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Dimensional reductions of DFT and mirror symmetry for Calabi–Yau three-folds and $K3 \times T^2$

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Abstract

We perform dimensional reductions of type IIA and type IIB double field theory in the flux formulation on Calabi–Yau three-folds and on $K3 \times T^2$. In addition to geometric and non-geometric three-index fluxes and Ramond–Ramond fluxes, we include generalized dilaton fluxes. We relate our results to the scalar potentials of corresponding four-dimensional gauged supergravity theories, and we verify the expected behavior under mirror symmetry. For Calabi–Yau three-folds we extend this analysis to the full bosonic action including kinetic terms.

1. Introduction

One of the important problems in string phenomenology is moduli stabilization. Moduli are massless scalar fields which arise when compactifying string theory and which are inconsistent with experimental observations. A way to address this issue is to turn on background fluxes on the internal manifold (see, e.g. [1–3] for reviews on the topic). At string tree-level, this creates a scalar potential that can stabilize the moduli parametrizing the vacuum degeneracy. It was, however, found that successive application of T-duality transformations to backgrounds with fluxes gives rise to geometrically ill-defined objects [4,5] which play an essential role in obtaining full

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moduli stabilization. Constructing phenomenologically realistic models from flux compactifications therefore requires suitable frameworks allowing for a mathematical description of such "non-geometric" backgrounds.

One natural approach is to relax the Calabi–Yau condition and only assume the existence of a nowhere vanishing spinor on the compactification manifold. As a consequence, Calabi–Yau manifolds are replaced by more general $SU(3)$ structure manifolds, which had previously been shown to arise as mirror symmetry duals of Calabi–Yau backgrounds with non-vanishing Neveu–Schwarz–Neveu-Schwarz (NS–NS) fluxes [6–8]. Focusing on type II theories and going one step further, this idea can be generalized by assuming the existence of a pair of non-vanishing spinors, one for each of the ten-dimensional supercharges. This is the underlying idea of compactifications on $SU(3) \times SU(3)$ structure manifolds. Such compactifications have been extensively studied in [6–18]. Interestingly, the latter show a natural connection to Hitchin's generalized geometry [19,20], where in this picture $SU(3) \times SU(3)$ appears as the structure group of the generalized tangent bundle $TM^6 \oplus T^*M^6$ of the internal manifold M^6.

In this paper, we will go another step further and consider compactifications of type II actions in the framework of double field theory (DFT) [21–25] (see also [26–28] for pedagogical reviews). In addition to the generalized tangent bundle, in DFT spacetime itself is doubled, allowing for a description of ten-dimensional supergravities in which T-duality becomes a manifest symmetry. In particular, it has been shown that there exists a "flux formulation" [29] of DFT in which geometric as well as non-geometric background fluxes arise naturally as constituents of the action and can locally be described as operators acting on differential forms.

It was found that compactifications and Scherk–Schwarz reductions of DFT yield the scalar potential of electrically gauged $\mathcal{N}=4$ supergravity in four dimensions [30–32]. More recently, a connection between Calabi–Yau compactifications of DFT and the scalar potential of four-dimensional $\mathcal{N}=2$ gauged supergravity was derived explicitly [33]. The purpose of the present paper is to add to the picture by generalizing the considered setting of [33] to a wider class of compactification manifolds and non-vanishing dilaton fluxes. We furthermore extend the formalism to dimensional reductions of the full DFT action by including the kinetic terms. This will eventually enable us to show how in DFT IIA $\leftrightarrow$ IIB Mirror Symmetry is restored due to the simultaneous presence of geometric and non-geometric fluxes.

In this paper we discuss the technical details of our analysis in some length, and therefore want to briefly summarize the main results of our work. In particular, the paper is organized as follows:

- In section 2, we provide a brief review on the framework of DFT. The section is concluded by a short presentation of the flux formulation and related notions which will be important for this paper.
- In section 3, we compactify the purely internal part of the type IIA and IIB DFT action on a Calabi–Yau three-fold. In doing so, we mainly rely on the elaborations of [33] although we slightly generalize this approach. Both results are related to the scalar potential of four-dimensional $\mathcal{N}=2$ gauged supergravity constructed in [34], and a first manifestation of Mirror Symmetry is discussed.
- In section 4, the discussion of section 3 is repeated for the compactification manifold $K3 \times T^2$ with the inclusion of dilaton fluxes. The necessary steps to generalize the Calabi–Yau setting are highlighted, and the special geometric properties of $K3 \times T^2$ are discussed in detail. The resulting four-dimensional scalar potential is related to the framework of [34], and a set of mirror mappings is constructed. A DFT origin of the $\mathcal{N}=4$ gauged super-

gravity scalar potential has already been elaborated in the previous works [30,31] using Scherk–Schwarz reductions, however, here we follow a different approach by employing the formalism of generalized Calabi–Yau geometry [19] and generalized K3 surfaces [35], giving rise to a scalar potential formulated in the language of $\mathcal{N}=2$ gauged supergravity. While the result shows characteristic features of its $\mathcal{N}=4$ counterpart, its relation to those of [30,31] seems to be nontrivial and will be investigated in future work.

- In section 5, we extend the setting of section 3 by including the kinetic terms. We use a generalized Kaluza–Klein ansatz [30,31,36] and treat the NS–NS and Ramond–Ramond (R–R) sectors separately. For the former, we will mostly rely on the results of section 3 and on the standard literature on Calabi–Yau compactifications of type II theories. The latter is more involved and gives rise to democratic type II supergravities with all known NS–NS fluxes (including the non-geometric ones) and R–R fluxes turned on. We first reduce the ten-dimensional equations of motion, following a similar pattern as done in [37] for manifolds with $SU(3)\times SU(3)$ structure. The resulting four-dimensional equations of motion can then be shown to originate from the four-dimensional $\mathcal{N}=2$ gauged supergravity action constructed in [34], where a subset of the axions appearing in the standard formulation is dualized to two-forms in order to account for both electric and magnetic charges. This will finally enable us to once more read off a set of mirror mappings between the full reduced type IIA and IIB actions.
- Section 6 concludes the discussion by summarizing the results and giving an outlook on open questions and possible future developments.

Throughout this work, we consider a doubled analogue of the spacetime manifold $M^{10} = M^{1,3}\times M^6$, where $M^{1,3}$ denotes a four-dimensional Lorentzian manifold and M^6 is an arbitrary Calabi–Yau three-fold or $K3\times T^2$. Moreover, we will apply the framework of special geometry in order to describe the complex structure and Kähler class moduli spaces of M^6. Due to the large number of distinct indices used in this paper, we provide an accessible indexing system in appendix A.

2. Basics of double field theory

This section will provide a brief overview on the notions of DFT, which form the basis of our upcoming considerations. For more details, we would like to refer the reader to [26–28].

2.1. Doubled spacetime

The basic idea of DFT is to enhance ordinary supergravity theories with additional structures in a way that T-duality becomes a manifest symmetry. Motivated by the insights from toroidal compactifications of the bosonic string, one doubles the dimension of the D-dimensional spacetime manifold M by introducing additional *winding coordinates* $\tilde{x}_{\hat{m}}$ conjugate to the winding number $\tilde{p}^{\hat{m}}$ (just as the normal spacetime coordinates $x^{\hat{m}}$ relate to the momenta $p_{\hat{m}}$) and arrange them in doubled coordinates

$$X^{\hat{M}} = \left(\tilde{x}_{\hat{m}}, x^{\hat{m}}\right), \; P_{\hat{M}} = \left(\tilde{p}^{\hat{m}}, p_{\hat{m}}\right) \qquad \text{with} \qquad \hat{m}=1,\dots D \text{ and } \hat{M}=1,\dots 2D. \tag{2.1}$$

The corresponding derivatives are denoted by

$$\partial_{\hat{m}} = \frac{\partial}{\partial x^{\hat{m}}}, \quad \tilde{\partial}^{\hat{m}} = \frac{\partial}{\partial \tilde{x}_{\hat{m}}}. \tag{2.2}$$

The spacetime manifold is locally equipped with the *generalized tangent bundle*

$$E = TM \oplus T^*M \tag{2.3}$$

and the $O(D, D, \mathbb{R})$ *invariant structure*

$$\eta_{\hat{M}\hat{N}} = \begin{pmatrix} 0 & \delta^{\hat{m}}{}_{\hat{n}} \\ \delta_{\hat{m}}{}^{\hat{n}} & 0 \end{pmatrix} = \eta^{\hat{M}\hat{N}} \tag{2.4}$$

defining the standard inner product of doubled vectors and taking the same role as the Minkowski metric in general relativity. The spacetime metric $\hat{g}_{\hat{m}\hat{n}}$ and the Kalb–Ramond field $\hat{B}_{\hat{m}\hat{n}}$ are repackaged into the *generalized metric*

$$\hat{\mathcal{H}}_{\hat{M}\hat{N}} = \begin{pmatrix} \hat{g}^{\hat{m}\hat{n}} & -\hat{g}^{\hat{m}\hat{p}}\hat{B}_{\hat{p}\hat{n}} \\ \hat{B}_{\hat{m}\hat{p}}\hat{g}^{\hat{p}\hat{n}} & \hat{g}_{\hat{m}\hat{n}} - \hat{B}_{\hat{m}p}\hat{g}^{\hat{p}\hat{q}}\hat{B}_{\hat{q}\hat{n}} \end{pmatrix}, \tag{2.5}$$

whose structure is closely related to the Buscher rules for T-duality transformations [38,39]. It defines a function $\hat{\mathcal{H}}_{\hat{M}\hat{N}}(X)$ of the doubled coordinates and parametrizes the coset space $\frac{O(D,D,\mathbb{R})}{O(D,\mathbb{R})\times O(D,\mathbb{R})}$. Similarly to general relativity, indices in DFT are raised and lowered by the $O(D, D, \mathbb{R})$ invariant metric $\eta_{\hat{M}\hat{N}}$ and $\eta^{\hat{M}\hat{N}}$, respectively. In particular, one obtains the relation

$$\hat{\mathcal{H}}^{\hat{M}\hat{N}} = \eta^{\hat{M}\hat{P}}\hat{\mathcal{H}}_{\hat{P}\hat{Q}}\eta^{\hat{Q}\hat{N}}, \tag{2.6}$$

implying the existence of a *generalized vielbein* $\hat{\mathcal{E}}^{\hat{A}}{}_{\hat{M}}$ satisfying

$$\hat{\mathcal{H}}_{\hat{M}\hat{N}} = \hat{\mathcal{E}}^{\hat{A}}{}_{\hat{M}}\hat{\mathcal{E}}^{\hat{B}}{}_{\hat{N}}S_{\hat{A}\hat{B}}. \tag{2.7}$$

Here, $\hat{M}, \hat{N}$ denote curved spacetime indices, and $\hat{A}, \hat{B}$ are flat tangent space indices. One can thus choose

$$S_{\hat{A}\hat{B}} = \begin{pmatrix} s^{\hat{a}\hat{b}} & 0 \\ 0 & s_{\hat{a}\hat{b}} \end{pmatrix}, \tag{2.8}$$

where $s_{\hat{a}\hat{b}}$ denotes the flat D-dimensional Minkowski metric. Using the vielbein $\hat{e}^{\hat{a}}{}_{\hat{m}}$ defined by the relation

$$\hat{g}_{\hat{m}\hat{n}} = \hat{e}^{\hat{a}}{}_{\hat{m}}s_{\hat{a}\hat{b}}\hat{e}^{\hat{b}}{}_{\hat{n}}, \tag{2.9}$$

$\hat{\mathcal{E}}^{\hat{A}}{}_{\hat{M}}$ can be parametrized as

$$\hat{\mathcal{E}}^{\hat{A}}{}_{\hat{M}} = \begin{pmatrix} \hat{e}_{\hat{a}}{}^{\hat{m}} & -\hat{e}_{\hat{a}}{}^{\hat{p}}\hat{B}_{\hat{p}\hat{m}} \\ 0 & \hat{e}^{\hat{a}}{}_{\hat{m}} \end{pmatrix}. \tag{2.10}$$

An action for DFT is determined by requiring invariance of the theory under local doubled diffeomorphisms

$$X^{\hat{M}} = \left(\tilde{x}_{\hat{m}}, x^{\hat{m}}\right) \to \left(\tilde{x}_{\hat{m}} + \tilde{\xi}_{\hat{m}}\left(X^{\hat{M}}\right), x^{\hat{m}} + \xi^{\hat{m}}\left(X^{\hat{M}}\right)\right) \tag{2.11}$$

and global $O(D, D, \mathbb{R})$ transformations. In conjunction with the requirement of the algebra of infinitesimal diffeomorphisms to be closed, the latter give rise to the so-called *strong constraint*

$$\eta^{\hat{M}\hat{N}}\partial_{\hat{M}}\hat{\Phi}\partial_{\hat{N}}\hat{\Psi} = 0, \tag{2.12}$$

where both $\hat{\Phi}$ and $\hat{\Psi}$ denote arbitrary fields or gauge parameters. One possible solution is given by setting $\widetilde{\partial}^{\hat{m}} = 0$, in which case the dual coordinates become unphysical and the theory reduces to ordinary supergravity. This also reveals an interpretation of T-duality transformations as rotations of a "physical section" in doubled spacetime.

2.2. *Flux formulation of double field theory*

There exist two physically equivalent formulations of DFT, differing only by terms that are either total derivatives or vanish by the strong constraint. For the purpose of this paper, working with the so-called *flux formulation* [30,31,40] (see also [21,22] for early developments) will be more convenient since it provides a natural (local) description of geometric as well as non-geometric background fluxes.

2.2.1. *NS–NS sector*

As starting point for the NS–NS sector, we consider the action [30,31,40]

$$S_{\text{NS–NS}} = \frac{1}{2}\int d^{20}X e^{-2\hat{d}}\Big[\hat{\mathcal{F}}_{\hat{M}\hat{N}\hat{P}}\hat{\mathcal{F}}_{\hat{M}'\hat{N}'\hat{P}'}\Big(\frac{1}{4}\hat{\mathcal{H}}^{\hat{M}\hat{M}'}\eta^{\hat{N}\hat{N}'}\eta^{\hat{P}\hat{P}'} - \frac{1}{12}\hat{\mathcal{H}}^{\hat{M}\hat{M}'}\hat{\mathcal{H}}^{\hat{N}\hat{N}'}\hat{\mathcal{H}}^{\hat{P}\hat{P}'} - \frac{1}{6}\eta^{\hat{M}\hat{M}'}\eta^{\hat{N}\hat{N}'}\eta^{\hat{P}\hat{P}'}\Big) + \hat{\mathcal{F}}_{\hat{M}}\hat{\mathcal{F}}_{\hat{M}'}\Big(\eta^{\hat{M}\hat{M}'} - \hat{\mathcal{H}}^{\hat{M}\hat{M}'}\Big)\Big], \tag{2.13}$$

where the *generalized dilaton* $\hat{d}$ is defined by the relation

$$e^{-2\hat{d}} = \sqrt{\hat{g}}e^{-2\phi}. \tag{2.14}$$

Employing flat coordinates and using the *generalized Weizenböck connection*

$$\hat{\Omega}_{\hat{A}\hat{B}\hat{C}} = \hat{\mathcal{E}}_{\hat{A}}{}^{\hat{I}}\left(\partial_{\hat{I}}\hat{\mathcal{E}}_{\hat{B}}{}^{\hat{J}}\right)\hat{\mathcal{E}}_{\hat{C}\hat{J}} \tag{2.15}$$

the *generalized fluxes* $\hat{\mathcal{F}}_{\hat{A}}$ and $\hat{\mathcal{F}}_{\hat{A}\hat{B}\hat{C}}$ can be written as

$$\hat{\mathcal{F}}_{\hat{A}} = \hat{\Omega}^{\hat{B}}{}_{\hat{B}\hat{A}} + 2\hat{\mathcal{E}}_{\hat{A}}{}^{\hat{I}}\partial_{\hat{I}}\hat{d} \qquad \text{and} \qquad \hat{\mathcal{F}}_{\hat{A}\hat{B}\hat{C}} = 3\hat{\Omega}_{[\underline{\hat{A}\hat{B}\hat{C}}]}, \tag{2.16}$$

where the squared brackets denote the antisymmetrization operator defined in appendix A. It will be explained in subsection 2.3.1 how these are related to the generalized fluxes with curved indices. When performing dimensional reduction, an obvious first step is to rewrite the action in terms of objects representing four-dimensional fields and assume all fields with external legs to be independent of the internal coordinates. We will do this by applying a generalized Kaluza–Klein ansatz for DFT [30,31,36], for which we split the coordinates into external and internal parts

$$X^{\hat{M}} = \left(\tilde{x}_\mu, x^\mu, Y^{\check{I}}\right), \qquad X^{\hat{A}} = \left(\tilde{x}_e, x^e, Y^{\check{A}}\right), \tag{2.17}$$

where we used the collective notation $Y^{\check{I}} = \left(\tilde{y}_{\check{i}}, y^{\check{i}}\right)$ and $Y^{\check{A}} = \left(\tilde{y}_{\check{a}}, y^{\check{a}}\right)$ for the latter. In order to preserve rigid $O(6,6,\mathbb{R})$ symmetry, we impose the section condition only on the external coordinates, therefore assuming also independence of all fields and gauge parameters of the external dual coordinates $\tilde{x}_\mu$, while leaving the dependence of purely internal fields on the doubled coordinates $Y^{\check{I}}, Y^{\check{A}}$ untouched.

