5\eta_6^3\|_\infty^{1/4}} \mathrm{d}\eta_1 \cdots \mathrm{d}\eta_6 \ll B(\log B)^4.$$

Using [**DF13**, lemma 3·5 (2)] and the (3·5) to bound the integral over η_8, we obtain

$$V_3(B) - V_2(B) \ll \int_{\substack{\|\eta_1\|_\infty,\dots,\|\eta_6\|_\infty \geqslant 1 \\ \|\eta_7\|_\infty < 1,\ (3\cdot 1),\ (3\cdot 16)}} \frac{B^{1/2}}{\|\eta_1\eta_5\eta_7\|_\infty^{1/2}} \mathrm{d}\eta_1 \cdots \mathrm{d}\eta_7 \ll B(\log B)^4.$$

Finally, using [**DF13**, lemma 3·5 (4)] and (3·5) to bound the integral over η_7 and η_8, we have

$$V_4(B) - V_3(B) \ll \int_{\substack{\|\eta_1\|_\infty,\|\eta_2\|_\infty,\|\eta_4\|_\infty,\|\eta_5\|_\infty,\|\eta_6\|_\infty \geqslant 1 \\ \|\eta_3\|_\infty < 1,\ (3\cdot 15)}} \frac{B^{3/4}}{\|\eta_1^2\eta_3\eta_4^2\eta_5\eta_6^3\|_\infty^{1/4}} \mathrm{d}\eta_1 \cdots \mathrm{d}\eta_6 \ll B(\log B)^4.$$

Using Lemma 16 and Lemma 17, this implies Theorem 1 for S_2.

4. *The quartic del Pezzo surface of type* $\mathbf{D}_4$

4·1. *Passage to a universal torsor*

We use the notation from [**Der06**].

For any given $\mathbf{C} = (C_0, \dots, C_5) \in \mathcal{C}^6$, we define $u_{\mathbf{C}} := \mathfrak{N}(C_0^3 C_1^{-1} \cdots C_5^{-1})$ and

$$\begin{aligned} &\mathcal{O}_1 := C_2C_3^{-1} && \mathcal{O}_2 := C_1C_2^{-1} && \mathcal{O}_3 := C_0C_1^{-1}C_2^{-1}C_5^{-1} \\ &\mathcal{O}_4 := C_3C_4^{-1} && \mathcal{O}_5 := C_5 && \mathcal{O}_6 := C_4 \\ &\mathcal{O}_7 := C_0C_1^{-1} && \mathcal{O}_8 := C_0C_5^{-1} && \mathcal{O}_9 := C_0^2C_1^{-1}C_2^{-1}C_3^{-1}C_4^{-1}. \end{aligned}$$

Let

$$\mathcal{O}_{j*} := \begin{cases} \mathcal{O}_j^{\neq 0}, & j \in \{1, \dots, 6\}, \\ \mathcal{O}_j, & j \in \{7, 8, 9\}. \end{cases}$$

For $\eta_j \in \mathcal{O}_j$, let

$$I_j := \eta_j \mathcal{O}_j^{-1}.$$

For $B \geqslant 0$, let $\mathcal{R}(B)$ be the set of all $(\eta_1, \dots, \eta_8) \in \mathbb{C}^8$ with $\eta_4\eta_6 \neq 0$ and

$$\left\|\eta_1^2\eta_2\eta_3^2\eta_4\eta_5^2\eta_8\right\|_\infty \leqslant B, \tag{4·1}$$

$$\left\|\eta_1^4\eta_2^2\eta_3^3\eta_4^3\eta_5^2\eta_6^2\right\|_\infty \leqslant B, \tag{4·2}$$

$$\left\|\eta_1^3\eta_2^2\eta_3^2\eta_4^2\eta_5\eta_6\eta_7\right\|_\infty \leqslant B, \tag{4·3}$$

$$\left\|\eta_1^2\eta_2\eta_3^2\eta_4\eta_5^2\eta_8 + \eta_1^2\eta_2^2\eta_3\eta_4\eta_7^2\right\|_\infty \leqslant B, \tag{4·4}$$

$$\left\|\frac{\eta_3\eta_5^2\eta_8^2 + \eta_2\eta_7^2\eta_8}{\eta_4\eta_6^2}\right\|_\infty \leqslant B \tag{4·5}$$

and let $M_{\mathbf{C}}(B)$ be the set of all

$$(\eta_1, \dots, \eta_9) \in \mathcal{O}_{1*} \times \cdots \times \mathcal{O}_{9*}$$

