Equilibria Under Knightian Price Uncertainty

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Abstract

We study economies in which agents face Knightian uncertainty about state prices. Knightian uncertainty leads naturally to nonlinear expectations. We introduce a corresponding equilibrium concept with sublinear prices and prove that equilibria exist under weak conditions. In general, such equilibria lead to Pareto inefficient allocations; the equilibria coincide with Arrow–Debreu equilibria only if the values of net trades are ambiguity–free in the mean. In economies without aggregate uncertainty, inefficiencies are generic. We introduce a constrained efficiency concept, *uncertainty–neutral efficiency*; equilibrium allocations under price uncertainty are efficient in this constrained sense. Arrow–Debreu equilibria turn out to be non–robust with respect to the introduction of Knightian uncertainty.

1 Introduction

We study economies with Knightian uncertainty about state prices. Knightian uncertainty describes the situation in which the probability distribution of relevant outcomes is not known exactly. In such a situation, it is natural to work with a nonadditive notion of expectation derived from a set of probability distributions. We introduce a corresponding nonlinear equilibrium concept, *Knight–Walras equilibrium*, where the forward price of a contingent consumption plan is given by the maximal expected value of the net consumption value.

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In our economy, agents trade contingent plans on a forward market at time 0 as in Debreu's original model of trade under uncertainty. The market is complete in the sense that all contingent plans are available. However, Knightian uncertainty induces an imperfection in the price formation of the market, resulting in sublinear prices.

Knightian uncertainty is described by a set of priors $\mathbb{P}$ which is common knowledge; we think of it as imprecise, but objective probabilistic information about the outcome distribution over Ω at time 1, as in Ellsberg's thought experiments. Following Walras and Debreu, we do not explicitly model the price formation process, but rather model its outcome by a sublinear expectation. The invisible hand of the market uses the maximal expected value over the set of priors $\mathbb{P}$ to price contingent claims. One might think of a cautious market maker who has imprecise probabilistic information about the states of the world and computes the maximal expected present value over a set of models to hedge Knightian uncertainty.

We establish existence of Knight–Walras equilibrium for general preferences including the well studied classes of smooth ambiguity preferences of Klibanoff, Marinacci, and Mukerji (2005) and variational preferences of Maccheroni, Marinacci, and Rustichini (2006). The proof uses an adaptation of Debreu's game–theoretic approach which has an interesting economic interpretation. Debreu works with a Walrasian auctioneer who maximizes the expected value of aggregate excess demand. In our proof, we introduce a further *Knightian* price player who chooses the relevant probability distribution. Under Knightian uncertainty, we can thus view Adam Smith's "invisible hand of the market" as consisting of two auctioneers where one of them chooses the (state) price and the other one the relevant probability distribution where "relevant" is to be understood as the prior under which the expected value of net trades is maximal.

Given that we have a nonlinear price system, one might ask if agents can generate arbitrage gains; any reasonable notion of equilibrium should exclude such arbitrage, of course. In our context, there is no financial market, so the arbitrage notion of a costless portfolio with positive gains does not apply here. We consider two natural notions of arbitrage for our sublinear prices. Following Aliprantis, Florenzano, and Tourky (2005), an arbitrage is a positive consumption plan with zero price. We show that this is precluded in Knight–Walras equilibrium. Alternatively, in our sublinear context, one could think of making gains by splitting a consumption bundle into two or more plans. Such arbitrage gains are neither possible; in fact, the convexity of our price functional is the reason for the absence of such arbitrage.

In case of pure risk, i.e., when the set of probability distributions consists of a singleton, the new notion coincides with the classic notion of an Arrow–

Debreu equilibrium under risk. A main objective of our paper is to study the differences that Knightian uncertainty of state prices create compared to the Arrow–Debreu equilibrium concept. We show that Arrow–Debreu and Knight–Walras equilibria coincide if and only if the values of the individual net demands are *ambiguity–free in mean*, i.e., when there in no ambiguity about the mean value of net demands.

We then ask how restrictive this condition is. To this end, we study the well–known class of economies without aggregate uncertainty and uncertainty–averse agents who share a common subjective belief at certainty. This class of preferences is introduced in Rigotti, Shannon, and Strzalecki (2008) and covers the well–studied class of pessimistic multiple prior agents (Gilboa and Schmeidler (1989)), smooth ambiguity models (Klibanoff, Marinacci, and Mukerji (2005)), and multiplier preferences (Hansen and Sargent (2001)). Without aggregate uncertainty, Rigotti, Shannon, and Strzalecki (2008) show that an interior allocation is efficient if and only if each agent is fully insured in equilibrium, extending previous results by Billot, Chateauneuf, Gilboa, and Tallon (2000) and Chateauneuf, Dana, and Tallon (2000) for specific classes of preferences[1]. We show that generically in endowments, these Arrow–Debreu equilibria are not Knight–Walras equilibria. Intuitively, it will be rarely the case that agent's net demand is ambiguity–free in mean when individual endowments are subject to Knightian uncertainty.

Based on the generic non–equivalence of equilibria under Knightian price uncertainty to Arrow–Debreu equilibrium, we show that Knight–Walras equilibria under no aggregate uncertainty are generically inefficient.

Efficiency is thus the exception under Knightian price uncertainty. We introduce a notion of constrained efficiency which we call *Uncertainty–Neutral Efficiency*. An allocation is constrained efficient if it is impossible to improve the allocation by trading in an ambiguity–free way. The fictitious social planner is thus restricted to redistributions with ambiguity–free net values. We show that Knight–Walras equilibrium allocations are uncertainty–neutral efficient.

We subsequently continue to explore the nature of Knight–Walras equilibria in economies without aggregate uncertainty. It turns out that Arrow–Debreu equilibria are not robust with respect to the introduction of Knightian uncertainty in prices. Even with a small amount of Knightian uncertainty,

[1] De Castro and Chateauneuf (2011) extend these results to the case of no aggregate ambiguity, i.e. $\sum e_i \in \mathbb{L}$. Strzalecki and Werner (2011) introduce the notion of a conditional subjective beliefs to study efficient allocations in general. In particular, efficient allocations are measurable with respect to aggregate endowment if agents share a common conditional belief. A further discussion of efficient allocations on the interim stage can be found in Kajii and Ui (2006) and Martins-da Rocha (2010).

the unique Knight–Walras equilibrium has no trade, in sharp contrast to the full insurance allocation of the Arrow–Debreu equilibrium. One might see this result as an equilibrium extension of the seminal result of Dow and Werlang (1992) who show that ambiguity–averse agents stay away from the asset demand for a whole range of prices.

At the end of this introduction, we discuss nonlinearities in price systems that have appeared in other economic environments. Our equilibrium and efficiency results apply to these models in as far as the sublinear structure of prices is shared.

While we favor the interpretation that the sublinearity of prices is a result of Knightian uncertainty, our model encompasses and generalizes various other models of market imperfections that have been discussed in the literature. Sublinear prices arise in incomplete financial markets, in insurance markets, or in markets with transaction costs. Our subsequent results apply to those models as well, therefore.

Araujo, Chateauneuf, and Faro (2012) discuss sublinear functionals which satisfy similar axioms as our Knightian expectation. They show that the sublinear price functional is equal to the superhedging price of an exogenous incomplete and arbitrage–free financial market if and only if mean–ambiguity free and undominated claims coincide, where a claim is called undominated if there is no superior consumption plan with the same price. Since we do not impose such a condition, our setup is more general than the setup created by incomplete financial markets. A further difference is the following. Since Araujo et al. consider properties of superhedging prices in financial markets, they work directly with the set of martingale measures while we rather work with the set of probability measures which describe the Knightian uncertainty of outcomes. In the case where our spot consumption price is simultaneously a density for all priors, and the above condition on the equality of mean–ambiguity free and undominated claims holds true, the results of Araujo et al. would thus allow to construct an incomplete financial market with our sublinear prices.

Jouini and Kallal (1995) discuss transaction cost models; they show that financial markets with bid–ask spreads can be characterized by a set of measures under which the expected payoff of securities remains in the bid–ask interval. This generalized class of martingale measures also leads naturally to sublinear prices. It is an open question if one can construct for a given sublinear price functional exogenous financial markets with transaction cost that would lead to the same pricing functional. We conjecture that a similar restriction as in Araujo, Chateauneuf, and Faro (2012) would need to be satisfied as well.

Castagnoli, Maccheroni, and Marinacci (2002) discuss sublinear prices in insurance markets. In particular, they characterize insurance prices which can be written as the sum of a fair premium, i.e. the usual linear expected net present value of the potential damage, and an ambiguity premium. Their characterization thus shares a certain conceptual analogy to our above interpretation of the sublinear price functional. Knightian uncertainty leads to sublinear insurance prices. Our results below thus also shed some light on (in)efficiency of insurance markets.

The papers cited above on incomplete markets, transaction costs, and insurance premia discuss properties related to sublinear functionals, but do not study equilibrium. Our paper completes this gap in the literature. Nonlinear forward prices are also discussed in Beißner (2012) and Cerreia-Vioglio, Maccheroni, and Marinacci (2015). Aliprantis, Tourky, and Yannelis (2001) consider nonlinear prices from an abstract point of view. More recently, Richter and Rubinstein (2015) put forward an equilibrium notion based on convex geometry for discrete commodities.

The remainder of the paper is organized as follows. Section 2 introduces the concept of Knight–Walras equilibria. Existence is established in the following section 3. Section 4 analyzes the relation to Arrow–Debreu equilibria. Subsequently, we study efficient allocations under Knightian uncertainty and discuss uncertainty–neutral efficiency. Section 6 investigates the sensitivity of Arrow–Debreu equilibria with respect to Knightian uncertainty before we conclude in Section 7. The Appendix collects proofs.

2 Knight–Walras Equilibrium

2.1 Expectations and Forward Prices

We consider a static economy under uncertainty with a finite state space Ω. In risky environments, or for probabilistically sophisticated agents, expectations are given by probability measures on Ω; under Knightian uncertainty, one is naturally led to sublinear expectations. Let us fix our notion of expectation first.