For the ten-dimensional metric and Kalb–Ramond field, we employ the splitting [30]

$$\hat{g}_{\hat{m}\hat{n}} = \begin{pmatrix} g_{\mu\nu} + g_{\check{k}\check{l}} A^{\check{k}}{}_{\mu} A^{\check{l}}{}_{\nu} & A^{\check{k}}{}_{\mu} g_{\check{k}\check{j}} \\ g_{\check{i}\check{k}} A^{\check{k}}{}_{\nu} & g_{\check{i}\check{j}} \end{pmatrix}, \qquad \hat{B}_{\hat{m}\hat{n}} = \begin{pmatrix} B_{\mu\nu} & -B_{\mu\check{j}} \\ B_{\check{i}\nu} & B_{\check{i}\check{j}} \end{pmatrix} \tag{2.18}$$

and arrange the parts with mixed external and internal indices in a generalized Kaluza–Klein vector

$$\mathcal{A}^{\check{I}}{}_{\mu} = \begin{pmatrix} B_{\check{i}\mu} \\ -A^{\check{i}}{}_{\mu} \end{pmatrix}. \tag{2.19}$$

Inserting this ansatz into (2.13), the NS–NS contribution to the action can be reformulated as [30,31,36]

$$\begin{aligned} S_{\text{NS–NS}} = \frac{1}{2} \int d^4x d^{12}Y \sqrt{g^{(4)}}\sqrt{g^{(6)}} e^{-2\phi} \Big[& \\ & \widetilde{R}^{(4)} + 4 g^{\mu\nu} D_\mu \phi D_\nu \phi - \frac{1}{4} g^{\mu\nu} g^{\rho\sigma} \mathcal{H}_{\check{I}\check{J}} \widetilde{\mathcal{F}}^{\check{I}}{}_{\mu\rho} \widetilde{\mathcal{F}}^{\check{J}}{}_{\nu\sigma} \\ & - \frac{1}{12} g^{\mu\nu} g^{\rho\sigma} g^{\tau\lambda} \widetilde{\mathcal{H}}_{\mu\rho\tau} \widetilde{\mathcal{H}}_{\nu\sigma\lambda} + g^{\mu\nu} \frac{1}{8} D_\mu \mathcal{H}_{\check{I}\check{J}} D_\nu \mathcal{H}^{\check{I}\check{J}} \\ & + \mathcal{F}_{\check{I}\check{J}\check{K}} \mathcal{F}_{\check{I}'\check{J}'\check{K}'} \Big(-\frac{1}{12} \mathcal{H}^{\check{I}\check{I}'} \mathcal{H}^{\check{J}\check{J}'} \mathcal{H}^{\check{K}\check{K}'} + \frac{1}{4} \mathcal{H}^{\check{I}\check{I}'} \eta^{\check{J}\check{J}'} \eta^{\check{K}\check{K}'} - \frac{1}{6} \eta^{\check{I}\check{I}'} \eta^{\check{J}\check{J}'} \eta^{\check{K}\check{K}'} \Big) \\ & + \mathcal{F}_{\check{I}} \mathcal{F}_{\check{I}'} \big(\eta^{\check{I}\check{I}'} - \mathcal{H}^{\check{I}\check{I}'} \big) \Big] \end{aligned} \tag{2.20}$$

where the quantities without hats denote the internal parts and where we defined the field strengths

$$\begin{aligned} \widetilde{\mathcal{F}}^{\check{I}}{}_{\mu\nu} &= 2\partial_{[\underline{\mu}} \mathcal{A}^{\check{I}}{}_{\underline{\nu}]} - \mathcal{F}^{\check{I}}{}_{\check{J}\check{K}} \mathcal{A}^{\check{J}}{}_{\mu} \mathcal{A}^{\check{K}}{}_{\nu} + 2\mathcal{F}_{\check{J}} \mathcal{A}^{\check{J}}{}_{[\underline{\mu}} \mathcal{A}^{\check{I}}{}_{\underline{\nu}]} - 2\mathcal{F}^{\check{I}} B_{\mu\nu}, \\ \widetilde{\mathcal{H}}_{\mu\nu\rho} &= 3\partial_{[\underline{\mu}} B_{\underline{\nu\rho}]} - 3\partial_{[\underline{\mu}} \mathcal{A}^{\check{K}}{}_{\underline{\nu}} \mathcal{A}_{\underline{\rho}]\check{K}} - 6\mathcal{F}_{\check{K}} \mathcal{A}^{\check{K}}{}_{[\underline{\mu}} B_{\underline{\nu\rho}]} - \mathcal{F}_{\check{I}\check{J}\check{K}} \mathcal{A}^{\check{I}}{}_{\mu} \mathcal{A}^{\check{J}}{}_{\nu} \mathcal{A}^{\check{K}}{}_{\rho} \end{aligned} \tag{2.21}$$

and the covariant derivatives

$$\begin{aligned} D_\mu \mathcal{H}_{\check{I}\check{J}} = \partial_\mu \mathcal{H}_{\check{I}\check{J}} &+ \mathcal{A}^{\check{K}}{}_{\mu} \mathcal{F}_{\check{K}\check{I}}{}^{\check{L}} \mathcal{H}_{\check{J}\check{L}} + \mathcal{A}^{\check{K}}{}_{\mu} \mathcal{F}_{\check{K}\check{J}}{}^{\check{L}} \mathcal{H}_{\check{I}\check{L}} \\ &- \mathcal{A}_{\mu\check{I}} \mathcal{H}_{\check{J}\check{K}} \mathcal{F}^{\check{K}} - \mathcal{A}_{\mu\check{J}} \mathcal{H}_{\check{I}\check{K}} \mathcal{F}^{\check{K}} + \mathcal{F}_{\check{I}} \mathcal{H}_{\check{J}\check{K}} \mathcal{A}^{\check{K}}{}_{\mu} + \mathcal{F}_{\check{J}} \mathcal{H}_{\check{I}\check{K}} \mathcal{A}^{\check{K}}{}_{\mu}, \\ D_\mu \phi = \partial_\mu \phi &- \mathcal{F}_{\check{K}} \mathcal{A}^{\check{K}}{}_{\mu}. \end{aligned} \tag{2.22}$$

2.2.2. R–R sector

A similar analysis has been done for the R–R sector in [41–45]. Recalling that the fields transform as $O(10, 10)$ spinors by construction, we expand

$$\hat{\mathfrak{G}} = \sum_n \frac{1}{n!} \hat{\mathfrak{G}}^{(n)}_{\hat{m}_1 \ldots \hat{m}_n} \hat{e}_{\hat{a}_1}{}^{\hat{m}_1} \ldots e_{\hat{a}_n}{}^{\hat{m}_n} \Gamma^{\hat{a}_1 \ldots \hat{a}_n} |0\rangle , \tag{2.23}$$

where $\Gamma^{\hat{a}_1 \ldots \hat{a}_n}$ denotes the totally antisymmetrized product of n gamma-matrices. The R–R gauge potentials can be combined into a spinor

$$\hat{\mathcal{C}} = \begin{cases} \sum_{n=0}^{4} \hat{C}_{2n+1} & \text{for type IIA theory} \\ \sum_{n=0}^{4} \hat{C}_{2n} & \text{for type IIB theory,} \end{cases} \tag{2.24}$$

which can be used to write

$$\hat{\mathfrak{G}} = \begin{cases} G_0 + \not{\nabla}\hat{\mathcal{C}} & \text{for type IIA theory} \\ \not{\nabla}\hat{\mathcal{C}} & \text{for type IIB theory,} \end{cases} \tag{2.25}$$

with the *generalized fluxed Dirac operator*

$$\not{\nabla} = \Gamma^{\hat{A}} \hat{\mathcal{E}}_{\hat{A}}{}^{\hat{M}} \partial_{\hat{M}} - \frac{1}{2}\Gamma^{\hat{A}} \hat{\mathcal{F}}_{\hat{A}} - \frac{1}{6}\Gamma^{\hat{A}\hat{B}\hat{C}} \hat{\mathcal{F}}_{\hat{A}\hat{B}\hat{C}}. \tag{2.26}$$

The zero-form R–R flux G_0 in the type IIA case arises as dual of the background field strength of $\hat{C}_9$. A pseudo-action for the R–R sector can be obtained by summing over all relevant components of a particular theory,

$$S_{\text{R–R}} = \frac{1}{2}\int d^4x d^{12}Y \left(-\frac{1}{2} \hat{\mathfrak{G}} \wedge \star \hat{\mathfrak{G}} \right). \tag{2.27}$$

Since all fields $\hat{C}_n$ of a certain theory appear explicitly, this has to be supplemented by duality constraints. Denoting the ten-dimensional n-form contributions by $\hat{\mathfrak{G}}_n$, these take the form [46]

$$\hat{\mathfrak{G}}_n = (-1)^{\lfloor \frac{n}{2} \rfloor} \star \hat{\mathfrak{G}}_n, \tag{2.28}$$

where the floor operator $\lfloor \cdot \rfloor$ gives as output the least integer that is greater than or equal to the argument.

2.3. Fluxes in doubled geometry

This section will focus on the scalar potential component of (2.20) and introduce a DFT interpretation of the NS–NS fluxes. This has first been investigated in [33], and much of this section will be based on this work.

2.3.1. Fluxes as fluctuations about the Calabi–Yau background

The main idea is to treat the generalized fluxes (2.16) as manifestations of small deviations from the Calabi–Yau background, arising from perturbations of the internal vielbeins

$$\mathcal{E}^{\check{A}}{}_{\check{I}} = \mathring{\mathcal{E}}^{\check{A}}{}_{\check{I}} + \overline{\mathcal{E}}^{\check{A}}{}_{\check{I}} + \mathcal{O}\left(\overline{\mathcal{E}}^2\right), \tag{2.29}$$

where $\mathring{\mathcal{E}}^{\check{A}}{}_{\check{I}}$ describes the Calabi–Yau background and $\overline{\mathcal{E}}^{\check{A}}{}_{\check{I}}$ the fluctuations. Inserting this expansion into the generalized fluxes (2.16), we can write

$$\mathcal{F}_{\check{A}} = \mathring{\mathcal{F}}_{\check{A}} + \overline{\mathcal{F}}_{\check{A}} + \mathcal{O}\left(\overline{\mathcal{E}}^2\right), \qquad \mathcal{F}_{\check{A}\check{B}\check{C}} = \mathring{\mathcal{F}}_{\check{A}\check{B}\check{C}} + \overline{\mathcal{F}}_{\check{A}\check{B}\check{C}} + \mathcal{O}\left(\overline{\mathcal{E}}^2\right). \tag{2.30}$$

As the notation implies, $\mathring{\mathcal{F}}_{\check{A}}$ and $\mathring{\mathcal{F}}_{\check{A}\check{B}\check{C}}$ depend only on $\mathring{\mathcal{E}}^{\check{A}}{}_{\check{I}}$ and do not contribute to the scalar potential since $\mathring{\mathcal{E}}^{\check{A}}{}_{\check{I}}$ satisfies the DFT equations of motion. By contrast, $\overline{\mathcal{F}}_{\check{A}}$ and $\overline{\mathcal{F}}_{\check{A}\check{B}\check{C}}$ depend linearly on the fluctuations $\overline{\mathcal{E}}^{\check{A}}{}_{\check{I}}$ and therefore have to be taken into account.

In the following, we will use the background component $\mathring{\mathcal{E}}^{\check{A}}{}_{\check{I}}$ of the vielbein to switch between flat and curved indices (defining, e.g. $\overline{\mathcal{F}}_{\check{I}\check{J}\underline{\check{K}}} = \mathring{\mathcal{E}}^{\check{A}}{}_{\check{I}}\mathring{\mathcal{E}}^{\check{B}}{}_{\check{J}}\mathring{\mathcal{E}}^{\check{C}}{}_{\check{K}}\overline{\mathcal{F}}_{\check{A}\check{B}\check{C}}$). For the case of constant expectation values, the three-indexed object $\overline{\mathcal{F}}_{\check{I}\check{J}\check{K}}$ has been shown to encode the known geometric and non-geometric NS–NS fluxes by

$$\overline{\mathcal{F}}_{\check{i}\check{j}\check{k}} = H_{\check{i}\check{j}\check{k}}, \quad \overline{\mathcal{F}}^{\check{i}}{}_{\check{j}\check{k}} = F^{\check{i}}{}_{\check{j}\check{k}}, \quad \overline{\mathcal{F}}_{\check{i}}{}^{\check{j}\check{k}} = Q_{\check{i}}{}^{\check{j}\check{k}}, \quad \overline{\mathcal{F}}^{\check{i}\check{j}\check{k}} = R^{\check{i}\check{j}\check{k}}. \tag{2.31}$$

Similarly, we define for the trace-terms and generalized dilaton fluxes (cf. the first relation of (2.16))

$$\overline{\mathcal{F}}_{\check{i}} = 2Y_{\check{i}} + F^{\check{m}}{}_{\check{m}\check{i}}, \quad \overline{\mathcal{F}}^{\check{i}} = 2Z^{\check{i}} + Q_{\check{m}}{}^{\check{m}\check{i}}. \tag{2.32}$$

As was discussed in [47], writing out the generalized metric $\mathcal{H}$ in terms of the internal metric and Kalb–Ramond field gives rise to certain combinations of the latter with the fluxes, for which it is convenient to use the shorthand notation

$$\begin{aligned}
\mathfrak{H}_{\check{i}\check{j}\check{k}} &= H_{\check{i}\check{j}\check{k}} + 3F^{\check{m}}{}_{[\underline{\check{i}}\underline{\check{j}}}B_{\check{m}\underline{\check{k}}]} + 3Q_{[\underline{\check{i}}}{}^{\check{m}\check{n}}B_{\check{m}\underline{\check{j}}}B_{\check{n}\underline{\check{k}}]} + R^{\check{m}\check{n}\check{p}}B_{\check{m}[\underline{\check{i}}}B_{\check{n}\underline{\check{j}}}B_{\check{p}\underline{\check{k}}]},\\
\mathfrak{F}^{\check{i}}{}_{\check{j}\check{k}} &= F^{\check{i}}{}_{\check{j}\check{k}} + 2Q_{[\underline{\check{j}}}{}^{\check{m}\check{i}}B_{\check{m}]\underline{\check{k}}} + R^{\check{m}\check{n}\check{i}}B_{\check{m}[\underline{\check{j}}}B_{\check{n}\underline{\check{k}}]},\\
\mathfrak{Q}_{\check{k}}{}^{\check{i}\check{j}} &= Q_{\check{k}}{}^{\check{i}\check{j}} + R^{\check{m}\check{i}\check{j}}B_{\check{m}\check{k}},\\
\mathfrak{R}^{\check{i}\check{j}\check{k}} &= R^{\check{i}\check{j}\check{k}},\\
\mathfrak{Y}_{\check{i}} &= Y_{\check{i}} + Z^{\check{m}}B_{\check{m}\check{i}},\\
\mathfrak{Z}^{\check{i}} &= Z^{\check{i}}.
\end{aligned} \tag{2.33}$$

2.3.2. Operator interpretation of fluxes

It will be useful to interpret the geometric and non-geometric fluxes as operators acting on differential forms. Employing a local basis $\left(dx^1,\ldots dx^6\right)$ and the contractions $(\iota_1,\ldots\iota_6)$ satisfying $\iota_{\check{i}}dx^{\check{j}} = \delta_{\check{i}}{}^{\check{j}}$, we define [48–50]

$$\begin{aligned}
H\wedge: \quad & \Omega^p(CY_3) \longrightarrow \Omega^{p+3}(CY_3)\\
& \omega_p \mapsto \frac{1}{3!}H_{\check{i}\check{j}\check{k}}\,dx^{\check{i}}\wedge dx^{\check{j}}\wedge dx^{\check{k}}\wedge\omega_p,\\
F\circ: \quad & \Omega^p(CY_3) \longrightarrow \Omega^{p+1}(CY_3)\\
& \omega_p \mapsto \frac{1}{2!}F^{\check{k}}{}_{\check{i}\check{j}}\,dx^{\check{i}}\wedge dx^{\check{j}}\wedge\iota_{\check{k}}\wedge\omega_p,\\
Q\bullet: \quad & \Omega^p(CY_3) \longrightarrow \Omega^{p-1}(CY_3)\\
& \omega_p \mapsto \frac{1}{2!}Q_{\check{i}}{}^{\check{j}\check{k}}\,dx^{\check{i}}\wedge\iota_{\check{j}}\wedge\iota_{\check{k}}\wedge\omega_p,\\
R\llcorner: \quad & \Omega^p(CY_3) \longrightarrow \Omega^{p-3}(KCY_3)\\
& \omega_p \mapsto \frac{1}{3!}R^{\check{i}\check{j}\check{k}}\,\iota_{\check{i}}\wedge\iota_{\check{j}}\wedge\iota_{\check{k}}\wedge\omega_p,\\
Y\wedge: \quad & \Omega^p(CY_3) \longrightarrow \Omega^{p+1}(CY_3)\\
& \omega_p \mapsto Y_{\check{i}}\,dx^{\check{i}}\wedge\omega_p,
\end{aligned} \tag{2.34}$$

$$
\begin{aligned}
Z\blacktriangledown :\quad & \Omega^p(CY_3) \longrightarrow \Omega^{p-1}(CY_3) \\
& \omega_p \mapsto Z^{\check{i}}\, \iota_{\check{i}} \wedge \omega_p ,
\end{aligned}
$$

the last two of which denote the newly introduced generalized dilaton fluxes first considered in a non-DFT context in [51,52] (see also [53,54] for a generalized-geometry perspective). These operators can be combined with the exterior derivative $\hat{\mathrm{d}}$ to define the *twisted differential*

$$
\hat{\mathcal{D}} = \hat{\mathrm{d}} - H\wedge - F\circ - Q\bullet - R\llcorner - Y\wedge - Z\blacktriangledown . \tag{2.35}
$$