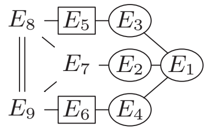

Fig. 3. Configuration of curves on $\widetilde{S}_3$.

that satisfy the *height conditions*

$$(\eta_1, \ldots, \eta_8) \in \mathcal{R}(u_{\mathbf{C}} B),$$

the *torsor equation*

$$\eta_2\eta_7^2 + \eta_3\eta_5^2\eta_8 + \eta_4\eta_6^2\eta_9 = 0 \tag{4·6}$$

and the *coprimality conditions*

$$I_j + I_k = \mathcal{O}_K \text{ for all distinct nonadjacent vertices } E_j, E_k \text{ in Figure 3.} \tag{4·7}$$

LEMMA 18. *We have*

$$N_{U_3,H}(B) = \frac{1}{\omega_K^6} \sum_{\mathbf{C} \in \mathcal{C}^6} |M_{\mathbf{C}}(B)|.$$

Proof. Again, the lemma is a specialization of [**DF13**, claim 4·1], and we prove it in an analogous way as Lemma 8. Let $E_3^{(0)} := \{y_1 = 0\}$, $E_7^{(0)} := \{y_2 = 0\}$, $E_8^{(0)} := \{y_0 = 0\}$, $E_9^{(0)} := \{-y_0y_1 - y_2^2 = 0\}$ in $\mathbb{P}^2_K$. We prove [**DF13**, claim 4·2] for the following sequence of blow-ups:

(i) blow up $E_3^{(0)} \cap E_7^{(0)} \cap E_9^{(0)}$, giving $E_2^{(1)}$;
(ii) blow up $E_2^{(1)} \cap E_3^{(1)} \cap E_9^{(1)}$, giving $E_1^{(2)}$;
(iii) blow up $E_1^{(2)} \cap E_9^{(2)}$, giving $E_4^{(3)}$;
(iv) blow up $E_4^{(3)} \cap E_9^{(3)}$, giving $E_6^{(4)}$;
(v) blow up $E_3^{(4)} \cap E_8^{(4)}$, giving $E_5^{(5)}$.

The inverse $\pi \circ \rho^{-1} : \mathbb{P}^2_K \dashrightarrow S_3$ of the projection $\rho \circ \pi^{-1} : S_3 \dashrightarrow \mathbb{P}^2_K$, $(x_0 : \cdots : x_4) \mapsto (x_0 : x_1 : x_2)$ is given by

$$(y_0 : y_1 : y_2) \longmapsto (y_0y_1^2 : y_1^3 : y_1^2y_2 : -y_1(y_0y_1 + y_2^2) : -y_0(y_0y_1 + y_2^2)).$$

With the map Ψ from [**DF13**, claim 4·2] sending $(\eta_1, \ldots, \eta_9)$ to

$$(\eta_1^2\eta_2\eta_3^2\eta_4\eta_5^2\eta_8,\ \eta_1^4\eta_2^2\eta_3^3\eta_4^3\eta_5^2\eta_6^2,\ \eta_1^3\eta_2^2\eta_3^2\eta_4^2\eta_5\eta_6\eta_7,\ \eta_1^2\eta_2\eta_3\eta_4^2\eta_6^2\eta_9,\ \eta_8\eta_9),$$

we see that the assumptions of [**DF13**, lemma 4·3] are satisfied, so [**DF13**, claim 4·2] holds for $i = 0$.

In the first two steps of the above chain of blow-ups, we are in the situation of [**DF13**, remark 4·5], so certain coprimality conditions need to be checked by hand. However, up to changing some indices, our situation in steps (1) and (2) is exactly the same as in Lemma 8, so the arguments given there apply to our lemma as well. Steps (3), (4), (5) are again covered by [**DF13**, lemma 4·4], which proves [**DF13**, claim 4·2]. From this, we deduce [**DF13**, claim 4·1] as in [**DF13**, lemma 9·1].

