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CONCEPTUAL BASIS FOR CALCULATIONS OF ABSORBED-DOSE DISTRIBUTIONS

Recommendations of the
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9. Geometric Considerations

The multiplicity of factors entering dosimetric calculations will rarely permit rigorous analytical solutions. As described in Sections 6 to 8, complex numerical integrations or Monte-Carlo computations are frequently required. Geometrical considerations are, then, mainly required for the utilization of whatever spatial symmetries may be inherent in a problem.

However, a common class of problems allows a rigorous treatment of the geometry of source and receptor regions and is, therefore, to be discussed first. Such problems are the determination of mean absorbed doses or absorbed-dose distributions in receptors (target regions) that are exposed to ionizing radiation from one or more source regions that contain uniformly distributed radioactive nuclides. The special condition for this class of problems requires that source and target regions are part of a uniform extended medium with boundaries sufficiently far away as to be disregardable. Often, this requirement is adequately met in dose calculations for internal emitters for which the approach considered in this section is especially designed. For unbounded uniform media, the computations are greatly simplified because multidimensional integrals over the volumes of interest can be reduced to one-dimensional integrals that contain only two functions. One of the functions, the isotropic point-source kernel, characterizes the radiation; the other function, the point-pair distance distribution, characterizes the geometry of the source and receptor regions and their relative positions.

In the following subsection, the essential definitions and general formulae will be given, then the point-source kernels will be described, and, finally, the point-pair distance distributions and related functions will be considered and will be illustrated by numerical examples. [For specific applications in the medical use of radionuclides, see MIRD (1988).]

9.1 Absorbed Dose in Receptor Regions

Assume a source region, A, that contains a uniformly distributed radionuclide with specific activity (activity per unit mass) α and consider further a receptor region, B, that is exposed to the radiation from A. The regions A and B may consist of several parts. They may

also partly or fully overlap. As stated above, regions A and B are treated as part of an infinite uniform medium of density ρ.

Let $\overline{E}(x)$ be the average energy imparted per transition to the medium at a distance up to x from a point source; this is called the *integral isotropic point-source kernel.* The derivative $e(x) = \dfrac{d\overline{E}(x)}{dx}$ is termed the *differential point-source kernel.* Isotropic point-source kernels have been computed for a variety of radiations and will be discussed in some detail in Subsection 9.3.

The absorbed-dose rate at a distance x from a volume element dV with specific activity (activity per unit mass) α is, then,

$$d\dot{D} = \alpha \frac{e(x)}{4\pi x^2} dV. \tag{9.1}$$

The absorbed-dose rate at a point due to the source region A is, accordingly,

$$\dot{D} = \alpha \int_A \frac{e(x)}{4\pi x^2} dV_A, \tag{9.2}$$

where x is the distance from the point to the volume element dV_A.

The average dose rate in the receptor region B due to radiation from the source region A is, therefore,

$$\overline{\dot{D}}_{AB} = \frac{\alpha}{V_B} \int_B \int_A \frac{e(x)}{4\pi x^2} dV_A \, dV_B, \tag{9.3}$$

where V_B is the volume of the receptor region B.

This integral, or similar integrals over two volumes with arguments that depend *only* on the distance between the two differential volume elements, are common in dosimetric calculations. It is important, therefore, that the 6-dimensional integral can be replaced by a one-dimensional integral. Berger (1970) has shown that the solution can be expressed in terms of the isotropic point-source kernel and the *point-pair distance distribution.*

The point-pair distance distribution and the related quantity, geometric reduction factor, will be considered in Section 9.4 for specified geometries. At present, it is sufficient to consider the general formulae. The point-pair distance distribution, $p_{AB}(x)$, is defined as the probability density of distances between random points uniformly distributed in A and random points uniformly distributed in B. Thus, $p_{AB}(x)\,dx$ is the probability that a random point selected in A and a random point selected in B are separated by a distance between x and $x+dx$.

Let $h(x)$ be a function of the distance x; then, one can reduce the 6-dimensional integral over the regions A and B to a one-dimensional integral

$$\int_A \int_B h(x)\, dV_A\, dV_B = V_A\, V_B \int_0^\infty h(x)\, p_{AB}(x)\, dx, \qquad (9.4)$$

where x is the distance between the two differential volume elements dV_A and dV_B, and V_A and V_B are the volumes of the regions A and B. Here, and in the following, the limits of integration in the one-dimensional integral are 0 and ∞. It is understood that $p_{AB}(x)$ may be zero in certain ranges of x, so that, effectively, the integral ranges from x_{min} to x_{max}.