Notice that the exterior derivative is that of the full ten-dimensional spacetime manifold. In the following, we will often distinguish between internal and external components, for which it makes sense to split the exterior derivative as

$$
\hat{\mathrm{d}} = \mathrm{d} + \mathrm{d}_{CY_3} \tag{2.36}
$$

and define a purely internal twisted differential $\mathcal{D}$ with respect to d_{CY_3}. For later convenience, we can furthermore define analogous operators for the Fraktur fluxes (2.33), including the Fraktur twisted differential $\hat{\mathfrak{D}}$. As shown for a simplified setting in [33], requiring nilpotency $\hat{\mathcal{D}}^2 = 0$ of the twisted differential (and similarly for $\hat{\mathfrak{D}}$) gives rise to the Bianchi identities

$$
\begin{aligned}
0 &= H_{\check{m}[\underline{\check{i}\check{j}}} F^{\check{m}}{}_{\underline{\check{k}\check{l}}]} - \frac{2}{3}\partial_{[\underline{\check{i}}} H_{\underline{\check{j}\check{k}\check{l}}]} , \\
0 &= F^{\check{m}}{}_{[\underline{\check{j}\check{k}}} F^{\check{l}}{}_{\underline{\check{i}}]\check{m}} + H_{\check{m}[\underline{\check{i}\check{j}}} Q_{\underline{\check{k}}]}{}^{\check{m}\check{l}} + \partial_{[\underline{\check{k}}} F^{\check{l}}{}_{\underline{\check{i}\check{j}}]} , \\
0 &= F^{\check{m}}{}_{[\underline{\check{i}\check{j}}]} Q_{\check{m}}{}^{[\underline{\check{k}\check{l}}]} - 4 F^{[\underline{\check{k}}}{}_{\check{m}[\underline{\check{i}}} Q_{\underline{\check{j}}]}{}^{\underline{\check{l}}]\check{m}} + H_{\check{m}\check{i}\check{j}} R^{\check{m}\check{k}\check{l}} - 2\partial_{[\underline{\check{i}}} Q_{\underline{\check{j}}]}{}^{\check{k}\check{l}} , \\
0 &= Q_{\check{m}}{}^{[\underline{\check{j}\check{k}}} Q_{\check{l}}{}^{\underline{\check{i}}]\check{m}} + R^{\check{m}[\underline{\check{i}\check{j}}} F^{\underline{\check{k}}]}{}_{\check{m}\check{l}} - \frac{1}{3}\partial_{\check{l}} R^{\check{i}\check{j}\check{k}} , \\[6pt]
0 &= R^{\check{m}[\underline{\check{i}\check{j}}} Q_{\check{m}}{}^{\underline{\check{k}\check{l}}]} , \\
0 &= R^{\check{m}\check{n}[\underline{\check{i}}} F^{\underline{\check{j}}]}{}_{\check{m}\check{n}} - R^{\check{m}[\underline{\check{i}\check{j}}]} Y_{\check{m}} - Z^{\check{m}} Q_{\check{m}}{}^{[\underline{\check{i}\check{j}}]} , \\
0 &= R^{\check{i}\check{m}\check{n}} H_{\check{j}\check{m}\check{n}} - F^{\check{i}}{}_{\check{m}\check{n}} Q_{\check{j}}{}^{\check{m}\check{n}} - 2 Q_{\check{j}}{}^{\check{m}\check{i}} Y_{\check{m}} + 2 Z^{\check{m}} F^{\check{i}}{}_{\check{m}\check{j}} - 2\partial_{\check{j}} Z^{\check{i}} , \\
0 &= Q_{[\underline{\check{i}}}{}^{\check{m}\check{n}} H_{\underline{\check{j}}]\check{m}\check{n}} - F^{\check{m}}{}_{[\underline{\check{i}\check{j}}]} Y_{\check{m}} - Z^{\check{m}} H_{\check{m}[\underline{\check{i}\check{j}}]} + 2\partial_{[\underline{\check{i}}} Y_{\check{j}]} , \\[6pt]
0 &= 6 R^{\check{m}\check{n}\check{p}} H_{\check{m}\check{n}\check{p}} + Z^{\check{m}} Y_{\check{m}} ,
\end{aligned} \tag{2.37}
$$

where the derivative terms vanish in the setting discussed in this paper and were included only for the sake of completeness. This form of the Bianchi identities generalizes the result of [33] and matches with the relations presented earlier in [29] when taking into account the definitions (2.32) and assuming independence of the dual coordinates.

Another central role will be played by the generalized primitivity constraints

$$
H_{\check{i}\check{a}\check{\bar{a}}} g^{\check{a}\check{\bar{a}}} = 0, \qquad F^{\check{i}}{}_{\check{a}\check{\bar{a}}} g^{\check{a}\check{\bar{a}}} = 0, \qquad Q_{\check{i}}{}^{\check{a}\check{\bar{a}}} g_{\check{a}\check{\bar{a}}} = 0, \qquad R^{\check{i}\check{a}\check{\bar{a}}} g_{\check{a}\check{\bar{a}}} = 0, \tag{2.38}
$$

which extend the corresponding condition for H arising from supersymmetry considerations in traditional approaches to flux compactifications. Indeed, the first condition is equivalent to requiring the interior product $H \lrcorner J$ of H and the Kähler form J to vanish. Analogous formulations are possible for the remaining fluxes by taking the interior product with $F\lrcorner$ to be with respect to the subscript indices and defining analogous contraction-like operators $Q^{\urcorner}$, $R^{\urcorner}$ for the superscript indices of the non-geometric fluxes. The primitivity constraints can then be recast in the coordinate-independent forms

$$H\lrcorner J = 0, \qquad F\lrcorner J = 0, \qquad Q^{\urcorner} J = 0, \qquad R^{\urcorner} J = 0. \tag{2.39}$$

Notice that the interior product of non-geometric fluxes looks very similar to the corresponding operators defined in (2.34), but contracts only as many indices as there are in the differential form it acts on. This structure is motivated by that of the Hodge-star operator (A.6), and the relations (2.39) describe a generalization of the corresponding constraints used in [33]. As we will see in the next section, this slight relaxation is necessary in order to make the framework applicable to more general settings of flux compactifications.

2.3.3. Geometric tools

To conclude this section, let us briefly introduce the most essential geometric objects which will become important in the following discussion. A useful tool to handle the flux operators is the so-called the *Mukai-pairing* of two differential forms η and ρ. It is defined by

$$\langle \eta, \rho \rangle = [\eta \wedge \lambda(\rho)]_6 \,, \tag{2.40}$$

where $[\cdot]_6$ picks the six-form-component and the involution λ acts on an n-form ρ as

$$\lambda(\rho) = (-1)^{\lceil \frac{n}{2} \rceil} \rho. \tag{2.41}$$

The operator $\lceil \cdot \rceil$ denotes the ceiling function, giving as output the greatest integer that is less than or equal to the argument. Furthermore, we denote the purely external and internal components of Kalb–Ramond field $\hat{B}$ by

$$B = \frac{1}{2!} B_{\mu\nu}\, dx^{\mu} \wedge dx^{\nu} \qquad \text{and} \qquad b = \frac{1}{2!} B_{\check{i}\check{j}}\, dx^{\check{i}} \wedge dx^{\check{j}}, \tag{2.42}$$

respectively, and define the b-twisted Hodge-star operator $\star_b$ by [55–57]

$$\star_b \eta = e^{b} \wedge \star\lambda\left(e^{-b}\eta\right), \tag{2.43}$$

which allows for a natural extension of the framework to the Fraktur fluxes (2.33).

3. The scalar potential on a Calabi–Yau three-fold

We start our discussion by considering only the purely internal parts of (2.20) and (2.27) on a Calabi–Yau three-fold CY_3. The type IIB setting was already discussed in [33], and here we generalize this analysis in order to prepare for our discussion in section 4. The aim of this section is to show that both the type IIA and IIB case correctly give rise to the scalar potential of four-dimensional $\mathcal{N} = 2$ gauged supergravity. We furthermore illustrate how the simultaneous presence of geometric and non-geometric fluxes allows for preservation of IIA ↔ IIB Mirror Symmetry in DFT.

Since we do not have to distinguish between different components of the internal manifold, we will drop the "checks" above internal indices ($\check{I}, \check{J}, \ldots \to I, J, \ldots$) for the rest of this section. We furthermore impose the strong constraint on the underlying Calabi–Yau background and the field fluctuations, assuming independence of the dual coordinates $\tilde{y}_i$. We will, however, not do so for the fluxes and only apply the weaker (quadratic) Bianchi identities (2.37), ensuring that the theory is capable of describing electric and magnetic gaugings and does not merely reduce to ordinary type II supergravities.

3.1. NS–NS sector

When substituting the expansions (2.30) into the purely internal terms of (2.20), those terms involving only the objects $\mathring{\mathcal{F}}_I$ and $\mathring{\mathcal{F}}_{IJK}$ describe the Calabi–Yau background and do not contribute to the scalar potential since $\mathring{\mathcal{E}}^A{}_I$ satisfies the DFT equations of motion. Furthermore, mixings between background values and fluctuations describe first order terms in the expansion about the minimum of the scalar potential and can be neglected as well. Considering the action up to second order in the deviations, we are therefore left with

$$S_{\text{NS–NS, scalar}} = \frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^{12}Y\sqrt{g^{(4)}}\sqrt{g_{CY_3}}e^{-2\phi}\Big[\overline{\mathcal{F}}_{IJK}\overline{\mathcal{F}}_{I'J'K'}\Big(-\frac{1}{12}\mathcal{H}^{II'}\mathcal{H}^{JJ'}\mathcal{H}^{KK'} + \frac{1}{4}\mathcal{H}^{II'}\eta^{JJ'}\eta^{KK'} - \frac{1}{6}\eta^{II'}\eta^{JJ'}\eta^{KK'}\Big) + \overline{\mathcal{F}}_I\overline{\mathcal{F}}_{I'}\Big(\eta^{II'} - \mathcal{H}^{II'}\Big)\Big]. \tag{3.1}$$

Inserting the relations (2.31) and (2.32), this can be rewritten in terms of the geometric and non-geometric fluxes as

$$\begin{aligned} S_{\text{NS–NS, scalar}} = \frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^{12}Y\sqrt{g^{(4)}}\sqrt{g_{CY_3}}e^{-2\phi}\Big[\\ -\frac{1}{12}\Big(\mathfrak{H}_{ijk}\mathfrak{H}_{i'j'k'}g^{ii'}g^{jj'}g^{kk'} + 3\mathfrak{F}^i{}_{jk}\mathfrak{F}^{i'}{}_{j'k'}g_{ii'}g^{jj'}g^{kk'} \\ + 3\mathfrak{Q}_i{}^{jk}\mathfrak{Q}_{i'}{}^{j'k'}g^{ii'}g_{jj'}g_{kk'} + \mathfrak{R}^{ijk}\mathfrak{R}^{i'j'k'}g_{ii'}g_{jj'}g_{kk'}\Big) \\ -\frac{1}{2}\Big(\mathfrak{F}^m{}_{ni}\mathfrak{F}^n{}_{mi'}g^{ii'} + \mathfrak{Q}_m{}^{ni}\mathfrak{Q}_n{}^{mi'}g_{ii'} - \mathfrak{H}_{mni}\mathfrak{Q}_{i'}{}^{mn}g^{ii'} - \mathfrak{F}^i{}_{mn}\mathfrak{R}^{mni'}g_{ii'}\Big) \\ -\Big(\mathfrak{F}^m{}_{mi} + 2\mathfrak{Y}_i\Big)\Big(\mathfrak{F}^{m'}{}_{m'i'} + 2\mathfrak{Y}_{i'}\Big)g^{ii'} \\ -\Big(\mathfrak{Q}_m{}^{mi} + 2\mathfrak{Z}^i\Big)\Big(\mathfrak{Q}_{m'}{}^{m'i'} + 2\mathfrak{Z}^{i'}\Big)g_{ii'}\Big], \end{aligned} \tag{3.2}$$

where the topological terms involving only the $O(6,6,\mathbb{R})$ invariant structure $\eta^{II'}$ cancel by the Bianchi identities (2.37). Now a key issue of this action is that the (generally unknown) metric g_{ij} of CY_3 appears explicitly. In traditional Calabi–Yau compactifications, this can be remedied by applying differential form notation and expanding the fields in terms of the cohomology bases. While this framework is not readily applicable to the setting of this paper, we can resolve this problem by employing the operator interpretation (2.34) in order to build a bridge to the special geometry of the Calabi–Yau moduli spaces. To keep the calculation as general as possible, we will include cohomologically trivial terms for the first part of this section and set them to zero only right before performing the dimensional reduction.

3.1.1. Single flux settings

As already demonstrated in [33], it is convenient to first assume vanishing internal B-field components and consider only one flux turned on at a time. It can then easily be shown that the constructed reformulation is still applicable in more general settings.

Pure H-flux

Due to its differential form nature, the discussion of the pure H -flux setting is particularly simple and requires only the tools of standard differential geometry. The corresponding Lagrangian of (3.2) takes the form

$$\mathcal{L}_{\text{NS–NS, scalar, }H} = \frac{e^{-2\phi}}{4} H_{ijk} H_{i'j'k'} g^{ii'} g^{jj'} g^{kk'}. \tag{3.3}$$

It is obvious that this can be written as

$$\star\mathcal{L}_{\text{NS–NS, scalar, }H} = -\frac{e^{-2\phi}}{2} H \wedge \star H, \tag{3.4}$$

where we the three-form H is related to the first operator of (2.34) by formally defining $H := H \wedge \mathbf{1}_{CY_3}$.

Pure F-flux

The NS–NS scalar potential Lagrangian in the pure F-flux scenario reads

$$\begin{aligned}&\mathcal{L}_{\text{NS–NS, scalar, }F}\\ &= -\frac{e^{-2\phi}}{4}\left(F^{i}{}_{jk} F^{i'}{}_{j'k'} g_{ii'} g^{jj'} g^{kk'} + 2F^{m}{}_{ni} F^{n}{}_{mi'} g^{ii'} + 4F^{m}{}_{mi} F^{m}{}_{mi'} g^{ii'}\right).\end{aligned} \tag{3.5}$$

While the three-form interpretation of H does not apply to F, we can construct a similar object by letting the operator $F\circ$ act on the Kähler form J of CY_3. We then obtain

$$\begin{aligned}&-\frac{1}{2}(F\circ J)\wedge\star(F\circ J)\\ &= \left[\frac{1}{4} F^{m}{}_{ij} F^{m'}{}_{i'j'} g_{mm'} g^{ii'} g^{jj'} - \frac{1}{2} F^{m}{}_{ij} F^{m'}{}_{i'j'} I^{j'}{}_{m} I^{j}{}_{m'} g^{ii'}\right]\star\mathbf{1}_{CY_3}\end{aligned} \tag{3.6}$$

and find that only the first terms of (3.5) and (3.6) match, while the second term

$$\begin{aligned}&-\frac{1}{2} F^{m}{}_{ij} F^{m'}{}_{i'j'} I^{j'}{}_{m} I^{j}{}_{m'} g^{ii'}\\ &= \left(F^{c}{}_{ab} F^{b}{}_{\bar{a}c} + F^{\bar{c}}{}_{a\bar{b}} F^{\bar{b}}{}_{\bar{a}\bar{c}} - F^{\bar{c}}{}_{ab} F^{b}{}_{\bar{a}\bar{c}} - F^{c}{}_{a\bar{b}} F^{\bar{b}}{}_{\bar{a}c}\right) g^{a\bar{a}}\end{aligned} \tag{3.7}$$

comes with reversed signs for the last two components. To see how this can be compensated for, notice that appropriate contraction of indices in the second Bianchi identity of (2.37) yields (for vanishing Q-flux) the relation

$$F^{k}{}_{a\bar{b}} F^{\bar{b}}{}_{\bar{a}k} + F^{k}{}_{\bar{b}\bar{a}} F^{\bar{b}}{}_{ak} + F^{k}{}_{\bar{a}a} F^{\bar{b}}{}_{\bar{b}k} = 0. \tag{3.8}$$

