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Lieb–Thirring and Cwickel–Lieb–Rozenblum inequalities for perturbed graphene with a Coulomb impurity

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Abstract. We study the two dimensional massless Coulomb–Dirac operator restricted to its positive spectral subspace and prove estimates on the negative eigenvalues created by electromagnetic perturbations.

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Contents

1. Introduction

As theoretically predicted [27, 7] and recently experimentally observed [18, 15], electrostatic potentials can create bound states in graphene, which corresponds to emergence of a quantum dot. We consider the case of a graphene sheet with an attractive Coulomb impurity perturbed by a weak electromagnetic potential and

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provide bounds on the energies of the bound states using a model similar to those of [8].

We begin by considering a graphene sheet with an attractive Coulomb impurity of strength ν. For energies near the conical point of the energy-quasi-momentum dispersion relation, the Hamiltonian of an electron in such material is effectively given by the massless Coulomb–Dirac operator (see [24] and Section IV of [5]). This operator acts in $\mathsf{L}^2(\mathbb{R}^2, \mathbb{C}^2)$ and is associated to the differential expression

$$d^\nu := -\mathrm{i}\boldsymbol{\sigma} \cdot \nabla - \nu|\cdot|^{-1}. \tag{1}$$

Here the units are chosen so that the Fermi velocity v_F equals 1, and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2) = \left(\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{smallmatrix}\right)\right)$ is the vector of Pauli matrices. For $\nu \in [0, 1/2]$ (which we assume throughout in the following) we work with the distinguished self-adjoint operator D^ν in $\mathsf{L}^2(\mathbb{R}^2, \mathbb{C}^2)$ associated to (1) (see [22, 32] and (33) below). The supercritical case of $\nu > 1/2$ is not considered here. In that case the canonical choice of a particular self-adjoint realisation among many possible is not well established.

We now state the main results of the paper. Scalar operators like $\sqrt{-\Delta}$ are applied to vector-valued functions component-wise without reflecting this in the notation.

Theorem 1. (1) *For every* $\nu \in [0, 1/2)$ *there exists* $C_\nu > 0$ *such that*

$$|D^\nu| \geqslant C_\nu \sqrt{-\Delta} \tag{2}$$

holds.

(2) *For any* $\lambda \in [0, 1)$ *there exists* $K_\lambda > 0$ *such that*

$$|D^{1/2}| \geqslant K_\lambda l^{\lambda-1}(-\Delta)^{\lambda/2} - l^{-1} \tag{3}$$

holds for any $l > 0$.

Note that for $\nu \in (0, 1/2]$ the inequality $(D^\nu)^2 \geqslant C(-\Delta)$ is false for any $C > 0$, since by Corollary 16 the operator domain of D^ν is not contained in $\mathsf{H}^1(\mathbb{R}^2, \mathbb{C}^2)$.

The operator inequality (3) is related to the estimate for the fractional Schrödinger operator with Coulomb potential in $\mathsf{L}^2(\mathbb{R}^2)$. For any $t \in (0, 1/2)$ there exists $M_t > 0$ such that

$$(-\Delta)^{1/2} - \frac{2(\Gamma(3/4))^2}{(\Gamma(1/4))^2|\cdot|} \geqslant M_t l^{2t-1}(-\Delta)^t - l^{-1} \tag{4}$$

holds for all $l > 0$, see (1.3) in [11] (and Theorem 2.3 in [28] for an analogous result in three dimensions).

Since the negative energy states of D^ν belong to the fully occupied valence band of graphene (Dirac sea) [31, 5], the space of physically available electronic states is $\mathfrak{H}_+^\nu := P_+^\nu \mathsf{L}^2(\mathbb{R}^2, \mathbb{C}^2)$, where P_+^ν is the spectral projector of D^ν to the half-line $[0, \infty)$. We now consider perturbations of D^ν by electromagnetic potentials, which are assumed to be weak enough so that the state space is essentially unchanged.

Corollary 2. *Suppose that* $(\nu, \gamma) \in ([0, 1/2] \times [0, \infty)) \setminus \{(1/2, 0)\}$. *Let V be a non-negative measurable* (2×2)*-matrix function with* $\mathrm{tr}(V^{2+\gamma}) \in \mathsf{L}^1(\mathbb{R}^2)$. *Let $\mathfrak{w}$ be a real-valued quadratic form in $\mathfrak{H}_+^\nu$ with the domain containing* $P_+^\nu \mathfrak{D}(|D^\nu|^{1/2})$. *Assume that there exists $C > 0$ such that*

$$0 \leqslant \mathfrak{w}[\varphi] \leqslant C(\||D^\nu|^{1/2}\varphi\|^2 + \|\varphi\|^2), \quad \textit{for all } \varphi \in P_+^\nu \mathfrak{D}(|D^\nu|^{1/2}). \tag{5}$$

Then the quadratic form

$$\begin{aligned} &\mathfrak{d}^\nu(\mathfrak{w}, V) \colon P_+^\nu \mathfrak{D}(|D^\nu|^{1/2}) \longrightarrow \mathbb{R}, \\ &\mathfrak{d}^\nu(\mathfrak{w}, V)[\varphi] := \||D^\nu|^{1/2}\varphi\|^2 + \mathfrak{w}[\varphi] - \int_{\mathbb{R}^2} \langle \varphi(\mathbf{x}), V(\mathbf{x})\varphi(\mathbf{x})\rangle \mathrm{d}\mathbf{x} \end{aligned}$$

is closed and bounded from below in $\mathfrak{H}_+^\nu$.

According to Theorem 10.1.2 in [3], there exists a unique self-adjoint operator $D^\nu(\mathfrak{w}, V)$ in $\mathfrak{H}_+^\nu$ associated to $\mathfrak{d}^\nu(\mathfrak{w}, V)$. In the following two theorems we study the negative spectrum of $D^\nu(\mathfrak{w}, V)$. Note that the eigenvalues of $D^\nu(\mathfrak{w}, V)$ can be interpreted as bound states of a quantum dot.

For numbers and self-adjoint operators we use the notation $x_\pm := \max\{\pm x, 0\}$ for the positive and negative parts of x.

Theorem 3. *Let* $\nu \in [0, 1/2)$. *There exists* $C_\nu^{\mathrm{CLR}} > 0$ *such that*

$$\mathrm{rank}(D^\nu(\mathfrak{w}, V))_- \leqslant C_\nu^{\mathrm{CLR}} \int_{\mathbb{R}^2} \mathrm{tr}(V(\mathbf{x}))^2 \, \mathrm{d}\mathbf{x}. \tag{6}$$

Analogues of Theorem 3 are widely known for many bounded from below self-adjoint operators as Cwickel–Lieb–Rozenblum inequalities (see [26, 6, 19] for the original contributions and [12] and references therein for further developments). In particular, in Example 3.3 of [12] it is proved that the estimate

$$\mathrm{rank}((-\Delta)^t - V)_- \leqslant (4\pi t)^{-1}(1-t)^{(t-2)/t} \int_{\mathbb{R}^2} \mathrm{tr}(V(\mathbf{x}))^{1/t} \mathrm{d}\mathbf{x} \tag{7}$$

holds for all $0 < t < 1$. Our proof of Theorem 3 is based on Theorem 1 and (7).

Theorem 4. *Let $\nu \in [0, 1/2]$ and $\gamma > 0$. There exists $C^{\mathrm{LT}}_{\nu,\gamma} > 0$ such that*

$$\mathrm{tr}(D^\nu(\mathfrak{w}, V))^\gamma_- \leqslant C^{\mathrm{LT}}_{\nu,\gamma} \int_{\mathbb{R}^2} \mathrm{tr}(V(\mathbf{x}))^{2+\gamma}\, \mathrm{d}\mathbf{x}. \tag{8}$$

Theorem 4 is a form of Lieb–Thirring inequality, (see [20] for the original result and [16] for a review of further developments). In another publication [21] we prove that $D^{1/2}(0, V)$ has a negative eigenvalue for any non-trivial $V \geqslant 0$. This situation is associated with the existence of a virtual level at zero, as observed for example for the operator $\big(-\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{1}{4r^2}\big)$ in $\mathsf{L}^2(\mathbb{R}_+)$ (see [9], Proposition 3.2). In particular, the bound (6) cannot hold for $\nu = 1/2$. In this case Theorem 4 is an equivalent of Hardy–Lieb–Thirring inequality (see [9, 11, 13]).

Certain estimates for the optimal constants in Theorems 1–4 can be extracted from the proofs provided. This results in explicit, but quite involved expressions.

The article is organised in the following way. We start with some auxiliary results in Section 2, where we prepare useful representations of operators of interest with the help of certain unitary transforms. One of such representations allows us to provide a rigorous definition (33) of D^ν. In Section 3 we study the operator $(-\Delta)^{1/2} - \alpha|\cdot|^{-1}$ in the representation, in which it can be relatively easily compared with $|D^\nu|$. Such comparison is done in the two critical channels of the angular momentum decomposition in Section 4. For the non-critical channels we obtain a lower bound on $|D^\nu|$ in terms of $(-\Delta)^{1/2}$ in Section 5. In the subsequent Section 6 we prove a channel-wise improvement of (4). Finally, in Section 7 we complete the proofs of Theorems 1–4 and Corollary 2.

2. Mellin, Fourier and related transforms in polar coordinates

In this section we introduce several unitary transformations which will be useful in the subsequent analysis. We also formulate and prove several technical results needed in the subsequent sections. Let $(r, \theta), (p, \omega) \in [0, \infty) \times [0, 2\pi)$ be the polar coordinates in $\mathbb{R}^2$ in coordinate and momentum spaces, respectively.

Fourier transform. We use the standard unitary Fourier transform in $\mathsf{L}^2(\mathbb{R}^2)$ given in the polar coordinates for $\varphi \in \mathsf{L}^1(\mathbb{R}^2) \cap \mathsf{L}^2(\mathbb{R}^2)$ by

$$(\mathcal{F}\varphi)(p, \omega) := \frac{1}{2\pi} \int_0^\infty \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}pr\cos(\omega-\theta)} \varphi(r, \theta)\, \mathrm{d}\theta\, r\mathrm{d}r. \tag{9}$$

Lemma 5. *For $m \in \mathbb{Z}$ and $\psi \in \mathsf{C}_0^\infty([0,\infty))$ the Fourier transform of*

$$\Psi^{(m)}(r,\theta) := r^{-1/2}\psi(r)\mathrm{e}^{\mathrm{i}m\theta} \tag{10}$$

is given in the polar coordinates by

$$\mathcal{F}(\Psi^{(m)})(p,\omega) = (-\mathrm{i})^m \mathrm{e}^{\mathrm{i}m\omega} \int_0^\infty \sqrt{r} J_m(pr)\psi(r)\mathrm{d}r. \tag{11}$$

Proof. According to [1], 10.9.2 and 10.2.2

$$\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}pr\cos(\omega-\theta)}\mathrm{e}^{\mathrm{i}m\theta}\mathrm{d}\theta = 2\pi \mathrm{i}^m J_m(-pr)\mathrm{e}^{\mathrm{i}m\omega} = 2\pi(-\mathrm{i})^m J_m(pr)\mathrm{e}^{\mathrm{i}m\omega}. \tag{12}$$

Substituting (10) into (9) and using (12) we obtain (11). □

Mellin transform. Let $\mathcal{M}$ be the unitary Mellin transform, first defined on $\mathsf{C}_0^\infty(\mathbb{R}_+)$ by

$$(\mathcal{M}\psi)(s) := \frac{1}{\sqrt{2\pi}} \int_0^\infty r^{-1/2-\mathrm{i}s}\psi(r)\mathrm{d}r, \tag{13}$$

and then extended to a unitary operator $\mathcal{M}\colon \mathsf{L}^2(\mathbb{R}_+) \to \mathsf{L}^2(\mathbb{R})$, see e.g. [17].