4·2. *Summations*

4·2·1. *The first summation over η_8 with dependent η_9*

LEMMA 19. *Let $\boldsymbol{\eta}' := (\eta_1, \dots, \eta_7)$ and $\mathbf{I}' := (I_1, \dots, I_7)$. Then*

$$|M_{\mathbf{C}}(B)| = \frac{2}{\sqrt{|\Delta_K|}} \sum_{\boldsymbol{\eta}' \in \mathcal{O}_{1*} \times \cdots \times \mathcal{O}_{7*}} \theta_8(\mathbf{I}') V_8(\mathfrak{N} I_1. \dots, \mathfrak{N} I_7; B) + O_{\mathbf{C}}(B(\log B)^2),$$

where

$$V_8(t_1, \dots, t_7; B) := \frac{1}{t_4 t_6^2} \int_{(\sqrt{t_1}, \dots, \sqrt{t_7}, \eta_8) \in \mathcal{R}(B)} \mathrm{d}\eta_8$$

and

$$\theta_8(\mathbf{I}') := \prod_{\mathfrak{p}} \theta_{1,\mathfrak{p}}(J_{\mathfrak{p}}(\mathbf{I}')).$$

Here, $J_{\mathfrak{p}}(\mathbf{I}') := \{j \in \{1, \dots, 7\} : \mathfrak{p} \mid I_j\}$ and

$$\theta_{1,\mathfrak{p}}(J) := \begin{cases} 1 & \text{if } J = \emptyset, \{5\}, \{6\}, \{7\}, \\ 1 - \frac{1}{\mathfrak{N}\mathfrak{p}} & \text{if } J = \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 7\}, \{3, 5\}, \{4, 6\}, \\ 1 - \frac{2}{\mathfrak{N}\mathfrak{p}} & \text{if } J = \{1\}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We apply [**DF13**, proposition 5·3] with $(A_1, A_0) := (2, 7)$, $(B_1, B_2, B_0) := (3, 5, 8)$, $(C_1, C_2, C_0) := (4, 6, 9)$, $D := 1$, $u_{\mathbf{C}} B$ instead of B, and Π_1, Π_2 as suggested in [**DF13**, remark 5·2].

As in Lemma 3, we see that the main term arising from [**DF13**, proposition 5·3] is the main term in the lemma, so it remains to deal with the error term.

For given $\boldsymbol{\eta}'$ and B, the set of all $\eta_8 \in \mathbb{C}$ with $(\eta_1, \dots, \eta_8) \in \mathcal{R}(u_{\mathbf{C}} B)$ is contained in the union of two balls of radius

$$R(\boldsymbol{\eta}'; u_{\mathbf{C}} B) \ll_{\mathbf{C}} \begin{cases} (B \mathfrak{N}(I_4 I_6^2 I_2^{-1} I_7^{-2}))^{1/2} & \text{if } \eta_7 \neq 0, \\ (B / \mathfrak{N}(I_1^2 I_2 I_3^2 I_4 I_5^2))^{1/2} & \text{if } \eta_7 = 0. \end{cases}$$

Indeed, this follows from [**DF13**, lemma 3·4 (1)], applied to (4·5), if $\eta_7 \neq 0$ and from (4·1) if $\eta_7 = 0$.

Hence, the error term is

$$\ll \sum_{\boldsymbol{\eta}', (4\cdot 8), (4\cdot 9)} 2^{\omega_K(I_1 I_4) + \omega_K(I_1 I_2 I_3)} \left(\frac{R(\boldsymbol{\eta}'; u_{\mathbf{C}} B)}{\mathfrak{N}(I_4 I_6^2)^{1/2}} + 1 \right),$$

where, using (4·2), (4·3), the sum runs over all $\boldsymbol{\eta}'$ with

$$\mathfrak{N}(I_1^4 I_2^2 I_3^3 I_4^3 I_5^2 I_6^2) \leqslant B \text{ and} \tag{4·8}$$

$$\mathfrak{N}(I_1^3 I_2^2 I_3^2 I_4^2 I_5 I_6 I_7) \leqslant B. \tag{4·9}$$

Let us first estimate the sum over all $\boldsymbol{\eta}'$ with $\eta_7 \neq 0$. We may sum over the I_j instead of the