Let $\mathbb{X} = \mathbb{R}^\Omega$ be the commodity space of contingent plans for our economy. We call $\mathbb{E} : \mathbb{X} \to \mathbb{R}$ a *(Knightian) expectation* if it satisfies the following properties:

1. $\mathbb{E}$ preserves constants: $\mathbb{E}[m] = m$ for all $m \in \mathbb{R}$,

2. $\mathbb{E}$ is monotone: $\mathbb{E}[x] \leq \mathbb{E}[y]$ for all $x, y \in \mathbb{X}$ with $x \leq y$,

3. $\mathbb{E}$ is sub-additive: $\mathbb{E}[x + y] \leq \mathbb{E}[x] + \mathbb{E}[y]$ for all $x, y \in \mathbb{X}$,

4. $\mathbb{E}$ is homogeneous: $\mathbb{E}[\lambda x] = \lambda \mathbb{E}[x]$ for $\lambda > 0$ and $x \in \mathbb{X}$,

5. $\mathbb{E}$ is relevant: $\mathbb{E}[-x] < 0$ for all $x \in \mathbb{X}_+ \setminus \{0\}$.

In the sequel, we denote by Δ the set of all probability measures on Ω. It is well known[2] that $\mathbb{E}$ is uniquely characterized by a convex and compact set $\mathbb{P} \subset \Delta$ of probability measures on Ω such that

$$\mathbb{E}[x] = \max_{P \in \mathbb{P}} E^P[x] \tag{1}$$

for all $x \in \mathbb{X}$; E^P denotes the usual linear expectation here. Relevance implies that the representing set $\mathbb{P}$ in (1) consists of measures with full support in the sense that for every $P \in \mathbb{P}$ we have $P(\omega) > 0$ for every $\omega \in \Omega$.

The sublinear expectation $\mathbb{E}$ leads naturally to a concept of (forward) price for contingent plans: let $\psi : \Omega \to \mathbb{R}_+$ be a positive state–price. The forward price for a contingent plan $x \in \mathbb{X}$ is

$$\Psi(x) = \mathbb{E}[\psi x]\,,$$

in analogy to the usual forward (or risk–adjusted) price under risk. We call $\Psi : \mathbb{X} \to \mathbb{R}$ *coherent price system.*

Remark 1 *Incomplete nominal financial markets lead to sublinear price functionals as discussed in Araujo, Chateauneuf, and Faro (2012). In that case, C can be identified with the sublinear expectation given by the set of equivalent martingale measures of the financial market.*

Our setup is more general as we start from a set of priors which describes the Knightian uncertainty of the world. If we have $E^P[\psi] = 1$ for all $P \in \mathbb{P}$, we can define a set of pricing measures

$$\mathbb{Q} := \Big\{ Q \in \Delta : Q(\omega) = \psi(\omega)P(\omega) \text{ for all } \omega \in \Omega \text{ and some } P \in \mathbb{P} \Big\}. \tag{2}$$

and

$$C(x) = \max_{Q \in \mathbb{Q}} E^Q[x]$$

is the cost functional of Araujo, Chateauneuf, and Faro (2012). Under certain conditions on C, one can then construct a financial market which has exactly this set of equivalent martingale measures.

[2] See Lemma 3.5 in Gilboa and Schmeidler (1989), Peng (2004), Artzner, Delbaen, Eber, and Heath (1999), or Föllmer and Schied (2011).

2.2 The Economy with Sublinear Forward Prices

We introduce now an economy with sublinear forward prices. Uncertainty is described by the state space Ω and the Knightian expectation $\mathbb{E}$, respectively the representing set of priors $\mathbb{P}$. We discuss the role of $\mathbb{E}$ in detail below. There is one physical commodity for consumption; an extension to finitely many commodities is straightforward.

Definition 1 *A* Knightian economy *(on Ω) is a triple*

$$\mathcal{E} = \left(I, (e_i, U_i)_{i \in \mathbb{I}}, \mathbb{E}\right)$$

where $I \geq 1$ denotes the number of agents, $e_i \in \mathbb{X}_+ = \{c \in \mathbb{X} : c(\omega) \geq 0 \text{ for all } \omega \in \Omega\}$ is the endowment of agent i, $U_i : \mathbb{X}_+ \to \mathbb{R}$ agent i's utility function, and $\mathbb{E}$ is a Knightian expectation.

As we fix the agents $\mathbb{I} = \{1, \ldots, I\}$ throughout the paper, we will sometimes use the shorthand notation $\mathcal{E}^{\mathbb{P}}$ to emphasize the dependence of the economy on the Knightian expectation $\mathbb{E}$ that is generated by $\mathbb{P}$.

Definition 2 *We call a pair (ψ, c) of a state–price $\psi : \Omega \to \mathbb{R}_+$ and an allocation $c = (c_i)_{i=1,\ldots,I} \in \mathbb{X}_+^I$ a* Knight–Walras equilibrium *if*

1. *the allocation c is feasible, i.e. $\sum_{i=1}^{I}(c_i - e_i) \leq 0$,*

2. *for each agent i, c_i is optimal in the* Knight-Walras *budget set*

$$\mathbb{B}(\psi, e_i) = \{c \in \mathbb{X}_+ : \mathbb{E}[\psi(c - e_i)] \leq 0\}, \tag{3}$$

   i.e. if $U_i(d) > U_i(c_i)$ then $d \notin \mathbb{B}(\psi, e_i)$.

We discuss some immediate properties of the new concept.

Example 1 1. *When $\mathbb{P} = \{P_0\}$ is a singleton, the budget constraint is linear; in this case, Knight–Walras and Arrow–Debreu equilibria coincide. In particular, equilibrium allocations are efficient.*

Note that in expected utility economies, the probability measure P_0 plays a minor role in equilibrium. As Harrison and Kreps (1979) have pointed out, the role of P_0 consists of determining the null sets and the commodity space of the economy. Indeed, the probability P_0 and the state price ψ determine a linear mapping $x \mapsto E^{P_0}[\psi x]$ for $x \in \mathbb{X}$; it is thus common in General Equilibrium Theory to look only at linear functionals Ψ of the form $\Psi(x) = \sum_{\omega \in \Omega} q(\omega)x(\omega)$ for some q. As long as P_0 has full support, the two approaches are equivalent with $P_0(\omega)\psi(\omega) = q(\omega)$.

2. At the other extreme, when $\mathbb{P} = \Delta$ consists of all probability measures, and the state-price ψ is strictly positive, the budget sets consist of all plans c with $c \leq e_i$ in all states. We are economically in the situation where all spot markets at time 1 operate separately and there is no possibility to transfer wealth over states. As a consequence, with strictly monotone utility functions, no trade is an equilibrium for every strictly positive state price ψ. Equilibrium allocations are inefficient, in general, and equilibrium prices are indeterminate.

2.3 Discussion of Sublinear Prices

In our economy, agents trade contingent plans on a forward market at time 0 as in Debreu's original model of trade under uncertainty. Note that the market is complete in the sense that all contingent plans are available. However, Knightian uncertainty induces an imperfection in the price formation of the market, resulting in sublinear prices.

The set of priors $\mathbb{P}$ is common knowledge of market participants; we think of it as imprecise probabilistic information about the outcome distribution over Ω at time 1, as in Ellsberg's thought experiments. As in Walras and Debreu's models, we do not explicitly model the price formation process, but rather model its outcome by the sublinear expectation $\mathbb{E}$. The invisible hand of the market uses the maximal expected value over a set of models to price contingent claims. One might think of a cautious market maker who has imprecise probabilistic information about the states of the world, described by $\mathbb{P}$. The market maker then computes the maximal expected present value over this set of models to stay on the safe side. Such an approach is also favored by recent approaches to the regulation of financial markets.

The price $\psi(\omega)$ is the spot price of the real consumption good in state ω at time 1. For simplicity, we consider the case of one physical good, but the extension to any finite number is straightforward; with d physical goods, $\psi(\omega)$ and $c(\omega)$ would take values in $\mathbb{R}^d$ and $\psi \cdot c$ would be the scalar product. The price ψ is used by the market to value contingent claims; the actual trade is then carried out via the contracts made at time 0 and markets do not re–open at time 1.

The Knight–Walras budget set $\mathbb{B}(\psi, e_i)$ in (3) is the intersection of budget sets under linear prices of the form $E^P[\psi \cdot]$, that is,

$$\mathbb{B}(\psi, e_i) = \bigcap_{P \in \mathbb{P}} \mathbb{B}^P(\psi, e_i),$$

where $\mathbb{B}^P(\psi, e_i) = \left\{ c \in \mathbb{X}_+ : E^P[\psi(c - e_i)] \leq 0 \right\}$ denotes the budget set in an Arrow–Debreu economy under $\mathbb{P} = \{P\}$. Hence, agents in the Knightian economy $\mathcal{E}^{\mathbb{P}}$ only consider those consumption bundles that are robustly affordable against the price uncertainty $\mathbb{P}$.

While we favor the interpretation that the sublinearity of prices is a result of Knightian uncertainty, our model encompasses and generalizes various other models of market imperfections that have been discussed in the literature. Sublinear (forward) prices appear in incomplete financial markets, in insurance markets, or in markets with transaction costs.

Araujo, Chateauneuf, and Faro (2012) discuss sublinear functionals which satisfy similar axioms as our Knightian expectation. They show that the sublinear price functional is equal to the superhedging price of an exogenous incomplete and arbitrage–free financial market if and only if the subspace of claims whose expectation does not depend on a specific prior $P \in \mathbb{P}$ coincides with the subspace of undominated claims under $\mathbb{E}$. In their setup, a claim x is called undominated if there is no claim $y > x$ with the same price. In incomplete financial markets, exactly the hedgable claims are undominated; claims that do not belong to the marketed subspace are dominated. In turn, if one starts with a sublinear pricing functional, this latter condition is sufficient to construct an incomplete financial market whose superhedging price functional is equal to the given sublinear price functional. Since we do not impose such a condition, our setup is more general than the setup created by incomplete financial markets. Moreover, as Araujo, Chateauneuf, and Faro (2012) consider the properties of superhedging prices in financial markets, they work directly with the set of martingale measures, or set $\psi = 1$. In the case where ψ is simultaneously a density for all $P \in \mathbb{P}$, and the above condition on the equality of mean–ambiguity free and undominated claims are equal, the results of Araujo, Chateauneuf, and Faro (2012) would thus allow to construct an incomplete financial market with our sublinear prices, compare also Example 1.

Jouini and Kallal (1995) discuss transaction cost models; they show that financial markets with bid–ask spreads can be characterized by a set of measures under which the expected payoff of securities remains in the bid–ask interval. This generalized class of martingale measures thus leads naturally to sublinear prices. It is an open question if one can construct for a given sublinear price functional exogenous financial markets with transaction cost that would lead to the same pricing functional. We conjecture that a similar restriction as in Araujo, Chateauneuf, and Faro (2012) would need to be satisfied as well.