Definition 6. *For $\lambda \in \mathbb{R}\setminus\{0\}$ let $\mathfrak{D}^\lambda$ be the set of functions $\psi \in \mathsf{L}^2(\mathbb{R})$ such that there exists Ψ analytic in the strip $\mathfrak{S}^\lambda := \{z \in \mathbb{C}\colon \operatorname{Im} z/\lambda \in (0,1)\}$ with the properties*

(1) $\mathsf{L}^2\text{-}\lim\limits_{t\to+0} \Psi(\cdot + \mathrm{i}t\lambda) = \psi(\cdot)$;

(2) *there exists* $\mathsf{L}^2\text{-}\lim\limits_{t\to 1-0} \Psi(\cdot + \mathrm{i}t\lambda)$;

(3) $\sup\limits_{t\in(0,1)} \int_{\mathbb{R}} |\Psi(s+\mathrm{i}t\lambda)|^2 \mathrm{d}s < \infty.$

For $\lambda \in \mathbb{R}$ let the operator of multiplication by r^λ in $\mathsf{L}^2(\mathbb{R}_+, \mathrm{d}r)$ be defined on its maximal domain $\mathsf{L}^2(\mathbb{R}_+, (1+r^{2\lambda})\mathrm{d}r)$. Applying the lemma of [30] (Section 5.4, page 125) to justify the translations of the integration contour between different values of t under Assumption 3 of Definition 6 we obtain

Theorem 7. *Let $\lambda \in \mathbb{R}\setminus\{0\}$. Then the identity*

$$\mathfrak{D}^\lambda = \mathcal{M}\mathsf{L}^2(\mathbb{R}_+, (1+r^{2\lambda})\mathrm{d}r)$$

holds, and for any $\psi \in \mathfrak{D}^\lambda$ the function Ψ from Definition 6 satisfies

$$\Psi(z) = (\mathcal{M} r^{\operatorname{Im} z}\mathcal{M}^*\psi)(\operatorname{Re} z), \qquad \textit{for all } z \in \mathfrak{S}^\lambda.$$

We conclude that r^λ acts as a complex shift in the Mellin space. Indeed, for $\lambda \in \mathbb{R}$ let $R^\lambda\colon \mathfrak{D}^\lambda \to \mathsf{L}^2(\mathbb{R})$ be the linear operator defined by

$$R^\lambda\psi := \begin{cases} \mathsf{L}^2\text{-}\lim\limits_{t\to 1-0} \Psi(\cdot + \mathrm{i}t\lambda), & \lambda \neq 0; \\ \psi, & \lambda = 0, \end{cases}$$

with Ψ as in Definition 6. It follows from Theorem 7 that R^λ is well defined and that

$$\mathcal{M}r^\lambda\mathcal{M}^* = R^\lambda \tag{14}$$

holds (see [17], Section II).

The following lemma will be needed later.

Lemma 8. *Let J_m be the Bessel function with $m \in \mathbb{Z}$. The relation*

$$\Big(\mathcal{M}((-\mathrm{i})^m \int_0^\infty \sqrt{\cdot r}\, J_m(\cdot r)\psi(r)\mathrm{d}r)\Big)(s) = \Xi_m(s)(\mathcal{M}\psi)(-s)$$

holds for every $\psi \in C_0^\infty([0,\infty))$ and $s \in \mathbb{R}$ with

$$\Xi_m(s) := (-\mathrm{i})^{|m|} 2^{-\mathrm{i}s} \frac{\Gamma((|m|+1-\mathrm{i}s)/2)}{\Gamma((|m|+1+\mathrm{i}s)/2)}. \tag{15}$$

Proof. It is enough to prove the statement for $m \in \mathbb{N}_0$, since $J_{-m} = (-1)^m J_m$, see 10.4.1 in [1]. According to 10.22.43 in [1],

$$\lim_{R\to\infty} (-\mathrm{i})^m \int_0^R t^{-\mathrm{i}s} J_m(t)\,\mathrm{d}t = \Xi_m(s). \tag{16}$$

It follows that

$$\sup_{L>0} \Big|\int_0^L t^{-\mathrm{i}s} J_m(t)\,\mathrm{d}t\Big| < \infty.$$

The claim now follows from the representation

$$\begin{aligned}\Big(\mathcal{M}((-\mathrm{i})^m &\int_0^\infty \sqrt{\cdot r}\, J_m(\cdot r)\psi(r)\mathrm{d}r)\Big)(s) \\ &= \lim_{R\to\infty} \frac{(-\mathrm{i})^m}{\sqrt{2\pi}} \int_0^R p^{-\mathrm{i}s} \int_{\operatorname{supp}\psi} \sqrt{r}\, J_m(pr)\psi(r)\,\mathrm{d}r\,\mathrm{d}p\end{aligned}$$

by Fubini's theorem, dominated convergence and (16). □

Remark 9. *For any $m \in \mathbb{Z}$ the function Ξ_m introduced in (15) allows an analytic continuation to $\mathbb{C} \setminus (-\mathrm{i}(1+|m|+2\mathbb{N}_0))$, whereas*

$$\Xi_m^{-1}(\cdot) = \overline{\Xi_m(\bar{\cdot})} \tag{17}$$

allows an analytic continuation to $\mathbb{C} \setminus (\mathrm{i}(1+|m|+2\mathbb{N}_0))$.

Lemma 10. *For* $(m, \lambda) \in \mathbb{Z} \times [0, 1]$ *and any* $\psi \in \mathfrak{D}^\lambda \supset \mathfrak{D}^1$ *with*

$$\Xi_m^{-1}\psi = \overline{\Xi_m(\bar{\cdot})}\psi \in \mathfrak{D}^\lambda \tag{18}$$

the commutation rule

$$R^\lambda \Xi_m^{-1}\psi = \Xi_m^{-1}(\cdot + \mathrm{i}\lambda) R^\lambda \psi \tag{19}$$

applies. Except for $(m, \lambda) = (0, 1)$ *condition* (18) *is automatically fulfilled for all* $\psi \in \mathfrak{D}^\lambda$.

Proof. It follows from Remark 9 that Ξ_m^{-1} is analytic in $\mathfrak{S}^1$ and, for $(m, \lambda) \neq (0, 1)$, in a complex neighbourhood of $\mathfrak{S}^\lambda$. With the help of the Stirling asymptotic formula

$$\Gamma(z) = \sqrt{\frac{2\pi}{z}} \left(\frac{z}{\mathrm{e}}\right)^z (1 + O(|z|^{-1})) \quad \text{for all } z \in \mathbb{C} \text{ with } |\arg z| < \pi - \delta,\ \delta > 0 \tag{20}$$

(see e.g. 5.11.3 in [1]) we conclude that the asymptotics

$$|\Xi_m^{-1}(z)| = |\overline{\Xi_m(\bar{z})}| = |\operatorname{Re} z|^{-\operatorname{Im} z}(1 + O(|z|^{-1})) \text{ holds for } z \in \mathfrak{S}^1 \text{ as } |z| \to \infty. \tag{21}$$

This implies that

$$\Xi_m^{-1} \text{ is analytic and bounded in } \mathfrak{S}^\lambda \text{ for all } (m, \lambda) \in (\mathbb{Z} \times [0, 1]) \setminus \{(0, 1)\}, \tag{22}$$

and the last statement of the lemma follows.

Since $\psi \in \mathfrak{D}^\lambda$, there exists Ψ as in Definition 6. Analogously, by (18) there exists Φ analytic in $\mathfrak{S}^\lambda$ corresponding to $\varphi := \Xi_m^{-1}\psi$ as in Definition 6. Then φ, $\psi \in \mathfrak{D}^{\lambda/2}$ and by (22)

$$\Phi(\cdot + \mathrm{i}\lambda/2) = R^{\lambda/2}\varphi = R^{\lambda/2}\Xi_m^{-1}\psi = \Xi_m^{-1}(\cdot + \mathrm{i}\lambda/2)\Psi(\cdot + \mathrm{i}\lambda/2)$$

holds on $\mathbb{R}$. Thus Φ and $\Xi_m^{-1}\Psi$ must coincide on their joint domain of analyticity $\mathfrak{S}^\lambda$. Since $R^\lambda \Xi_m^{-1}\psi = \mathsf{L}^2\text{-}\lim_{t \to 1-0} \Phi(\cdot + \mathrm{i}t\lambda)$ exists, it must coincide as a function on $\mathbb{R}$ with

$$\begin{aligned} \mathsf{L}^2\text{-}\lim_{t \to 1-0} \Xi_m^{-1}(\cdot + \mathrm{i}t\lambda)\Psi(\cdot + \mathrm{i}t\lambda) &= \Xi_m^{-1}(\cdot + \mathrm{i}\lambda)\, \mathsf{L}^2\text{-}\lim_{t \to 1-0} \Psi(\cdot + \mathrm{i}t\lambda) \\ &= \Xi_m^{-1}(\cdot + \mathrm{i}\lambda) R^\lambda \psi, \end{aligned} \tag{23}$$

where the first equality in (23) can be justified by passing to an almost everywhere convergent subsequence. □

For $\lambda = 1$, multiplying (19) by Ξ_m we conclude:

Corollary 11. *For $m \in \mathbb{Z}$ and $\psi \in \mathfrak{D}^1$ (satisfying $\Xi_0^{-1}\psi \in \mathfrak{D}^1$ if $m = 0$) the identity*

$$\Xi_m R^1 \Xi_m^{-1} \psi = V_{|m|-1/2}(\cdot + \mathrm{i}/2) R^1 \psi$$

holds with

$$V_j(z) := \frac{\Gamma((j+1+\mathrm{i}z)/2)\Gamma((j+1-\mathrm{i}z)/2)}{2\Gamma((j+2+\mathrm{i}z)/2)\Gamma((j+2-\mathrm{i}z)/2)}, \tag{24}$$

for $j \in \mathbb{N}_0 - 1/2$ and $z \in \mathbb{C} \setminus \mathrm{i}(\mathbb{Z} + 1/2)$.

We will need the following properties of V_j.

Lemma 12. *For every $j \in \mathbb{N}_0 - 1/2$ the function (24) is analytic in $\mathbb{C} \setminus \mathrm{i}(\mathbb{Z} + 1/2)$ and has the following properties:*

(1) *$V_j(z) = V_j(-z)$, for all $z \in \mathbb{C} \setminus \mathrm{i}(\mathbb{Z} + 1/2)$;*

(2) *$V_j(s)$ is positive and strictly monotonously decreasing for $s \in \mathbb{R}_+$;*

(3) *$V_j(\mathrm{i}\zeta)$ is positive and strictly monotonously increasing for $\zeta \in [0, 1/2)$;*

(4) *the relation*

$$(z^2 + (j+1)^2) V_j(z) = (V_{j+1}(z))^{-1} \tag{25}$$

holds for all $z \in \mathbb{C} \setminus \mathrm{i}(\mathbb{Z} + 1/2)$.

Proof. (2) For $z \in \mathbb{C} \setminus (-\mathbb{N}_0)$ let $\psi(z) := \Gamma'(z)/\Gamma(z)$ be the digamma function. Differentiating (24) and using Formula 5.7.7 in [1] we obtain

$$\begin{aligned} V_j'(s) &= V_j(s) \operatorname{Im}(\psi((j+2+\mathrm{i}s)/2) - \psi((j+1+\mathrm{i}s)/2)) \\ &= 2s V_j(s) \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{s^2 + (k+j+1)^2} < 0, \quad \text{for all } s > 0. \end{aligned}$$

(3) Analogously to (2), we compute

$$\mathrm{i}V_j'(\mathrm{i}\zeta) = 2\zeta V_j(\mathrm{i}\zeta) \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+j+1)^2 - \zeta^2} > 0, \quad \text{for all } \zeta \in [0, 1/2).$$

(4) Follows directly from (24) and the recurrence relation $\Gamma(z+1) = z\Gamma(z)$ (valid for all $z \in \mathbb{C} \setminus (-\mathbb{N}_0)$). ☐

Angular decomposition. We can represent arbitrary $u \in \mathsf{L}^2(\mathbb{R}^2)$ in the polar coordinates as

$$u(r,\theta) = \frac{1}{\sqrt{2\pi}} \sum_{m\in\mathbb{Z}} r^{-1/2} u_m(r) \mathrm{e}^{\mathrm{i}m\theta}$$

with

$$u_m(r) := \sqrt{\frac{r}{2\pi}} \int_0^{2\pi} u(r,\theta)\mathrm{e}^{-\mathrm{i}m\theta}\mathrm{d}\theta.$$

The map

$$\mathcal{W}\colon \mathsf{L}^2(\mathbb{R}^2) \longrightarrow \bigoplus_{m\in\mathbb{Z}} \mathsf{L}^2(\mathbb{R}_+), \quad u \longmapsto \bigoplus_{m\in\mathbb{Z}} u_m \tag{26}$$

is unitary.

For the proof of the following lemma (based on Lemmata 2.1, 2.2 of [4]) see the proof of Theorem 2.2.5 in [2].

Lemma 13. *For $m \in \mathbb{Z}$ and $z \in (1,\infty)$ let*

$$Q_{|m|-1/2}(z) := 2^{-|m|-1/2} \int_{-1}^{1} (1-t^2)^{|m|-1/2}(z-t)^{-|m|-1/2}\,\mathrm{d}t \tag{27}$$

be the Legendre function of the second kind, see [35], *Section* 15.3. *Let the quadratic form $\mathfrak{q}_m$ be defined on $\mathsf{L}^2(\mathbb{R}_+,(1+p^2)^{1/2}\mathrm{d}p)$ by*

$$\mathfrak{q}_m[g] := \pi^{-1} \iint_{\mathbb{R}_+^2} \overline{g(p)}\, Q_{|m|-1/2}\Big(\frac{1}{2}\Big(\frac{q}{p}+\frac{p}{q}\Big)\Big) g(q)\,\mathrm{d}q\,\mathrm{d}p.$$

Then for every f in the Sobolev space $\mathsf{H}^{1/2}(\mathbb{R}^2)$ the relation

$$\int_{\mathbb{R}^2} |\mathbf{x}|^{-1}|f(\mathbf{x})|^2\,\mathrm{d}\mathbf{x} = \sum_{m\in\mathbb{Z}} \mathfrak{q}_m[(\mathcal{F}f)_m]$$

holds.