η_j and obtain

$$\begin{aligned}
&\ll_{\mathbf{C}} \sum_{\mathbf{I}', (4\cdot 8), (4\cdot 9)} 2^{\omega_K(I_1 I_4)+\omega_K(I_1 I_2 I_3)} \left(\frac{B^{1/2}}{\mathfrak{N}(I_2 I_7^2)^{1/2}} + 1 \right) \\
&\ll \sum_{\substack{I_1,\ldots,I_5,I_7 \\ \mathfrak{N}I_j \leqslant B}} \left(\frac{2^{\omega_K(I_1 I_4)+\omega_K(I_1 I_2 I_3)} B}{\mathfrak{N}I_1^2 \mathfrak{N}I_2^{3/2} \mathfrak{N}I_3^{3/2} \mathfrak{N}I_4^{3/2} \mathfrak{N}I_5 \mathfrak{N}I_7} + \frac{2^{\omega_K(I_1 I_4)+\omega_K(I_1 I_2 I_3)} B}{\mathfrak{N}I_1^3 \mathfrak{N}I_2^2 \mathfrak{N}I_3^2 \mathfrak{N}I_4^2 \mathfrak{N}I_5 \mathfrak{N}I_7} \right) \\
&\ll B(\log B)^2.
\end{aligned}$$

Now we assume that $\eta_7 = 0$ and sum over the remaining variables. We obtain the upper bound

$$\begin{aligned}
&\ll_{\mathbf{C}} \sum_{I_1,\ldots,I_6, (4\cdot 8)} 2^{\omega_K(I_1 I_4)+\omega_K(I_1 I_2 I_3)} \left(\frac{B^{1/2}}{\mathfrak{N}I_1 \mathfrak{N}I_2^{1/2} \mathfrak{N}I_3 \mathfrak{N}I_4 \mathfrak{N}I_5 \mathfrak{N}I_6} + 1 \right) \\
&\ll \sum_{I_1, I_3, \ldots, I_6} \left(\frac{2^{\omega_K(I_1 I_4)+\omega_K(I_1 I_3)} B^{3/4} \log B}{\mathfrak{N}I_1^2 \mathfrak{N}I_3^{7/4} \mathfrak{N}I_4^{7/4} \mathfrak{N}I_5^{3/2} \mathfrak{N}I_6^{3/2}} + \frac{2^{\omega_K(I_1 I_4)+\omega_K(I_1 I_3)} B^{1/2} \log B}{\mathfrak{N}I_1^2 \mathfrak{N}I_3^{3/2} \mathfrak{N}I_4^{3/2} \mathfrak{N}I_5 \mathfrak{N}I_6} \right) \\
&\ll B^{3/4} \log B.
\end{aligned}$$

4·2·2. *The second summation over η_7*

LEMMA 20. *Write $\boldsymbol{\eta}'' := (\eta_1, \ldots, \eta_6)$. We have*

$$\begin{aligned}
|M_{\mathbf{C}}(B)| = \left(\frac{2}{\sqrt{|\Delta_K|}} \right)^2 \sum_{\boldsymbol{\eta}'' \in \mathcal{O}_{1*} \times \cdots \times \mathcal{O}_{6*}} \mathcal{A}(\theta_8(\mathbf{I}'), I_7) V_7(\mathfrak{N}I_1, \ldots, \mathfrak{N}I_6; B) \\
+ O_{\mathbf{C}}(B(\log B)^2),
\end{aligned}$$

where, for $t_1, \ldots, t_6 \geqslant 1$,

$$V_7(t_1, \ldots, t_6; B) := \frac{\pi}{t_4 t_6^2} \int_{(\sqrt{t_1}, \ldots, \sqrt{t_7}, \eta_8) \in \mathcal{R}(B)} \mathrm{d}t_7 \, \mathrm{d}\eta_8,$$

with a positive variable t_7 and a complex variable η_8.