Castagnoli, Maccheroni, and Marinacci (2002) discuss sublinear prices in insurance markets. In particular, they characterize insurance prices which can be written as the sum of a fair premium, i.e. the usual linear expected value of the potential damage, and an ambiguity premium of the form

$\text{Amb}_{\mathbb{P}}(x) = \sup_{P,Q\in\mathbb{P}} \left|E^P[x] - E^Q[x]\right|$. Their characterization thus shares a certain conceptual analogy to our above interpretation of the sublinear price functional. Knightian uncertainty leads to sublinear insurance prices.

The papers cited above discuss properties related to sublinear functionals, but do not study equilibrium. Our paper completes this gap in the literature.

2.4 On Utility Functions under Uncertainty

For our analysis, we will use two sets of assumptions on preferences and endowments. First, we shall specify a very general set of properties inspired by the classic assumptions made in General Equilibrium Theory.

Assumption 1 *Each agent's endowment e_i is strictly positive. Each utility function $U_i : \mathbb{X}_+ \to \mathbb{R}$ is*

- *continuous,*
- *monotone, i.e. if $c \geq c'$ then $U_i(c) \geq U_i(c')$,*
- *semi–strictly quasi–concave, i.e. for all $c, c' \in \mathbb{X}_+$ with $U(c) > U(c')$ we have for all $\lambda \in (0,1)$*

$$U(\lambda c + (1-\lambda)c') > U(c')\,.$$

- *and non–satiated, i.e. for $c \in \mathbb{X}_+$ there exists $c' \in \mathbb{X}_+$ with $U_i(c') > U_i(c)$.*

Every concave utility function is semi–strictly quasi–concave. Semi–strict quasi–concavity and non–satiation imply local non–satiation; for $c \in \mathbb{X}_+$ and $\epsilon > 0$, non–satiation allows to choose $c' \in \mathbb{X}_+$ with $U_i(c') > U_i(c)$. We can find $\lambda \in (0,1)$ such that $c'' = \lambda c + (1-\lambda)c'$ satisfies $\|c - c''\| < \epsilon$; by semi–strict quasi–concavity, $U_i(c'') > U_i(c)$. There are utility functions which are monotone, semi–strictly quasi–concave, non–satiated, but not strictly monotone. An example are multiple priors utilities as, in its simplest form, $U_i(x) = \min_{\omega\in\Omega} x(\omega)$. As far as existence is concerned, the assumption of strict monotonicity of endowments can be weakened to a typical "cheaper point assumption". As our focus is on the effects of Knightian uncertainty, we do not carry out this small generalization.

Preferences that allow for the perception of ambiguity have been extensively explored in recent years. The following example lists some natural utility functions in $\mathcal{E}^{\mathbb{P}}$ which have been axiomatized in the literature.

Example 2 *1. Multiple–prior expected utilities (Gilboa and Schmeidler (1989)) take the form*

$$U_i(c) = -\mathbb{E}[-u_i(c)] = \min_{P \in \mathbb{P}} E^P[u_i(c)] \tag{4}$$

for a suitable (continuous, strictly increasing, strictly concave) Bernoulli utility function $u_i : \mathbb{R}_+ \to \mathbb{R}$.

Subjective reactions to the imprecise probabilistic information $\mathbb{P}$ *in the spirit of Gajdos, Hayashi, Tallon, and Vergnaud (2008) can be described by preferences of the form*

$$U_i(c) = \min_{P \in \Phi_i(\mathbb{P})} E^P[u_i(c)]$$

for a selection $\Phi_i(\mathbb{P}) \subset \mathbb{P}$*. Note that a singleton* $\Phi_i(\mathbb{P}) = \{P_i\}$ *leads to ambiguity–neutral subjective expected utility agents.*

2. The smooth model of Klibanoff, Marinacci, and Mukerji (2005) has

$$U_i(c) = \int_{\mathbb{P}} \phi_i \left(E^P[u_i(c)] \right) \mu_i(\mathrm{d}P)$$

for a continuous, monotone, strictly concave ambiguity index $\phi_i : \mathbb{R} \to \mathbb{R}$ *and a second–order prior* μ_i*, a measure with support included in* $\mathbb{P}$.

3. Dana and Riedel (2013) introduce anchored preferences of the form

$$U_i(c) = \min_{P \in \mathbb{P}} E^P[u_i(c) - u(e_i)] \, .$$

These preferences have recently been axiomatized by Faro (2015). With multiple prior utilities, they belong to the larger class of variational preferences (Maccheroni, Marinacci, and Rustichini (2006)) of the form

$$U_i(c) = \inf_{P \in \mathbb{P}} E^P u_i(c) + \gamma(P)$$

for a suitable penalty function $\gamma : \mathbb{P} \to \mathbb{R}_+ \cup \{\infty\}$.

The above preferences share two common features which are typical for ambiguity–averse preferences. We gather them in the following assumption. In order to do so, the concept of *subjective beliefs* introduced by Rigotti, Shannon, and Strzalecki (2008) is useful. The set of subjective beliefs $\pi_i(c)$ of agent i at $c \in \mathbb{X}_+$ is given by

$$\pi_i(c) = \Big\{ Q \in \Delta : E^Q[y] \geq E^Q[c] \text{ for all } y \text{ with } U_i(y) \geq U_i(c) \Big\}. \tag{5}$$

The set $\pi_i(c)$ consists of the normalized supports of the upper contour sets of U_i at the consumption plan c; it contains all beliefs for which the agent is unwilling to trade net consumption plans with zero expected net payoff.

Assumption 2
- *The utility functions U_i are concave and strictly monotone.*
- *Each U_i is* translation invariant at certainty: *For all $h \in \mathbb{X}$ and all constant bundles $c, c' > 0$, if $U_i(c + \lambda h) \geq U_i(c)$ for some $\lambda > 0$, then there exists $\lambda' > 0$ such that $U_i(c' + \lambda' h) \geq U_i(c')$.*
- *Preferences are consistent with the set of priors $\mathbb{P}$, i.e. we have $\pi_i \subset \mathbb{P}$, and agents share some common subjective belief at certainty: $\bigcap_{i=1}^{I} \pi_i \neq \emptyset$.*

Concavity is slightly more restrictive than semi–strict quasi–concavity, but satisfied by most models. In the same spirit, strict monotonicity is a slightly more restrictive condition than mere monotonicity, but will be satisfied in most applications. Translation invariance at certainty is introduced in Rigotti, Shannon, and Strzalecki (2008). It ensures that subjective beliefs are constant across constant bundles, and thus independent of the particular constant c (Proposition 8 in their paper). We denote the subjective belief of agent i at any constant bundle $m > 0$ by $\pi_i = \pi_i(m)$.

Translation invariance at certainty is satisfied by the common utility functions that model uncertainty aversion, including the ambiguity–neutral expected utility case, the smooth ambiguity model, multiple priors, and variational preferences that we listed above. The second part of the assumption ensures that the subjective beliefs at constant bundles are consistent with the set of priors $\mathbb{P}$ that describes the Knightian uncertainty of the economy. As we assume that $\mathbb{P}$ is common knowledge, it is a natural assumption that the subjective beliefs are consistent with the given imprecise probabilistic information and that the agents share some belief about possible priors. According to Rigotti, Shannon, and Strzalecki (2008), efficient allocations under no aggregate uncertainty are full insurance allocations if and only if agents share some common subjective belief at certainty.

3 Existence of Knight–Walras Equilibria

In this section, we establish existence of a Knight–Walras equilibrium. If agents have single–valued demand, one can modify a standard proof, as, e.g. in Hildenbrand and Kirman (1988), to prove existence. Under Knightian uncertainty, natural examples arise where demand can be set–valued. A point in case are ambiguity–averse, yet risk neutral agents with Gilboa–Schmeidler preferences. If we include this general case, one needs to work more. We think that the proof, beyond the natural interest in generality, provides additional insights into the working of markets under Knightian uncertainty as we explain below.

Theorem 1 *Under Assumption 1, Knight-Walras equilibria* (ψ, c) *exist.*

A standard existence proof of Arrow–Debreu equilibrium uses a game–theoretic approach. One introduces a price player who maximizes the expected value of aggregate excess demand over state prices. Let us call this type of player a *Walrasian price player.* The consumers maximize their utility given the budget constraint. The equilibrium of the game is an Arrow–Debreu equilibrium. Our method to prove existence follows this game–theoretic approach. Due to Knightian uncertainty, we have to introduce a second, *Knightian*, price player. This player maximizes the expected value of aggregate excess demand over the priors $P \in \mathbb{P}$, taking the state price as given. The Walrasian price player in the Knight–Walras equilibrium acts in the same way as in the Arrow–Debreu equilibrium.

Note that even though our market features sublinear prices, a weak form of Walras' law holds true. As semi–strict concavity and non–satiation imply local non–satiation, the budget constraint is binding for each agent i at an optimal consumption plan c_i; by sublinearity, net aggregate demand $\zeta = \sum_{i=1}^{I} c_i - e_i$ satisfies $\Psi(\zeta) \leq \sum_{i=1}^{I} \Psi(c_i - e_i) = 0$.

Given that we have a nonlinear price system, one might ask if agents can generate arbitrage gains; any reasonable notion of equilibrium should exclude such arbitrage, of course. In our context, there is no financial market, so the arbitrage notion of a costless portfolio with positive gains does not apply here. We consider two natural notions of arbitrage for our sublinear prices. Following Aliprantis, Florenzano, and Tourky (2005), an arbitrage is a consumption plan $c \in \mathbb{X}_+ \setminus \{0\}$ with $\Psi(c) = 0$. Alternatively, in our sublinear context, one could think of making gains by splitting a consumption bundle into two or more plans. Note that the gain from selling a plan c is the negative of "buying" the plan $-c$, i.e. $-\Psi(-c)$. The following proposition shows that neither form of arbitrage is possible in Knight–Walras equilibrium.