Multiplying this by $g^{a\bar{a}}$, we find after taking into account the corresponding primitivity constraint of (2.38)

$$F^{c}{}_{a\bar{b}} F^{\bar{b}}{}_{\bar{a}c} g^{a\bar{a}} = F^{\bar{c}}{}_{ab} F^{b}{}_{\bar{a}\bar{c}} g^{a\bar{a}} \tag{3.9}$$

Using this, adding the expression

$$\frac{1}{2}(\Omega\wedge F\circ J)\wedge\star(\overline{\Omega}\wedge F\circ J) = -2\left[F^{\bar{c}}{}_{ab} F^{c}{}_{\bar{a}\bar{b}} g_{c\bar{c}} g^{a\bar{a}} g^{b\bar{b}} - 2F^{\bar{c}}{}_{ab} F^{b}{}_{\bar{a}\bar{c}} g^{a\bar{a}}\right]\star\mathbf{1}_{CY_3} \tag{3.10}$$

involving the holomorphic three-form Ω of CY_3 gives the correct second term of (3.6), but also comes with an additional contribution that has to be canceled. We once more resolve this by adding

$$-\frac{1}{2}\,(F\circ\Omega)\wedge\star\big(F\circ\overline{\Omega}\big)=\left[2F^{\bar{c}}{}_{ab}F^{c}{}_{\bar{a}\bar{b}}g_{c\bar{c}}g^{a\bar{a}}g^{b\bar{b}}+\frac{1}{2}F^{m}{}_{mi}F^{m}{}_{mi'}g^{ii'}\right]\star\mathbf{1}_{CY_3}. \tag{3.11}$$

Finally, the missing trace-term can be obtained by substituting the primitivity constraint (cf. (2.38)) into the only remaining non-trivial expression related the Calabi–Yau structure forms,

$$-\frac{1}{2}\left(F\circ\frac{1}{2}J^2\right)\wedge\star\left(F\circ\frac{1}{2}J^2\right)=\left[\frac{1}{2}F^{m}{}_{mi}F^{m}{}_{mi'}g^{ii'}\right]\star\mathbf{1}_{CY_3}, \tag{3.12}$$

and we find in total

$$\begin{aligned}&\star\,\mathcal{L}_{\text{NS–NS, scalar, }F}\\&=-\frac{e^{-2\phi}}{2}\bigg[(F\circ J)\wedge\star(F\circ J)+\left(F\circ\frac{1}{2}J^2\right)\wedge\star\left(F\circ\frac{1}{2}J^2\right)\\&\qquad+(F\circ\Omega)\wedge\star\big(F\circ\overline{\Omega}\big)-(\Omega\wedge F\circ J)\wedge\star\big(\overline{\Omega}\wedge F\circ J\big)\bigg].\end{aligned} \tag{3.13}$$

Notice that this poses a slight generalization of the corresponding expression found in [33] due to the presence of additional trace-terms of F. In particular, the reformulation only works when employing only the relaxed primitivity constraints (2.38), (2.39).

Pure Q-flux

The analysis of the pure Q-flux setting follows a very similar pattern as for the F-flux, and we will only sketch the basic idea here. By proceeding completely analogously to the F-flux case, one can show that the Lagrangian can be reformulated as

$$\begin{aligned}&\star\,\mathcal{L}_{\text{NS–NS, scalar, }Q}\\&=-\frac{e^{-2\phi}}{2}\bigg[\left(Q\bullet\frac{1}{2}J^2\right)\wedge\star\left(Q\bullet\frac{1}{2}J^2\right)+\left(Q\bullet\frac{1}{3!}J^3\right)\wedge\star\left(Q\bullet\frac{1}{3!}J^3\right)\\&\qquad+(Q\bullet\Omega)\wedge\star\big(Q\bullet\overline{\Omega}\big)-\left(\Omega\wedge Q\bullet\frac{1}{2}J^2\right)\wedge\star\left(\overline{\Omega}\wedge Q\bullet\frac{1}{2}J^2\right)\bigg],\end{aligned} \tag{3.14}$$

where the only nontrivial step is to take into account the relation

$$Q_k{}^{a\bar{b}}Q_{\bar{b}}{}^{\bar{a}k}+Q_k{}^{\bar{b}\bar{a}}Q_{\bar{b}}{}^{ak}+Q_k{}^{\bar{a}a}Q_{\bar{b}}{}^{\bar{b}k}=0 \tag{3.15}$$

obtained by appropriately contracting the fourth Bianchi identity of (2.37), which can eventually be recast in the form

$$g_{a\bar{a}}Q_{\bar{b}}{}^{ac}Q_c{}^{\bar{a}\bar{b}}=g_{a\bar{a}}Q_b{}^{a\bar{c}}Q_{\bar{c}}{}^{\bar{a}b} \tag{3.16}$$

and used to identify certain contributions arising from the first and third term of (3.14). Again, the result describes a slight generalization of the one found in [33], and matching for the trace-terms requires one to use the relaxes primitivity constraints (2.38), (2.39).

Pure R-flux

Similarly to the symmetry between the pure F- and Q-flux settings, the reformulation of pure R-flux case shows a strong resemblance of the pure H-flux setting, and it seems natural to consider the term $R \llcorner \frac{1}{3!} J^3$. This expression can be handled best by exploiting the relation

$$\frac{1}{3!} J^3 = \star \mathbf{1}_{CY_3} = \frac{\sqrt{g_{CY_3}}}{6!} \varepsilon_{i_1 \dots i_6} dx^{i_1} \wedge \dots \wedge dx^{i_6}, \tag{3.17}$$

to show that

$$R \llcorner \left(\frac{1}{3!} J^3 \right) = - \frac{\sqrt{g_{CY_3}}}{3!3!} R^{ijk} \varepsilon_{ijklmn} dx^l \wedge dx^m \wedge dx^n. \tag{3.18}$$

Inserting the relation (A.2) for $D = 3$ and $p = 3$, we then find

$$\star \mathcal{L}_{\text{NS–NS, scalar, } R} = - \frac{e^{2\phi}}{2} \left(R \llcorner \frac{1}{3!} J^3 \right) \wedge \star \left(R \llcorner \frac{1}{3!} J^3 \right). \tag{3.19}$$

Pure Y- and Z-flux

While the nature of the generalized dilaton fluxes Y and Z differs from that of their (three-indexed) geometric and non-geometric counterparts, including them into the framework presented here requires only minor modifications. The idea is again to consider all possible combinations of flux operators with the holomorphic three-form Ω or powers of the Kähler-form J. Direct computation of the corresponding expressions then shows that the Lagrangian (3.2) for the (combined) pure Y- and Z-flux settings can be rewritten as

$$\begin{aligned} \star \mathcal{L}_{\text{NS–NS, scalar, } Y} = - \frac{e^{-2\phi}}{2} \Big[& \big(Y \wedge \mathbf{1}_{CY_3} \big) \wedge \star \big(Y \wedge \mathbf{1}_{CY_3} \big) + (Y \wedge J) \wedge \star (Y \wedge J) \\ & + \left(Y \wedge \frac{1}{2} J^2 \right) \wedge \star \left(Y \wedge \frac{1}{2} J^2 \right) + (Y \wedge \Omega) \wedge \star \big(Y \wedge \overline{\Omega} \big) \Big] \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} \star \mathcal{L}_{\text{NS–NS, scalar, } Z} = - \frac{e^{-2\phi}}{2} \Big[& (Z \blacktriangledown J) \wedge \star (Z \blacktriangledown J) + \left(Z \blacktriangledown \frac{1}{2} J^2 \right) \wedge \star \left(Z \blacktriangledown \frac{1}{2} J^2 \right) \\ & + \big(Z \blacktriangledown \star \mathbf{1}_{CY_3} \big) \wedge \star \big(Z \blacktriangledown \star \mathbf{1}_{CY_3} \big) + (Y \wedge \Omega) \wedge \star \big(Y \wedge \overline{\Omega} \big) \Big], \end{aligned} \tag{3.21}$$

respectively. Notice that, although there do exist corresponding non-trivial expressions, we did not include any mixings between J and Ω. The reason for this discrepancy will become clear when considering more general settings in the next subsection.

3.1.2. Generalization

H-, F-, Q- and R-fluxes

Before turning to the most general setting, it makes sense to first consider the case of all three-indexed fluxes H, F, Q, R being present and vanishing one-indexed fluxes Y and Z. It was shown in [33] that the Lagrangian (3.2) can then be written as

$$\star\mathcal{L}_{\text{NS–NS, scalar, }HFQR} = -e^{-2\phi}\Big[\frac{1}{2}\chi\wedge\star\overline{\chi}+\frac{1}{2}\Psi\wedge\star\overline{\Psi} \\ -\frac{1}{4}(\Omega\wedge\chi)\wedge\star\big(\overline{\Omega}\wedge\overline{\chi}\big)-\frac{1}{4}(\Omega\wedge\overline{\chi})\wedge\star\big(\overline{\Omega}\wedge\chi\big)\Big], \tag{3.22}$$

where

$$\chi = \mathcal{D}e^{iJ}, \qquad \Psi = \mathcal{D}\Omega \tag{3.23}$$

and the twisted differential $\mathcal{D}$ defined in (2.35) (with vanishing Y- and Z-components). Taking into account the generalized primitivity constraints (2.38), it is easy to check that this formula correctly reproduces the single flux settings. Concerning the mixings between different fluxes, a minimal requirement for matching with the original Lagrangian (3.2) is that all mixings between different fluxes except for the HQ- and FR-combinations vanish. Since the only nontrivial contributions of (3.22) to the integral over CY_3 are the ones proportional to its volume form $\star\mathbf{1}_{CY_3}$, the relevant combinations of differential forms to check are those where both constituents share the same degree. This in particular excludes all components of the poly-form Ψ. Furthermore, those terms arising from quadratic combinations of χ involving precisely one even and one odd power of iJ cancel due to the complex conjugation operator reversing the signs only for imaginary differential forms. A simple computation shows that the remaining terms of (3.22) are the desired HQ- and FR-combinations, which read

$$T_{HQ} = -H\wedge\star\Big(Q\bullet\frac{1}{2}J^2\Big)+\text{Re}\,(\Omega\wedge H)\wedge\star\Big(\overline{\Omega}\wedge Q\bullet\frac{1}{2}J^2\Big), \\ T_{FR} = -F\circ J\wedge\star\Big(R\llcorner\frac{1}{3!}J^3\Big)+\text{Re}\,(\Omega\wedge F\circ J)\wedge\star\Big(\overline{\Omega}\wedge R\llcorner\frac{1}{3!}J^3\Big). \tag{3.24}$$

To show that these correctly reproduce the mixing terms of (3.2), one can again follow a similar pattern as in the single flux settings, and we refer the reader to the original work [33] for detailed calculations. The most important step here is to once more make use of the second and fourth Bianchi identities of (2.37) in order to relate the above expressions to the original action, which will in particular offset additional contributions arising from modifications of the relations (3.8) and (3.15) we used in the pure F- and Q-flux settings.

Including the Y- and Z-fluxes

When trying to incorporate the generalized dilaton fluxes Y and Z into the framework, one immediate problem is that the relation (3.22) does not even hold for the single flux settings. This is due to the appearance of additional mixings between e^{iJ} and Ω arising from the expressions in the second line, which cancel half of the desired terms and leave an overall mismatch by a factor of $\frac{1}{2}$. We resolve this by slightly modifying the expression in such a way that only the Y- and Z- terms are affected: Using the Mukai-pairing defined in (2.40), we find the more general Lagrangian

$$\mathcal{L}_{\text{NS–NS, scalar}} = -e^{-2\phi}\Big[\frac{1}{2}\,\|\langle\chi,\star\overline{\chi}\rangle\|+\frac{1}{2}\,\big\|\big\langle\Psi,\star\overline{\Psi}\big\rangle\big\|-\frac{1}{4}\,\|\langle\chi,\Omega\rangle\|^2-\frac{1}{4}\,\big\|\big\langle\chi,\overline{\Omega}\big\rangle\big\|^2\Big], \tag{3.25}$$

where the norm $\|\cdot\|$ is with respect to the scalar product (A.7) and χ and Ψ are defined as in (3.23), the twisted differential taking its general form (2.35). It is easy to check by direct computation and use of the primitivity constraints (2.38) that (3.25) reduces to the previously described special cases when setting the corresponding subsets of fluxes to zero. Of the newly appearing mixing terms, the non-vanishing ones are precisely the FY- and QZ-combinations, which correctly give rise to the trace-dilaton-mixings found in the last two lines of (3.2).

Notice that this formulation of the scalar potential shows a stronger resemblance of its generalized geometry counterpart found in [37] for compactifications of type II supergravities on manifolds with general $SU(3) \times SU(3)$ structures.

3.1.3. Including the Kalb–Ramond field

In a final step, the above results are once more generalized to the setting of a non-vanishing internal Kalb–Ramond field b. As can be inferred from the structure of the Lagrangian (3.2), this can be achieved by simply replacing

$$H \to \mathfrak{H}, \quad F \to \mathfrak{F}, \quad Q \to \mathfrak{Q}, \quad R \to \mathfrak{R}, \quad Y \to \mathfrak{Y}, \quad Z \to \mathfrak{Z} \tag{3.26}$$

and, thus, for the twisted differential

$$\mathcal{D} \to \mathfrak{D} = \mathrm{d} - \mathfrak{H}\wedge - \mathfrak{F}\circ - \mathfrak{Q}\bullet - \mathfrak{R}\llcorner - \mathfrak{Y}\wedge - \mathfrak{Z}\blacktriangledown. \tag{3.27}$$

Mathematically, the Kähler and complex structures of Calabi–Yau manifolds with non-vanishing b-field are described by the modified poly-forms

$$e^{i\mathfrak{J}} \to e^{b+iJ}, \qquad \Omega \to e^{b}\Omega. \tag{3.28}$$

At a later point, it will be convenient to absorb the factor e^b into the twisted differential. We therefore consider the relation [33]

$$\mathfrak{D} = e^{-b}\mathcal{D}e^{b} - \frac{1}{2}\left(\mathfrak{Q}_i{}^{mn}B_{mn}dx^i + \mathfrak{R}^{imn}B_{mn}\iota_i\right), \tag{3.29}$$

which can be derived by direct computation and using closure of b. Imposing primitivity constraints analogous to (2.38) for the Fraktur fluxes and the modified Calabi–Yau structure forms (3.28),

$$\mathfrak{Q}\urcorner\mathfrak{J} = 0, \qquad \mathfrak{R}\urcorner\mathfrak{J} = 0,$$

we furthermore obtain the relations

$$\begin{aligned} Q_i{}^{mn}B_{mn} + i\,R^{mnp}B_{im}J_{np} + R^{mnp}B_{im}B_{np} &= 0, \\ R^{mnp}B_{np} + i\,R^{mnp}J_{np} &= 0, \end{aligned} \tag{3.30}$$

showing that the terms in the brackets of (3.29) vanish and, in fact,

$$\mathfrak{D} = e^{-b}\mathcal{D}e^{b}. \tag{3.31}$$

We thus find for the NS–NS scalar potential in the most general case

$$\mathcal{L}_{\text{NS–NS, scalar}} = -e^{-2\phi}\left[\frac{1}{2}\,\|\langle\chi,\star\overline{\chi}\rangle\| + \frac{1}{2}\,\left\|\langle\Psi,\star\overline{\Psi}\rangle\right\| - \frac{1}{4}\,\|\langle\chi,\Omega\rangle\|^2 - \frac{1}{4}\,\left\|\langle\chi,\overline{\Omega}\rangle\right\|^2\right] \tag{3.32}$$

with

$$\chi = e^{-b}\mathcal{D}e^{b+iJ}, \qquad \Psi = e^{-b}\mathcal{D}\left(e^{b}\Omega\right). \tag{3.33}$$

3.2. R–R sector

Reformulating the scalar potential contribution of the R–R action (2.27) is more straightforward as one encounters only differential form terms. We will do this separately for the type IIA and IIB cases.