The natural Hilbert space for spin-$1/2$ particles is $\mathsf{L}^2(\mathbb{R}^2,\mathbb{C}^2)$. Moreover, the natural angular momentum decomposition associated to (1) is not given by (26), but by

$$\mathcal{A} := \mathcal{S}\mathcal{W}\begin{pmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{pmatrix},$$

where the unitary operator $\mathcal{S}$ is defined as

$$\mathcal{S}\colon \bigoplus_{m\in\mathbb{Z}} \mathsf{L}^2(\mathbb{R}_+,\mathbb{C}^2) \longrightarrow \bigoplus_{\varkappa\in\mathbb{Z}+1/2} \mathsf{L}^2(\mathbb{R}_+,\mathbb{C}^2), \quad \bigoplus_{m\in\mathbb{Z}} \begin{pmatrix} \varphi_m \\ \psi_m \end{pmatrix} \longmapsto \bigoplus_{\varkappa\in\mathbb{Z}+1/2} \begin{pmatrix} \varphi_{\varkappa-1/2} \\ \psi_{\varkappa+1/2} \end{pmatrix}. \tag{28}$$

For $\nu \in [0, 1/2]$ and $\varkappa \in \mathbb{Z}+1/2$ we define the operators $D^{\nu}_{\varkappa,\max}$ in $\mathsf{L}^2(\mathbb{R}_+, \mathbb{C}^2)$ by the differential expressions

$$d^{\nu}_{\varkappa} = \begin{pmatrix} -\frac{\nu}{r} & -\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\varkappa}{r} \\ \frac{\mathrm{d}}{\mathrm{d}r} - \frac{\varkappa}{r} & -\frac{\nu}{r} \end{pmatrix} \tag{29}$$

on their maximal domains

$$\mathfrak{D}(D^{\nu}_{\varkappa,\max}) := \{u \in \mathsf{L}^2(\mathbb{R}_+, \mathbb{C}^2) \cap \mathsf{AC}_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{C}^2)\colon d^{\nu}_{\varkappa} u \in \mathsf{L}^2(\mathbb{R}_+, \mathbb{C}^2)\}.$$

Let $D^{\nu}_{\max}$ be the maximal operator in $\mathsf{L}^2(\mathbb{R}^2, \mathbb{C}^2)$ corresponding to (1) on the domain

$$\begin{aligned}\mathfrak{D}(D^{\nu}_{\max}) := \{u \in \mathsf{L}^2(\mathbb{R}^2, \mathbb{C}^2)\colon &\text{ there exists } w \in \mathsf{L}^2(\mathbb{R}^2, \mathbb{C}^2) \text{ such that} \\ &\langle u, d^{\nu} v\rangle = \langle w, v\rangle \text{ holds for all } v \in \mathsf{C}^{\infty}_0(\mathbb{R}^2 \setminus \{0\}, \mathbb{C}^2)\}.\end{aligned}$$

The following Lemma follows from Section 7.3.3 in [29].

Lemma 14. *The operator $D^{\nu}_{\max}$ preserves the fibres of the half-integer angular momentum decomposition and satisfies*

$$\mathcal{A}\, D^{\nu}_{\max}\, \mathcal{A}^* = \bigoplus_{\varkappa \in \mathbb{Z}+1/2} D^{\nu}_{\varkappa,\max}.$$

In the following lemma we construct particular self-adjoint restrictions of $D^{\nu}_{\varkappa,\max}$.

Lemma 15. *For $\nu \in [0, 1/2]$ and $\varkappa \in (\mathbb{Z} + 1/2)$ let*

$$\mathfrak{C}^{\nu}_{\varkappa} := \mathsf{C}^{\infty}_0(\mathbb{R}_+, \mathbb{C}^2) \dot{+} \begin{cases} \mathrm{span}\{\psi^{\nu}_{\varkappa}\}, & \textit{for } \varkappa = \pm 1/2 \textit{ and } \nu \in (0, 1/2]; \\ \{0\}, & \textit{otherwise}, \end{cases} \tag{30}$$

with

$$\psi^{\nu}_{\varkappa}(r) := \sqrt{2\pi} \begin{pmatrix} \nu \\ \sqrt{\varkappa^2 - \nu^2} - \varkappa \end{pmatrix} r^{\sqrt{\varkappa^2-\nu^2}} \mathrm{e}^{-r}, \qquad r \in \mathbb{R}_+. \tag{31}$$

Then the restriction of $D^{\nu}_{\varkappa,\max}$ to $\mathfrak{C}^{\nu}_{\varkappa}$ is essentially self-adjoint in $\mathsf{L}^2(\mathbb{R}_+, \mathbb{C}^2)$. We define $D^{\nu}_{\varkappa}$ to be the self-adjoint operator in $\mathsf{L}^2(\mathbb{R}_+, \mathbb{C}^2)$ obtained as the closure of this restriction.

Proof. For $\nu \in [0, 1/2]$, $\varkappa \in \mathbb{Z} + 1/2$ let $D^{\nu}_{\varkappa,\min}$ be the closure of the restriction of $D^{\nu}_{\varkappa,\max}$ to $\mathsf{C}^{\infty}_{0}(\mathbb{R}_+, \mathbb{C}^2)$. To determine the defect indices of $D^{\nu}_{\varkappa,\min}$ we observe that the fundamental solution of the equation $d^{\nu}_{\varkappa}\varphi = 0$ in $\mathbb{R}_+$ is a linear combination of two functions:

$$\varphi^{\nu}_{\varkappa,\pm}(r) := \begin{cases} \begin{pmatrix} 1/2 \pm 1/2 \\ 1/2 \mp 1/2 \end{pmatrix} r^{\pm\varkappa}, & \text{for } \nu = 0; \\ \begin{pmatrix} \nu \\ \pm\sqrt{\varkappa^2 - \nu^2} - \varkappa \end{pmatrix} r^{\pm\sqrt{\varkappa^2-\nu^2}}, & \text{for } 0 < \nu^2 < \varkappa^2; \end{cases}$$

$$\varphi^{\nu}_{\varkappa,+} := \begin{pmatrix} \nu \\ -\varkappa \end{pmatrix} \quad \text{and} \quad \varphi^{\nu}_{\varkappa,0}(r) := \begin{pmatrix} \nu \ln r \\ 1 - \varkappa \ln r \end{pmatrix}, \quad \text{for } \nu^2 = \varkappa^2 = 1/4.$$

Now we apply Theorems 1.4 and 1.5 of [33]. Since $\varphi^{\nu}_{\varkappa,+} \notin \mathsf{L}^2((1,\infty))$ for any $\varkappa$ and ν, the differential expression (29) is in the limit point case at infinity. For $\varkappa^2 - \nu^2 \geqslant 1/4$ we have $\varphi^{\nu}_{\varkappa,-} \notin \mathsf{L}^2((0,1))$ and hence (29) is in the limit point case at zero. In this case the defect indices of $D^{\nu}_{\varkappa,\min}$ are zero and thus $D^{\nu}_{\varkappa,\min}$ is self-adjoint.

For $\varkappa^2 - \nu^2 < 1/4$, i.e. $\varkappa = \pm 1/2$ and $\nu \in (0, 1/2]$, any solution of $d^{\nu}_{\varkappa}\varphi = 0$ belongs to $\mathsf{L}^2((0,1))$ and hence (29) is in the limit circle case at zero with the deficiency indices of $D^{\nu}_{\varkappa,\min}$ being $(1,1)$. In this case every one-dimensional extension of $D^{\nu}_{\varkappa,\min}$ which is a restriction of $D^{\nu}_{\varkappa,\max}$ is self-adjoint (see e.g. [3], Section 4.4.1). Theorem 1.5(2) in [33] implies

$$\lim_{\varepsilon\to+0} \langle \varphi(\varepsilon), \mathrm{i}\sigma_2\psi(\varepsilon)\rangle_{\mathbb{C}^2} = 0 \quad \text{for all } \varphi \in \mathfrak{D}(D^{\nu}_{\varkappa,\min}),\ \psi \in \mathfrak{D}(D^{\nu}_{\varkappa,\max}). \tag{32}$$

Choosing $\psi(r) := \mathrm{e}^{-r}\varphi^{\nu}_{\varkappa,-}(r)$ for $\nu^2 < \varkappa^2$ and $\psi(r) := \mathrm{e}^{-r}\varphi^{\nu}_{\varkappa,0}(r)$ for $\nu^2 = \varkappa^2$ in (32), we conclude that $\psi^{\nu}_{\varkappa} \notin \mathfrak{D}(D^{\nu}_{\varkappa,\min})$. Thus the closure of the restriction of $D^{\nu}_{\varkappa,\max}$ to $\mathfrak{C}^{\nu}_{\varkappa}$ is a one-dimensional extension of $D^{\nu}_{\varkappa,\min}$, hence a self-adjoint operator. □

For $\nu \in (0, 1/2]$ we now define

$$D^{\nu} := \mathcal{A}^* \Big(\bigoplus_{\varkappa\in\mathbb{Z}+1/2} D^{\nu}_{\varkappa} \Big) \mathcal{A}. \tag{33}$$

By Lemma 14, D^{ν} is a self-adjoint operator in $\mathsf{L}^2(\mathbb{R}^2, \mathbb{C}^2)$ corresponding to (1). Lemma 15 implies:

Corollary 16. *Let $\delta_{\cdot,\cdot}$ be the Kronecker symbol. The set*

$$\mathfrak{C}^\nu := \mathsf{C}_0^\infty(\mathbb{R}^2 \setminus \{0\}, \mathbb{C}^2) \dot{+} \operatorname{span}\{\Psi_+^\nu, \Psi_-^\nu\}$$

with

$$\begin{aligned}\Psi_\pm^\nu(r,\theta) &:= \Big(\mathcal{A}^* \bigoplus_{\varkappa\in\mathbb{Z}+1/2} \delta_{\varkappa,\pm 1/2}\psi^\nu_{\pm 1/2}\Big)(r,\theta)\\ &= \begin{pmatrix} \nu e^{i(\pm 1/2 - 1/2)\theta} \\ -i(\sqrt{1/4-\nu^2} \mp 1/2) e^{i(\pm 1/2 + 1/2)\theta}\end{pmatrix} r^{\sqrt{1/4-\nu^2}-1/2} e^{-r}\end{aligned}$$

is an operator core for D^ν.

Remark 17. *For a particular class of non-semibounded operators a distinguished self-adjoint realisation can be selected by requiring the positivity of the Schur complement (see* [10]*). In this sense $D^\nu + \operatorname{diag}(1,-1)$ is a distinguished self-adjoint realisation of the massive Coulomb–Dirac operator as proven in* [22].

MWF-transform. We now introduce the unitary transform

$$\mathcal{T}\colon \mathsf{L}^2(\mathbb{R}^2) \longrightarrow \bigoplus_{m\in\mathbb{Z}} \mathsf{L}^2(\mathbb{R}), \quad \mathcal{T} := \mathcal{M}\mathcal{W}\mathcal{F}, \tag{34}$$

where $\mathcal{M}$ acts fibre-wise. A direct calculation using Lemmata 5 and 8 gives

$$\mathcal{T}\varphi = \bigoplus_{m\in\mathbb{Z}} \mathcal{T}_m \varphi_m, \tag{35}$$

where for $m \in \mathbb{Z}$ the operators $\mathcal{T}_m\colon \mathsf{L}^2(\mathbb{R}_+) \to \mathsf{L}^2(\mathbb{R})$ are given by

$$(\mathcal{T}_m\phi)(s) := \Xi_m(s)(\mathcal{M}\phi)(-s) \quad \text{for any } \phi \in \mathsf{L}^2(\mathbb{R}_+). \tag{36}$$

In the following two lemmata we study the actions of several operators in the MWF-representation.

Lemma 18. *The relations*

$$(\mathcal{S}\mathcal{T})(-i\boldsymbol{\sigma}\cdot\nabla)(\mathcal{S}\mathcal{T})^* = \bigoplus_{\varkappa\in\mathbb{Z}+1/2} (R^1 \otimes \sigma_1) \tag{37}$$

and for any $\lambda \in \mathbb{R}$

$$\mathcal{T}(-\Delta)^{\lambda/2}\mathcal{T}^* = \bigoplus_{m\in\mathbb{Z}} R^\lambda \tag{38}$$

hold.