Combining Equation (9.3) and (9.4), one obtains

$$\bar{\dot{D}}_{AB} = \alpha\, V_A \int_0^\infty \frac{e(x)\, p_{AB}(x)}{4\pi x^2}\, dx = \frac{a}{\rho} \int_0^\infty \frac{e(x)\, p_{AB}(x)}{4\pi x^2}\, dx, \qquad (9.5)$$

where a is the total activity in A, and the right hand expression must be used if A has zero volume.

A point-pair distance distribution usually cannot be determined if one deals with unbounded regions. It is then practical to use a modified quantity, called the *geometric reduction factor* (Berger, 1970)[7],

$$U_{AB}(x) = V_B\, p_{AB}(x)/4\pi x^2, \qquad V_B \neq 0. \qquad (9.6)$$

If B has zero volume, $U_{AB}(x)$ is not defined; but one can then utilize

$$U_{BA}(x) = V_A\, p_{AB}(x)/4\pi x^2, \qquad V_A \neq 0. \qquad (9.7)$$

One can show that the geometric reduction factor, $U_{AB}(x)$, is equal to the probability that a random point of A ends up in B, if shifted in a random direction by distance x; a more detailed discussion of the point-pair distance distribution and the geometrical reduction factor is given in Section 9.4.

With the geometrical reduction factor one can rewrite Equation (9.5) as

$$\bar{\dot{D}}_{AB} \begin{cases} = \dfrac{a}{V_B} \int e(x)\, U_{AB}(x)\, dx, & \text{for } V_B \neq 0,\ V_A \text{ finite} \\ = \alpha \int e(x)\, U_{BA}(x) dx, & \text{for other situations, but with } V_A > 0. \end{cases} \qquad (9.8)$$

[7]In the case of an extended region A, the term *average* geometric reduction factor has formerly been employed. The proximity function, $s_{AB}(x) = V_B \cdot p_{AB}(x)$, can also be used, and is closely related to $U_{AB}(x)$.

If the receptor region B is unbounded, the mean dose rate may not be meaningful; one may then consider the rate of energy imparted to B as

$$\dot{\epsilon}_{AB} \begin{cases} = a \int e(x) U_{AB}(x) dx, & \text{for } V_B > 0,\ V_A \text{ finite} \\ = \alpha\, V_B \int e(x)\, U_{BA}(x) dx, & \text{for other situations.} \end{cases} \tag{9.9}$$

As before, a and α are total activity and specific activity in A, respectively

These equations permit a variety of dosimetric computations, provided the receptor and target regions are part of an infinite or sufficiently extended uniform medium. The derivation of the quantities $e(x)$ and $U_{AB}(x)$ is usually a far simpler problem than the direct solution of Equation (9.3). More importantly, the formalism just described permits a convenient separation of radiation-transport calculations into two independent parts: the point-source kernel $e(x)$ and the geometrical factor $U_{AB}(x)$, which is independent of radiation properties and depends merely on the configuration of the source and receptor regions.

Examples for the two functions are given in Appendix C; specific problems may require separate numerical computations or Monte-Carlo calculations of the two functions. Also, it should be noted that much of the formalism described in this subsection is equivalent to the method of absorbed fractions described in ICRU Report 32 (ICRU, 1979).

9.2 Reciprocity Theorem

The definition of the point-pair distance distribution is symmetrical in A and B, *i.e.*, $p_{AB}(x) = p_{BA}(x)$. It follows, from Equation (9.5), that the mean dose rate in B from the activity a in A is equal to the mean dose rate in A from the activity a in B. This is the reciprocity theorem.

On the other hand, if the specific activity α is kept the same, the energy imparted to B from A is equal to the energy imparted to A from B. The reciprocity theorem applies, also, if one of the regions is a point; the mean dose rate in B from a point source A of activity a is equal to the absorbed dose rate at A, if the activity a is uniformly distributed in B.

It must be noted that the reciprocity theorem holds under the condition of A and B being part of a sufficiently extended uniform

medium. Approximate validity can still hold for non-uniform media. For the consideration of non-uniform media the density $\rho(\vec{x})$ will depend on position. If the specific activity (activity per unit mass) in A is constant, Equation (9.3) is to be replaced by the relation

$$\dot{D}_{AB} = \frac{a}{M_A M_B} \int_A \int_B \dot{D}(\vec{x}_A, \vec{x}_B)\, \rho(\vec{x}_A)\, \rho(\vec{x}_B)\, d^3x_A\, d^3x_B, \quad (9.10)$$

where M_A and M_B are the mass of A and B, and where the dose rate, $\dot{D}(\vec{x}_A, \vec{x}_B)$, from a unit activity at point $\vec{x}_A$ to point $\vec{x}_B$ depends on the composition and density of the medium throughout the vicinity of $\vec{x}_A$ and $\vec{x}_B$ that is relevant to the radiation transport from $\vec{x}_A$ to $\vec{x}_B$.