Proof. We apply [**DF13**, proposition 6·1] as suggested in [**DF13**, section 6] in the case $b_0 = 1$. We have

$$|M_{\mathbf{C}}(B)| = \frac{2}{\sqrt{|\Delta_K|}} \sum_{\boldsymbol{\eta}'' \in \mathcal{O}_{1*} \times \cdots \times \mathcal{O}_{6*}} \sum_{\eta_7 \in \mathcal{O}_7} \vartheta(I_7) g(\mathfrak{N}I_7) + O_{\mathbf{C}}(B(\log B)^2), \tag{4·10}$$

where $\vartheta(I_7) := \theta_8(\mathbf{I}')$ and $g(t) := V_8(\mathfrak{N}I_1, \ldots, \mathfrak{N}I_6, t; B)$.

By [**DF13**, lemma 5·4, lemma 2·2], the function ϑ satisfies [**DF13**, (6·1)] with $C := 0$ and $c_\vartheta := 2^{\omega_K(I_1 I_3 \cdots I_6)}$.

By (4·3), we have $g(t) = 0$ whenever $t > t_2 := B/(\mathfrak{N}I_1^3 \mathfrak{N}I_2^2 \mathfrak{N}I_3^2 \mathfrak{N}I_4^2 \mathfrak{N}I_5 \mathfrak{N}I_6)$, and by [**DF13**, lemma 3·4 (2)] applied to (4·5), we obtain

$$g(t) \ll \frac{1}{\mathfrak{N}I_4 \mathfrak{N}I_6^2} \cdot \frac{(\mathfrak{N}I_4 \mathfrak{N}I_6^2 B)^{1/2}}{(\mathfrak{N}I_3 \mathfrak{N}I_5^2)^{1/2}} = \frac{B^{1/2}}{\mathfrak{N}I_3^{1/2} \mathfrak{N}I_4^{1/2} \mathfrak{N}I_5 \mathfrak{N}I_6} =: c_g.$$

By [**DF13**, proposition 6·1], the sum over η_7 in (4·10) is just

$$\vartheta((0))g(0) + \frac{2\pi}{\sqrt{|\Delta_K|}}\mathcal{A}(\vartheta(\mathfrak{a}), \mathfrak{a}, \mathcal{O}_K)\int_{t\geqslant 1} g(t)\,\mathrm{d}t \\ + O\left(\frac{2^{\omega_K(I_1 I_3\cdots I_6)}B^{1/2}}{\mathfrak{N}I_3^{1/2}\mathfrak{N}I_4^{1/2}\mathfrak{N}I_5\mathfrak{N}I_6}\cdot\frac{B^{1/2}}{\mathfrak{N}I_1^{3/2}\mathfrak{N}I_2\mathfrak{N}I_3\mathfrak{N}I_4\mathfrak{N}I_5^{1/2}\mathfrak{N}I_6^{1/2}}\right).$$

Due to (4·2), $\vartheta((0))g(0)$ and $\int_0^1 g(t)\,\mathrm{d}t$ are dominated by the error term, so the main term in the lemma is correct.

Let us consider the error term. Both the sum and the integral are zero whenever $\boldsymbol{\eta}''$ violates (4·2). We may sum over the $(I_1, \ldots, I_6)$ satisfying (4·8) instead of the $\boldsymbol{\eta}''$, so the error term is

$$\ll \sum_{\mathbf{I}'',\ \mathfrak{N}I_j\leqslant B} 2^{\omega_K(I_1 I_3\cdots I_6)}\left(\frac{B}{\mathfrak{N}I_1^{3/2}\mathfrak{N}I_2\mathfrak{N}I_3^{3/2}\mathfrak{N}I_4^{3/2}\mathfrak{N}I_5^{3/2}\mathfrak{N}I_6^{3/2}}\right) \\ \ll B\log B.$$

LEMMA 21. *If* $\mathbf{I}''$ *runs over all six-tuples* $(I_1, \ldots, I_6)$ *of nonzero ideals of* $\mathcal{O}_K$ *then we have*

$$N_{U_3,H}(B) = \left(\frac{2}{\sqrt{|\Delta_K|}}\right)^2 \sum_{\mathbf{I}''}\mathcal{A}(\theta_8(\mathbf{I}'), I_7)V_7(\mathfrak{N}I_1, \ldots, \mathfrak{N}I_6; B) + O(B(\log B)^2).$$

Proof. This is analogous to [**DF13**, lemma 9·4].