Proof. For any $(\varphi, \psi) \in \mathsf{H}^1(\mathbb{R}^2, \mathbb{C}^2)$, applying (34), (26), and (28) we obtain

$$\begin{aligned}
\mathcal{S}\mathcal{T}(-\mathrm{i}\boldsymbol{\sigma}\cdot\nabla)\begin{pmatrix}\varphi\\ \psi\end{pmatrix} &= \mathcal{S}\mathcal{M}\mathcal{W}\begin{pmatrix}0 & p\mathrm{e}^{-\mathrm{i}\omega}\\ p\mathrm{e}^{\mathrm{i}\omega} & 0\end{pmatrix}\begin{pmatrix}\mathcal{F}\varphi\\ \mathcal{F}\psi\end{pmatrix}\\
&= \bigoplus_{\varkappa\in\mathbb{Z}+1/2}\mathcal{M}p\begin{pmatrix}(\mathcal{F}\psi)_{\varkappa+1/2}\\ (\mathcal{F}\varphi)_{\varkappa-1/2}\end{pmatrix}\\
&= \Big(\bigoplus_{\varkappa\in\mathbb{Z}+1/2}(\mathcal{M}p\mathcal{M}^*)\otimes\sigma_1\Big)\mathcal{S}\mathcal{M}\mathcal{W}\mathcal{F}\begin{pmatrix}\varphi\\ \psi\end{pmatrix},
\end{aligned}$$

which according to (14) and (34) leads to (37). To get (38), the same argument applies with p^λ instead of $\begin{pmatrix}0 & p\mathrm{e}^{-\mathrm{i}\omega}\\ p\mathrm{e}^{\mathrm{i}\omega} & 0\end{pmatrix}$ and $\mathcal{S}$ removed. □

Lemma 19. *The relation*

$$\mathcal{T}|\cdot|^{-1}\mathcal{T}^* = \bigoplus_{m\in\mathbb{Z}}\Xi_m R^1 \Xi_m^{-1} \tag{39}$$

holds.

Proof. For any $\varphi \in \mathsf{L}^2(\mathbb{R}^2, (1+|\mathbf{x}|^{-2})\,\mathrm{d}\mathbf{x})$ applying (35), (36), (14), and (17) we obtain for almost every $s \in \mathbb{R}$

$$\begin{aligned}
(\mathcal{T}(|\cdot|^{-1}\varphi))(s) &= \bigoplus_{m\in\mathbb{Z}}\Xi_m(s)(\mathcal{M}((\cdot)^{-1}\varphi_m))(-s)\\
&= \bigoplus_{m\in\mathbb{Z}}\Xi_m(s)(R^{-1}\mathcal{M}\varphi_m)(-s)\\
&= \bigoplus_{m\in\mathbb{Z}}\Xi_m(s)(R^1((\mathcal{M}\varphi_m)(-\cdot)))(s)\\
&= \bigoplus_{m\in\mathbb{Z}}\Xi_m(s)(R^1\Xi_m^{-1}\mathcal{T}_m\varphi_m)(s).
\end{aligned}$$

This together with (35) gives (39). □

U-transform. For $\varkappa \in \mathbb{Z}+1/2$ let the unitary operators

$$\mathcal{U}_\varkappa\colon \mathsf{L}^2(\mathbb{R}_+, \mathbb{C}^2) \longrightarrow \mathsf{L}^2(\mathbb{R}, \mathbb{C}^2)$$

be defined by

$$\mathcal{U}_\varkappa\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} := \Big(\mathcal{S}\mathcal{T}\mathcal{A}^*\bigoplus_{\tilde\varkappa\in\mathbb{Z}+1/2}\delta_{\tilde\varkappa,\varkappa}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}\Big)_\varkappa = \begin{pmatrix}\mathcal{T}_{\varkappa-1/2}\psi_1\\ -\mathrm{i}\mathcal{T}_{\varkappa+1/2}\psi_2\end{pmatrix}. \tag{40}$$

A straightforward calculation involving (36), (13), (24), (25) and the elementary properties of the gamma function delivers

Lemma 20. *For* $\nu \in (0, 1/2]$ *let*

$$\beta := \sqrt{1/4 - \nu^2}.$$

The functions (31) *from the operator core* $\mathfrak{C}^{\nu}_{\pm 1/2}$ *of* $D^{\nu}_{\pm 1/2}$ *satisfy the relation*

$$\mathcal{U}_{\pm 1/2}\psi^{\nu}_{\pm 1/2} = \chi^{\nu}_{\pm}\begin{pmatrix} 1 \\ \nu V_{\mp 1/2}(\mathrm{i}\beta) \end{pmatrix} + \begin{pmatrix} \xi^{\nu}_{\pm} \\ \eta^{\nu}_{\pm} \end{pmatrix}$$

with

$$\chi^{\nu}_{\pm} := \nu \Xi_{\pm 1/2 - 1/2}(\mathrm{i}(\beta + 1/2)) \frac{(\cdot - \mathrm{i})\Gamma(\mathrm{i}\cdot + \beta + 1/2)}{\mathrm{i}(\beta - 1/2)}, \tag{41}$$

$$\xi^{\nu}_{\pm} := \nu \Gamma(\mathrm{i}\cdot + \beta + 1/2)\Big(\Xi_{\pm 1/2 - 1/2}(\cdot) - \frac{(\cdot - \mathrm{i})\Xi_{\pm 1/2 - 1/2}(\mathrm{i}(\beta + 1/2))}{\mathrm{i}(\beta - 1/2)}\Big), \tag{42}$$

$$\eta^{\nu}_{\pm} := \frac{\mathrm{i}\nu^2 \Gamma(\mathrm{i}\cdot + \beta + 1/2)}{\beta \pm 1/2}\Big(\Xi_{\pm 1/2 + 1/2}(\cdot) - \frac{(\cdot - \mathrm{i})\Xi_{\pm 1/2 + 1/2}(\mathrm{i}(\beta + 1/2))}{\mathrm{i}(\beta - 1/2)}\Big). \tag{43}$$

3. Fourier–Mellin theory of the relativistic massless Coulomb operator in two dimensions

For $\alpha \in \mathbb{R}$ consider the symmetric operator

$$\widetilde{H}^{\alpha} := (-\Delta)^{1/2} - \alpha|\cdot|^{-1}$$

in $\mathsf{L}^2(\mathbb{R}^2)$ on the domain

$$\mathfrak{D}(\widetilde{H}^{\alpha}) := \mathsf{H}^1(\mathbb{R}^2) \cap \mathsf{L}^2(\mathbb{R}^2, |\mathbf{x}|^{-2}\,\mathrm{d}\mathbf{x}).$$

According to Lemmata 18 and 19 and Corollary 11 we have

$$\mathcal{T}\widetilde{H}^{\alpha}\mathcal{T}^* = \bigoplus_{m \in \mathbb{Z}} (1 - \alpha V_{|m| - 1/2}(\cdot + \mathrm{i}/2))R^1 =: \bigoplus_{m \in \mathbb{Z}} \widetilde{H}^{\alpha}_m, \tag{44}$$

where the right hand side is an orthogonal sum of operators in $\mathsf{L}^2(\mathbb{R})$ densely defined on

$$\mathfrak{D}(\widetilde{H}^{\alpha}_m) := \{\varphi \in \mathfrak{D}^1 : \Xi^{-1}_m \varphi \in \mathfrak{D}^1\}. \tag{45}$$

According to Lemma 10, $\mathfrak{D}(\widetilde{H}^{\alpha}_m) = \mathfrak{D}^1$ for $m \in \mathbb{Z} \setminus \{0\}$. On the other hand, $\mathfrak{D}(\widetilde{H}^{\alpha}_0) \neq \mathfrak{D}^1$ for any $\alpha \neq 0$, which corresponds to the absence of Hardy inequality in two dimensions.

Lemma 21. *For $m \in \mathbb{Z}$ and $\alpha \in \mathbb{R}$ the operator $\widetilde{H}_m^\alpha$ is symmetric. It is bounded below (and non-negative) in $\mathsf{L}^2(\mathbb{R})$ if and only if*

$$\alpha \leqslant \alpha_m := \frac{1}{V_{|m|-1/2}(0)} = \frac{2\Gamma^2((2|m|+3)/4)}{\Gamma^2((2|m|+1)/4)}. \tag{46}$$

Proof. For $m \in \mathbb{Z}$ and $\varphi \in \mathfrak{D}(\widetilde{H}_m^\alpha) \subset \mathfrak{D}^1$ using Corollary 11 and Lemma 10 we compute

$$\begin{aligned}\langle \varphi, \widetilde{H}_m^\alpha \varphi\rangle &= \langle \varphi, R^1\varphi\rangle - \alpha\langle \varphi, \Xi_m R^1 \Xi_m^{-1}\varphi\rangle \\ &= \int_{-\infty}^{+\infty} (1-\alpha V_{|m|-1/2}(s))|(R^{1/2}\varphi)(s)|^2 \mathrm{d}s.\end{aligned} \tag{47}$$

Since the right hand side is real-valued, $\widetilde{H}_m^\alpha$ is symmetric. By Lemma 12(1, 2) the condition (46) is equivalent to the non-negativity of $1-\alpha V_{|m|-1/2}(s)$ for all $s \in \mathbb{R}$. For $\alpha > \alpha_m$, $1-\alpha V_{|m|-1/2}$ is negative on an open interval $(-\rho, \rho)$ (ρ depends on m and α). For $n \in \mathbb{N}$ the functions

$$\varphi_n := \tilde{\varphi}_n / \|\tilde{\varphi}_n\|_{\mathsf{L}^2(\mathbb{R})}, \quad \text{with } \tilde{\varphi}_n(s) := \mathrm{e}^{-n(s-\mathrm{i}/2)^2}(1-\mathrm{e}^{-n^2(s-\mathrm{i})^2})$$

are normalised in $\mathsf{L}^2(\mathbb{R})$ and belong to $\mathfrak{D}(\widetilde{H}_m^\alpha)$. Using Lemma 12 we estimate

$$1-\alpha V_{|m|-1/2} \leqslant (1-\alpha V_{|m|-1/2}(\rho/2))\mathbf{1}_{[-\rho/2,\rho/2]} + \mathbf{1}_{\mathbb{R}\setminus(-\rho/2,\rho/2)} \tag{48}$$

as a function on $\mathbb{R}$. It follows from (48) that for n big enough (47) becomes negative with $\varphi := \varphi_n$. But then replacing $\varphi_n(s)$ by $\lambda^{\mathrm{i}s}\varphi_n(s)$ (still normalised and belonging to $\mathfrak{D}(\widetilde{H}_m^\alpha)$ for all $\lambda \in \mathbb{R}_+$) we can make the quadratic form (47) arbitrarily negative. □

Given $m \in \mathbb{Z}$, Lemma 21 allows us for $\alpha \leqslant \alpha_m$ to pass from the symmetric operator $\widetilde{H}_m^\alpha$ to the self-adjoint operator H_m^α by Friedrichs extension [14]. The following description of the domains of H_m^α with $m \neq 0$, $\alpha \in (0, \alpha_m]$ follows analogously to Corollary 2 in [17] (see also Section 2.2.3 of [2]) with the help of Lemma 12:

Lemma 22. *Let $m \in \mathbb{Z} \setminus \{0\}$.*

(1) *For $\alpha < V_{|m|-1/2}^{-1}(\mathrm{i}/2)$ the operator $\widetilde{H}_m^\alpha$ is self-adjoint.*

(2) *For $\alpha = V_{|m|-1/2}^{-1}(\mathrm{i}/2)$ the operator $\widetilde{H}_m^\alpha$ is essentially self-adjoint.*

(3) *For $\alpha \in (V_{|m|-1/2}^{-1}(\mathrm{i}/2), \alpha_m]$ the Friedrichs extension H_m^α of $\widetilde{H}_m^\alpha$ is the restriction of*

$$(\widetilde{H}_m^\alpha)^* = R^1(1-\alpha V_{|m|-1/2}(\cdot - \mathrm{i}/2)) \tag{49}$$

to

$$\mathfrak{D}(H_m^\alpha) = \mathfrak{D}^1 \dotplus \operatorname{span}\{(\cdot - \mathrm{i}/2 + \mathrm{i}\zeta_{m,\alpha})^{-1}\},$$

where $\zeta_{m,\alpha}$ *is the unique solution of*

$$1 - \alpha V_{|m|-1/2}(-\mathrm{i}\zeta_{m,\alpha}) = 0 \tag{50}$$

in $(-1/2, 0]$.

In the case $m = 0$ the functions $V_{-1/2}(\cdot \pm \mathrm{i}/2)$ are not bounded on $\mathbb{R}$, which makes the argument of [17] not directly applicable (as both factors in (49) are unbounded). Instead of providing an exact description of $\mathfrak{D}(H_0^\alpha)$ we prove a simpler result.

Lemma 23. *For* $\alpha \in (0, \alpha_0]$ *the domain of the Friedrichs extension* H_0^α *of* $\widetilde{H}_0^\alpha$ *satisfies*

$$\mathfrak{D}(H_0^\alpha) \supseteq \mathfrak{D}(\widetilde{H}_0^\alpha) \dotplus \operatorname{span}\{\varphi_0^\alpha\}$$

with

$$\varphi_0^\alpha(s) := \frac{s - \mathrm{i}}{(s - 2\mathrm{i})(s - \mathrm{i}/2 + \mathrm{i}\zeta_{0,\alpha})}$$

and $\zeta_{0,\alpha}$ *defined as in* (50). *Moreover,*

$$(H_0^\alpha \varphi_0^\alpha)(s) = \frac{s(1 - \alpha V_{-1/2}(s + \mathrm{i}/2))}{(s - \mathrm{i})(s + \mathrm{i}/2 + \mathrm{i}\zeta_{0,\alpha})} \tag{51}$$

holds for all $s \in \mathbb{R} \setminus \{0\}$.