3.2.1. Type IIA theory

Starting from the purely internal component of (2.27) and substituting the definitions (2.25) and (2.24), we find for the internal components of the poly-form $\hat{\mathfrak{G}}^{(\mathrm{IIA})}$

$$\begin{aligned}
\mathfrak{G}_0^{(\mathrm{IIA})} &= G_0 - \mathfrak{Q}\bullet C_1 - \mathfrak{R}\llcorner C_3 - \mathfrak{Z}\blacktriangledown C_1, \\
\mathfrak{G}_2^{(\mathrm{IIA})} &= G_2 - B\wedge G_0 - \mathfrak{F}\circ C_1 - \mathfrak{Q}\bullet C_3 - \mathfrak{R}\llcorner C_5 - \mathfrak{Y}\wedge C_1 - \mathfrak{Z}\blacktriangledown C_3, \\
\mathfrak{G}_4^{(\mathrm{IIA})} &= G_4 - B\wedge G_2 + \frac{1}{2}B^2\wedge G_0 - \mathfrak{H}\wedge C_1 - \mathfrak{F}\circ C_3 - \mathfrak{Q}\bullet C_5 - \mathfrak{Y}\wedge C_3 - \mathfrak{Z}\blacktriangledown C_5 \\
\mathfrak{G}_6^{(\mathrm{IIA})} &= G_6 - B\wedge G_4 + \frac{1}{2}B^2\wedge G_2 - \frac{1}{3!}B^3\wedge G_0 - \mathfrak{H}\wedge C_3 - \mathfrak{F}\circ C_5 - \mathfrak{Y}\wedge C_5,
\end{aligned} \tag{3.34}$$

immediately revealing that the Lagrangian takes the form

$$\star\mathcal{L}_{\text{R–R}}^{(\mathrm{IIA})} = -\frac{1}{2}\mathfrak{G}^{(\mathrm{IIA})}\wedge\star\mathfrak{G}^{(\mathrm{IIA})}. \tag{3.35}$$

Here, $\mathfrak{G}^{(\mathrm{IIA})}$ denotes the purely internal part of $\hat{\mathfrak{G}}^{(\mathrm{IIA})}$ given by

$$\mathfrak{G}^{(\mathrm{IIA})} = e^{-b}\mathcal{G}^{(\mathrm{IIA})} + e^{-b}\mathcal{D}\left(e^{b}\mathcal{C}^{(\mathrm{IIA})}\right), \tag{3.36}$$

with

$$\begin{aligned}
\mathcal{C}^{(\mathrm{IIA})} &= C_1 + C_3 + C_5 + C_7 + C_9, \\
\mathcal{G}^{(\mathrm{IIA})} &= G_0 + G_2 + G_4 + G_6
\end{aligned} \tag{3.37}$$

comprising the purely internal components of the C_{2n+1}-fields (including those which become massive in the process of compactification) and the background R–R fluxes G_{2n}. Notice that the former are to be understood as fluctuations $\overline{C}_{2n+1}$, and one can equivalently write (3.36) as $\mathfrak{G}^{(\mathrm{IIA})} = G_0 + e^{-b}\mathcal{D}\left[e^{b}\left(\mathring{\mathcal{C}}^{(\mathrm{IIA})} + \overline{\mathcal{C}}^{(\mathrm{IIA})}\right)\right]$. The former formulation will, however, be more convenient since it allows one to treat all R–R fluxes on equal footing and obtain the same structure for the type IIA and IIB settings.

3.2.2. Type IIB theory

The analysis of the type IIB setting is completely analogous to the type IIA case, and one eventually arrives at

$$\star\mathcal{L}_{\text{R–R}}^{(\mathrm{IIB})} = -\frac{1}{2}\mathfrak{G}^{(\mathrm{IIB})}\wedge\star\mathfrak{G}^{(\mathrm{IIB})} \tag{3.38}$$

with

$$\mathfrak{G}^{(\mathrm{IIB})} = e^{-b}\mathcal{G}^{(\mathrm{IIB})} + e^{-b}\mathcal{D}\left(e^{b}\mathcal{C}^{(\mathrm{IIB})}\right) \tag{3.39}$$

and

$$\begin{aligned} \mathcal{G}^{(\mathrm{IIB})} &= G_1 + G_3 + G_5, \\ \mathcal{C}^{(\mathrm{IIB})} &= C_0 + C_2 + C_4 + C_6 + C_8. \end{aligned} \tag{3.40}$$

3.3. Dimensional reduction

The reformulated scalar potential described in (3.32), (3.35) and (3.38) depends only on the Kähler form and the holomorphic three-form of CY_3 and can be evaluated by utilizing the framework of special geometry for the Calabi–Yau moduli spaces.

3.3.1. Special geometry of Calabi–Yau three-folds

Since we are interested only in those fields which do not acquire mass in the course of the compactification, we would like to follow the standard procedure of Calabi–Yau compactifications and expand the appearing fields in terms of the cohomology bases of CY_3. In the setting discussed here, this additionally requires a way to describe the action of the flux operators (2.34) on the field expansions. We therefore start by reviewing the topological properties of Calabi–Yau manifolds and proceed by constructing a framework that incorporates the flux operators of DFT.

Even cohomology

The nontrivial even cohomology groups are precisely $H^{n,n}(CY_3)$ with $n = 0, 1, 2, 3$. We denote the corresponding bases by

$$\begin{aligned} &\left\{\mathbf{1}^{(6)}\right\} \in H^{0,0}(CY_3), \\ &\{\omega_{\mathsf{i}}\} \in H^{1,1}(CY_3), \\ &\left\{\widetilde{\omega}^{\mathsf{i}}\right\} \in H^{2,2}(CY_3), \qquad\qquad \text{with } \mathsf{i} = 1, \dots h^{1,1} \\ &\left\{\frac{\sqrt{g_{CY_3}}}{\mathcal{K}} \star \mathbf{1}^{(6)}\right\} \in H^{3,3}(CY_3), \end{aligned} \tag{3.41}$$

where $\mathcal{K}$ is the volume of CY_3. For later convenience, it makes sense to set $\omega_0 = \star\mathbf{1}^{(6)}$ and $\widetilde{\omega}^0 = \mathbf{1}^{(6)}$, allowing us to use the collective notation

$$\begin{aligned} \omega_{\mathsf{I}} &= \left(\omega_0, \quad \omega_{\mathsf{i}}\right), \qquad\qquad \text{with } \mathsf{I} = 0, \dots h^{1,1} \\ \widetilde{\omega}^{\mathsf{I}} &= \left(\widetilde{\omega}^0, \quad \widetilde{\omega}^{\mathsf{i}}\right). \end{aligned} \tag{3.42}$$

This structure is motivated by the action of the involution operator (2.41). We choose the two bases such that the normalization condition

$$\int_{CY_3} \omega_{\mathsf{I}} \wedge \widetilde{\omega}^{\mathsf{J}} = \delta_{\mathsf{I}}{}^{\mathsf{J}} \tag{3.43}$$

holds. For the Kähler form J of CY_3 and the Kalb–Ramond field $\hat{B}$, we use the expansions

$$J = v^{\mathsf{i}}\omega_{\mathsf{i}} \qquad \text{and} \qquad \hat{B} = B + b = B + b^{\mathsf{i}}\omega_{\mathsf{i}}, \tag{3.44}$$

where B denotes the external component of $\hat{B}$ living in $M^{1,3}$ and b its internal counterpart. The internal expansion coefficients b^{i} can be combined with v^{i} to define the complexified Kähler form

$$\mathfrak{J} = \left(b^{\mathsf{i}} + i v^{\mathsf{i}}\right)\omega_{\mathsf{i}} =: t^{\mathsf{i}}\omega_{\mathsf{i}}. \tag{3.45}$$

We furthermore introduce the shorthand notation

$$\begin{aligned}\mathcal{K}_{\mathsf{ijk}} &= \int_{CY_3} \omega_{\mathsf{i}} \wedge \omega_{\mathsf{j}} \wedge \omega_{\mathsf{k}}, \\ \mathcal{K}_{\mathsf{ij}} &= \int_{CY_3} \omega_{\mathsf{i}} \wedge \omega_{\mathsf{j}} \wedge J = \mathcal{K}_{\mathsf{ijk}} v^{\mathsf{k}}, \\ \mathcal{K}_{\mathsf{i}} &= \int_{CY_3} \omega_{\mathsf{i}} \wedge J \wedge J = \mathcal{K}_{\mathsf{ijk}} v^{\mathsf{j}} v^{\mathsf{k}}, \\ \mathcal{K} &= \frac{1}{3!} \int_{CY_3} J \wedge J \wedge J = \frac{1}{6} \mathcal{K}_{\mathsf{ijk}} v^{\mathsf{i}} v^{\mathsf{j}} v^{\mathsf{k}},\end{aligned} \tag{3.46}$$

where the $\mathcal{K}_{\mathsf{ijk}}$, $\mathcal{K}_{\mathsf{ij}}$ and $\mathcal{K}_{\mathsf{i}}$ are called intersection numbers. Using this, one can eventually expand the first poly-form of (3.33) in terms of the complexified Kähler class moduli

$$e^{B+iJ} = e^{\mathfrak{J}} = \tilde{\omega}^0 + t^{\mathsf{i}} \omega_{\mathsf{i}} + \frac{1}{2!} \left(\mathcal{K}_{\mathsf{ijk}} t^{\mathsf{i}} t^{\mathsf{j}}\right) \tilde{\omega}^{\mathsf{k}} + \frac{1}{3!} \left(\mathcal{K}_{\mathsf{ijk}} t^{\mathsf{i}} t^{\mathsf{j}} t^{\mathsf{k}}\right) \omega_0, \tag{3.47}$$

where all powers of order ≥ 4 vanish on CY_3.

Odd cohomology

The nontrivial odd cohomology groups are given by $H^{3,0}(CY_3)$, $H^{2,1}(CY_3)$, $H^{1,2}(CY_3)$ and $H^{0,3}(CY_3)$. For these we introduce the collective basis

$$\left\{\alpha_{\mathsf{A}}, \beta^{\mathsf{A}}\right\} \in H^3(CY_3) \qquad \text{with } \mathsf{A} = 0, \ldots h^{1,2}, \tag{3.48}$$

which can be normalized to satisfy

$$\int_{CY_3} \alpha_{\mathsf{A}} \wedge \beta^{\mathsf{B}} = \delta_{\mathsf{A}}{}^{\mathsf{B}}. \tag{3.49}$$

The complex structure moduli are encoded by the holomorphic three-form Ω of CY_3, which we expand in terms of the periods X^{A} and F_{A} as

$$\Omega = X^{\mathsf{A}} \alpha_{\mathsf{A}} - F_{\mathsf{A}} \beta^{\mathsf{A}}. \tag{3.50}$$

Notice that there is a minus sign in front of the β^{A}. Throughout this paper, we will apply this convention to all odd cohomology expansions of fields, while the signs are exchanged for field strengths. The periods F_{A} are functions of X^{A} and can be determined from a holomorphic pre-potential F by $F_{\mathsf{A}} = \frac{\partial F}{\partial X^{\mathsf{A}}}$. Defining $F_{\mathsf{AB}} = \frac{\partial F_{\mathsf{A}}}{\partial X^{\mathsf{B}}}$, one can write the period matrix $\mathcal{M}_{\mathsf{AB}}$ as

$$\mathcal{M}_{\mathsf{AB}} = \overline{F}_{\mathsf{AB}} + 2i \frac{\operatorname{Im}(F_{\mathsf{AC}}) X^{\mathsf{C}} \operatorname{Im}(F_{\mathsf{BD}}) X^{\mathsf{D}}}{X^{\mathsf{E}} \operatorname{Im}(F_{\mathsf{EF}}) X^{\mathsf{F}}}, \tag{3.51}$$

which is related to the cohomology bases (3.48) by

$$\begin{aligned}\int_{CY_3} \alpha_{\mathsf{A}} \wedge \star\alpha_{\mathsf{B}} &= -\left[(\operatorname{Im}\mathcal{M}) + (\operatorname{Re}\mathcal{M}) (\operatorname{Im}\mathcal{M})^{-1} (\operatorname{Re}\mathcal{M})\right]_{\mathsf{AB}}, \\ \int_{CY_3} \alpha_{\mathsf{A}} \wedge \star\beta^{\mathsf{B}} &= -\left[(\operatorname{Re}\mathcal{M}) (\operatorname{Im}\mathcal{M})^{-1}\right]_{\mathsf{A}}{}^{\mathsf{B}},\end{aligned} \tag{3.52}$$

$$\int_{CY_3} \beta^{\mathsf{A}} \wedge \star\beta^{\mathsf{B}} = -\Big[\mathrm{Im}\mathcal{M}^{-1}\Big]^{AB}.$$

Gauge coupling matrices

Denoting some arbitrary poly-form field A which can be expanded in terms of the nontrivial cohomology bases of CY_3 by

$$A = A^{\mathsf{I}}\omega_{\mathsf{I}} + A_{\mathsf{I}}\tilde{\omega}^{\mathsf{I}} + A^{\mathsf{A}}\alpha_{\mathsf{A}} - A_{\mathsf{A}}\beta^{\mathsf{A}}, \tag{3.53}$$

one can define a collective notation by

$$A^{\mathbb{I}} = \big(\, A^{\mathsf{I}}, \quad A_{\mathsf{I}}\,\big)^T \qquad \text{and} \qquad A^{\mathbb{A}} = \big(\, A^{\mathsf{A}}, \quad -A_{\mathsf{A}}\,\big)^T. \tag{3.54}$$

Again, notice that we will use reversed signs for the third cohomology group in case of field strengths. Similarly, we define the collective cohomology bases

$$\Sigma_{\mathbb{I}} = \big(\, \omega_{\mathsf{I}}, \quad \tilde{\omega}^{\mathsf{I}}\,\big) \qquad \text{and} \qquad \Xi_{\mathbb{A}} = \big(\, \alpha_{\mathsf{A}}, \quad \beta^{\mathsf{A}}\,\big) \tag{3.55}$$

and the matrix

$$\mathbb{M}_{\mathbb{A}\mathbb{B}} = \int_{CY_3} \begin{pmatrix} -\langle \alpha_{\mathsf{A}}, \star_b \alpha_{\mathsf{B}}\rangle & \langle \alpha_{\mathsf{A}}, \star_b \beta^{\mathsf{B}}\rangle \\ \langle \beta^{\mathsf{A}}, \star_b \alpha_{\mathsf{B}}\rangle & -\langle \beta^{\mathsf{A}}, \star_b \beta^{\mathsf{B}}\rangle \end{pmatrix}, \tag{3.56}$$

which can be expressed in terms of the period matrix (3.52) as

$$\mathbb{M} = \begin{pmatrix} \mathbb{1} & -\mathrm{Re}\mathcal{M} \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} \mathrm{Im}\mathcal{M} & 0 \\ 0 & \mathrm{Im}\mathcal{M}^{-1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ -\mathrm{Re}\mathcal{M} & \mathbb{1} \end{pmatrix}. \tag{3.57}$$

For later convenience, we parametrize the even cohomology analogue

$$\mathbb{N}_{\mathbb{I}\mathbb{J}} = \int_{CY_3} \begin{pmatrix} \langle \omega_{\mathsf{I}}, \star_b \omega_{\mathsf{J}}\rangle & \langle \omega_{\mathsf{I}}, \star_b \tilde{\omega}^{\mathsf{J}}\rangle \\ \langle \tilde{\omega}^{\mathsf{I}}, \star_b \omega_{\mathsf{J}}\rangle & \langle \tilde{\omega}^{\mathsf{I}}, \star_b \tilde{\omega}^{\mathsf{J}}\rangle \end{pmatrix} \tag{3.58}$$

as

$$\mathbb{N} = \begin{pmatrix} \mathbb{1} & -\mathrm{Re}\mathcal{N} \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} \mathrm{Im}\mathcal{N} & 0 \\ 0 & \mathrm{Im}\mathcal{N}^{-1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ -\mathrm{Re}\mathcal{N} & \mathbb{1} \end{pmatrix}, \tag{3.59}$$

where $\mathcal{N}_{\mathsf{IJ}}$ denotes the corresponding period matrix of the special Kähler manifold spanned by the complexified Kähler class moduli. A detailed discussion of its structure can be found in [58].

Using the notation (3.42), one can also see that the Mukai-pairing (2.40) induces a symplectic structure by

$$\int_{CY_3} \langle \Sigma_{\mathbb{I}}, \Sigma_{\mathbb{J}}\rangle = (S_{\text{even}})_{\mathbb{I}\mathbb{J}} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \in Sp\left(2h^{1,1}+2, \mathbb{R}\right) \tag{3.60}$$

and

$$\int_{CY_3} \langle \Xi_{\mathbb{A}}, \Xi_{\mathbb{B}}\rangle = (S_{\text{odd}})_{\mathbb{A}\mathbb{B}} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \in Sp\left(2h^{1,2}+2, \mathbb{R}\right). \tag{3.61}$$

For simplicity, we will omit the subscripts "even" and "odd" from now on. The dimension can, however, easily be inferred from the context or read off from the indices when using component notation.

3.3.2. Fluxes and cohomology bases

In the previous subsections we treated the fluxes as operators in a local coordinate basis, but for our subsequent analysis we need to relate these operators to actions on the cohomology basis elements (3.41) and (3.48). For toroidal compactification this transition from the coordinate basis to the cohomology is straightforward to derive, but for more general manifolds this is still an open question. However, as in [18], we can propose an action of the fluxes on the cohomology and check whether it leads to the expected results. For the three-index fluxes in the present context this has been done in [33], but for the Y- and Z-fluxes it is not clear how to interpret them on a Calabi–Yau three-fold. We therefore set Y and Z to zero for the remainder of this section.