For the reciprocity theorem to hold, $\dot{D}(\vec{x}_A, \vec{x}_B)$ must equal $\dot{D}(\vec{x}_B, \vec{x}_A)$, i.e., an activity at point $\vec{x}_A$ must produce the same dose at point $\vec{x}_B$, as the same activity at $\vec{x}_B$ produces at $\vec{x}_A$. For scattered radiation, this identity will commonly not apply. However, for the unscattered primary radiation, it can usually be assumed. It applies strictly for the unscattered radiation if all interaction coefficients are proportional to density; in this latter case $\dot{D}(\vec{x}_A, \vec{x}_B)$ is equal to a function $h(s)/x^2$, where $h(s)$ depends only on the type of radiation and on the integral, s, of the density along the straight line between $\vec{x}_A$ and $\vec{x}_B$.

9.3 Isotropic Point-Source Kernels

Point-source kernels have been calculated for a variety of radiations; examples of the results are given below. First, two particularly simple examples will be treated, although these are of more didactic than pragmatic interest.

For photons of sufficiently low energy, E_0, only the photoelectric effect plays a role and, if energy transport by the photoelectrons, Auger electrons and fluorescent radiation is disregarded, one obtains

$$\overline{E}(x) = E_0(1 - \exp(-\mu x)),$$

and (9.11)

$$e(x) = \mu E_0 \exp(-\mu x).$$

In a first approximation, the tracks of α-particles can be treated as straight lines with continuous energy loss equal to the stopping power. With these assumptions, and if energy transport by δ-rays is disregarded, one has

$$\overline{E}(x) = E(R_0) - E(R_0 - x)$$

and (9.12)

$$e(x) = S(R_0 - x),$$

where $E(R)$ is the energy of the particle with residual range R and $S(R)$ is the stopping power at the same energy and remaining range.

Figure 9.1 gives the resulting integral, isotropic point-source kernels normalized to the initial energy E_0 for different energies, E_0, of the α-particles. The differential kernels are frequently called Bragg-curves.

The derivation of isotropic point-source kernels for electrons must account for various factors such as curvature of the particle track, energy-loss straggling and energy transport by secondary electrons. Such computations have been performed by Berger (1971, 1973) and by a number of other authors who have used Monte-Carlo simulations of charged particle tracks.

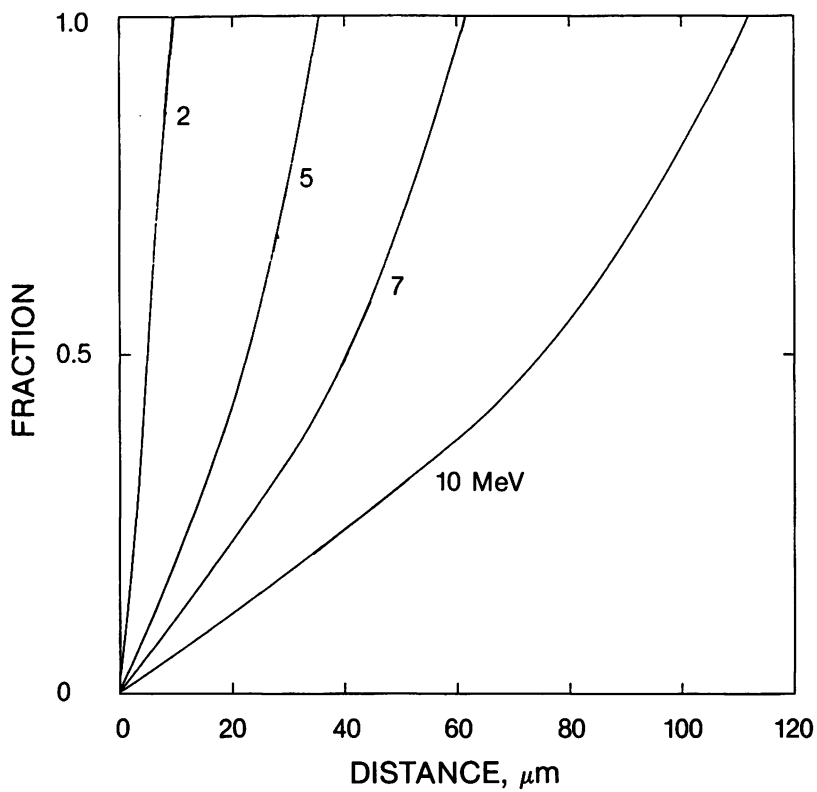

Fig. 9.1. Integral, isotropic point-source kernels, normalized to the initial energy, E_0, for α particles in water. The ordinate is the fraction of the emitted energy that is deposited along the particle track, while the abscissa is the distance that the particle has traveled. The electronic stopping power is taken from Ziegler (1980).