4·2·3. *The remaining summations*

LEMMA 22. *We have*

$$N_{U_3,H}(B) = \left(\frac{2}{\sqrt{|\Delta_K|}}\right)^8\left(\frac{h_K}{\omega_K}\right)^6\theta_0 V_0(B) + O(B(\log B)^4\log\log B),$$

where θ_0 *is as in* (1·7) *and*

$$V_0(B) := \int_{\substack{(\eta_1,\ldots,\eta_8)\in\mathcal{R}(B)\\ \|\eta_1\|_\infty,\ldots,\|\eta_6\|_\infty\geqslant 1}} \frac{1}{\|\eta_4\eta_6^2\|_\infty}\,\mathrm{d}\eta_1\cdots\mathrm{d}\eta_8,$$

with complex variables $\eta_1, \ldots, \eta_8$.

Proof. By [**DF13**, lemma 3·4 (5)] applied to (4·5), we have

$$V_7(t_1, \ldots, t_6; B) \ll \frac{B^{3/4}}{t_2^{1/2}t_3^{1/4}t_4^{1/4}t_5^{1/2}t_6^{1/2}} = \frac{B}{t_1\cdots t_6}\left(\frac{B}{t_1^4t_2^2t_3^3t_4^3t_5^2t_6^2}\right)^{-1/4}.$$

We apply [**DF13**, proposition 7·3] with $r = 5$ and use polar coordinates.

4·3. *Proof of Theorem* 1 *for* S_3

Let $\alpha(\widetilde{S}_3) := \frac{1}{34560}$ and

$$\omega_\infty(\widetilde{S}_3) := \frac{12}{\pi}\int_{\|z_0z_1^2\|_\infty, \|z_1^3\|_\infty, \|z_1^2z_2\|_\infty, \|z_1(z_0z_1+z_2^2)\|_\infty, \|z_0(z_0z_1+z_2^2)\|_\infty\leqslant 1} \mathrm{d}z_0\,\mathrm{d}z_1\,\mathrm{d}z_2.$$

LEMMA 23. *Let $\mathcal{R}(B)$ be as in* (4·1)–(4·5). *Define*

$$V_0'(B) := \int_{\substack{(\eta_1,\dots,\eta_8)\in\mathcal{R}(B)\\ \|\eta_1\|_\infty,\|\eta_2\|_\infty,\|\eta_4\|_\infty,\|\eta_5\|_\infty,\|\eta_6\|_\infty\geqslant 1\\ \|\eta_1^4\eta_2^2\eta_4^3\eta_5^2\eta_6^2\|_\infty\leqslant B}} \frac{1}{\|\eta_4\eta_6^2\|_\infty}\,\mathrm{d}\eta_1\cdots\mathrm{d}\eta_8,$$

where $\eta_1,\dots,\eta_8$ are complex variables. Then

$$\pi^6\alpha(\widetilde{S}_3)\omega_\infty(\widetilde{S}_3)B(\log B)^5 = 4V_0'(B). \tag{4·11}$$

Proof. Let $\eta_1,\eta_2,\eta_4,\eta_5,\eta_6\in\mathbb{C}$, $B>0$, and define $l := (B\|\eta_1^2\eta_2\eta_4^3\eta_5\eta_6^4\|_\infty)^{1/2}$. Let η_3,η_7,η_8 be complex variables. After the coordinate transformation $z_0 = l^{-1/3}\eta_5\cdot\eta_8$, $z_1 = l^{-1/3}\eta_1^2\eta_2\eta_4^2\eta_5\eta_6^2\cdot\eta_3$, $z_2 = l^{-1/3}\eta_1\eta_2\eta_4\eta_6\cdot\eta_7$, we have

$$\omega_\infty(\widetilde{S}_3) = \frac{12}{\pi}\frac{\|\eta_1\eta_2\eta_4\eta_5\eta_6\|_\infty}{B}\int_{(\eta_1,\dots,\eta_8)\in\mathcal{R}(B)}\frac{1}{\|\eta_4\eta_6^2\|_\infty}\,\mathrm{d}\eta_3\,\mathrm{d}\eta_7\,\mathrm{d}\eta_8. \tag{4·12}$$