Let us now become more concrete and note that the H-flux can be expanded in the basis (3.48) as

$$H = -\tilde{h}^{\mathsf{A}}\alpha_{\mathsf{A}} + h_{\mathsf{A}}\beta^{\mathsf{A}} \tag{3.62}$$

and that it acts as a wedge product with a three-form. While there is no such obvious relation for the remaining fluxes, one can extract useful structures by letting them act on the basis elements. Following [18], we consider the following action of the twisted differential $\mathcal{D}$ on the cohomology of the Calabi–Yau three-fold

$$\begin{aligned} \mathcal{D}\alpha_{\mathsf{A}} &= O_{\mathsf{A}}{}^{\mathsf{I}}\omega_{\mathsf{I}} + \mathcal{O}_{\mathsf{AI}}\tilde{\omega}^{\mathsf{I}}, & \mathcal{D}\beta^{\mathsf{A}} &= \tilde{P}^{\mathsf{AI}}\omega_{\mathsf{I}} + \tilde{P}^{\mathsf{A}}{}_{\mathsf{I}}\tilde{\omega}^{\mathsf{I}}, \\ \mathcal{D}\omega_{\mathsf{I}} &= -\tilde{P}^{\mathsf{A}}{}_{\mathsf{I}}\alpha_{\mathsf{A}} + O_{\mathsf{AI}}\beta^{\mathsf{A}}, & \mathcal{D}\tilde{\omega}^{\mathsf{I}} &= \tilde{P}^{\mathsf{AI}}\alpha_{\mathsf{A}} - O_{\mathsf{A}}{}^{\mathsf{I}}\beta^{\mathsf{A}}, \end{aligned} \tag{3.63}$$

where we used the collective notation (3.42) to set

$$\begin{aligned} &O_{\mathsf{A}0} = r_{\mathsf{A}},\ \tilde{P}^{\mathsf{A}}{}_{0} = \tilde{r}^{\mathsf{A}}, \\ &O_{\mathsf{A}}{}^{0} = h_{\mathsf{A}}, \tilde{P}^{\mathsf{A}0} = \tilde{h}^{\mathsf{A}}. \end{aligned} \tag{3.64}$$

Similarly to the previous sections, one can arrange the flux coefficients in a collective notation that will greatly simplify calculations at a later point. We define the matrices

$$\mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}} = \begin{pmatrix} -\tilde{P}^{\mathsf{A}}{}_{\mathsf{I}} & \tilde{P}^{\mathsf{AI}} \\ O_{\mathsf{AI}} & -O_{\mathsf{A}}{}^{\mathsf{I}} \end{pmatrix}, \qquad \widetilde{\mathcal{O}}^{\mathbb{I}}{}_{\mathbb{A}} = \begin{pmatrix} (O^T)^{\mathsf{I}}{}_{\mathsf{A}} & (\tilde{P}^T)^{\mathsf{IA}} \\ (O^T)_{\mathsf{IA}} & (\tilde{P}^T)_{\mathsf{I}}{}^{\mathsf{A}} \end{pmatrix}, \tag{3.65}$$

such that the action of the twisted differential on the cohomology bases can be expressed in the shorthand notation

$$\mathcal{D}(\Sigma^T)_{\mathbb{I}} = (\mathcal{O}^T)_{\mathbb{I}}{}^{\mathbb{A}}(\Xi^T)_{\mathbb{A}}, \qquad \mathcal{D}(\Xi^T)_{\mathbb{A}} = (\widetilde{\mathcal{O}}^T)_{\mathbb{A}}{}^{\mathbb{I}}(\Sigma^T)_{\mathbb{I}}. \tag{3.66}$$

They can be related by

$$\widetilde{\mathcal{O}} = -S^{-1}\mathcal{O}^T S. \tag{3.67}$$

Nilpotency of the twisted differential furthermore implies that the relations

$$\mathcal{D}^2(\Sigma^T)_{\mathbb{I}} = 0 \qquad \text{and} \qquad \mathcal{D}^2(\Xi^T)_{\mathbb{A}} = 0 \tag{3.68}$$

have to be satisfied, giving rise to the constraints

$$\widetilde{\mathcal{O}}^{\mathbb{I}}{}_{\mathbb{A}}\mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}} = 0, \qquad \mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}}\widetilde{\mathcal{O}}^{\mathbb{I}}{}_{\mathbb{A}} = 0, \tag{3.69}$$

which take the role of a cohomology version of (2.37) and will be important in section 5.

3.3.3. Integrating over the internal space – NS–NS sector

Proceeding in the same manner as for ordinary type II supergravity theories, we now expand the fields of the scalar potential in the cohomology bases (3.42) and (3.48) in order to filter out those terms which become massive in four dimensions. For the NS–NS poly-forms, we utilize the expansions (3.47) and (3.50) to arrange coefficients in vectors

$$\begin{aligned} V^{\mathbb{I}} &= \left(\frac{1}{3!}\mathcal{K}_{\mathsf{ijk}}t^{\mathsf{i}}t^{\mathsf{j}}t^{\mathsf{k}}, \quad t^{\mathsf{i}}, \quad 1, \quad \frac{1}{2!}\mathcal{K}_{\mathsf{ijk}}t^{\mathsf{i}}t^{\mathsf{j}} \right)^T \\ W^{\mathbb{A}} &= \left(X^{\mathsf{A}}, \quad -F_{\mathsf{A}} \right)^T \end{aligned} \tag{3.70}$$

of dimension $(2h^{1,1}+2)$ and $(2h^{1,2}+2)$, respectively, enabling us to use the shorthand notation

$$e^{b+iJ} = \Sigma_{\mathbb{I}} V^{\mathbb{I}}, \qquad \Omega = \Xi_{\mathbb{A}} W^{\mathbb{A}}. \tag{3.71}$$

Using the flux matrices (3.65) and the relations (3.66), the poly-forms χ and Ψ can now be expressed as

$$\begin{aligned} \chi &= e^{-b}\,\Xi_{\mathbb{A}}\mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}}V^{\mathbb{I}}, \\ \Psi &= e^{-b}\,\Sigma_{\mathbb{I}}\widetilde{\mathcal{O}}^{\mathbb{I}}{}_{\mathbb{A}}W^{\mathbb{A}}. \end{aligned} \tag{3.72}$$

When integrating the NS–NS action (3.32) over CY_3, the first two terms of (3.72) combine to the matrices (3.56) and (3.58), and one eventually obtains for the scalar potential

$$\begin{aligned} V_{\text{scalar, NS–NS}} = e^{-2\phi}\Big[& V^{\mathbb{I}}(\mathcal{O}^T)_{\mathbb{I}}{}^{\mathbb{A}}\mathbb{M}_{\mathbb{A}\mathbb{B}}\mathcal{O}^{\mathbb{B}}{}_{\mathbb{J}}V^{\mathbb{J}} + W^{\mathbb{A}}(\widetilde{\mathcal{O}}^T)_{\mathbb{A}}{}^{\mathbb{I}}\mathbb{N}_{\mathbb{I}\mathbb{J}}\widetilde{\mathcal{O}}^{\mathbb{J}}{}_{\mathbb{B}}\overline{W}^{\mathbb{B}} \\ & - \frac{1}{4\mathcal{K}}\overline{W}^{\mathbb{A}}S_{\mathbb{A}\mathbb{B}}\mathcal{O}^{\mathbb{B}}{}_{\mathbb{I}}\left(V^{\mathbb{I}}\overline{V}^{\mathbb{J}} + \overline{V}^{\mathbb{I}}V^{\mathbb{J}}\right)(\mathcal{O}^T)_{\mathbb{J}}{}^{\mathbb{C}}(S^T)_{\mathbb{C}\mathbb{D}}\overline{W}^{\mathbb{D}}\Big]. \end{aligned} \tag{3.73}$$

3.3.4. Integrating over the internal space – R–R sector

Following the same pattern for the R–R sector, we start by discarding the cohomologically trivial C-fields and expand

$$\begin{aligned} e^B\mathcal{C}^{(\text{IIA})} &= C^{(3)\mathsf{A}}\alpha_{\mathsf{A}} - C^{(3)}{}_{\mathsf{A}}\beta^{\mathsf{A}}, \\ e^B\mathcal{C}^{(\text{IIB})} &= C^{(0)}{}_0\tilde{\omega}^0 + C^{(2)\mathsf{I}}\omega_{\mathsf{I}} + C^{(4)}{}_{\mathsf{I}}\tilde{\omega}^{\mathsf{I}} + C^{(6)0}\omega_0. \end{aligned} \tag{3.74}$$

The expansion coefficients are again arranged in vectors

$$\begin{aligned} \mathsf{C}_0^{\mathbb{A}} &= \left(C^{(3)\mathsf{A}}, \quad C^{(3)\mathsf{A}} \right) && \text{(type IIA theory)}, \\ \mathsf{C}_0^{\mathbb{I}} &= \left(C^{(6)0}, \quad C^{(2)\mathsf{I}}, \quad C^{(0)}{}_0, \quad C^{(4)}{}_{\mathsf{I}}\tilde{\omega}^{\mathsf{I}} \right) && \text{(type IIB theory)}, \end{aligned} \tag{3.75}$$

where the subscript index “0” denotes the number of external components and is introduced for consistency with section 5. Similarly, we write for the non-trivial R–R fluxes

$$\begin{aligned} \mathcal{G}^{(\text{IIA})} &= G^{(0)}{}_0\tilde{\omega}^0 + G^{(2)\mathsf{I}}\omega_{\mathsf{I}} + G^{(4)}{}_{\mathsf{I}}\tilde{\omega}^{\mathsf{I}} + G^{(6)0}\omega_0, \\ \mathcal{G}^{(\text{IIB})} &= -G^{(3)\mathsf{A}}\alpha_{\mathsf{A}} + G^{(3)}{}_{\mathsf{A}}\beta^{\mathsf{A}}, \end{aligned} \tag{3.76}$$

and

$$\begin{aligned} \mathsf{G}^{\mathbb{I}}_{\text{flux}} &= \left(G^{(6)0}, \quad G^{(2)\mathsf{I}}, \quad G^{(0)}{}_0, \quad G^{(4)}{}_{\mathsf{I}} \right) && \text{(type IIA theory)}, \\ \mathsf{G}^{\mathbb{A}}_{\text{flux}} &= \left(G^{(3)\mathsf{A}}, \quad G^{(3)}{}_{\mathsf{A}} \right) && \text{(type IIB theory)}, \end{aligned} \tag{3.77}$$

allowing us to reformulate the poly-forms (3.36) and (3.39) as

$$\begin{aligned}\mathfrak{G}^{(\mathrm{IIA})} &= e^{-b}\Sigma_{\mathbb{I}}\left(\mathsf{G}^{\mathbb{I}}_{\text{flux}} + \widetilde{\mathcal{O}}^{\mathbb{I}}{}_{\mathbb{A}}\mathsf{C}^{\mathbb{A}}_0\right),\\ \mathfrak{G}^{(\mathrm{IIB})} &= e^{-b}\Xi_{\mathbb{A}}\left(\mathsf{G}^{\mathbb{A}}_{\text{flux}} + \mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}}\mathsf{C}^{\mathbb{I}}_0\right).\end{aligned} \tag{3.78}$$

Integrating (3.35) and (3.38) over CY_3 and once more utilizing the relations (3.56) and (3.58), we eventually arrive at

$$\begin{aligned}V^{(\mathrm{IIA})}_{\text{scalar, R–R}} &= \frac{1}{2}\left(\mathsf{G}^{\mathbb{I}}_{\text{flux}} + \widetilde{\mathcal{O}}^{\mathbb{I}}{}_{\mathbb{A}}\mathsf{C}^{\mathbb{A}}_0\right)\mathbb{N}_{\mathbb{I}\mathbb{J}}\left(\mathsf{G}^{\mathbb{J}}_{\text{flux}} + \widetilde{\mathcal{O}}^{\mathbb{J}}{}_{\mathbb{B}}\mathsf{C}^{\mathbb{B}}_0\right),\\ V^{(\mathrm{IIB})}_{\text{scalar, R–R}} &= \frac{1}{2}\left(\mathsf{G}^{\mathbb{A}}_{\text{flux}} + \mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}}\mathsf{C}^{\mathbb{I}}_0\right)\mathbb{M}_{\mathbb{A}\mathbb{B}}\left(\mathsf{G}^{\mathbb{B}}_{\text{flux}} + \mathcal{O}^{\mathbb{B}}{}_{\mathbb{J}}\mathsf{C}^{\mathbb{J}}_0\right).\end{aligned} \tag{3.79}$$

3.3.5. Mirror symmetry

Since DFT incorporates all fluxes of the T-duality chain presented in [4,5], it is to be expected that IIA ↔ IIB Mirror Symmetry is restored in this setting. Indeed, comparing the results (3.79) for the type IIA and IIB cases, it is easy to verify that the theories are related to each other as

$$\begin{aligned}\mathbb{M}_{\mathbb{A}\mathbb{B}} &\leftrightarrow \mathbb{N}_{\mathbb{I}\mathbb{J}}, & h^{1,1} &\leftrightarrow h^{1,2},\\ V^{\mathbb{I}} &\leftrightarrow W^{\mathbb{A}}, & S_{\mathbb{I}\mathbb{J}} &\leftrightarrow S_{\mathbb{A}\mathbb{B}}\\ \mathsf{C}^{\mathbb{I}}_0 &\leftrightarrow \mathsf{C}^{\mathbb{A}}_0, & \mathsf{G}^{\mathbb{I}}_{\text{flux}} &\leftrightarrow \mathsf{G}^{\mathbb{A}}_{\text{flux}},\\ \mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}} &\leftrightarrow \widetilde{\mathcal{O}}^{\mathbb{I}}{}_{\mathbb{A}}.\end{aligned} \tag{3.80}$$

These transformations strongly resemble those appearing in traditional Calabi–Yau compactifications of supergravity theories [59,60]: The first two lines resemble an exchange of roles between the Kähler class and complex structure moduli spaces, while line three describes an obvious replacement of the theory-specific R–R fields. The last line encodes mappings between the fluxes, which in particular contain exchanges between the geometric and non-geometric ones, once more illustrating how the latter are required for preservation of IIA ↔ IIB Mirror Symmetry. Taken as a whole, this implies that type IIA DFT compactified on a Calabi–Yau three-fold CY_3 is physically equivalent to its type IIB analogue compactified on a mirror Calabi–Yau three-fold $\widetilde{CY}_3$, with the Hodge-diamonds of the two manifolds being related by a reflection along their diagonal axes.

Note that the relations involving the expansion coefficients can be lifted to ten dimensions, allowing for a more compact notation

$$\chi \leftrightarrow \Psi, \qquad \hat{\mathfrak{G}}^{(\mathrm{IIA})} \leftrightarrow \hat{\mathfrak{G}}^{(\mathrm{IIB})} \tag{3.81}$$

of the mirror mappings as an exchange of the poly-forms (3.33), (3.36) and (3.39) we used to reformulate the DFT action. Similarly to component notation, we see that they precisely correspond to an exchange of the terms encoding the complexified Kähler-class (χ) and complex structure (Ψ) moduli, besides a mapping between the IIA and IIB R–R objects. In particular, the structure of the theory remains invariant under Mirror Symmetry.

4. The scalar potential on $K3 \times T^2$

We next repeat the process of dimensional reduction for DFT on $K3 \times T^2$ and thereby show how the framework presented in the previous section can straightforwardly be generalized to more complex cases of flux compactifications. Much of the following discussion is completely analogous to the Calabi–Yau setting, and we will therefore focus on the specific features of

$K3 \times T^2$ instead. We will furthermore simplify computations by setting cohomologically trivial terms to zero right at the beginning of the calculation from now on.

In order to distinguish between $K3$ and T^2 indices, we split the "checked" indices $\check{I}, \check{J}, \ldots$ into $I, J, \ldots$ labeling $K3$ coordinates and $R, S \ldots$ labeling T^2 coordinates. Their complex-geometric (undoubled) analogues are denoted by $a, \bar{a}, b, \bar{b}$ and $g, \bar{g}, h, \bar{h}$, respectively. For convenience, we accordingly split the flux operators (2.34) into their distinct cohomologically nontrivial components,

$$\begin{aligned}
H\wedge: \quad & \Omega^p\left(K3\times T^2\right) \longrightarrow \Omega^{p+3}\left(K3\times T^2\right) \\
& \qquad \omega_p \mapsto \frac{1}{2!} H_{ijr}\, dx^i \wedge dx^j \wedge dx^r \wedge \omega_p, \\
F\circ: \quad & \Omega^p\left(K3\times T^2\right) \longrightarrow \Omega^{p+1}\left(K3\times T^2\right) \\
& \qquad \omega_p \mapsto \left(\frac{1}{2!} F^r{}_{ij}\, dx^i \wedge dx^j \wedge \iota_r + F^j{}_{ir}\, dx^i \wedge dx^r \wedge \iota_j\right) \wedge \omega_p, \\
Q\bullet: \quad & \Omega^p\left(K3\times T^2\right) \longrightarrow \Omega^{p-1}\left(K3\times T^2\right) \\
& \qquad \omega_p \mapsto \left(\frac{1}{2!} Q_r{}^{ij}\, dx^r \wedge \iota_i \wedge \iota_j + Q_i{}^{jr}\, dx^i \wedge \iota_j \wedge \iota_r\right) \wedge \omega_p, \\
R\llcorner: \quad & \Omega^p\left(K3\times T^2\right) \longrightarrow \Omega^{p-3}\left(K3\times T^2\right) \\
& \qquad \omega_p \mapsto \frac{1}{3!} R^{ijr}\, \iota_i \wedge \iota_j \wedge \iota_r \wedge \omega_p, \\
Y\wedge: \quad & \Omega^p\left(K3\times T^2\right) \longrightarrow \Omega^{p+1}\left(K3\times T^2\right) \\
& \qquad \omega_p \mapsto Y_r\, dx^r \wedge \omega_p, \\
Z\blacktriangledown: \quad & \Omega^p\left(K3\times T^2\right) \longrightarrow \Omega^{p-1}\left(K3\times T^2\right) \\
& \qquad \omega_p \mapsto Z^r\, \iota_r \wedge \omega_p.
\end{aligned} \tag{4.1}$$

Finally, we again impose the strong constraint only for the background and the field fluctuations, while applying the Bianchi identities (2.37) for the fluxes.