Figure 9.2 gives the resulting integral, isotropic point-source kernels normalized to the mean initial electron energy for beta rays and monoenergetic electrons of various energies. The point kernels are given in a scaled form, i.e., as a function of the variable x/x_{90}, where *x is the distance from the source and* x_{90} is the radius of the sphere about the source within which 90 per cent of the emitted energy is deposited. One notices that the scaled, integral point kernels are not very sensitive to the energies of the beta rays and electrons.

High energy photons require a consideration of the scattered photon contribution and, depending on the nature of the problem, also a consideration of energy transport by charged secondaries. Figure

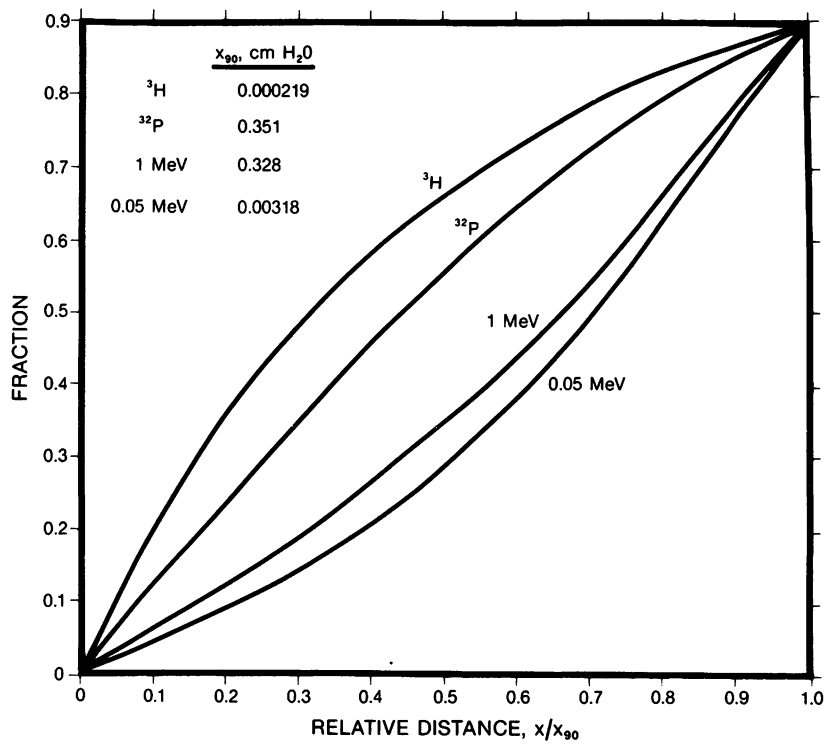

Fig. 9.2. Integral, isotropic point-source kernels, normalized to the mean initial electron energy for beta rays and monoenergetic electrons. The ordinate is the fraction of the emitted energy that is deposited within a sphere of radius x about the point source. The abscissa is the relative distance, x/x_{90}, where x_{90} is the radius of the sphere within which 90 per cent of the energy is deposited. The results are for interactions with water, and x_{90} distances are shown in the tabulation in the figure. The curves were developed from data in Berger (1971).

9.3 gives a numerical example for a photon, isotropic point-source kernel. Neutrons present a similar problem where the consideration of the scattered radiation is essential.

9.4 Point-Pair Distance Distributions and Geometric Reduction Factors

The point-pair distance distribution and the related quantity, geometric reduction factor, can, in principle, be computed for any configuration of interest. However, the presence of complicated multiple integrals may make the numerical work difficult. Monte-Carlo methods have been developed that simplify calculation of multiple inte-