Since the negative curves $[E_1],\dots,[E_6]$ generate the effective cone of $\widetilde{S}_3$, and $[-K_{\widetilde{S}_3}] = [4E_1+2E_2+3E_3+3E_4+2E_5+2E_6]$, [**DF13**, lemma 8·1] gives

$$\alpha(\widetilde{S}_3)(\log B)^5 = \frac{1}{3\pi^5}\int_{\substack{\|\eta_1\|_\infty,\|\eta_2\|_\infty,\|\eta_4\|_\infty,\|\eta_5\|_\infty,\|\eta_6\|_\infty\geqslant 1\\ \|\eta_1^4\eta_2^2\eta_4^3\eta_5^2\eta_6^2\|_\infty\leqslant B}}\frac{\mathrm{d}\eta_1\,\mathrm{d}\eta_2\,\mathrm{d}\eta_4\,\mathrm{d}\eta_5\,\mathrm{d}\eta_6}{\|\eta_1\eta_2\eta_4\eta_5\eta_6\|_\infty}. \tag{4·13}$$

The lemma follows by substituting (4·12) and (4·13) in (4·11).

To finish our proof, we compare $V_0(B)$ from Lemma 22 with $V_0'(B)$ defined in Lemma 23. We show that, starting from $V_0(B)$, we can add the condition $\|\eta_1^4\eta_2^2\eta_4^3\eta_5^2\eta_6^2\|_\infty\leqslant B$ and remove $\|\eta_3\|_\infty\geqslant 1$ with negligible error. First, we note that (4·2), together with $\|\eta_3\|_\infty\geqslant 1$ implies the condition $\|\eta_1^4\eta_2^2\eta_4^3\eta_5^2\eta_6^2\|_\infty\leqslant B$, so we can add it to the domain of integration for $V_0(B)$ without changing the result.

Using [**DF13**, lemma 3·4 (3)] applied to (4·5) to bound the integral over η_7,η_8, we see that an upper bound for $V_0'(B)-V_0(B)$ is given by

$$\ll \int_{\substack{\|\eta_1\|_\infty,\|\eta_2\|_\infty,\|\eta_4\|_\infty,\|\eta_5\|_\infty,\|\eta_6\|_\infty\geqslant 1\\ \|\eta_3\|_\infty<1,\ \|\eta_1^4\eta_2^2\eta_3^3\eta_4^2\eta_5^2\eta_6^2\|_\infty\leqslant B}}\frac{B^{3/4}}{\|\eta_2^2\eta_3\eta_4\eta_5^2\eta_6^2\|_\infty^{1/4}}\,\mathrm{d}\eta_1\cdots\mathrm{d}\eta_6 \ll B(\log B)^4.$$

Using Lemma 22 and Lemma 23, this implies Theorem 1 for S_3.

5. *The quartic del Pezzo surface of type* $\mathbf{D}_5$

The surface S_4 defined by (1·5) is an equivariant compactification of $\mathbb{G}_a^2$ (as remarked in [**BB07**]; see [**DL10**, Lemma 6] for details), hence Manin's conjecture for S_4 over arbitrary number fields is a special case of [**CLT02**]. Alternatively, our methods lead to Manin's conjecture for S_4 over imaginary quadratic fields as stated in Theorem 1 with

$$\alpha(\widetilde{S}_4) := \frac{1}{345600} \text{ and}$$
$$\omega_\infty(\widetilde{S}_4) := \frac{12}{\pi}\int_{\|z_0^3\|_\infty,\|z_0z_1^2\|_\infty,\|z_0^2z_1\|_\infty,\|z_0^2z_2\|_\infty,\|z_0z_2^2+z_1^3\|_\infty\leqslant 1}\mathrm{d}z_0\,\mathrm{d}z_1\,\mathrm{d}z_2.$$

We remark that the parameterization of rational points is as in [**DF13**, claim 4·1], and that the order of summations can be chosen as in [**BB07**, section 5]. The details can be found in the first preprint `arXiv:1304.3352v1` of this article, but upon the referee's suggestion, we omit them here.