Proposition 1 *Let* $(\psi, (\hat{c}_i)_{i \in \mathbb{I}})$ *be a Knight–Walras equilibrium. Under Assumption 2, the following absence of arbitrage conditions hold true.*

1. *We have* $\Psi(c) > 0$ *for all* $c \in \mathbb{X}_+ \setminus \{0\}$.
2. *Let* $x = y + z$ *for* $x, y, z \in \mathbb{X}$. *Buying (selling)* x *and selling (buying)* y *and* z *separately yields no profits. We have*

$$\Psi(x) \geq -\Big(\Psi(-y) + \Psi(-z)\Big) \quad \textit{and} \quad \Psi(y) + \Psi(z) \geq -\Psi(-x).$$

4 (Non–) Equivalence to Arrow–Debreu Equilibrium

If the expectation $\mathbb{E}$ is linear, Knight–Walras equilibria are Arrow–Debreu equilibria; by the first welfare theorem, equilibrium allocations are thus efficient. With incomplete Knightian preferences in the sense of Bewley (2002), the Arrow–Debreu equilibria of the linear economies $\mathcal{E}^{\{P\}}$ are also equilibria under Knightian uncertainty; see Rigotti and Shannon (2005) and Dana and Riedel (2013). It seems thus natural to ask whether such a result might hold true for our Knightian economies.

In a first step, we show that Knight–Walras equilibria are Arrow–Debreu equilibria if and only if the expected net consumption values of all agents does not depend on the specific prior in the representing set $\mathbb{P}$ int he sense of the following definition. We then show for the particular transparent example of no aggregate uncertainty that this property is generically not satisfied in Knight–Walras equilibrium.

Definition 3 *We call $\xi \in \mathbb{X}$ ($\mathbb{P}$)–ambiguity free in mean if ξ has the same expectation for all $P \in \mathbb{P}$, i.e. there is a constant $k \in \mathbb{R}$ with $E^P[\xi] = k$ for all $P \in \mathbb{P}$. We denote the set of plans which are ambiguity–free in mean by $\mathbb{L}$ or $\mathbb{L}^{\mathbb{P}}$.*

Note that ξ is ambiguity–free in mean if and only if we have

$$\mathbb{E}[-\xi] = -\mathbb{E}[\xi]\,.$$

We will use this fact sometimes below[3].

We can now clarify when Arrow–Debreu equilibria of a particular linear economy $\mathcal{E}^{\{P\}}$ are also Knight–Walras equilibria.

Theorem 2 *Fix a prior $P \in \mathbb{P}$. Let $(\psi, (c_i))$ be an Arrow–Debreu equilibrium for the (linear) economy $\mathcal{E}^{\{P\}}$. The following assertions are equivalent:*

1. *$(\psi, (c_i))$ is a Knight–Walras equilibrium for $\mathcal{E}^{\mathbb{P}}$.*
2. *The value of net demands $\xi_i = \psi(c_i - e_i)$ are $\mathbb{P}$–ambiguity free in the mean for all agents i.*

[3] The concept has appeared before in De Castro and Chateauneuf (2011) and Riedel and Beißner (2014). For the notion of unambiguous *events*, see also Epstein and Zhang (2001). A stronger notion would require that the probability distribution of a plan is the same under all priors in $\mathbb{P}$; Ghirardato, Maccheroni, and Marinacci (2004) call such plans "crisp acts".

Let us consider the particularly transparent case of no aggregate uncertainty. We shall show that generically in endowments, Arrow–Debreu equilibria are not Knight–Walras equilibria.

Theorem 3 *Assume that $\mathbb{E}$ is not linear. Under no aggregate uncertainty and Assumption 1 and 2, generically in endowments, Arrow–Debreu equilibria of $\mathcal{E}^{\{P\}}$ for some $P \in \mathbb{P}$ are not Knight–Walras equilibria of $\mathcal{E}^{\mathbb{P}}$.*

More precisely: let $M = \left\{(e_i)_{i=1,\ldots,I} \in \mathbb{X}^I_{++} : \sum e_i = 1\right\}$ be the set of economies without aggregate uncertainty normalized to 1. Let N be the subset of elements (e_i) of M for which there exists $P \in \mathbb{P}$ and an Arrow–Debreu equilibrium $(\psi, (c_i))$ of the economy $\mathcal{E}^{\{P\}}$ which is also a Knight–Walras equilibrium of the economy $\mathcal{E}^{\mathbb{P}}$. N is a Lebesgue null subset of the $(I-1)\cdot \#\Omega$–dimensional manifold M.

A key step in the proof of the above theorem is the insight of Lemma 1 below that the subspace of mean–ambiguity free contingent plans $\mathbb{L}$ has a strictly smaller dimension that the full space $\mathbb{R}^\Omega$ if there is Knightian uncertainty. Under our assumptions, Arrow–Debreu equilibrium allocations are full insurance allocations; after changing the measure, the first order conditions allow to identify the state price with the constant 1. Since $\mathbb{L}$ contains the constant functions, one can show that an Arrow–Debreu equilibrium is a Knight-Walras equilibrium only if the endowments are in $\mathbb{L}$.

Lemma 1

1. *The set $\mathbb{L}$ of plans $\xi \in \mathbb{X}$ which are $\mathbb{P}$–ambiguity–free in mean forms a subspace of $\mathbb{X}$ which includes all constant functions. If $\#\mathbb{P} > 1$, $\mathbb{L}$ has a strictly smaller dimension than $\mathbb{X}$.*
2. *For $\eta \in \mathbb{X}$ and $\xi \in \mathbb{L}$ we have $\mathbb{E}[\eta + \xi] = \mathbb{E}[\eta] + \mathbb{E}[\xi]$.*

5 Efficiency

The previous section shows that Knight–Walras and Arrow–Debreu equilibria rarely coincide under no aggregate uncertainty. The question thus arises if the first welfare theorem holds true. To tackle this question, we discuss the welfare properties of Knight–Walras equilibria in the light of recent results on efficient allocations under Knightian uncertainty.

5.1 Pareto Efficiency and Knight–Walras Equilibria

Rigotti, Shannon, and Strzalecki (2008) use the concept of *subjective beliefs* that we introduced in Assumption 2 to characterize efficient allocations in the economy $\mathcal{E}^{\mathbb{P}}$. They show that an interior allocation $(c_1, \ldots, c_I) \in (\mathbb{X}_{++})^I$ is efficient if and only if the agents share a common subjective belief, i.e. $\cap_{i \in \mathbb{I}} \pi_i(c_i) \neq \emptyset$ with $\pi_i(c_i)$ being the set of subjective beliefs of agent i at consumption plan c_i defined in (5).

Without aggregate uncertainty and when agents have multiple prior or Choquet expected utility preferences, the above condition entails that an interior allocation is efficient if and only if it fully insures each agent (Billot, Chateauneuf, Gilboa, and Tallon (2000); Chateauneuf, Dana, and Tallon (2000)). De Castro and Chateauneuf (2011) extend these results to the case of no aggregate ambiguity in mean, i.e. $\sum e_i \in \mathbb{L}$. Strzalecki and Werner (2011) introduce the notion of a conditional subjective beliefs to study efficient allocations in general. In particular, efficient allocations are measurable with respect to aggregate endowment if agents share a common conditional belief[4].

In analogy to subjective beliefs, we now introduce the concept of *effective pricing measures*; for a coherent price system $\Psi : \mathbb{X} \to \mathbb{R}$ we call

$$\varphi_\Psi(\xi) \quad = \quad \left\{ Q \in \Delta : E^Q[\xi] \geq E^Q[\eta] \text{ for all } \eta \text{ with } \Psi(\eta) \leq \Psi(\xi) \right\}$$

the set of effective pricing measures at $\xi \in \mathbb{X}$.

Proposition 2 *For any $\xi \in \mathbb{X}$, we have*

$$\varphi_\Psi(\xi) = \left\{ Q \in \Delta : Q = \lambda\psi \cdot P \text{ for some } \lambda > 0 \text{ and some } P \in \operatorname{argmax}_{P' \in \mathbb{P}} E^{P'}[\psi\xi] \right\}.$$

The concepts of subjective beliefs and effective pricing measures allow to characterize Knight–Walras equilibria in a compact way.

Theorem 4 *Let (c_i) be a feasible interior allocation in $\mathcal{E}^{\mathbb{P}}$. Under Assumptions 1 and 2, $(\psi, (c_i))$ is a Knight–Walras equilibrium for $\mathcal{E}^{\mathbb{P}}$ if and only if*

$$\pi_i(c_i) \cap \varphi_\Psi(c_i - e_i) \neq \emptyset \qquad \text{for all } i \in \mathbb{I}.$$

With the help of the previous theorem, we show that Knight–Walras equilibria usually fail to be efficient when there is no aggregate uncertainty.

Theorem 5 *Next to Assumptions 1 and 2, assume that the utility functions U_i are differentiable at certainty. Under no aggregate uncertainty, generically in endowments, Knight–Walras equilibrium allocations of $\mathcal{E}^{\mathbb{P}}$ are inefficient.*

[4] A further discussion of efficient allocations on the interim stage can be found in Kajii and Ui (2006) and Martins-da Rocha (2010).

More precisely: let $M = \left\{ (e_i)_{i=1,\dots,I} \in \mathbb{X}_{++}^I : \sum e_i = 1 \right\}$ *be the set of economies without aggregate uncertainty normalized to* 1. *Let* N_e *be the subset of elements* (e_i) *of* M *for which there exists a Knight–Walras equilibrium* $(\psi, (c_i))$ *such that* (c_i) *is efficient.* N_e *is a Lebesgue null subset of the* $(I-1)\cdot\#\Omega$*–dimensional manifold* M.

By the above theorem, Knight–Walras equilibria have inefficient allocations for large classes of economies. In particular, for the widely used classes of smooth ambiguity preferences (Klibanoff, Marinacci, and Mukerji (2005)) and of multiplier preferences (Hansen and Sargent (2001)), Knight–Walras equilibria of economies without aggregate uncertainty are generically inefficient.

5.2 Uncertainty–Neutral Efficiency

In general, Knight–Walras equilibria are inefficient. We introduce now a concept of constrained efficiency for our Knightian framework. If the Walrasian auctioneer aims for *robust* rules, he might consider only values of net trades that are independent of the specific priors in $\mathbb{P}$.

We might also consider a situation of cooperative negotiation among the agents. In a framework of Knightian uncertainty described by the set of priors $\mathbb{P}$, different priors may matter for different agents. For multiple prior agents, e.g., different priors are usually relevant for buyers and sellers of a contingent claim.

The preceding reasoning suggests the following concept of constrained efficiency.

Definition 4 *Let* $\mathcal{E} = \left(I, (e_i, U_i)_{i\in\mathbb{I}}, \mathbb{E}\right)$ *be a Knightian economy. Let* $c = (c_i)_{i\in\mathbb{I}}$ *be a feasible allocation. Let* ψ *be a state–price density. We call the allocation* c uncertainty neutral efficient *(given* ψ *and* $\mathbb{P}$*) if there is no other feasible allocation* $d = (d_i)_{i\in\mathbb{I}}$ *with*

$$\eta_i = \psi\,(d_i - e_i) \in \mathbb{L}^{\mathbb{P}}$$

and $U_i(d_i) > U_i(c_i)$ *for all* $i \in \mathbb{I}$.