Proof. According to Theorem 5.38 in [34], H_0^α is the restriction of $(\widetilde{H}_0^\alpha)^*$ to $\mathfrak{D}(H_0^\alpha) := \mathfrak{Q}_0^\alpha \cap \mathfrak{D}((\widetilde{H}_0^\alpha)^*)$, where $\mathfrak{Q}_0^\alpha$ is the closure of $\mathfrak{D}(\widetilde{H}_0^\alpha)$ in the norm of the quadratic form of $\widetilde{H}_0^\alpha + 1$.

Since $\mathsf{C}_0^\infty(\mathbb{R}^2 \setminus \{0\}) \subset \mathfrak{D}(\widetilde{H}^\alpha)$ is dense in $\mathsf{H}^{1/2}(\mathbb{R}^2)$, the representation (44) shows that $\mathfrak{D}(\widetilde{H}_0^\alpha)$ is dense in $\mathfrak{D}^{1/2}$ with respect to the graph norm of $R^{1/2}$ for all $\alpha \in (0, \alpha_0]$. Lemma 21 implies the inequalities

$$\langle \varphi, R^1 \varphi \rangle \geqslant \langle \varphi, \widetilde{H}_0^\alpha \varphi \rangle \geqslant (1 - \alpha/\alpha_0) \langle \varphi, R^1 \varphi \rangle$$

for all $\alpha \in (0, \alpha_0]$ and $\varphi \in \mathfrak{D}(\widetilde{H}_0^\alpha)$. Thus $\mathfrak{Q}_0^\alpha = \mathfrak{D}^{1/2} \subset \mathfrak{Q}_0^{\alpha_0}$ for $\alpha \in (0, \alpha_0)$ and the right hand side of (47) coincides with the closure of the quadratic form of $\widetilde{H}_0^\alpha$ on every $\varphi \in \mathfrak{D}^{1/2}$ for $\alpha \in (0, \alpha_0]$.

For $n \in \mathbb{N}$ let $\psi_n(s) := (s - \mathrm{i})(s - 2\mathrm{i})^{-1}(s - \mathrm{i}/2 - \mathrm{i}/n)^{-1} \in \mathfrak{D}^{1/2} \subset \mathfrak{Q}_0^{\alpha_0}$. Computing the right hand side of (47) on $\varphi := \psi_n - \psi_m$ with $m \leqslant n$ we obtain

$$\int_{-\infty}^{+\infty} (1 - \alpha_0 V_{-1/2}(s)) |R^{1/2}(\psi_n - \psi_m)(s)|^2 \mathrm{d}s \leqslant \int_{-\infty}^{+\infty} \frac{(1 - \alpha_0 V_{-1/2}(s))}{s^2(m^2 s^2 + 1)} \mathrm{d}s.$$

By Lemma 12 and monotone convergence we conclude that $(\psi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathfrak{Q}_0^{\alpha_0}$ which converges to $\varphi_0^{\alpha_0}$ in $\mathsf{L}^2(\mathbb{R})$. Thus $\varphi_0^{\alpha_0}$ belongs to $\mathfrak{Q}_0^{\alpha_0}$.

For every $\varphi \in \mathfrak{D}(\widetilde{H}_0^{\alpha})$ with $\alpha \in (0, \alpha_0]$ taking into account the relations

$$\varphi_0^{\alpha} \in \mathfrak{D}^{1/4}, \ \Xi_0^{-1}\varphi_0^{\alpha} \in \mathfrak{D}^{1/4} \text{ and } (1-\alpha V_{-1/2}(\cdot - \mathrm{i}/4))\varphi_0^{\alpha}(\cdot + \mathrm{i}/4) \in \mathfrak{D}^{3/4}$$

(recall (50) and (21)) and using Corollary 11 and Lemma 10 we obtain

$$\begin{aligned}
\langle \varphi_0^{\alpha}, \widetilde{H}_0^{\alpha}\varphi\rangle &= \langle \varphi_0^{\alpha}, (R^1 - \alpha\,\Xi_0 R^1 \Xi_0^{-1})\varphi\rangle \\
&= \langle R^{1/4}\varphi_0^{\alpha}, R^{3/4}\varphi\rangle - \alpha\langle R^{1/4}\Xi_0^{-1}\varphi_0^{\alpha}, R^{3/4}\Xi_0^{-1}\varphi\rangle \\
&= \langle (1-\alpha V_{-1/2}(\cdot - \mathrm{i}/4))\varphi_0^{\alpha}(\cdot + \mathrm{i}/4), R^{3/4}\varphi\rangle \\
&= \langle (1-\alpha V_{-1/2}(\cdot + \mathrm{i}/2))\varphi_0^{\alpha}(\cdot + \mathrm{i}), \varphi\rangle.
\end{aligned}$$

It follows that $\varphi_0^{\alpha} \in \mathfrak{D}((\widetilde{H}_0^{\alpha})^*)$ and (51) holds for all $\alpha \in (0, \alpha_0]$. □

We now make a crucial observation concerning the functions (31) transformed in Lemma 20.

Lemma 24. *Let* $\nu \in (0, 1/2]$. *The functions* (41), (42), *and* (43) *satisfy*:

(1) $\xi_{\pm}^{\nu}$ *and* $\eta_{\pm}^{\nu}$ *belong to* $\mathfrak{D}^1$;

(2) $\Xi_0^{-1}\xi_{+}^{\nu}$ *and* $\Xi_0^{-1}\eta_{-}^{\nu}$ *belong to* $\mathfrak{D}^1$;

(3) $\chi_{\pm}^{\nu}$ *belong to* $\mathfrak{D}(H_0^{(V_{-1/2}(\mathrm{i}\beta))^{-1}})$ *and*

$$H_0^{(V_{-1/2}(\mathrm{i}\beta))^{-1}}\chi_{\pm}^{\nu} = (1-(V_{-1/2}(\mathrm{i}\beta))^{-1}V_{-1/2}(\cdot\, + \mathrm{i}/2))\chi_{\pm}^{\nu}(\cdot\, + \mathrm{i}); \tag{52}$$

(4) $\chi_{\pm}^{\nu}$ *belong to* $\mathfrak{D}(H_1^{(V_{1/2}(\mathrm{i}\beta))^{-1}})$.

Proof. (1) By Remark 9 and since the gamma function is analytic in $\mathbb{C}\setminus(-\mathbb{N}_0)$ with a simple pole at zero, $\xi_{\pm}^{\nu}$ and $\eta_{\pm}^{\nu}$ are analytic in a complex neighbourhood of the strip $\mathfrak{S}^1$. Thus, for every $\rho > 0$, $\xi_{\pm}^{\nu}$ and $\eta_{\pm}^{\nu}$ are bounded on $\mathfrak{A}_\rho := \{z \in \mathbb{C}\colon \operatorname{Re} z \in [-\rho, \rho], \operatorname{Im} z \in [0, 1]\}$. On $\mathfrak{S}^1 \setminus \mathfrak{A}_\rho$ substituting the asymptotics (20) into (42), (43), and (15) (or using (17) and (21)) and choosing ρ big enough we obtain the properties (1)–(3) of Definition 6.

(2) Both $\Xi_0^{-1}\xi_{+}^{\nu}$ and $\Xi_0^{-1}\eta_{-}^{\nu}$ are analytic in a complex neighbourhood of $\mathfrak{S}^1$. We can thus repeat the proof of (1) taking (21) into account.

(3) By Lemma 23, it suffices to show that

$$\chi^\nu_\pm + \mathrm{i}\nu\frac{2\beta-3}{2\beta-1}\Xi_{\pm 1/2-1/2}(\mathrm{i}(\beta+1/2))\varphi_0^{(V_{-1/2}(\mathrm{i}\beta))^{-1}} \in \mathfrak{D}(\widetilde{H}_0^{(V_{-1/2}(\mathrm{i}\beta))^{-1}}), \quad (53)$$

see (45). This follows analogously to (1), since $\zeta_{0,(V_{-1/2}(\mathrm{i}\beta))^{-1}} := -\beta$ is the solution of (50). Formula (52) follows from (53), (44), and (51).

(4) The proof is analogous to (3). Since $\zeta_{1,(V_{1/2}(\mathrm{i}\beta))^{-1}} := -\beta$ is the solution of (50) we conclude that

$$\chi^\nu_\pm + \mathrm{i}\nu\Xi_{\pm 1/2-1/2}(\mathrm{i}(\beta+1/2))(\cdot - \mathrm{i}/2 + \mathrm{i}\zeta_{1,(V_{1/2}(\mathrm{i}\beta))^{-1}})^{-1}$$

belongs to $\mathfrak{D}(\widetilde{H}_1^{(V_{1/2}(\mathrm{i}\beta))^{-1}})$ characterised in Lemma 22. □

4. Critical channels estimate

For $\nu \in (0, 1/2]$ we introduce the (2×2)-matrix-valued function on $\mathbb{R}$:

$$M^\nu_\pm(s) := \begin{pmatrix} -\nu V_{\mp 1/2}(s+\mathrm{i}/2) & 1 \\ 1 & -\nu V_{\pm 1/2}(s+\mathrm{i}/2) \end{pmatrix}.$$

Lemma 25. *For any $\Psi \in \mathfrak{C}^\nu_{\pm 1/2}$ there exists a decomposition*

$$\mathfrak{U}_{\pm 1/2}\Psi = \begin{pmatrix} \zeta \\ \upsilon \end{pmatrix} + a\chi^\nu_\pm \begin{pmatrix} 1 \\ \nu V_{\mp 1/2}(\mathrm{i}\beta) \end{pmatrix} \quad (54)$$

with $\zeta \in \mathfrak{D}(\widetilde{H}_{1/2\mp 1/2}^{(V_{\mp 1/2}(\mathrm{i}\beta))^{-1}})$, $\upsilon \in \mathfrak{D}(\widetilde{H}_{1/2\pm 1/2}^{(V_{\pm 1/2}(\mathrm{i}\beta))^{-1}})$ and $a \in \mathbb{C}$. Moreover, the representation

$$\mathfrak{U}_{\pm 1/2} D^\nu_{\pm 1/2}\Psi = M^\nu_\pm \begin{pmatrix} R^1\zeta + a\chi^\nu_\pm(\cdot + \mathrm{i}) \\ R^1\upsilon + a\nu V_{\mp 1/2}(\mathrm{i}\beta)\chi^\nu_\pm(\cdot+\mathrm{i}) \end{pmatrix}$$

holds.

Proof. The decomposition (54) follows from definition (30), Lemma 20, definition (45) and Lemma 24. For any $(\varpi, \varsigma) \in \mathsf{C}_0^\infty(\mathbb{R}_+, \mathbb{C}^2)$ using (33) and (40), Lemmata 18 and 19, Corollary 11, (52), and (49) we obtain

$$
\begin{aligned}
&\left\langle D^{\nu}_{\pm 1/2}\Psi, \begin{pmatrix}\varpi\\ \varsigma\end{pmatrix}\right\rangle\\
&\quad= \left\langle \Psi, D^{\nu}_{\pm 1/2}\begin{pmatrix}\varpi\\ \varsigma\end{pmatrix}\right\rangle\\
&\quad= \left\langle \Psi, \left(\mathcal{A}(\mathcal{S}\mathcal{T})^*\mathcal{S}\mathcal{T}D^{\nu}(\mathcal{S}\mathcal{T})^*\mathcal{S}\mathcal{T}\mathcal{A}^* \bigoplus_{\varkappa\in\mathbb{Z}+1/2} \delta_{\varkappa,\pm 1/2}\begin{pmatrix}\varpi\\ \varsigma\end{pmatrix}\right)_{\varkappa=\pm 1/2}\right\rangle\\
&\quad= \left\langle \left(\begin{pmatrix}\zeta\\ \upsilon\end{pmatrix} + a\chi^{\nu}_{\pm}\begin{pmatrix}1\\ \nu V_{\mp 1/2}(\mathrm{i}\beta)\end{pmatrix}\right),\right.\\
&\qquad\qquad \left.\begin{pmatrix}-\nu\Xi_{1/2\mp 1/2}R^1\Xi^{-1}_{1/2\mp 1/2} & R^1\\ R^1 & -\nu\Xi_{1/2\pm 1/2}R^1\Xi^{-1}_{1/2\pm 1/2}\end{pmatrix}\mathfrak{U}_{\pm 1/2}\begin{pmatrix}\varpi\\ \varsigma\end{pmatrix}\right\rangle\\
&\quad= \left\langle M^{\nu}_{\pm}R^1\begin{pmatrix}\zeta\\ \upsilon\end{pmatrix}, \mathfrak{U}_{\pm 1/2}\begin{pmatrix}\varpi\\ \varsigma\end{pmatrix}\right\rangle\\
&\qquad + a\left\langle \chi^{\nu}_{\pm}\begin{pmatrix}\nu V_{\mp 1/2}(\mathrm{i}\beta)\\ 1\end{pmatrix}, \begin{pmatrix}H^{(V_{\mp 1/2}(\mathrm{i}\beta))^{-1}}_{1/2\mp 1/2} & 0\\ 0 & H^{(V_{\pm 1/2}(\mathrm{i}\beta))^{-1}}_{1/2\pm 1/2}\end{pmatrix}\mathfrak{U}_{\pm 1/2}\begin{pmatrix}\varpi\\ \varsigma\end{pmatrix}\right\rangle\\
&\quad= \left\langle \mathfrak{U}^*_{\pm 1/2}M^{\nu}_{\pm}\begin{pmatrix}R^1\zeta + a\chi^{\nu}_{\pm}(\cdot + \mathrm{i})\\ R^1\upsilon + a\nu V_{\mp 1/2}(\mathrm{i}\beta)\chi^{\nu}_{\pm}(\cdot + \mathrm{i})\end{pmatrix}, \begin{pmatrix}\varpi\\ \varsigma\end{pmatrix}\right\rangle.
\end{aligned}
$$