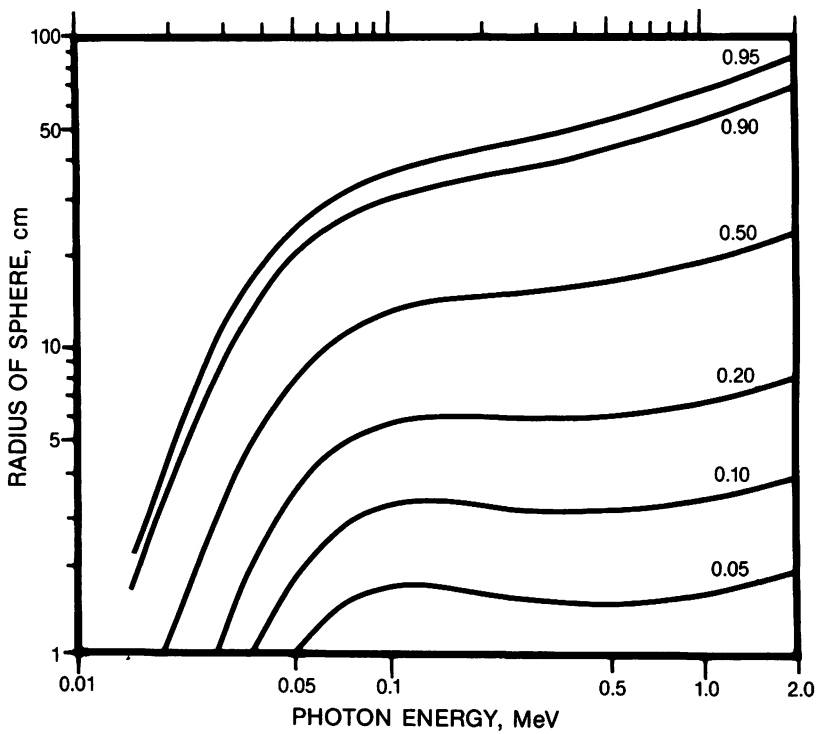

Fig. 9.3. Energy deposition around monoenergetic, point-isotropic sources of photons with energies up to 2 MeV in an unbounded water medium. The figure shows the radii of spheres around the point source within which the indicated fractions of the emitted photon energies are absorbed. Several different plots are possible; in this case, the radii of the spheres are plotted as functions of the source photon energy. The curves were developed from data in Berger (1968).

grals. Analytical solutions exist, however, for a variety of simple geometries.

Two important cases can be distinguished and will be identified by the terms *autologous* and *heterologous*. The term autologous refers to the condition that source region and target coincide, i.e., A = B. The term heterologous refers to cases where source region and target differ, although they may partly overlap.

A special case is the condition that either the source region or the target region is a point while the other region is an extended domain. In certain cases, this situation can be treated rigorously, and, then, it is possible to obtain relatively simple integrals in the calculation of the spatial distribution of absorbed dose around extended source regions.

In the autologous case, the point-pair distance distributions are a familiar concept of geometrical probability, and they have—although for different applications, or from a purely mathematical standpoint—been treated in the literature [see, for example, Kendall and Moran (1963), Weil (1983), Stoyan *et al.* (1987)]. There is also a relation between the geometric reduction factor, $U(x)$, for a convex body and the chord-length distributions, $f(x)$, that result when the body is randomly traversed by straight lines that are uniformly and isotropically distributed (Kellerer, 1971, 1984):

$$f(x) = (4V/S)\,\frac{d^2U(x)}{dx^2}, \tag{9.13}$$

where V and S are volume and surface of the body, and $4V/S$ is its mean chord length. Furthermore, for a convex body, $U(x)$ is the sum distribution of the lengths of rays in random directions from a random point in A to the surface of A.

For both the autologous and the heterologous cases, the two quantities $p_{AB}(x)$ and $U_{AB}(x)$ are equivalent. The definition of the point-pair distance distribution $p_{AB}(x)$ as the probability density of distances between pairs of random points leads to solutions for the spheroid and for the cylinder [Equations (C.7 and C.10) in Appendix C]. The separate definition of $U_{AB}(x)$ in Subsection 9.1 can be formulated in a way that is also suitable for computations: If $A(x,\theta)$ is the translate of A by a distance x in the direction θ, then $U_{AB}(x)$ is numerically equal to the volume of the intersection $A(x,\theta) \cap B$ uniformly averaged over all directions and divided by V_A (for related considerations see Enns and Ehlers, 1978). This form of the definition leads to the solution for the sphere or for two concentric spheres A and B [see Equations (C.3 and C.15) in Appendix C]. The relation

$$U_S(x) = (V_A U_A(x) + V_B\, U_B(x) - 2\, V_A\, U_{AB}(x)) \,/\, (V_A - V_B) \tag{9.14}$$

is then used to obtain the geometric reduction factor for the spherical shell S = A − B [see Equation (C.4)].

Finally, it may be noted that the geometric reduction factor can also be defined in two-dimensional space. In this case, it can be calculated from the point-pair distance distribution by the relation

$$U_{AB}(x) = S_B p_{AB}(x) / 2\pi x, \tag{9.15}$$

where S_B is the area of B.