Our notion of uncertainty–neutral efficiency shares some similarities with other notions of constrained efficiency, but is slightly stronger. Suppose that $\mathbb{L}^{\mathbb{Q}}$ is the marketed subspace of an incomplete financial market as in Remark 1. In this case, a feasible allocation $c = (c_i)_{i\in\mathbb{I}}$ is $\mathbb{L}^{\mathbb{Q}}$–constrained efficient if the net consumption bundle is in the marketed subspace, $(c_i - e_i) \in \mathbb{L}^{\mathbb{Q}}$ for all

$i \in \mathbb{I}$, and there is no other feasible allocation $d = (d_i)_{i\in\mathbb{I}}$ with $(d_i - e_i) \in \mathbb{L}^{\mathbb{Q}}$ and $U_i(d_i) > U_i(c_i)$ for all $i \in \mathbb{I}$, compare Magill and Quinzii (2002). We do not impose the first condition that the net consumption plan be in $\mathbb{L}^{\mathbb{Q}}$, and thus uncertainty–neutral efficiency with respect to $(\psi, \mathbb{P})$ is a stronger condition than $\mathbb{L}^{\mathbb{Q}}$–constrained efficiency.

Knight–Walras equilibria satisfy our robust version of efficiency.

Theorem 6 *Let (ψ, c) be a Knight–Walras equilibrium of the Knightian economy $\mathcal{E} = \left(I, (e_i, U_i)_{i\in\mathbb{I}}, \mathbb{E}\right)$. Then c is uncertainty neutral efficient (given ψ and $\mathbb{E}$).*

6 Sensitivity of Arrow–Debreu Equilibria with respect to Knightian Price Uncertainty

In this section we explore first the robustness of Arrow–Debreu equilibria with respect to the introduction of a small amount of Knightian uncertainty when agents have multiple–prior utilities. With no aggregate uncertainty, equilibria change in a discontinuous way with small uncertainty perturbations; whereas agents attain full insurance under pure risk, no trade (and thus no insurance) occurs in equilibrium with a tiny amount of Knightian uncertainty. We then take the opposite view and consider growing uncertainty. When uncertainty is sufficiently large, no trade is again the unique equilibrium.

Throughout this section, we fix continuously differentiable, strictly concave, and strictly increasing Bernoulli utility functions $u_i : \mathbb{R}_+ \to \mathbb{R}$ and write for a given set of priors $\mathbb{P}$

$$U_i^{\mathbb{P}}(c) = \min_{P\in\mathbb{P}} E^P[u_i(c)]$$

for the associated multiple–prior utility function. Let us start with an example where the introduction of a tiny amount of uncertainty changes the equilibrium allocation drastically.

Example 3 *Let $\Omega = \{1, 2\}$. Let the set of priors be $\mathbb{P}_\epsilon = \{p \in \Delta : p_1 \in [\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon]\}$ for some $\epsilon \in [0, 1/2)$.*

For $\epsilon > 0$, a consumption plan is ambiguity–free in mean if and only if it is full insurance; we have $\mathbb{L}^{\mathbb{P}} = \{c \in \mathbb{X} : c(1) = c(2)\}$.

Let there be no aggregate ambiguity, without loss of generality $e = 1$ in both states. Let there be two agents $I = 2$ (with multiple–prior utilities as *stated above) and uncertain endowments, e.g.* $e_1 = (1/3, 2/3)$ *and* $e^2 = (2/3, 1/3)$.

In a Knight–Walras equilibrium, the state price has to be strictly positive because of strictly monotone utility functions $U_i^{\mathbb{P}}$. *Since we have two agents, the budget constraint implies that*

$$0 = \mathbb{E}[\psi(c_1 - e_1)] = \mathbb{E}[\psi(c^2 - e^2)]$$

or

$$0 = \mathbb{E}[\psi(c_1 - e_1)] = \mathbb{E}[(-\psi(c_1 - e_1))] \,.$$

Hence, $\psi(c_1 - e_1)$ *is mean–ambiguity free, thus constantly equal to zero here. Since* ψ *is strictly positive,* $c_1 = e_1$ *follows. There is no trade in Knight–Walras equilibrium for every* $\epsilon > 0$. *In sharp contrast, agents achieve full insurance in every Arrow–Debreu equilibrium of any linear economy* $\mathcal{E}^{\{P\}}$.

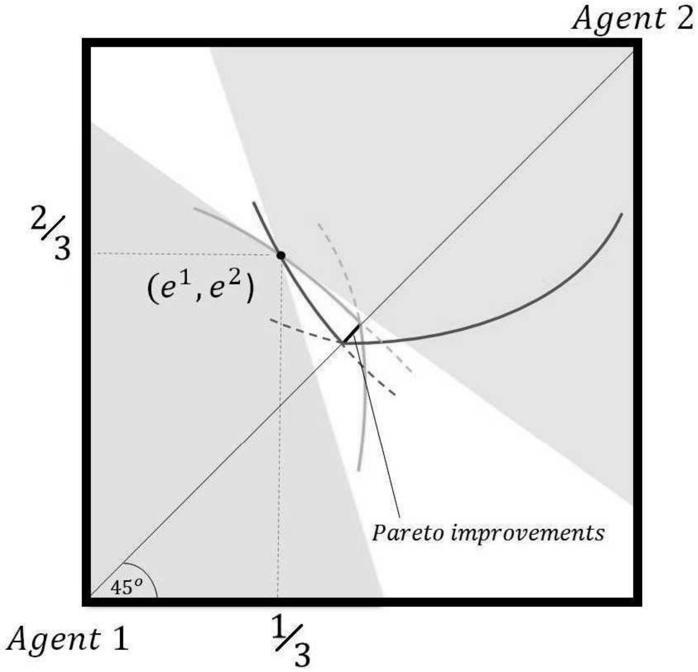

Figure 1: An Edgeworth box for Example 3

The example and Figure 2 uses the fact that we are in a simple world with two states and two agents. In general, the situation will be more involved. Nevertheless, the discontinuity when passing from a risk economy to $\mathcal{E}^{\{P\}}$ to a Knightian economy $\mathcal{E}^{\mathbb{E}}$ remains.

Let us now consider economies of the form

$$\mathcal{E}^{\mathbb{E}} = \Big(I, \big(e_i, U_i^{\mathbb{P}}\big)_{i=1\dots,I}, \mathbb{E}\Big)$$

with strictly positive initial endowment allocation $e = (e_1, \dots, e_I) \in \mathbb{X}_{++}^I$. Here, $\mathbb{E}$ denotes the Knightian expectation induced by the set of priors $\mathbb{P}$. We assume that aggregate endowment is constant, $\bar{e} = \sum_{i\in\mathbb{I}} e_i \in \mathbb{R}_{++}$.

Let $\mathbb{K}(\Delta)$ be the set of closed and convex subsets of $\text{int}(\Delta)$ equipped with the usual Hausdorff metric $\mathtt{d}_H$. Define the *Knight–Walras (KW) equilibrium correspondence* $\mathcal{KW} : \mathbb{K}(\Delta) \times \mathbb{X}_+^I \rightrightarrows \mathbb{X}_+^{I+1}$ via

$$\mathcal{KW}(\mathbb{P}) = \Big\{(\psi, c) \in \mathbb{X}_+^{I+1} : (\psi, c) \text{ is a KW–equilibrium in } \mathcal{E}^{\mathbb{P}}\Big\}.$$

According to Theorem 1, the set of KW–equilibria $\mathcal{KW}(\mathbb{P})$ in the economy is nonempty.

Theorem 7 *Let $\mathbb{P} : [0, 1) \to \mathbb{K}(\Delta)$ be a continuous function with $\mathbb{P}(0) = \{P_0\}$ for some $P_0 \in \text{int}(\Delta)$. For $0 < \epsilon < 1$, assume $P_0 \in \text{int}\,\mathbb{P}(\epsilon)$ and $(e_i) \notin (\mathbb{L}^{\mathbb{P}(\epsilon)})^I$.*

The Knight–Walras equilibrium correspondence

$$\epsilon \mapsto \mathcal{KW}(\mathbb{P}(\epsilon), e)$$

is discontinuous in zero.

For $0 \le \epsilon < 1$, define, as in Subsection 2.1, the Knightian expectation $\mathbb{E}_\epsilon[X] = \mathbb{E}^{\mathbb{P}(\epsilon)}[X] = \max_{P\in\mathbb{P}(\epsilon)} E^P[X]$.

The previous result shows that a tiny amount of Knightian uncertainty can substantially change equilibria. We now consider the opposite case of growing Knightian uncertainty and impose no assumption on the aggregate endowment $\bar{e} = \sum e_i$. We show that no trade is the only equilibrium if Knightian uncertainty is large enough, thus generalizing our initial Example 1.2.

Next we state a simple result on uniqueness of Knight–Walras equilibria, when no–trade is an equilibrium.

Lemma 2 *If (ψ, e) is a Knight–Walras equilibrium, then e is the unique Knight–Walras equilibrium allocation.*

Next we increase the Knightian uncertainty in the economy $\mathcal{E}^{\mathbb{P}}$. As the following result shows, if ambiguity becomes sufficiently large then there is no trade in equilibrium. Recall that we keep the standing assumption on multiple prior–utility functions.

Theorem 8 *If Knightian uncertainty is sufficiently large, every Knight–Walras–equilibrium is a no–trade equilibrium: There is a $\mathbb{P}' \in \mathbb{K}(\Delta)$ such that for every $\mathbb{P}'' \in \mathbb{K}(\Delta)$ with $\mathbb{P}'' \supset \mathbb{P}'$ we have*

$$\mathcal{KW}(\mathbb{P}'') = \mathbb{X}_{\mathbb{P}''} \times \{e\}$$

for

$$\mathbb{X}_{\mathbb{P}''} = \left\{ \psi \in \mathbb{X}_{++} \,\middle|\, \exists \mu \in \mathbb{R}^I_{++} : u_i{}'(e_i) \cdot \arg\min_{P \in \mathbb{P}''} E^P[u_i(e_i)] \cap \mu_i \psi \cdot \mathbb{P}'' \neq \emptyset, \; \forall i \in \mathbb{I} \right\}.$$

7 Conclusion

Knightian uncertainty leads naturally to nonlinear expectations derived from a set of priors. This led us to study a new equilibrium concept, Knight–Walras equilibrium, where prices are sublinear. We established existence of such equilibrium points and studied its efficiency properties. While one cannot expect fully efficient allocations, in general, the allocation of a Knight–Walras equilibrium satisfies a restricted efficiency criterion: if the authority is restricted to ambiguity–neutral trades, it cannot improve upon a Knight–Walras equilibrium allocation.