By density of $\mathsf{C}_0^\infty(\mathbb{R}_+, \mathbb{C}^2)$ the claim follows. ∎

Lemma 26. *For $\nu \in (0, 1/2]$ define the functions*

$$K^{\nu}_{\pm}(s) := |1 - (V_{\pm 1/2}(\mathrm{i}\beta))^{-1}V_{\pm 1/2}(s + \mathrm{i}/2)|^2$$

on $\mathbb{R}\setminus\{0\}$. Then there exists a constant $\eta_\nu > 0$ such that the lower bound

$$(M^{\nu}_{\pm})^*M^{\nu}_{\pm} \geqslant \eta_\nu \operatorname{diag}(K^{\nu}_{\mp}, K^{\nu}_{\pm}) \tag{55}$$

holds point-wise on $\mathbb{R}\setminus\{0\}$.

Proof. It is enough to establish (55) for M^{ν}_{+} and then use the relation $M^{\nu}_{-} = \sigma_1 M^{\nu}_{+}\sigma_1$. We introduce the shorthand $V := V_{1/2}(\mathrm{i}\beta) = \nu^{-2}(V_{-1/2}(\mathrm{i}\beta))^{-1}$ (see (25) for the second equality). For any $s \in \mathbb{R}\setminus\{0\}$, estimating

$$K^{\nu}_{\pm}(s) \leqslant 2(1 + (V_{\pm 1/2}(\mathrm{i}\beta))^{-2}|V_{\pm 1/2}(s + \mathrm{i}/2)|^2)$$

and using (24) we obtain

$$K_+^\nu(s) \leqslant 2(1 + (1 + s^2)^{-1} V^{-2})$$

and

$$K_-^\nu(s) \leqslant 2(1 + \nu^4 V^2 s^{-2}).$$

Analogously we get

$$(M_+^\nu(s))^* M_+^\nu(s) = \begin{pmatrix} 1 + \nu^2 s^{-2} & -\dfrac{\nu(1-2\mathrm{i}s)}{s^2 + \mathrm{i}s} P(s) \\ -\dfrac{\nu(1+2\mathrm{i}s)}{s^2 - \mathrm{i}s} \overline{P}(s) & 1 + \nu^2 (1 + s^2)^{-1} \end{pmatrix}$$

with

$$P(s) := \frac{\Gamma((1 + \mathrm{i}s)/2)\Gamma(-\mathrm{i}s/2)}{\Gamma((1 - \mathrm{i}s)/2)\Gamma(\mathrm{i}s/2)}, \quad |P(s)| = 1.$$

Thus for any $\eta > 0$ the inequality

$$\det((M_+^\nu(s))^* M_+^\nu(s) - \frac{\eta}{2} \operatorname{diag}(K_-^\nu(s), K_+^\nu(s))) \geqslant \frac{\mathcal{A}s^4 + \mathcal{B}s^2 + \mathcal{C}}{s^2(1 + s^2)V^2} \tag{56}$$

holds with

$$\mathcal{A} := V^2(1 - \eta)^2,$$

$$\mathcal{B} := V^2(1 - 2\nu^2) - (1 + 2V^2 + 2\nu^2 V^2 + \nu^4 V^4)\eta + (1 + V^2 + \nu^4 V^4)\eta^2,$$

$$\mathcal{C} := \nu^4 V^2 - \nu^2(1 + V^2 + \nu^2 V^4 + \nu^4 V^4)\eta + \nu^4 V^2 (1 + V^2)\eta^2.$$

There exists $\eta_\nu > 0$ such that for any $\eta \in [0, 2\eta_\nu]$ the coefficients $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are strictly positive, hence also the right hand side of (56). Since for $\eta = 0$ both eigenvalues of $(M_+^\nu(s))^* M_+^\nu(s)$ are positive, both eigenvalues of

$$(M_+^\nu(s))^* M_+^\nu(s) - \eta \operatorname{diag}(K_-^\nu(s), K_+^\nu(s))$$

are non-negative for all $s \in \mathbb{R} \setminus \{0\}$ provided $\eta \in [0, \eta_\nu]$. □

Remark 27. *It is easy to see that*

$$\eta_\nu = \inf_{s \in \mathbb{R}\setminus\{0\}} \eta_-^\nu(s), \tag{57}$$

where $\eta_-^\nu(s)$ is the smallest of the two solutions η of

$$\det((M_+^\nu(s))^* M_+^\nu(s) - \eta \operatorname{diag}(K_-^\nu(s), K_+^\nu(s))) = 0.$$

Numerical analysis indicates that the infimum in (57) is achieved for $s = +0$ and is thus equal to

$$\frac{1}{2}\Bigg(\frac{\nu^2+1}{(1-V_{1/2}(\mathrm{i}\beta)^{-1})^2} + \nu^2 V_{-1/2}(\mathrm{i}\beta)^2 - \sqrt{\Big(\frac{\nu^2+1}{(1-V_{1/2}(\mathrm{i}\beta)^{-1})^2} + \nu^2 V_{-1/2}(\mathrm{i}\beta)^2\Big)^2 - \frac{4\nu^4 V_{-1/2}(\mathrm{i}\beta)^2}{(1-V_{1/2}(\mathrm{i}\beta)^{-1})^2}}\Bigg).$$

The final result of this section is

Lemma 28. *The inequality*

$$(D^\nu_{\pm 1/2})^2 \geqslant \eta_\nu (\mathfrak{U}^*_{\pm 1/2} \operatorname{diag}(H^{(V_{\mp 1/2}(\mathrm{i}\beta))^{-1}}_{1/2 \mp 1/2}, H^{(V_{\pm 1/2}(\mathrm{i}\beta))^{-1}}_{1/2 \pm 1/2}) \mathfrak{U}_{\pm 1/2})^2 \tag{58}$$

holds for any $\nu \in (0, 1/2]$ with η_ν defined in Lemma 26.

Proof. For arbitrary $\Psi \in \mathfrak{C}^\nu_{\pm 1/2}$ we use (54) to represent $\mathfrak{U}_{\pm 1/2}\Psi$. Applying Lemmata 25, 26, 23 and 22 together with equation (52) we get

$$\begin{aligned}
\|D^\nu_{\pm 1/2}\Psi\|^2 &= \left\| M^\nu_\pm \begin{pmatrix} R^1\zeta + a\chi^\nu_\pm(\cdot + \mathrm{i}) \\ R^1\upsilon + a\nu V_{\mp 1/2}(\mathrm{i}\beta)\chi^\nu_\pm(\cdot + \mathrm{i}) \end{pmatrix} \right\|^2 \\
&\geqslant \eta_\nu \left\| \begin{pmatrix} \Big(1 - \dfrac{V_{\mp 1/2}(\cdot + \mathrm{i}/2)}{V_{\mp 1/2}(\mathrm{i}\beta)}\Big)(R^1\zeta + a\chi^\nu_\pm(\cdot + \mathrm{i})) \\ \Big(1 - \dfrac{V_{\pm 1/2}(\cdot + \mathrm{i}/2)}{V_{\pm 1/2}(\mathrm{i}\beta)}\Big)(R^1\upsilon + a\nu V_{\mp 1/2}(\mathrm{i}\beta)\chi^\nu_\pm(\cdot + \mathrm{i})) \end{pmatrix} \right\|^2 \\
&= \eta_\nu \|\mathfrak{U}^*_{\pm 1/2} \operatorname{diag}(H^{(V_{\mp 1/2}(\mathrm{i}\beta))^{-1}}_{1/2 \mp 1/2}, H^{(V_{\pm 1/2}(\mathrm{i}\beta))^{-1}}_{1/2 \pm 1/2}) \mathfrak{U}_{\pm 1/2}\Psi\|^2.
\end{aligned}$$

Since $\mathfrak{C}^\nu_{\pm 1/2}$ is an operator core for $D^\nu_{\pm 1/2}$, we conclude (58). □

5. Non-critical channels estimate

Lemma 29. *For $\nu \in (0, 1/2]$ the operator inequalities*

$$(D^\nu_\varkappa)^2 \geqslant (1 - \nu(3(16 + \nu^2)^{1/2} - 5\nu)/8)(\mathfrak{U}^*_\varkappa R^1 \mathfrak{U}_\varkappa)^2 \tag{59}$$

hold true for all $\varkappa \in (\mathbb{Z} + 1/2) \setminus \{-1/2, 1/2\}$.

Proof. As in Lemma 28, it is enough to prove (59) on the functions from the operator core $\mathfrak{C}^\nu_\varkappa$ which, according to (30), coincides with $\mathsf{C}^\infty_0(\mathbb{R}_+, \mathbb{C}^2)$.

With the help of Lemma 18, (40) and (33) we get for every $\varphi \in \mathsf{C}^\infty_0(\mathbb{R}_+, \mathbb{C}^2)$

$$\begin{aligned}
\|\mathcal{U}^*_\varkappa R^1 \mathcal{U}_\varkappa \varphi\|^2 &= \Big\| \bigoplus_{\tilde\varkappa \in \mathbb{Z}+1/2} \delta_{\tilde\varkappa,\varkappa} R^1 \mathcal{U}_\varkappa \varphi \Big\|^2 \\
&= \Big\| \mathcal{S}\mathcal{T}(-\Delta)^{1/2}(\mathcal{S}\mathcal{T})^* \bigoplus_{\tilde\varkappa \in \mathbb{Z}+1/2} \delta_{\tilde\varkappa,\varkappa}\, \mathcal{U}_\varkappa \varphi \Big\|^2 \\
&= \Big\| \mathcal{A}(-\mathrm{i}\boldsymbol{\sigma}\cdot\nabla)\mathcal{A}^*(\mathcal{S}\mathcal{T}\mathcal{A}^*)^* \bigoplus_{\tilde\varkappa \in \mathbb{Z}+1/2} \delta_{\tilde\varkappa,\varkappa}\, \mathcal{U}_\varkappa \varphi \Big\|^2 \\
&= \|D^0_\varkappa \varphi\|^2.
\end{aligned}$$

It is thus enough to prove (59) with $D^0_\varkappa$ instead of $\mathcal{U}^*_\varkappa R^1 \mathcal{U}_\varkappa$.

For $b \in \mathbb{R}$ we introduce a family of matrix-functions

$$A^\nu_\varkappa(b,s) := \begin{pmatrix} \nu^2 + b(s^2 + (1/2 - \varkappa)^2) & 2\nu(\mathrm{i}s + \varkappa) \\ 2\nu(-\mathrm{i}s + \varkappa) & \nu^2 + b(s^2 + (\varkappa + 1/2)^2) \end{pmatrix}, \quad s \in \mathbb{R}.$$

A straightforward calculation using Lemma 15, (29) and (13) delivers

$$\|D^\nu_\varkappa \varphi\|^2 = \|\mathcal{M} D^\nu_\varkappa \varphi\|^2 = \int_{-\infty}^{\infty} \langle (R^{-1}\mathcal{M}\varphi)(s), A^\nu_\varkappa(1,s)(R^{-1}\mathcal{M}\varphi)(s)\rangle \, \mathrm{d}s.$$

Thus

$$\|D^\nu_\varkappa \varphi\|^2 - (1-b)\|D^0_\varkappa \varphi\|^2 = \int_{-\infty}^{\infty} \langle (R^{-1}\mathcal{M}\varphi)(s), A^\nu_\varkappa(b,s)(R^{-1}\mathcal{M}\varphi)(s)\rangle \, \mathrm{d}s \quad (60)$$

holds. The eigenvalues of $A^\nu_\varkappa(b,s)$ are given by

$$a^\nu_{\varkappa,\pm}(b,s) := \nu^2 + b/4 + \varkappa^2 b + s^2 b \pm (4\varkappa^2\nu^2 + 4\nu^2 s^2 + \varkappa^2 b^2)^{1/2}. \quad (61)$$

Note that $a^\nu_{\varkappa,\pm} = a^\nu_{-\varkappa,\pm}$.