4.1. Reformulating the action

The toolbox we used to reformulate the internal NS–NS action on CY_3 builds upon on the mathematical framework of generalized Calabi–Yau structures [19] and can be straightforwardly extended to arbitrary manifolds admitting such a one. For the case of $K3 \times T^2$, this can be done by utilizing the features of generalized $K3$ surfaces [35] and formally viewing T^2 as a complex torus with a generalized Calabi–Yau structure. We therefore exploit the product structure of $K3 \times T^2$ and consider the Kähler class and complex structure forms

$$e^{b+iJ} = e^{b_{K3}+iJ_{K3}} \wedge e^{b_{T^2}+iJ_{T^2}}, \qquad e^b \wedge \Omega = \left(e^{b_{K3}} \wedge \Omega_{K3}\right) \wedge \left(e^{b_{T^2}} \wedge \Omega_{T^2}\right), \tag{4.2}$$

respectively. The reformulation of the scalar potential part of the NS–NS sector (2.20) then follows a very similar pattern as in the Calabi–Yau case. As an instructive example, one can easily check that the only non-trivial contribution of the pure H-flux setting is given by

$$\star\mathcal{L}_{\text{NS–NS, scalar, }H} = \frac{e^{-2\phi}}{4} H_{ijr} H_{i'j'r'} g^{ii'} g^{jj'} g^{rr'} \star \mathbf{1}_{K3\times T^2}, \tag{4.3}$$

which can again be written as

$$\star\mathcal{L}_{\text{NS–NS, scalar, }H} = -\frac{e^{-2\phi}}{2} H \wedge \star H, \tag{4.4}$$

with H now defined as in (4.1). The F-flux allows for different nontrivial components and is therefore slightly more involved. From the initial action (2.20), we obtain

$$\begin{aligned}\mathcal{L}_{\text{NS–NS, scalar, }F} = -\frac{e^{-2\phi}}{4}\Big(& F^{r}{}_{ij} F^{r'}{}_{i'j'} g^{ii'} g^{jj'} g_{rr'} + 2F^{i}{}_{jr} F^{i'}{}_{j'r'} g_{ii'} g^{jj'} g^{rr'} \\ & + 2F^{m}{}_{nr} F^{n}{}_{mr'} g^{rr'} + 4F^{m}{}_{mr} F^{m'}{}_{m'r'} g^{rr'} + 4F^{r}{}_{mi} F^{m}{}_{ri'} g^{ii'}\Big),\end{aligned} \tag{4.5}$$

Denoting the first and second component of $F\circ$ by $F_1\circ$ respectively $F_2\circ$ (based on the split employed in (4.1)), the first term can be rewritten similarly to the H-flux contribution as

$$\begin{aligned}& -\frac{e^{-2\phi}}{4} F^{r}{}_{ij} F^{r'}{}_{i'j'} g^{ii'} g^{jj'} g_{rr'} \star \mathbf{1}_{K3\times T^2} \\ & = -\frac{e^{-2\phi}}{2} \left[F_1 \circ \left(\star\mathbf{1}_{K3} \wedge \mathbf{1}_{T^2}\right)\right] \wedge \star \left[F_1 \circ \left(\star\mathbf{1}_{K3} \wedge \mathbf{1}_{T^2}\right)\right],\end{aligned} \tag{4.6}$$

while a calculation analogous to the pure F-flux case in the Calabi–Yau setting yields for the next three terms

$$\begin{aligned}& -\frac{e^{-2\phi}}{4}\left(2F^{i}{}_{jr} F^{i'}{}_{j'r'} g_{ii'} g^{jj'} g^{rr'} + 2F^{m}{}_{nr} F^{n}{}_{mr'} g^{rr'} + 4F^{m}{}_{mr} F^{m'}{}_{m'r'} g^{rr'}\right) \star \mathbf{1}_{K3\times T^2} \\ & = -\frac{e^{-2\phi}}{2}\Big\{\left[F_2 \circ \left(iJ_{K3} \wedge \mathbf{1}_{T^2}\right)\right] \wedge \star \left[F_2 \circ \left(iJ_{K3} \wedge \mathbf{1}_{T^2}\right)\right] \\ & \quad + \left[F_2 \circ \left(\star\mathbf{1}_{K3} \wedge \mathbf{1}_{T^2}\right)\right] \wedge \star \left[F_2 \circ \left(\star\mathbf{1}_{K3} \wedge \mathbf{1}_{T^2}\right)\right] \\ & \quad + \left[F_2 \circ \left(\Omega_{K3} \wedge \Omega_{T^2}\right)\right] \wedge \star \left[F_2 \circ \left(\overline{\Omega}_{K3} \wedge \overline{\Omega}_{T^2}\right)\right] \\ & \quad - \left[\left(\Omega_{K3} \wedge \Omega_{T^2}\right) \wedge F_2 \circ \left(iJ_{K3} \wedge \mathbf{1}_{T^2}\right)\right] \wedge \star \left[\left(\overline{\Omega}_{K3} \wedge \overline{\Omega}_{T^2}\right) F_2 \circ \left(iJ_{K3} \wedge \mathbf{1}_{T^2}\right)\right]\Big\}\end{aligned} \tag{4.7}$$

and the final one

$$\begin{aligned}& -e^{-2\phi} F^{r}{}_{mi} F^{m}{}_{ri'} g^{ii'} \star \mathbf{1}_{K3\times T^2} \\ & = -e^{-2\phi}\Big\{\left[F_1 \circ \left(\mathbf{1}_{K3} \wedge iJ_{T^2}\right)\right] \wedge \star \left[F_2 \circ \left(iJ_{K3} \wedge \mathbf{1}_{T^2}\right)\right] \\ & \quad - \left[\left(\Omega_{K3} \wedge \Omega_{T^2}\right) \wedge F_1 \circ \left(\mathbf{1}_{K3} \wedge iJ_{T^2}\right)\right] \wedge \star \left[\left(\Omega_{K3} \wedge \Omega_{T^2}\right) F_2 \circ \left(iJ_{K3} \wedge \mathbf{1}_{T^2}\right)\right]\Big\},\end{aligned} \tag{4.8}$$

showing that the F-contribution to the scalar potential takes the form (3.13) already known from the Calabi–Yau setting. The discussion of the non-geometric and generalized dilaton fluxes as well as the R–R sector is analogous. For the most general setting, we eventually arrive at the familiar expressions (3.32), (3.35) and (3.38), with the fluxes adjusted according to (4.1) and e^{iJ} and Ω as in (4.2).

4.2. Dimensional reduction

We next proceed as usual by expanding the fields and fluxes in terms of the cohomology bases of $K3 \times T^2$ before integrating over the internal manifold.

4.2.1. Special geometry of $K3 \times T^2$

As in the Calabi–Yau case, it is convenient to treat the even and odd cohomology groups of the compactification manifolds separately in order to allow for a description of the Kähler class and complex structure moduli spaces as well as Mirror Symmetry. Since all nontrivial cohomology groups of $K3$ are of even degree, the property of a cohomologically nontrivial differential form on $K3 \times T^2$ being even or odd depends purely on its T^2 component.

Even cohomology

The even cohomology bases of T^2 are precisely the identity $\mathbf{1}_{T^2}$ for the zero-forms and $\star\mathbf{1}_{T^2}$ for the two-forms (the latter of which coincides with the normalized Kähler form),

$$\begin{aligned} &\left\{\mathbf{1}_{T^2}\right\} \in H^0\left(T^2\right), \\ &\left\{\frac{\sqrt{g_{T^2}}}{\mathcal{K}_{T^2}} \star \mathbf{1}_{T^2}\right\} \in H^2\left(T^2\right), \end{aligned} \tag{4.9}$$

and we denote them by v_0 respectively v_3 from now on. The bases of the $K3$ de Rham cohomology groups are given by

$$\begin{aligned} &\{\mathbf{1}_{K3}\} \in H^0(K3), \\ &\{\sigma_{\mathsf{u}}\} \in H^2(K3) \qquad\qquad \text{with } \mathsf{u} = 1, \ldots 22 \\ &\left\{\tfrac{\sqrt{g_{K3}}}{\mathcal{K}_{K3}} \star \mathbf{1}_{K3}\right\} \in H^4(K3), \end{aligned} \tag{4.10}$$

and we define $\sigma_0 = \mathbf{1}_{K3}$ and $\sigma_{23} = \star\mathbf{1}_{K3}$, enabling us to arrange the $K3$ bases in a collective notation

$$\sigma_{\mathsf{U}} = \begin{pmatrix} \sigma_0 & \sigma_{\mathsf{u}} & \sigma_{23} \end{pmatrix}. \tag{4.11}$$

We furthermore define η_{uv} to be the intersection metric

$$\eta_{\mathsf{uv}} = \int_{K3} \sigma_{\mathsf{u}} \wedge \sigma_{\mathsf{v}}. \tag{4.12}$$

Its signature $(3, 19)$ resembles the fact that there are three antiselfdual two-forms (the Kähler form, the holomorphic two-form and its antiholomorphic counterpart) and 19 selfdual ones. This metric can serve as a building block of a matrix

$$L_{\mathsf{UV}} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & \eta_{\mathsf{uv}} & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad L^{\mathsf{UV}} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & \eta^{\mathsf{uv}} & 0 \\ -1 & 0 & 0 \end{pmatrix}, \tag{4.13}$$

which we use to lower and raise cohomological $K3$ indices,

$$\sigma^{\mathsf{U}} = L^{\mathsf{UV}} \sigma_{\mathsf{V}}. \tag{4.14}$$

Putting all of the above objects together, we can define a collective basis for the even de Rham cohomology groups of $K3 \times T^2$ by

$$\begin{aligned} \omega_{\mathsf{I}} &= \begin{pmatrix} \omega_0 & \omega_{\mathsf{u}} & \omega_{23} \end{pmatrix} = \begin{pmatrix} v_0 \wedge \sigma_0 & v_0 \wedge \sigma_{\mathsf{u}} & v_0 \wedge \sigma_{23} \end{pmatrix}, \\ \widetilde{\omega}^{\mathsf{I}} &= \begin{pmatrix} \widetilde{\omega}^0 & \widetilde{\omega}^{\mathsf{u}} & \widetilde{\omega}^{23} \end{pmatrix} = \begin{pmatrix} v_3 \wedge \sigma^0 & v_3 \wedge \sigma^{\mathsf{u}} & v_3 \wedge \sigma^{23} \end{pmatrix}, \end{aligned} \tag{4.15}$$

where the labeling $\mathsf{I}, \mathsf{J}, \ldots$ was chosen to make it distinguishable from its odd counterpart. The basis elements satisfy the normalization condition

$$\int_{K3\times T^2} \omega_{\mathsf{I}} \wedge \widetilde{\omega}^{\mathsf{J}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \delta_{\mathsf{u}}{}^{\mathsf{v}} & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{4.16}$$

We again use the collective notation

$$\Sigma_{\mathbb{I}} = \begin{pmatrix} \omega_{\mathsf{I}} & \tilde{\omega}^{\mathsf{I}} \end{pmatrix}. \tag{4.17}$$

Analogously to the Calabi–Yau case, this basis defines a symplectic structure by

$$\int_{K3\times T^2} \big\langle \Sigma_{\mathbb{I}}, \Sigma_{\mathbb{J}} \big\rangle = (S_{\text{even}})_{\mathbb{I}\mathbb{J}} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \in Sp\,(48, \mathbb{R})\,. \tag{4.18}$$

In order to describe the Kähler class moduli space of $K3 \times T^2$, we combine the Kähler form J and the internal part b of the $\hat{B}$-field to the complexified Kähler form

$$\mathfrak{J} = b + iJ = \big(b_{T^2} + iJ_{T^2}\big) + (b_{K3} + iJ_{K3}) = \rho\widetilde{\omega}^0 + t^{\mathsf{u}}\omega_{\mathsf{u}}, \tag{4.19}$$

where the latter splitting can be applied due to the vanishing first Betti number of $K3$. The complex parameter $\rho = b^0 + iw^0$ encodes the volume modulus w^0 of T^2 as well as the component b^0 of $\hat{B}$ living purely in T^2. Analogously, the t^{u} denote the moduli w^{u} of J_{K3} and b^{u} spanning the complexified Kähler cone of $K3$. In the upcoming discussion, we will mainly encounter the poly-form $e^{\mathfrak{J}}$, which we will expand as $e^{\mathfrak{J}} = \Sigma_{\mathbb{I}} V^{\mathbb{I}}$ with

$$V^{\mathbb{I}} = \begin{pmatrix} 1, & t^{\mathsf{u}}, & t^{\mathsf{u}}t^{\mathsf{v}}\eta_{\mathsf{uv}}, & \rho t_{\mathsf{u}}t_{\mathsf{v}}\eta^{\mathsf{uv}}, & \rho t_{\mathsf{u}}, & \rho \end{pmatrix}^T. \tag{4.20}$$

Odd cohomology

A basis for the odd cohomology groups can be constructed in a similar manner by replacing the even basis elements of T^2 by two one-form basis elements

$$\{v_1, v_2\} \in H^1\left(T^2\right) \qquad \text{with} \int_{T^2} v_1 \wedge v_2 = 1 \tag{4.21}$$

and defining

$$\begin{aligned} \alpha_{\mathsf{A}} &= \begin{pmatrix} \alpha_0 & \alpha_{\mathsf{u}} & \alpha_{23} \end{pmatrix} = \begin{pmatrix} v_1 \wedge \sigma_0 & v_1 \wedge \sigma_{\mathsf{u}} & v_1 \wedge \sigma_{23} \end{pmatrix}, \\ \beta^{\mathsf{A}} &= \begin{pmatrix} \beta^0 & \beta^{\mathsf{u}} & \beta^{23} \end{pmatrix} = \begin{pmatrix} v_2 \wedge \sigma^0 & v_2 \wedge \sigma^{\mathsf{u}} & v_2 \wedge \sigma^{23} \end{pmatrix}. \end{aligned} \tag{4.22}$$

They satisfy the normalization condition

$$\int_{K3\times T^2} \alpha_{\mathsf{A}} \wedge \beta^{\mathsf{A}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \delta_{\mathsf{u}}{}^{\mathsf{v}} & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{4.23}$$

and can be arranged in a collective basis

$$\Xi_{\mathbb{A}} = \begin{pmatrix} \alpha_{\mathsf{A}} & \beta^{\mathsf{A}} \end{pmatrix} \tag{4.24}$$

to define a symplectic structure by

$$\int\limits_{K3\times T^2} \langle \Xi_{\mathbb{A}}, \Xi_{\mathbb{B}} \rangle = (S_{\text{odd}})_{\mathbb{I}\mathbb{J}} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \in Sp\,(48, \mathbb{R})\,. \tag{4.25}$$

Notice that we again incorporated a relative minus sign into the expansions in terms of the even and odd cohomology bases for later convenience. More specifically, we expand an arbitrary polyform field A as

$$A = A^{\mathbb{I}}\Sigma_{\mathbb{I}} + A^{\mathbb{A}}\Xi_{\mathbb{A}} = A^{\mathsf{I}}\omega_{\mathsf{I}} + A_{\mathsf{I}}\tilde{\omega}^{\mathsf{I}} + A^{\mathsf{A}}\alpha_{\mathsf{A}} - A_{\mathsf{A}}\beta^{\mathsf{A}}. \tag{4.26}$$

Similarly to the Kähler class case, the complex structure moduli space of $K3 \times T^2$ can be described by its holomorphic three-form Ω, which on its part can be split into a holomorphic one-form Ω_{T^2} living in T^2 and a holomorphic two-form Ω_{K3} living in $K3$. Viewing T^2 as a one-dimensional complex torus, the former encodes the modular (complex structure) parameter τ by

$$\Omega_{T^2} = v_1 - \tau v_2, \tag{4.27}$$

where

$$\tau = \int\limits_{T^2} \Omega_{T^2} \wedge v_1\,. \tag{4.28}$$

Similarly, the latter can be expanded as

$$\Omega_{K3} = T^{\mathsf{u}}\sigma_{\mathsf{u}}, \tag{4.29}$$

allowing us to expand the complete holomorphic three-form Ω in the basis (4.22). In the following, we will be mainly concerned with the expression $e^b\Omega$, which can be expanded as $e^b\Omega = \Xi_{\mathbb{A}} W^{\mathbb{A}}$ with

$$W^{\mathbb{A}} = \begin{pmatrix} 0, & T^{\mathsf{u}}, & T^{\mathsf{u}} b^{\mathsf{v}} \eta_{\mathsf{uv}}, & \tau T_{\mathsf{u}} b_{\mathsf{v}} \eta^{\mathsf{uv}}, & \tau T_{\mathsf{u}}, & 0 \end{pmatrix}^T. \tag{4.30}$$