The introduction of Knightian friction on the price side rather than the utility side can have strong effects. In a world without aggregate uncertainty, no–trade equilibria result even with a tiny amount of uncertainty. The abrupt change of equilibria with respect to Knightian uncertainty has potentially strong implications for consumption–based asset pricing results which rely on the assumption of probabilistically sophisticated agents and markets. In dynamic and continuous–time models, these questions remain to be explored.

A Existence

The proof of Theorem 1 follows Debreu's game–theoretic approach. We will prove existence first in the compactified or truncated economy in order to ensure a compact valued demand correspondence. The budget set $\mathbb{B}(\psi, e_i)$, defined in (3), is in general not compact within $\mathbb{X}$, so we truncate $\mathbb{B}$ by introducing $\overline{\mathbb{B}}(\psi, e_i) = \mathbb{B}(\psi, e_i) \cap [0, 2\bar{e}]$, where $\bar{e} = \sum_i e_i$ denotes the aggregate endowment and $[0, 2\bar{e}]$ denotes the compact order interval $\{x \in \mathbb{X} : 0 \leq x \leq 2\bar{e}\}$. The corresponding truncated economy $\mathcal{E}^{\mathbb{P}}$ is given by

$$\overline{\mathcal{E}} = \left(I, \left(e_i, \overline{U}_i\right)_{i \in \mathbb{I}}, \mathbb{E} \right),$$

where $\overline{U}_i : [0, 2\bar{e}] \to \mathbb{R}$ is the restriction of U_i to the truncated consumption set $[0, 2\bar{e}]$.

To prepare the proof of the existence of Knight–Walras equilibria, we begin with an investigation of the truncated Knight-Walras budget correspondence $\overline{\mathbb{B}}$. To prove the continuity of our budget correspondence, we follow the lines of Debreu (1982). We let

$$\Delta = \left\{ \psi \in \mathbb{X}_+ : \sum_{\omega \in \Omega} \psi(\omega) = 1 \right\} .$$

Lemma 3 *Under Assumption 1, we have the following properties for the budget sets in the truncated economy.*

1. *The budget sets $\overline{\mathbb{B}}(\psi, e_i)$ are nonempty, compact, and convex for all $\psi \in \Delta$.*
2. *The correspondence $\psi \mapsto \overline{\mathbb{B}}(\psi, e_i)$ is homogeneous of degree zero.*
3. *The correspondence $\psi \mapsto \overline{\mathbb{B}}(\psi, e_i)$ is continuous at any $\psi \in \Delta$.*

Proof of Lemma 3:

1. Since $0, e_i \in \mathbb{B}(\psi, e_i)$ and $0, e_i \in [0, 2\bar{e}]$, for every $\psi, e_i \in \mathbb{X}_+$, the truncated budget set $\overline{\mathbb{B}}$ is nonempty. The untruncated budget set $\mathbb{B}(\psi, e_i)$ is the intersection of budget sets under linear prices of the form $E^P[\psi \cdot]$, that is,
$$\mathbb{B}(\psi, e_i) = \bigcap_{P \in \mathbb{P}} \mathbb{B}^P(\psi, e_i),$$
where $\mathbb{B}^P(\psi, e_i) = \left\{ c \in \mathbb{X}_+ : E^P[\psi(c - e_i)] \leq 0 \right\}$ denotes the closed and convex budget in an Arrow–Debreu economy under $\mathbb{P} = \{P\}$. The arbitrary intersection of convex (closed) sets is again convex (closed) and so is $\mathbb{B}(\psi, e_i)$. Consequently, $\overline{\mathbb{B}}(\psi, e_i)$ is nonempty, compact and convex.

2. By definition, the Knightian expectation $\mathbb{E}$ is positively homogeneous. The result then follows by the same arguments as in the case with linear price systems.

3. The order interval $[0, 2\bar{e}]$ is a compact, convex, nonempty set in $\mathbb{X} = \mathbb{R}^\Omega$. We prove the continuity of $\overline{\mathbb{B}} : \Delta \rightrightarrows [0, 2\bar{e}]$.

   To establish upper hemi-continuity, it suffices to show the closed graph property, since $\overline{\mathbb{B}}$ is compact valued by part 1. The graph of the budget correspondence $gr(\overline{\mathbb{B}}) = \{(\psi, x) \in \Delta \times [0, 2\bar{e}] : x \in \overline{\mathbb{B}}(\psi, x)\}$ is closed since $\psi \mapsto \max_{P\in\mathbb{P}} E^P[\psi x]$ is continuous for all $x \in \mathbb{X}$, by an application of Berge's maximum theorem.

Now let us consider lower hemi-continuity. Let $\psi_n \to \psi$ and $x \in \overline{\mathbb{B}}(\psi, e_i)$. Let us denote by Ψ_n the price system induced by a normalized $\psi_n \in \Delta$. We consider two cases.

Case 1: If $\Psi(x - e_i) < 0$, then by continuity, for some $\bar{n} \in \mathbb{N}$, we have $\Psi_n(x - e_i) < 0$ for $n \geq \bar{n}$. We define the following converging sequence

$$x_n = \begin{cases} x'_n \in \overline{\mathbb{B}}(\psi_n, e_i) \text{ arbitrary}, & \text{if } n \leq \bar{n} \\ x, & n > \bar{n}. \end{cases}$$

Then $x_n \to x$ and $x_n \in \overline{\mathbb{B}}(\psi_n, e_i)$.

Case 2: We now consider the case $\Psi(x - e_i) = 0$. Note that $\Psi(x' - e_i) < 0$ for $x' = \frac{e_i}{2}$: since $\mathbb{E}$ is relevant and endowments are strictly positive by assumption 1, we get

$$\Psi(x' - e_i) = \frac{1}{2}\Psi(-e_i) = \frac{1}{2}\mathbb{E}(-\psi e_i) < 0.$$

For n large,

$$\Psi_n^{\cap} = \{y \in \mathbb{X} : \Psi_n(y - e_i) = 0\} \cap \{y \in \mathbb{X} : \exists \lambda \in \mathbb{R} : y = \lambda x + (1 - \lambda)x'\}$$

is nonempty. Since $\Psi_n^{\cap}$ is the closed subset of a line, $\bar{x}_n = \arg\min_{y \in \Psi_n^{\cap}} \|y - x\|$ is unique. Now, set

$$x_n = \begin{cases} \bar{x}_n, & \textit{if } \bar{x}_n \in [x', x] \\ x, & \textit{else.} \end{cases}$$

By construction, we have $x_n \in \overline{\mathbb{B}}(\psi_n, e_i)$ and $x_n \to x$ in $\mathbb{X}$.

□

Proof of Theorem 1:

We show first existence of an equilibrium in the truncated economy $\overline{\mathcal{E}} = \left(I, \left(e_i, \overline{U}_i\right)_{i\in\mathbb{I}}, \mathbb{E}\right)$ and verify later that this candidate is also an equilibrium in the original economy $\mathcal{E}^{\mathbb{P}}$.

The existence proof of an equilibrium in $\overline{\mathcal{E}}$ is divided into six steps.

1. *Continuity of the Budget correspondence:* By Assumption 1, each initial endowment e_i is strictly positive. The continuity of the correspondence $\overline{\mathbb{B}} : \Delta \rightrightarrows [0, 2\bar{e}]$ follows from Lemma 3.3.

2. *Properties of the demand correspondence:* Consider the (truncated) demand correspondence $\overline{X}_i : \Delta \Rightarrow [0, 2\bar{e}]$. By step 1, $\overline{\mathbb{B}}(\cdot, e_i) : \Delta \Rightarrow [0, 2\bar{e}]$ is continuous, hence by Berge's maximum theorem the demand

$$\overline{X}_i(\psi) = \arg \max_{x \in \overline{\mathbb{B}}(\psi, e_i)} \overline{U}_i(x)$$

is upper hemi-continuous, compact and non–empty valued, since $\overline{U}_i$ is continuous on $gr(\overline{\mathbb{B}})$. By quasi–concavity of U_i, $\overline{X}_i(\psi)$ is convex-valued.

3. *Walrasian Auctioneer:* Define the Walrasian price adjustment correspondence $W : [0, 2\bar{e}]^I \times \mathbb{P} \Rightarrow \Delta$ via

$$W(x_1, \ldots, x_I, P) = \arg \max_{\psi \in \Delta} E^P \left[\psi \sum_{i \in \mathbb{I}} \Big(x_i - e_i \Big) \right].$$

As W consists of the maximizers of a linear functional over a compact set, the correspondence is upper hemi-continuous by Berge's Maximum Theorem, and it attains convex, compact and nonempty values.

4. *Knightian Auctioneer:* Define the Knightian adjustment correspondence $K : [0, 2\bar{e}]^I \times \Delta \Rightarrow \mathbb{P}$ via

$$K(x_1, \ldots, x_I, \psi) = \arg \max_{P \in \mathbb{P}} E^P \left[\psi \sum_{i \in \mathbb{I}} \Big(x_i - e_i \Big) \right].$$

Once again, by Berge's Maximum Theorem, the correspondence is upper hemi-continuous. Since E^P is linear and $\mathbb{P}$ is convex, K is convex–, compact– and nonempty–valued.

5. *Existence of a Fixed-Point:* Set $\overline{X} = \left(\overline{X}_1, \ldots, \overline{X}_I\right)$. Putting things together we have the combined correspondence

$$\left[K\overline{X}W\right] : \mathbb{P} \times [0, 2\bar{e}]^I \times \Delta \Rightarrow \mathbb{P} \times [0, 2\bar{e}]^I \times \Delta$$

as a product of nonempty–, compact–, and convex –valued upper hemi-continuous correspondences (see step 2, 3 and 4). Consequently, a fixed–point

$$\left(\bar{P}, \bar{x}_1, \ldots \bar{x}_I, \bar{\psi}\right) \in \left[K\overline{X}W\right] \left(\bar{P}, \bar{x}_1, \ldots \bar{x}_I, \bar{\psi}\right)$$

exists by an application of Kakutani's fixed-point theorem.

6. *Feasibility:* We check the feasibility of the fixed-point allocation $\bar{x}$.