We now seek $b < 1$ such that the inequality $a^\nu_{\varkappa,-}(b,s) \geq 0$ holds for all $\varkappa \in \mathbb{N}_1 + 1/2$ and $s \in \mathbb{R}$. We claim that, for all other parameters being fixed, $a^\nu_{\varkappa,-}(b,s)$ is an increasing function of $\varkappa \in \mathbb{N}_1 + 1/2$ provided $b \geqslant 2\nu/\sqrt{15}$ holds. Indeed, extending (61) to $\varkappa \in \mathbb{R}$, we get

$$\begin{aligned}
a^\nu_{\varkappa+1,-}(b,s) - a^\nu_{\varkappa,-}(b,s) &= \int_\varkappa^{\varkappa+1} \frac{\partial a^\nu_{\tilde\varkappa,-}}{\partial\tilde\varkappa}(b,s)\,\mathrm{d}\tilde\varkappa \\
&= (2\varkappa+1)b - \int_\varkappa^{\varkappa+1} \frac{(4\nu^2+b^2)\tilde\varkappa}{\sqrt{(4\nu^2+b^2)\tilde\varkappa^2 + 4\nu^2 s^2}}\,\mathrm{d}\tilde\varkappa \\
&\geqslant 4b - \sqrt{4\nu^2+b^2} \\
&\geqslant 0.
\end{aligned}$$

Note that $a^\nu_{3/2,-}(b,s) = a^\nu_{3/2,-}(b,-s)$. For $s > 0$ and

$$b \geqslant \nu\sqrt{2(\sqrt{13}/3 - 1)} \qquad (> 2\nu/\sqrt{15})$$

we have

$$\begin{aligned}\frac{\partial a^\nu_{3/2,-}(b,s)}{\partial s} &= 2bs - \frac{4\nu^2 s}{\sqrt{9\nu^2 + 4\nu^2 s^2 + 9b^2/4}} \\ &\geqslant 2s(b - 2\nu^2/\sqrt{9\nu^2 + 9b^2/4}) \\ &\geqslant 0.\end{aligned}$$

Thus, provided

$$b \geqslant \nu(3\sqrt{16+\nu^2} - 5\nu)/8 \qquad (> \nu\sqrt{2(\sqrt{13}/3 - 1)} \quad \text{for all } \nu \in (0, 1/2])$$

holds, for any $s \in \mathbb{R}$ and $\varkappa \in \mathbb{N}_1 + 1/2$ we have

$$a^\nu_{\varkappa,-}(b,s) \geqslant a^\nu_{3/2,-}(b,0) = \nu^2 + 5b/2 - 3\sqrt{\nu^2 + b^2/4} \geqslant 0.$$

It follows now from (60) that

$$(D^\nu_\varkappa)^2 \geqslant (1 - \nu(3(16+\nu^2)^{1/2} - 5\nu)/8)(D^0_\varkappa)^2$$

(and hence (59)) holds for all $\nu \in (0, 1/2]$ and $\varkappa \in (\mathbb{Z} + 1/2) \setminus \{-1/2, 1/2\}$. □

6. Critical lower bounds

In this section we prove lower bounds analogous to the critical hydrogen inequality introduced in Theorem 2.3 of [28] and further developed in [11].

For $\gamma \in \mathbb{R}$ we introduce the quadratic form

$$\mathfrak{p}^\gamma[f] := \int_{\mathbb{R}_+} p^\gamma |f(p)|^2 \, \mathrm{d}p$$

on $\mathsf{L}^2(\mathbb{R}_+, (1 + p^\gamma)\mathrm{d}p)$.

The next theorem will imply a lower bound for the quadratic form of the critical operator $H^{\alpha_m}_m$. Recall the definition (46) of α_m and Lemma 13.

Theorem 30. *For any $m \in \mathbb{Z}$ and $\lambda \in (0,1)$ there exists $K_{m,\lambda} > 0$ such that for all $l > 0$ the inequality*

$$\mathfrak{p}^1 - \alpha_m \mathfrak{q}_m \geqslant K_{m,\lambda} l^{\lambda-1} \mathfrak{p}^\lambda - l^{-1}\mathfrak{p}^0 \tag{62}$$

holds on $\mathsf{L}^2(\mathbb{R}_+, (1+p)\mathrm{d}p)$.

Proof. Let $m \in \mathbb{Z}$, $\lambda \in (3/4, 1)$ and $f \in \mathsf{L}^2(\mathbb{R}_+, (1+p)\mathrm{d}p)$. Using the non-negativity of $Q_{|m|-1/2}$, the Cauchy–Bunyakovsky–Schwarz inequality and that

$$(q + l^{\lambda-1}q^{\lambda})^{-1} \leqslant q^{-1} - l^{\lambda-1}q^{\lambda-2} + l^{2(\lambda-1)}q^{2\lambda-3} \quad \text{holds for all } q, l > 0$$

(which follows from $(1+z)^{-1} \leqslant 1 - z + z^2$ for all $z \geqslant 0$ by letting $z := (lq)^{\lambda-1}$) we obtain

$$\begin{aligned}\mathfrak{q}_m[f] &= \frac{1}{\pi}\int_0^\infty\int_0^\infty \overline{f(p)} f(q) Q_{|m|-1/2}\Big(\frac{1}{2}\Big(\frac{p}{q}+\frac{q}{p}\Big)\Big)\mathrm{d}q\,\mathrm{d}p \\ &\leqslant \frac{1}{\pi}\int_0^\infty\int_0^\infty |f(p)|^2 Q_{|m|-1/2}\Big(\frac{1}{2}\Big(\frac{p}{q}+\frac{q}{p}\Big)\Big)\Big(\frac{p+l^{\lambda-1}p^{\lambda}}{q+l^{\lambda-1}q^{\lambda}}\Big)\mathrm{d}q\,\mathrm{d}p \\ &\leqslant \frac{1}{\pi}\int_0^\infty\int_0^\infty |f(p)|^2 Q_{|m|-1/2}\Big(\frac{1}{2}\Big(\frac{p}{q}+\frac{q}{p}\Big)\Big) \\ &\qquad\qquad (p+l^{\lambda-1}p^{\lambda})(q^{-1} - l^{\lambda-1}q^{\lambda-2} + l^{2(\lambda-1)}q^{2\lambda-3})\mathrm{d}q\,\mathrm{d}p.\end{aligned} \tag{63}$$

From (27) it is easy to find the asymptotics

$$Q_{|m|-1/2}((x+x^{-1})/2) \sim \frac{\pi^{1/2}\Gamma(|m|+1/2)}{\Gamma(|m|+1)}\begin{cases} x^{-|m|-1/2}, & \text{for } x \to +\infty; \\ x^{|m|+1/2}, & \text{for } x \to +0, \end{cases}$$

which implies that the function

$$\begin{aligned}V_{|m|-1/2}(z) &:= \frac{1}{\pi}\int_0^\infty Q_{|m|-1/2}((x+x^{-1})/2)x^{-\mathrm{i}z-1}\mathrm{d}x \\ &= \frac{p^{\mathrm{i}z}}{\pi}\int_0^\infty Q_{|m|-1/2}\Big(\frac{1}{2}\Big(\frac{p}{q}+\frac{q}{p}\Big)\Big)q^{-\mathrm{i}z-1}\mathrm{d}q\end{aligned}$$

is well defined and analytic in the strip $\{z \in \mathbb{C} \colon \operatorname{Im} z \in (-|m|-1/2, |m|+1/2)\}$. It also coincides there with the function defined in (24), as can be seen by comparing Lemma 13 with Lemmata 19, 10 and Corollary 11 or by a calculation as in Section VI of [17].

We can then rewrite the right hand side of (63) obtaining

$$\begin{aligned}\mathfrak{q}_m[f] = {} & V_{|m|-1/2}(0)\mathfrak{p}^1[f] + (V_{|m|-1/2}(0) - V_{|m|-1/2}(\mathrm{i}(\lambda-1)))l^{\lambda-1}\mathfrak{p}^{\lambda}[f] \\ & + (V_{|m|-1/2}(2\mathrm{i}(\lambda-1)) - V_{|m|-1/2}(\mathrm{i}(\lambda-1)))l^{2(\lambda-1)}\mathfrak{p}^{2\lambda-1}[f] \\ & + V_{|m|-1/2}(2\mathrm{i}(\lambda-1))l^{3(\lambda-1)}\mathfrak{p}^{3\lambda-2}[f].\end{aligned} \tag{64}$$

Lemmata 21 and 12 imply

$$V_{|m|-1/2}(0) = \alpha_m^{-1}; \tag{65a}$$

$$V_{|m|-1/2}(0) - V_{|m|-1/2}(\mathrm{i}(\lambda-1)) < 0; \tag{65b}$$

$$V_{|m|-1/2}(2\mathrm{i}(\lambda-1)) - V_{|m|-1/2}(\mathrm{i}(\lambda-1)) \geqslant 0; \tag{65c}$$

$$V_{|m|-1/2}(2\mathrm{i}(\lambda-1)) \geqslant 0. \tag{65d}$$

For every $\lambda \in (3/4, 1)$ and $\varepsilon_1, \varepsilon_2 > 0$ there exist $C_1, C_2 > 0$ such that the inequalities

$$l^{2(\lambda-1)} p^{2\lambda-1} \leqslant \varepsilon_1 p^\lambda l^{\lambda-1} + C_1 l^{-1}, \quad l^{3(\lambda-1)} p^{3\lambda-2} \leqslant \varepsilon_2 p^\lambda l^{\lambda-1} + C_2 l^{-1} \tag{66}$$

hold for all $p, l > 0$. Substituting (66) into (64) and taking (65) into account by choosing ε_1 and $\varepsilon_2 > 0$ small enough we obtain

$$\mathfrak{q}_m[f] \leqslant \mathfrak{p}^1[f]/\alpha_m - C_1(m,\lambda) l^{\lambda-1} \mathfrak{p}^\lambda[f] + C_2(m,\lambda) l^{-1} \mathfrak{p}^0[f] \tag{67}$$

with $C_1(m,\lambda)$, $C_2(m,\lambda) > 0$ for $\lambda \in (3/4, 1)$. For $\lambda \in (0, 3/4]$, $\lambda' \in (3/4, 1)$ we can find a constant $C_3(\lambda, \lambda') > 0$ with

$$l^{\lambda'} \mathfrak{p}^{\lambda'} \geqslant -C_3(\lambda, \lambda') \mathfrak{p}^0 + l^\lambda \mathfrak{p}^\lambda$$

and get (67) for λ from (67) for λ'. Rescaling l and using $\alpha_m > 0$ we arrive at (62). □

Corollary 31. *For $m \in \mathbb{Z}$ and $\lambda \in (0, 1)$ the inequality*

$$H_m^{\alpha_m} \geqslant K_{m,\lambda} l^{\lambda-1} R^\lambda - l^{-1} \tag{68}$$

holds for all $l > 0$ with $K_{m,\lambda}$ as in (62).