Acknowledgements. This collaboration was supported by the Center for Advanced Studies of LMU München. We thank the referee for useful comments.

REFERENCES

[BB07] R. DE LA BRETÈCHE and T. D. BROWNING. On Manin's conjecture for singular del Pezzo surfaces of degree 4. I. *Michigan Math. J.* **55**(1) (2007), 51–80.

[BD09] T. D. BROWNING and U. DERENTHAL. Manin's conjecture for a quartic del Pezzo surface with A_4 singularity. *Ann. Inst. Fourier (Grenoble)* **59**(3) (2009), 231–1265.

[BM90] V. V. BATYREV and YU. I. MANIN. Sur le nombre des points rationnels de hauteur borné des variétés algébriques. *Math. Ann.* **286**(1–3) (1990), 27–43.

[BT95] V. V. BATYREV and YU. TSCHINKEL. Rational points of bounded height on compactifications of anisotropic tori. *Int. Math. Res. Not. IMRN* **(12)** (1995), 591–635.

[BT98a] V. V. BATYREV and YU. TSCHINKEL. Manin's conjecture for toric varieties. *J. Algebraic Geom.* **7**(1) (1998), 15–53.

[BT98b] V. V. BATYREV and YU. TSCHINKEL. Tamagawa numbers of polarized algebraic varieties. *Astérisque* **(251)** (1998), 299–340. Nombre et répartition de points de hauteur bornée (Paris, 1996).

[CLT02] A. CHAMBERT–LOIR and YU. TSCHINKEL. On the distribution of points of bounded height on equivariant compactifications of vector groups. *Invent. Math.* **148**(2) (2002), 421–452.

[Der06] U. DERENTHAL. Singular Del Pezzo surfaces whose universal torsors are hypersurfaces. *Proc. Lond. Math. Soc.* (3)*, to appear*, arXiv:math.AG/0604194 (2006).

[Der09] U. DERENTHAL. Counting integral points on universal torsors. *Int. Math. Res. Not. IMRN* **(14)** (2009), 2648–2699.

[DF13] U. DERENTHAL and C. FREI. Counting imaginary quadratic points via universal torsors. *Compos. Math., to appear*, arXiv:1302.6151, (2013).

[DL10] U. DERENTHAL and D. LOUGHRAN. Singular del Pezzo surfaces that are equivariant compactifications. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* **377** (Issledovaniya po Teorii Chisel. 10) **241** (2010), 26–43.

[DL12] U. DERENTHAL and D. LOUGHRAN. Equivariant compactifications of two-dimensional algebraic groups. *Proc. Edinb. Math. Soc., to appear*, arXiv:1212.3518 (2012).

[DT07] U. DERENTHAL and YU. TSCHINKEL. Universal torsors over del Pezzo surfaces and rational points. In *Equidistribution in Number Theory, an Introduction*, volume 237 of *NATO Sci. Ser. II Math. Phys. Chem.* (Springer, Dordrecht, 2007), pages 169–196.

[FMT89] J. FRANKE, YU. I. MANIN and YU. TSCHINKEL. Rational points of bounded height on Fano varieties. *Invent. Math.* **95**(2) (1989), 421–435.

[Pey95] E. PEYRE. Hauteurs et mesures de Tamagawa sur les variétés de Fano. *Duke Math. J.* **79**(1) (1995), 101–218.

[TT12] S. TANIMOTO and YU. TSCHINKEL. Height zeta functions of equivariant compactifications of semi-direct products of algebraic groups. In *Zeta Functions in Algebra and Geometry*, volume 566 of *Contemp. Math.* (Amer. Math. Soc., Providence, RI, 2012), pages 119–157.