   By the budget constraint, the sublinearity of $c \mapsto \mathbb{E}[\bar{\psi}c]$ (since $\bar{\psi} \geq 0$), we get for the fixed point $\left(\bar{P}, \bar{x}_1, \dots \bar{x}_I, \bar{\psi}\right)$

$$\begin{aligned} 0 &\geq \sum_i \mathbb{E}\Big[\bar{\psi}(\bar{x}_i - e_i)\Big] \\ &\geq \mathbb{E}\Big[\bar{\psi}\sum_i(\bar{x}_i - e_i)\Big] \\ &= E^{\bar{P}}\Big[\bar{\psi}\sum_i(\bar{x}_i - e_i)\Big] \geq E^{\bar{P}}\Big[\psi\sum_i(\bar{x}_i - e_i)\Big]. \end{aligned} \tag{6}$$

   The first inequality follows from the definition of the budget set and $\bar{x}_i \in \overline{X}_i(\psi)$ for all $i \in \mathbb{I}$. The last inequality holds for all $\psi \in \Delta$ and by the positive homogeneity of linear expectations, it holds even for all $\psi \in \mathbb{X}_+$. We thus have $l\left(\sum_{i\in\mathbb{I}} \bar{x}_i - e_i)\right) \leq 0$ for all positive linear forms on $\mathbb{X}$. This implies $\sum_{i\in\mathbb{I}}(\bar{x}_i - e_i) \leq 0$.

   For the feasibility of the equilibrium allocation, the truncation is irrelevant.

7. *Maximality in $\mathcal{E}^{\mathbb{P}}$:* Since $\bar{x}_i \in \overline{X}_i(\bar{\psi})$, we have

$$\bar{x}_i \in \operatorname{argmax}_{x\in\mathbb{B}(\psi,e_i)\cap[0,2\bar{e}]} \bar{U}_i(x).$$

   We have to show that $\bar{x}_i$ also maximizes U_i on $\mathbb{B}(\psi, e_i)$. Suppose there is a $x \in \mathbb{B}(\psi, e_i)$ in the original budget set, such that $U_i(x) > U_i(\bar{x}_i)$. Then we have for some $\lambda \in (0,1)$, $\lambda x + (1-\lambda)\bar{x}_i \in \overline{\mathbb{B}}(\psi, e_i) = \mathbb{B}(\psi, e_i) \cap [0, 2\bar{e}]$. The semi–strict quasi–concavity of U_i yields $U_i(\lambda x + (1-\lambda)\bar{x}_i) > U_i(\bar{x}_i)$, a contradiction. Therefore, $(\bar{x}_1, \dots, \bar{x}_I, \bar{\psi})$ is also an equilibrium in the original economy $\mathcal{E}^{\mathbb{P}}$.

□

B Proofs

Proof of Proposition 1:

1. Suppose that for some $c \in \mathbb{X}_+\backslash\{0\}$ we have $\Psi(c) \leq 0$. Then sublinearity of Ψ and the equilibrium budget constraint imply that $\tilde{c} = \hat{c}_1 + c$ satisfies $\Psi(\tilde{c} - e_1) \leq \Psi(c) + \Psi(\hat{c}_1 - e_1) \leq 0$. Hence, $\tilde{c}$ is in the budget set of agent 1; by strict monotonicity, $U_1(\tilde{c}) > U_1(\hat{c}_1)$, a contradiction to the equilibrium conditions.

2. For $x, y, z \in \mathbb{X}$, assume that $x = y + z$ holds true. Buying the asset x at cost $\Psi(x)$ and selling y and z separately yields no profits. Let P_z minimize $E^P[\psi z]$ over $\mathbb{P}$. Then

$$\begin{aligned} -\Big(\Psi(-y) + \Psi(-z)\Big) &= \min_{P\in\mathbb{P}} E^P[\psi y] + \min_{P\in\mathbb{P}} E^P[\psi z] \\ &\leq E^{P_z}[\psi y] + E^{P_z}[\psi z] \\ &= E^{P_z}[\psi x] \\ &\leq \max_{P\in\mathbb{P}} E^P[\psi x] = \Psi(x). \end{aligned}$$

Similarly, selling the asset x short and buying y and z separately yields no profits because of the sublinearity of Ψ. Let P_x minimize $E^P[\psi z]$ over $\mathbb{P}$. Then

$$\begin{aligned} \Psi(y) + \Psi(z) &= \max_{P\in\mathbb{P}} E^P[\psi y] + \max_{P\in\mathbb{P}} E^P[\psi z] \\ &\geq E^{P_x}[\psi y] + E^{P_x}[\psi z] \\ &= E^{P_x}[\psi x] \\ &= \min_{P\in\mathbb{P}} E^P[\psi x] = -\Psi(-x). \end{aligned}$$

□

PROOF OF THEOREM 2: Let $(\psi, (c_i))$ be an Arrow–Debreu equilibrium for the (linear) economy $\mathcal{E}^{\{P\}}$. Then markets clear.

Suppose first that the value of net demands $\xi_i = \psi(c_i - e_i)$ are ambiguity–free in the mean for all agents i. We need to check that c_i is in agent i's budget set for the Knightian economy $\mathcal{E}^{\mathbb{P}}$, and optimal. By assumption, we have

$$E^Q[\psi(c_i - e_i)] = k$$

for all $Q \in \mathbb{P}$ for some constant k. As c_i is budget–feasible in $\mathcal{E}^{\mathbb{P}}$ and utility functions are locally non–satiated by Assumption 1, we have $k = 0$, i.e.

$$\mathbb{E}[\psi(c_i - e_i)] = E^P[\psi(c_i - e_i)] = 0\,.$$

As c_i is part of an Arrow–Debreu equilibrium, c_i is optimal in the linear budget set given by the prior P; this budget set contains the budget of the Knightian economy $\mathcal{E}^{\mathbb{P}}$, defined in (3). Hence, c_i is optimal for agent i in the Knightian economy. We conclude that $(\psi, (c_i))$ is a Knight–Walras equilibrium for $\mathcal{E}^{\mathbb{P}}$.

Now suppose that $(\psi, (c_i))$ is a Knight–Walras equilibrium. We need to check that all ξ_i have expectation zero under all $P \in \mathbb{P}$ for all i.

As utility functions are locally non–satiated, the budget constraint is binding for all agents, $\mathbb{E}[\xi_i] = 0$ for all i. It is enough to show that $\mathbb{E}[-\xi_i] = 0$ for all i (because this entails $\min_{P\in\mathbb{P}} E^P[\xi_i] = \max_{P\in\mathbb{P}} E^P[\xi_i] = 0$.) By sublinearity, we have $\mathbb{E}[-\xi_i] \geq 0$. Market clearing implies

$$\mathbb{E}[-\xi_i] = \mathbb{E}\left[\sum_{j\neq i} \xi_j\right] \leq \sum_{j\neq i} \mathbb{E}[\xi_j] = 0\,.$$

We conclude that $\mathbb{E}[-\xi_i] = 0$ for all i, as desired. □

Proof of Theorem 3: Let (e_i) be an allocation in N. Let $(\psi, (c_i))$ be an Arrow–Debreu equilibrium of the economy $\big(I, (e_i, U_i)_{i\in\mathbb{I}}, \{P\}\big)$ which is also a Knight–Walras equilibrium of $\mathcal{E}^{\mathbb{P}}$.

Due to our assumptions, Proposition 9 in Rigotti, Shannon, and Strzalecki (2008) yields that (c_i) is a full insurance allocation. Utility maximization in Arrow–Debreu equilibrium implies that for some $\lambda_i > 0$ we have $\lambda_i\psi \cdot P \in \pi_i$ for all i; in particular, there exists $Q \in \bigcap_{i=1}^{I} \pi_i \subset \mathbb{P}$ such that $\lambda_i \Psi(c_i - e_i) = E^Q[c_i - e_i]$. Therefore, the allocation (c_i) and the price $\tilde{\psi} = 1$ form an Arrow–Debreu equilibrium in the economy $\big(I, (e_i, U_i)_{i\in\mathbb{I}}, \{Q\}\big)$.

From Theorem 2, we then know that $(c_i - e_i) \in \mathbb{L}$. As c_i is constant, it belongs to $\mathbb{L}$; as $\mathbb{L}$ is a vector space by Lemma 1, we conclude $e_i = -(c_i - e_i) + c_i \in \mathbb{L}$. As the vector space $\mathbb{L}$ has strictly smaller dimension than $\mathbb{X}$, again by Lemma 1, we conclude that N is a null set in M. □

Proof of Lemma 1:

1. Let $\mathbb{L}$ denote the set of all contingent plans which are ambiguity–free in mean. Constant plans are obviously ambiguity–free in mean, hence $\mathbb{L}$ is not empty. As expectations are linear, the property of being ambiguity–free in mean is preserved by taking sums and scalar products. Hence, $\mathbb{L}$ is a subspace of $\mathbb{X}$.

   If $\#\mathbb{P} > 1$, we have $P_1, P_2 \in \mathbb{P}$ such that $P_1 - P_2 \neq 0 \in \mathbb{X}$. In abuse of notation $x \in \mathbb{X}$ is $\{P_1, P_2\}$–ambiguity–free in the mean, if

   $$\langle P_1, x\rangle = \langle P_2, x\rangle.$$

   This equation yields a hyperplane $H = \{x \in \mathbb{X} : \langle P_1 - P_2, x\rangle = 0\}$, with $0 \in H$. Consequently H is subvector space of $\mathbb{X}$ with strictly smaller dimension and contains all plans being $\{P_1, P_2\}$–ambiguity free in mean.

   The result follow from the first part and $\{P_1, P_2\} \subset \mathbb{P}$ implies $\mathbb{L} \subset H$.