Proof. For any $\varphi \in \mathfrak{D}(\widetilde{H}_m^{\alpha_m})$ we have

$$\langle \varphi, \widetilde{H}_m^{\alpha_m} \varphi \rangle = \langle \varphi, R^1 \varphi \rangle - \alpha_m \langle \varphi, V_{|m|-1/2}(\cdot + \mathrm{i}/2) R^1 \varphi \rangle. \tag{69}$$

By (14), the first term on the right hand side of (69) coincides with $\mathfrak{p}^1[\mathcal{M}^* \varphi]$. Letting

$$\Phi := \mathcal{T}^* \bigoplus_{n \in \mathbb{Z}} \delta_{n,m} \varphi$$

and using Lemma 19, Corollary 11 and Lemma 13 we obtain

$$\langle \varphi, V_{|m|-1/2}(\cdot + \mathrm{i}/2) R^1 \varphi \rangle = \langle \Phi, r^{-1} \Phi \rangle = \mathfrak{q}_m[\mathcal{M}^* \varphi].$$

Thus (69) can be written as

$$\langle \varphi, \widetilde{H}_m^{\alpha_m} \varphi \rangle = \mathfrak{p}^1[\mathcal{M}^* \varphi] - \alpha_m \mathfrak{q}_m[\mathcal{M}^* \varphi]$$

for any $\varphi \in \mathfrak{D}(\widetilde{H}_m^{\alpha_m})$. Using Theorem 30, (14) and that $H_m^{\alpha_m}$ is the Friedrichs extension of $\widetilde{H}_m^{\alpha_m}$ we conclude (68). □

7. Proofs of the main theorems

Proof of Theorem 1. (1) By Lemma 21

$$\langle \varphi, \widetilde{H}_m^\alpha \varphi \rangle \geqslant (1 - \alpha/\alpha_m)\langle \varphi, R^1 \varphi \rangle$$

holds for all $m \in \mathbb{Z}$, $\alpha \in [0, \alpha_m)$ and $\varphi \in \mathfrak{D}(\widetilde{H}_m^\alpha)$. Passing to the Friedrichs extension and using (46) we obtain

$$H_m^\alpha \geqslant (1 - \alpha V_{|m|-1/2}(0)) R^1. \tag{70}$$

By the operator monotonicity of the square root, Lemma 28 implies

$$|D_{\pm 1/2}^\nu| \geqslant \eta_\nu^{1/2} \mathcal{U}_{\pm 1/2}^* \operatorname{diag}(H_{1/2\mp 1/2}^{(V_{\mp 1/2}(\mathrm{i}\beta))^{-1}}, H_{1/2\pm 1/2}^{(V_{\pm 1/2}(\mathrm{i}\beta))^{-1}}) \mathcal{U}_{\pm 1/2}. \tag{71}$$

With (70) we conclude

$$|D_{\pm 1/2}^\nu| \geqslant \eta_\nu^{1/2} \min\left\{1 - \frac{V_{-1/2}(0)}{V_{-1/2}(\mathrm{i}\beta)}, 1 - \frac{V_{1/2}(0)}{V_{1/2}(\mathrm{i}\beta)}\right\} \mathcal{U}_{\pm 1/2}^* R^1 \mathcal{U}_{\pm 1/2}. \tag{72}$$

Lemma 29 implies, in its turn, the estimate

$$|D_\varkappa^\nu| \geqslant (1 - \nu(3(16 + \nu^2)^{1/2} - 5\nu)/8)^{1/2} \mathcal{U}_\varkappa^* R^1 \mathcal{U}_\varkappa \tag{73}$$

for all $\varkappa \in (\mathbb{Z} + 1/2) \setminus \{-1/2, 1/2\}$. Combining it with (72), (33), and Lemma 18 we arrive at

$$|D^\nu| \geqslant C_\nu \mathcal{A}^* \Big(\bigoplus_{\varkappa \in \mathbb{Z}+1/2} \mathcal{U}_\varkappa^* R^1 \mathcal{U}_\varkappa \Big) \mathcal{A} = C_\nu \mathcal{T}^* \Big(\bigoplus_{m \in \mathbb{Z}} R^1 \Big) \mathcal{T} = C_\nu \sqrt{-\Delta} \tag{74}$$

with

$$\begin{aligned} C_\nu := \min \Big\{ & \eta_\nu^{1/2} \Big(1 - \frac{V_{-1/2}(0)}{V_{-1/2}(\mathrm{i}\beta)}\Big), \\ & \eta_\nu^{1/2} \Big(1 - \frac{V_{1/2}(0)}{V_{1/2}(\mathrm{i}\beta)}\Big), \\ & (1 - \nu(3(16 + \nu^2)^{1/2} - 5\nu)/8)^{1/2} \Big\}. \end{aligned}$$

(2) Corollary 31 and (71) imply

$$|D_{\pm 1/2}^\nu| \geqslant \eta_\nu^{1/2} (\min\{K_{0,\lambda}, K_{1,\lambda}\} l^{\lambda - 1} \mathcal{U}_{\pm 1/2}^* R^\lambda \mathcal{U}_{\pm 1/2} - l^{-1}). \tag{75}$$

For $\varkappa \in (\mathbb{Z} + 1/2) \setminus \{-1/2, 1/2\}$ we combine (73) and the simple inequality

$$R^1 \geqslant \lambda^{-\lambda} (1 - \lambda)^{\lambda - 1} l^{\lambda - 1} R^\lambda - l^{-1}$$

which follows from the spectral theorem. This together with (75) implies (3) with

$$K_\lambda := \min\{\eta_{1/2}^{\lambda/2} K_{0,\lambda}, \eta_{1/2}^{\lambda/2} K_{1,\lambda}, \lambda^{-\lambda} (1 - \lambda)^{\lambda - 1} 2^{-5\lambda/2} (37 - 3\sqrt{65})^{\lambda/2}\}$$

by a calculation analogous to (74). □

Proof of Corollary 2. Under the assumptions of Corollary 2 for any $\varepsilon > 0$ there exists a decomposition

$$V = V_\varepsilon + B_\varepsilon \tag{76}$$

with

$$\| \operatorname{tr} V_\varepsilon^{2+\gamma} \|_{\mathsf{L}^1(\mathbb{R}^2)} < \varepsilon^{2+\gamma} \quad \text{and} \quad B_\varepsilon \in \mathsf{L}^\infty(\mathbb{R}^2, \mathbb{C}^{2\times 2}).$$

By Hölder and Sobolev inequalities there exists $C_S > 0$ such that for any $\varphi \in P_+^\nu \mathfrak{D}(|D^\nu|^{1/2})$ we get

$$\left| \int_{\mathbb{R}^2} \langle \varphi(\mathbf{x}), V_\varepsilon(\mathbf{x})\varphi(\mathbf{x})\rangle \mathrm{d}\mathbf{x} \right| \leqslant \varepsilon \|\varphi\|^2_{\mathsf{L}^{\frac{4+2\gamma}{1+\gamma}}(\mathbb{R}^2)} \leqslant \varepsilon C_S \|(-\Delta)^{1/(4+2\gamma)}\varphi\|^2. \tag{77}$$

Now (2) and the estimate $(-\Delta)^{1/(2+\gamma)} \leqslant (-\Delta)^{1/2} + 1$ imply

$$\|(-\Delta)^{1/(4+2\gamma)}\varphi\|^2 \leqslant C_\nu^{-1} \| |D^\nu|^{1/2}\varphi\|^2 + \|\varphi\|^2, \tag{78}$$

for any $\nu \in [0, 1/2)$, $\gamma \geqslant 0$. For $\nu = 1/2$ and $\gamma > 0$ we use (3) with $\lambda := 2/(2+\gamma)$, $l := K_{2/(2+\gamma)}^{(2+\gamma)/\gamma}$ obtaining

$$\|(-\Delta)^{1/(4+2\gamma)}\varphi\|^2 \leqslant \| |D^{1/2}|^{1/2}\varphi\|^2 + K_{2/(2+\gamma)}^{-(2+\gamma)/\gamma} \|\varphi\|^2. \tag{79}$$

Combining (76) and (77) with (78) or (79) we conclude that V is an infinitesimal form perturbation of $\mathfrak{d}^\nu(0,0)$ for all $(\nu,\gamma) \in ([0,1/2] \times [0,\infty)) \setminus \{(1/2,0)\}$. This together with (5) implies that $\mathfrak{d}^\nu(\mathfrak{w}, V)$ is bounded from below by some $-M \in \mathbb{R}$ and that

$$\mathfrak{d}^\nu(\mathfrak{w}, V)[\cdot] + (M+1)\|\cdot\|^2 \quad \text{and} \quad \mathfrak{d}^\nu(0,0)[\cdot] + \|\cdot\|^2$$

are equivalent norms on $P_+^\nu \mathfrak{D}(|D^\nu|^{1/2})$ (see e.g. the proof of Theorem X.17 in [25]). □

Proof of Theorem 3. Using the spectral theorem and (2) we obtain

$$\begin{aligned}
&\operatorname{rank}(D^\nu(\mathfrak{w}, V))_- \\
&\quad = \sup\dim\{\mathfrak{X} \text{ subspace of } P_+^\nu \mathfrak{D}(|D^\nu|^{1/2}): \\
&\qquad\qquad \mathfrak{d}^\nu(\mathfrak{w}, V)[\psi] < 0 \text{ for all } \psi \in \mathfrak{X}\setminus\{0\}\} \\
&\quad \leqslant \sup\dim\Big\{\mathfrak{X} \text{ subspace of } \mathsf{H}^{1/2}(\mathbb{R}^2,\mathbb{C}^2): \text{ for all } \psi \in \mathfrak{X}\setminus\{0\} \\
&\qquad\qquad \|(-\Delta)^{1/4}\psi\|^2 - C_\nu^{-1}\int_{\mathbb{R}^2} \langle \psi(\mathbf{x}), V(\mathbf{x})\psi(\mathbf{x})\rangle \mathrm{d}\mathbf{x} < 0 \text{ holds}\Big\} \\
&\quad = \operatorname{rank}((-\Delta)^{1/2} - C_\nu^{-1} V)_-,
\end{aligned}$$

where the operator on the right hand side is the one considered in Example 3.3 of [12]. The statement now follows from (7) with

$$C_\nu^{\mathrm{CLR}} := 4C_\nu^{-2}/\pi. \qquad \square$$

Proof of Theorem 4. For $\nu < 1/2$, the statement follows from Theorem 3 in the usual way. First, we pass to the integral representation

$$\begin{aligned} \operatorname{tr}(D^\nu(\mathfrak{w},V))_-^\gamma &= \gamma\int_0^\infty \operatorname{rank}(D^\nu(\mathfrak{w},V)+\tau)_-\tau^{\gamma-1}\mathrm{d}\tau \\ &\leqslant \gamma\int_0^\infty \operatorname{rank}(D^\nu(\mathfrak{w},(V-\tau)_+))_-\tau^{\gamma-1}\mathrm{d}\tau. \end{aligned} \tag{80}$$

Now, applying (6), we can estimate the right hand side of (80) by

$$\gamma C_\nu^{\mathrm{CLR}}\int_{\mathbb{R}^2}\int_0^\infty \operatorname{tr}(V(\mathbf{x})-\tau)_+^2\tau^{\gamma-1}\mathrm{d}\tau\,\mathrm{d}\mathbf{x}. \tag{81}$$

For $\mathbf{x}\in\mathbb{R}$ let $v_{1,2}(\mathbf{x})$ be the eigenvalues of $V(\mathbf{x})$. Computing the trace in the eigenbasis of $V(\mathbf{x})$ we obtain for all $\tau \geqslant 0$

$$\operatorname{tr}(V(\mathbf{x})-\tau)_+^2 = \sum_{j=1}^2 (v_j(\mathbf{x})-\tau)_+^2. \tag{82}$$

Substituting (82) into (81) and computing the integrals we derive (8) with

$$C_{\nu,\gamma}^{\mathrm{LT}} = \frac{2C_\nu^{\mathrm{CLR}}}{(\gamma+1)(\gamma+2)}, \qquad \text{for } \nu < 1/2.$$

For $\nu = 1/2$, the inequality (8) follows from (3) by a calculation similar to the one in the proof of Theorem 1.1 in [11]. Namely, proceeding analogously to the proof of Theorem 3, but using (3) instead of (2), we observe the inequalities

$$\operatorname{rank}(D^{1/2}(\mathfrak{w},V)+\tau)_- \leqslant \operatorname{rank}((-\Delta)^{\lambda/2} - K_\lambda^{-1}l^{1-\lambda}(V+(l^{-1}-\tau)))_- \tag{83}$$

for all $\lambda\in(0,1)$, $\tau, l > 0$. We now let $l := (\sigma\tau)^{-1}$ with $\sigma\in(0,1)$ and estimate the right hand side of (83) from above with the help of (7) by

$$(2\pi\lambda)^{-1}(1-\lambda/2)^{1-4/\lambda}K_\lambda^{-2/\lambda}(\sigma\tau)^{2(\lambda-1)/\lambda}\int_{\mathbb{R}^2}\operatorname{tr}(V(\mathbf{x})-(1-\sigma)\tau)_+^{2/\lambda}\,\mathrm{d}\mathbf{x}.$$

Substituting this into (80) and integrating in τ we get for $2/(2+\gamma) < \lambda < 1$

$$\operatorname{tr}(D^\nu(\mathfrak{w},V))_-^\gamma \leqslant C_{1/2,\gamma}^{\mathrm{LT}}(\lambda,\sigma)\int_{\mathbb{R}^2}\operatorname{tr}(V(\mathbf{x}))^{2+\gamma}\,\mathrm{d}\mathbf{x}$$

with

$$C^{\mathrm{LT}}_{1/2,\gamma}(\lambda,\sigma) := \gamma\Big(1-\frac{\lambda}{2}\Big)^{1-\frac{4}{\lambda}}\frac{\Gamma(2+\gamma-\frac{2}{\lambda})\Gamma(1+\frac{2}{\lambda})}{2\pi\lambda K_\lambda^{\frac{2}{\lambda}}\Gamma(3+\gamma)}\sigma^{2-\frac{2}{\lambda}}(1-\sigma)^{-\gamma-2+\frac{2}{\lambda}}.$$

The estimate (8) follows with

$$C^{\mathrm{LT}}_{1/2,\gamma} := \min_{\substack{\lambda\in(2/(2+\gamma),1)\\ \sigma\in(0,1)}} C^{\mathrm{LT}}_{1/2,\gamma}(\lambda,\sigma) = \min_{\lambda\in(2/(2+\gamma),1)} C^{\mathrm{LT}}_{1/2,\gamma}\Big(\lambda,\frac{2(1-\lambda)}{\lambda\gamma}\Big). \qquad \square$$

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