2. As ξ is ambiguity–free in mean, we have $E^P[\xi] = \mathbb{E}[\xi]$ for all $P \in \mathbb{P}$. As $\mathbb{E}$ is additive with respect to constants, we obtain

$$\mathbb{E}[X] + \mathbb{E}[\xi] = \max_{P\in\mathbb{P}} E^P[X] + \mathbb{E}[\xi] = \max_{P\in\mathbb{P}} \left(E^P[X] + \mathbb{E}[\xi]\right)$$
$$= \max_{P\in\mathbb{P}} \left(E^P[X] + E^P[\xi]\right) = \max_{P\in\mathbb{P}} E^P[X+\xi] = \mathbb{E}[X+\xi]\,.$$

□

PROOF OF PROPOSITION 2: From Proposition 2 of Rigotti, Shannon, and Strzalecki (2008), we have that a risk–neutral multiple–prior expected utility (see Example 2) with state dependent utility index $u(\omega, c) = \psi(\omega)u(c)$, satisfies

$$\pi^{MEU}(c) \quad = \quad \left\{ \frac{q}{\|q\|} : q = \psi P \text{ for some } P \in \arg\min_{P\in\mathbb{P}} E^P[\psi c] \right\}.$$

Using $\min(\cdot) = -\max(-\cdot)$ and the definition of φ_Ψ, the result follows. □

PROOF OF THEOREM 4:

The condition

$$\pi_i(c_i) \cap \varphi_\Psi(c_i - e_i) \neq \emptyset$$

is the necessary and sufficient first–order condition for the utility maximization problem of agent i in our non–differentiable setup. Let us denote the sub-differential of a convex function f at by $x \in \mathbb{X}$ by $\partial f(x) = \{Df(x) \in \mathbb{X} : f(y) - f(x) \geq Df(x)(y - x)\ \forall y \in y \in \mathbb{X}\}$. Clearly $-U_i$ and Ψ are convex. Optimality of c_i for agent $i's$ problem is then characterized by $0 \in \partial - U_i(c_i) + \partial\mu_i\Psi(c_i - e_i)$, for some $\mu_i \geq 0$, for all $i \in \mathbb{I}$. This yields $\partial - U_i(c_i) \cap \mu_i\partial\Psi(c_i - e_i) \neq \emptyset$. The assumption of an interior allocation makes each consumption c_i not binding to the positivity constraint $c_i \in \mathbb{X}_+$. Hence, $\mu_i > 0$ and the result follows after an appropriate normalization, since subjective beliefs π_i and subjective pricing measures φ are collinear with the respective to the sub-differentials $\partial - U_i$ and $\partial\Psi$.

□

PROOF OF THEOREM 5: Let (ψ, c) be a Knight–Walras equilibrium and assume that (c_i) is efficient.

Due to our assumptions and Proposition 9 in Rigotti, Shannon, and Strzalecki (2008), (c_i) is a full insurance allocation. As the utility functions are differentiable at certainty, the subjective belief π_i is a singleton; as the agents share a common subjective belief, we have $\pi_i = \{Q\}$ for some $Q \in \mathbb{P}$. By Proposition 2, $Q \in \varphi_\Psi(c_i - e_i)$; in particular,

$$E^Q[c_i - e_i] = 0$$

for all $i = 1, \ldots, I$. We conclude that $\tilde{\psi} = 1$ and (c_i) form an Arrow–Debreu equilibrium in the economy $\mathcal{E}^{\{Q\}}$ because (c_i) is feasible and satisfies the (necessary and sufficient) first–order condition of utility maximization under the Arrow–Debreu budget constraint. Theorem 3 concludes the proof. □

Proof of Theorem 6: Let (ψ, c) be a Knight–Walras equilibrium of the Knightian economy $\mathcal{E} = \left(I, (e_i, U_i)_{i\in\mathbb{I}}, \mathbb{P}\right)$. Suppose there is a feasible allocation $d = (d_i)_{i=1,\ldots,I}$ with $U_i(d_i) > U_i(c_i)$ for all $i \in \mathbb{I}$. From optimality, we have then $d_i \notin \mathbb{B}(\psi, e_i)$, or $\mathbb{E}[\eta_i] > 0$. Suppose furthermore $\eta_i = \psi\,(d_i - e_i) \in \mathbb{L}^{\mathbb{E}}$. Take any prior $P \in \mathbb{P}$. As the net excess demand is ambiguity–free in mean, we have

$$E^P[\eta_i] = \mathbb{E}[\eta_i] > 0\,.$$

As the expectation under P is linear, we obtain by summing up and feasibility of the allocation d

$$0 = E^P\left[\sum_{i=1}^{I} \psi(d_i - e_i)\right] = \sum_{i=1}^{I} E^P\left[\psi(d_i - e_i)\right] > 0\,,$$

a contradiction. □

Proof of Theorem 7: For $\epsilon = 0$, we are in an Arrow–Debreu economy without aggregate uncertainty. As a consequence, $\mathcal{KW}(\mathbb{P}(0), e)$ contains only full insurance allocations.

Fix $\epsilon > 0$. Let us first show that a mapping $X : \Omega \to \mathbb{R}$ is $\mathbb{P}(\epsilon)$–ambiguity–free in mean if and only if it is constant. Due to our assumptions, $\mathbb{P}(\epsilon)$ contains a ball (relatively to Δ) around P_0 of the form

$$B_\eta(P_0) = \{Q \in \Delta : \|Q - P_0\| < \eta\}$$

for some $\eta > 0$. We use here, without loss of generality, the maximum norm in $\mathbb{R}^\Omega$.

Suppose $E^Q[X] = k$ for some $k \in \mathbb{R}$ and all $Q \in \mathbb{P}(\epsilon)$. Let $\mathtt{1} = (1, 1, \ldots, 1) \in \mathbb{X}$ denote the vector with all components equal to 1. Let $Z \in \mathbb{X}$ satisfy $Z \cdot 1 = 0$ with $\|Z\| = 1$. Then $P_0 + \tilde{\eta} Z \in B_\eta(P_0) \subset \mathbb{P}(\epsilon)$ for all $0 < \tilde{\eta} < \eta$. Hence, we have

$$c = E^{P_0 + \eta Z}[X] = E^{P_0}[X + \tilde{\eta} Z \cdot X]\,.$$

As $0 < \tilde{\eta} < \eta$ is arbitrary, $Z \cdot X = 0$ for all Z with norm 1 and $Z \cdot \mathtt{1} = 0$ follows. By linearity, this extends to all Z with $Z \cdot \mathtt{1} = 0$; it follows that X is a multiple of 1, hence constant.

In the next step, we show that $(\psi, c) \in \mathcal{KW}(\mathbb{P}(\epsilon))$ implies $c = e$. Let (ψ, c) be a Knight–Walras equilibrium for the economy $\mathcal{E}^{\mathbb{P}(\epsilon)}$. Let $\xi_i = \psi(c_i - e_i)$ be the value of net trade for agent i. Then we have $\sum_{i \in \mathbb{I}} \xi_i = 0$ by market clearing in equilibrium. As the utility functions are strictly monotone, the budget constraint is binding, so $\mathbb{E}_\epsilon[\xi_i] = 0$ for all i. From subadditivity, we get $\mathbb{E}_\epsilon[-\xi_i] \geq 0$. On the other hand, from market clearing, subadditivity, and the binding budget constraint,

$$\mathbb{E}_\epsilon\left[-\xi_i\right] = \mathbb{E}_\epsilon\left[\sum_{j \neq i} \xi_j\right] \leq \sum_{j \neq i} \mathbb{E}_\epsilon[\xi_j] = 0\,.$$

We conclude that ξ_i is ambiguity–free in mean, thus constant. Due to the budget constraint, $\xi_i = 0$. As state prices must be strictly positive in equilibrium due to strictly monotone utility functions, we conclude that $c_i = e_i$.

The Knight–Walras equilibrium correspondence is thus discontinuous in zero. □

PROOF OF LEMMA 2: Suppose there is another Knight–Walras equilibrium allocation (ψ', x) with $\emptyset \neq \mathbb{J} = \{i \in \mathbb{I} : x_i \neq e_i\}$. We have $U_j^{\mathbb{P}}(x_j) \geq U_j^{\mathbb{P}}(e_j)$ for all $j \in \mathbb{J}$.

We show $\mathbb{E}[\psi'(x_j - e_j)] > 0$ for all $j \in \mathbb{J}$, which contradicts the budget feasibility of x_j. Take some $\epsilon > 0$ and note that $U_j^{\mathbb{P}}(x_j + \epsilon e_j) > U_j^{\mathbb{P}}(x_j)$ by strict monotonicity. As x_j is optimal in the budget set, we obtain $\mathbb{E}[\psi'(x_j + \epsilon e_j - e_j)] > 0$. Letting ϵ to zero, we have $\mathbb{E}[\psi'(x_j - e_j)] \geq 0$. Now suppose $\mathbb{E}[\psi'(x_k - e_k)] = 0$ for some $k \in \mathbb{J}$. Under the assumptions for this section, $U_k^{\mathbb{P}}$ is strictly concave, we derive for any $\mu \in (0, 1)$

$$U_k^{\mathbb{P}}(\mu x_k + (1-\mu)e_k) > \mu U_k^{\mathbb{P}}(x_k) + (1-\mu)U_k^{\mathbb{P}}(e_k) \geq U_k^{\mathbb{P}}(e_k).$$

We now obtain, by the positive homgeneity of $\mathbb{E}$

$$\begin{aligned} 0 &< \mathbb{E}\left[\psi'\left(\mu x_k + (1-\mu)e_i - e_k\right)\right] \\ &= \mathbb{E}\left[\psi' \mu (x_k - e_k)\right] \\ &= \mu \mathbb{E}\left[\psi'(x_k - e_k)\right] \\ &\leq 0\,, \end{aligned}$$

a contradiction. □

PROOF OF THEOREM 8: Since utility is strictly increasing, an equilibrium state price must be strictly positive.

Under full Knightian uncertainty, $\mathbb{P} = \Delta$, the budget set of agent i is $[0, e_i]$. By strict monotonicity and convexity of preferences, the better–off set

$\{x \in \mathbb{X}_+ : U_i^{\mathbb{P}}(x) \geq U_i^{\mathbb{P}}(e_i)\}$ can be supported by a hyperplane with a strictly positive normal vector π_i. Since $U_i^{\mathbb{P}}$ is of multiple–prior type, an increase of $\mathbb{P}$ to $\mathbb{P}' \in \mathbb{K}(\Delta)$ let the better–off set $\{x \in \mathbb{X}_+ : U_i^{\mathbb{P}'}(x) \geq U_i^{\mathbb{P}'}(e_i)\}$ shrink and $P^{\pi_i} = \frac{\pi_i}{\|\pi_i\|} \in \Delta$ remains a supporting prior.

For large $\mathbb{P}'$ such that $P^{\pi_i} \in \mathbb{P}'$ for all $i \in \mathbb{I}$, all individual first–order conditions are satisfied. e_i is then optimal in $\mathbb{B}^{\mathbb{P}'}(1, e_i)$. A larger $\mathbb{P}'' \supset \mathbb{P}'$ leaves this result unchanged. An application of Lemma 2 establishes uniqueness of the no–trade equilibrium. □

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