Gauge coupling matrices

As in the Calabi–Yau setting, we again define a gauge coupling matrix

$$\mathbb{M}_{\mathbb{A}\mathbb{B}} = \int\limits_{K3\times T^2} \begin{pmatrix} -\langle \alpha_{\mathsf{A}}, \star_b \alpha_{\mathsf{B}} \rangle & \langle \alpha_{\mathsf{A}}, \star_b \beta^{\mathsf{B}} \rangle \\ \langle \beta^{\mathsf{A}}, \star_b \alpha_{\mathsf{B}} \rangle & -\langle \beta^{\mathsf{A}}, \star_b \beta^{\mathsf{B}} \rangle \end{pmatrix}, \tag{4.31}$$

which can be written as

$$\mathbb{M}_{\mathbb{A}\mathbb{B}} = \frac{1}{\text{Im}\tau} \begin{pmatrix} |\tau|^2\, \widetilde{\mathbb{N}}_{\mathsf{AB}} & \text{Re}\tau\, \widetilde{\mathbb{N}}_{\mathsf{A}}{}^{\mathsf{B}} \\ \text{Re}\tau\, \widetilde{\mathbb{N}}^{\mathsf{A}}{}_{\mathsf{B}} & \widetilde{\mathbb{N}}^{\mathsf{AB}} \end{pmatrix}, \tag{4.32}$$

where

$$\widetilde{\mathbb{N}}_{\mathbb{A}\mathbb{B}} = \int\limits_{K3} \begin{pmatrix} \langle \sigma_{\mathsf{U}}, \star_{b_{K3}} \sigma_{\mathsf{V}} \rangle & \langle \sigma_{\mathsf{U}}, \star_{b_{K3}} \sigma^{\mathsf{V}} \rangle \\ \langle \sigma^{\mathsf{U}}, \star_{b_{K3}} \sigma_{\mathsf{V}} \rangle & \langle \sigma^{\mathsf{U}}, \star_{b_{K3}} \sigma^{\mathsf{V}} \rangle \end{pmatrix} \tag{4.33}$$

is the $K3$ analogue of (3.58) (recall that the indices $\mathsf{A}, \mathsf{B}, \ldots, \mathsf{I}, \mathsf{J}, \ldots$ and $\mathsf{U}, \mathsf{V}, \ldots$ run over the same values). Similarly, we define for the even cohomology groups

$$\mathbb{N}_{\mathbb{I}\mathbb{J}} = \int_{K3\times T^2} \begin{pmatrix} \langle \omega_{\mathsf{I}}, \star_b \omega_{\mathsf{J}} \rangle & \langle \omega_{\mathsf{I}}, \star_b \tilde{\omega}^{\mathsf{J}} \rangle \\ \langle \tilde{\omega}^{\mathsf{I}}, \star_b \omega_{\mathsf{J}} \rangle & \langle \tilde{\omega}^{\mathsf{I}}, \star_b \tilde{\omega}^{\mathsf{J}} \rangle \end{pmatrix}, \tag{4.34}$$

which can be reformulated as

$$\mathbb{N}_{\mathbb{I}\mathbb{J}} = \frac{1}{\mathrm{Im}\rho} \begin{pmatrix} |\rho|^2\, \widetilde{\mathbb{N}}_{\mathsf{IJ}} & \mathrm{Re}\rho \widetilde{\mathbb{N}}_{\mathsf{I}}{}^{\mathsf{J}} \\ \mathrm{Re}\rho \widetilde{\mathbb{N}}^{\mathsf{I}}{}_{\mathsf{J}} & \widetilde{\mathbb{N}}^{\mathsf{IJ}} \end{pmatrix}, \tag{4.35}$$

with $\widetilde{\mathbb{N}}_{\mathbb{I}\mathbb{J}}$ taking the same form as (4.33).

4.2.2. Fluxes and cohomology bases

To relate the flux operators (4.1) to the gaugings of four-dimensional supergravity, we once more proceed analogously to the Calabi–Yau setting. The action of the twisted differential (2.35) on the cohomology bases can be summarized by the relations

$$\mathcal{D}(\Sigma^T)_{\mathbb{I}} = (\mathcal{O}^T)_{\mathbb{I}}{}^{\mathbb{A}}(\Xi^T)_{\mathbb{A}}, \qquad \mathcal{D}(\Xi^T)_{\mathbb{A}} = (\widetilde{\mathcal{O}}^T)_{\mathbb{A}}{}^{\mathbb{I}}(\Sigma^T)_{\mathbb{I}}, \tag{4.36}$$

where the charge matrices

$$\mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}} = \begin{pmatrix} -\tilde{P}^{\mathsf{A}}{}_{\mathsf{I}} & \tilde{P}^{\mathsf{AI}} \\ O_{\mathsf{AI}} & -O_{\mathsf{A}}{}^{\mathsf{I}} \end{pmatrix}, \qquad \widetilde{\mathcal{O}}^{\mathbb{I}}{}_{\mathbb{A}} = \begin{pmatrix} (O^T)^{\mathsf{I}}{}_{\mathsf{A}} & (\tilde{P}^T)^{\mathsf{IA}} \\ (O^T)_{\mathsf{IA}} & (\tilde{P}^T)_{\mathsf{I}}{}^{\mathsf{A}} \end{pmatrix} \tag{4.37}$$

comprise the flux expansion coefficients. Their components read

$$\begin{aligned}
\tilde{P}^{\mathsf{A}}{}_{\mathsf{I}} &= \begin{pmatrix} (f+y)^0{}_0 & q^0{}_{\mathsf{u}} & 0 \\ h^{\mathsf{u}}{}_0 & (f+y)^{\mathsf{u}}{}_{\mathsf{u}} & q^{\mathsf{u}}{}_{23} \\ 0 & h^{23}{}_{\mathsf{u}} & (f+y)^{23}{}_{23} \end{pmatrix}, \\
\tilde{P}^{\mathsf{AI}} &= \begin{pmatrix} 0 & r^{0\mathsf{u}} & (q+z)^{0\,23} \\ r^{\mathsf{u}0} & (q+z)^{\mathsf{uu}} & f^{\mathsf{u}\,23} \\ (q+z)^{23\,0} & f^{23\,\mathsf{u}} & 0 \end{pmatrix}, \\
O_{\mathsf{AI}} &= \begin{pmatrix} 0 & h_{0\mathsf{u}} & (f+y)_{0\,23} \\ h_{\mathsf{u}0} & (f+y)_{\mathsf{uu}} & q_{\mathsf{u}\,23} \\ (f+y)_{23\,0} & q_{23\,\mathsf{u}} & 0 \end{pmatrix}, \\
O_{\mathsf{A}}{}^{\mathsf{I}} &= \begin{pmatrix} (q+z)_0{}^0 & f_0{}^{\mathsf{u}} & 0 \\ r_{\mathsf{u}}{}^0 & (q+z)_{\mathsf{u}}{}^{\mathsf{u}} & f_{\mathsf{u}}{}^{23} \\ 0 & r_{23}{}^{\mathsf{u}} & (q+z)_{23}{}^{23} \end{pmatrix},
\end{aligned} \tag{4.38}$$

once more satisfying the relation

$$\widetilde{\mathcal{O}} = -S^{-1}\mathcal{O}^T S. \tag{4.39}$$

The notation was chosen such that the small letters in the charge matrices indicate the fluxes they descend from. While their origin should be clear for most cases, there are some caveats for the F- and Q-fluxes: Here, the coefficients with unequal indices arise from the flux components with two sub- respectively superscript $K3$ indices, while the coefficients with matching indices originate from the components with one sub- and one superscript index in $K3$.

4.2.3. Integrating over the internal space

With everything formulated in the same framework as the Calabi–Yau setting, it is now an easy exercise to integrate over the internal manifold. Similar considerations as in subsection 3.3.3 and 3.3.4 eventually lead to the results

$$\begin{aligned} V^{(\mathrm{IIA})}_{\text{scalar, NS–NS}} = e^{-2\phi}\Big[& V^{\mathbb{I}}(\mathcal{O}^T)_{\mathbb{I}}{}^{\mathbb{A}}\mathbb{M}_{\mathbb{A}\mathbb{B}}\mathcal{O}^{\mathbb{B}}{}_{\mathbb{J}}V^{\mathbb{J}} + W^{\mathbb{A}}(\widetilde{\mathcal{O}}^T)_{\mathbb{A}}{}^{\mathbb{I}}\mathbb{N}_{\mathbb{I}\mathbb{J}}\widetilde{\mathcal{O}}^{\mathbb{J}}{}_{\mathbb{B}}\overline{W}^{\mathbb{B}} \\ & - \frac{1}{4\mathcal{K}}\overline{W}^{\mathbb{A}}S_{\mathbb{A}\mathbb{B}}\mathcal{O}^{\mathbb{B}}{}_{\mathbb{I}}\left(V^{\mathbb{I}}\overline{V}^{\mathbb{J}} + \overline{V}^{\mathbb{I}}V^{\mathbb{J}}\right)(\mathcal{O}^T)_{\mathbb{J}}{}^{\mathbb{C}}(S^T)_{\mathbb{C}\mathbb{D}}\overline{W}^{\mathbb{D}}\Big] \\ & + \frac{1}{2}\left(\mathsf{G}^{\mathbb{I}}_{\text{flux}} + \widetilde{\mathcal{O}}^{\mathbb{I}}{}_{\mathbb{A}}\mathsf{C}^{\mathbb{A}}_0\right)\mathbb{N}_{\mathbb{I}\mathbb{J}}\left(\mathsf{G}^{\mathbb{J}}_{\text{flux}} + \widetilde{\mathcal{O}}^{\mathbb{J}}{}_{\mathbb{B}}\mathsf{C}^{\mathbb{B}}_0\right), \end{aligned} \tag{4.40}$$

for the type IIA case and

$$\begin{aligned} V^{(\mathrm{IIB})}_{\text{scalar, NS–NS}} = e^{-2\phi}\Big[& V^{\mathbb{I}}(\mathcal{O}^T)_{\mathbb{I}}{}^{\mathbb{A}}\mathbb{M}_{\mathbb{A}\mathbb{B}}\mathcal{O}^{\mathbb{B}}{}_{\mathbb{J}}V^{\mathbb{J}} + W^{\mathbb{A}}(\widetilde{\mathcal{O}}^T)_{\mathbb{A}}{}^{\mathbb{I}}\mathbb{N}_{\mathbb{I}\mathbb{J}}\widetilde{\mathcal{O}}^{\mathbb{J}}{}_{\mathbb{B}}\overline{W}^{\mathbb{B}} \\ & - \frac{1}{4\mathcal{K}}\overline{W}^{\mathbb{A}}S_{\mathbb{A}\mathbb{B}}\mathcal{O}^{\mathbb{B}}{}_{\mathbb{I}}\left(V^{\mathbb{I}}\overline{V}^{\mathbb{J}} + \overline{V}^{\mathbb{I}}V^{\mathbb{J}}\right)(\mathcal{O}^T)_{\mathbb{J}}{}^{\mathbb{C}}(S^T)_{\mathbb{C}\mathbb{D}}\overline{W}^{\mathbb{D}}\Big] \\ & + \frac{1}{2}\left(\mathsf{G}^{\mathbb{A}}_{\text{flux}} + \mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}}\mathsf{C}^{\mathbb{I}}_0\right)\mathbb{M}_{\mathbb{A}\mathbb{B}}\left(\mathsf{G}^{\mathbb{B}}_{\text{flux}} + \mathcal{O}^{\mathbb{B}}{}_{\mathbb{J}}\mathsf{C}^{\mathbb{J}}_0\right) \end{aligned} \tag{4.41}$$

for the type IIB case. Comparing the results reveals the same set of Mirror Transformations (3.80) already known from the Calabi–Yau setting (including a self-reflection of the Hodge diamond. One can furthermore see from the structure of the $K3 \times T^2$ gauge coupling matrices (4.32) and (4.35) that the mappings $\mathbb{M}_{\mathbb{A}\mathbb{B}} \leftrightarrow \mathbb{N}_{\mathbb{I}\mathbb{J}}$ can be realized by

$$\tau \leftrightarrow \rho. \tag{4.42}$$

In the bases employed above, the explicit mirror mapping between the moduli fields is not obvious. However, for T^2 mirror symmetry acts as (4.42) – whereas for the $K3$-part there are 19 complex-structure moduli plus a complex scalar consisting of the $(2,0)$- and $(0,2)$-components of the B-field, which are interchanged with the 20 complexified Kähler moduli.

5. Obtaining the full action of $\mathcal{N} = 2$ gauged supergravity

We next show how the framework can be extended to the kinetic terms by deriving the full four-dimensional action of $\mathcal{N} = 2$ gauged supergravity from the Calabi–Yau setting. In doing so, we again set cohomologically trivial terms to zero at the beginning of the calculation. A more thorough analysis similar to section 3 and dimensional reductions on $K3 \times T^2$ are more involved due to the appearance of additional Kaluza–Klein-like terms and will be saved for future work.

5.1. NS–NS sector

Due to the vanishing first and fifth Betti numbers of Calabi–Yau three-folds, there do not exist any non-trivial one- or five-cycles on CY_3. It follows that all fields with effectively one or five free internal indices acquire mass in four dimensions and can be ignored in the low-energy limit. One immediate effect is that all components of the metric and the Kalb–Ramond field with mixed indices can be discarded, which drastically simplifies the expressions (2.21) and (2.22) building up the NS–NS contribution (2.20) to the action,

$$\widetilde{\mathcal{F}}^I{}_{\mu\nu} \to 0, \qquad \widetilde{\mathcal{H}}_{\mu\nu\rho} \to \partial_{[\underline{\mu}} B_{\underline{\nu\rho}]}, \qquad D_\mu \mathcal{H}_{IJ} \to \partial_\mu \mathcal{H}_{IJ}, \qquad \mathcal{F}_I \to 0, \tag{5.1}$$

leaving us with

$$\begin{aligned} S_{\text{NS–NS}} = \frac{1}{2} \int \mathrm{d}^4x \mathrm{d}^{12}Y \sqrt{g^{(4)}} \sqrt{g_{CY_3}} e^{-2\phi} \Big[\\ R^{(4)} + 4 g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{12} g^{\mu\nu} g^{\rho\sigma} g^{\tau\lambda} \partial_{[\underline{\mu}} B_{\underline{\rho\tau}]} \partial_{[\underline{\nu}} B_{\underline{\sigma\lambda}]} + \frac{1}{8} g^{\mu\nu} \partial_\mu \mathcal{H}_{IJ} \partial_\nu \mathcal{H}^{IJ} \\ + \mathcal{F}_{IJK} \mathcal{F}_{I'J'K'} \Big(-\frac{1}{12} \mathcal{H}^{II'} \mathcal{H}^{JJ'} \mathcal{H}^{KK'} + \frac{1}{4} \mathcal{H}^{II'} \eta^{JJ'} \eta^{KK'} - \frac{1}{6} \eta^{II'} \eta^{JJ'} \eta^{KK'} \Big) \Big]. \end{aligned} \tag{5.2}$$

The first three terms are known from normal type II supergravities, while the last two lines were shown to correctly give rise to the scalar potential of $\mathcal{N} = 2$ gauged supergravity in section 3. It is therefore to be expected that the remaining term $\frac{1}{8} g^{\mu\nu} \partial_\mu \mathcal{H}_{IJ} \partial_\nu \mathcal{H}^{IJ}$ gives rise to the kinetic terms of the Kähler class and complex structure moduli. Indeed, inserting (2.5) and using antisymmetry of the Kalb–Ramond field, one obtains

$$\frac{1}{8} g^{\mu\nu} \partial_\mu \mathcal{H}_{IJ} \partial_\nu \mathcal{H}^{IJ} = \frac{1}{4} g^{\mu\nu} \left(\partial_\mu g_{ij} \partial_\nu g^{ij} - g^{ik} g^{jl} \partial_\mu b_{ij} \partial_\nu b_{kl} \right). \tag{5.3}$$

The first term encodes the dynamics of the internal metric, which is fully described by its fluctuations. Similarly to Calabi–Yau compactifications of supergravity theories, these can be expanded in terms of the Kähler class and complex structure moduli. For the Kalb–Ramond field, one can proceed analogously by using the expansion (3.44), which combines with the Kähler class moduli to form the complexified Kähler moduli.

Using this as a starting point, the rest of the dimensional reduction follows the same principles as in Calabi–Yau compactifications of type II supergravities. A review of the topic in general can be found in chapter two of [58], a similar discussion concerning manifolds with $SU(3) \times SU(3)$ structure in [37,57]. After switching to Einstein frame via Weyl-rescaling

$$g_{\mu\nu} \to e^{-2\phi} g_{\mu\nu} \tag{5.4}$$

of the external metric, one eventually arrives at

$$S_{\text{NS–NS, kin}} = \int_{M^{1,3}} \frac{1}{2} R^{(4)} \star \mathbf{1}^{(4)} - \mathrm{d}\phi \wedge \star \mathrm{d}\phi - \frac{1}{2} e^{-4\phi} \mathrm{d}B \wedge \star \mathrm{d}B - g_{\mathsf{ij}} \mathrm{d}t^{\mathsf{i}} \wedge \star \mathrm{d}t^{\mathsf{j}} - g_{\mathsf{a}\bar{\mathsf{b}}} \mathrm{d}z^{\mathsf{a}} \wedge \star \mathrm{d}\bar{z}^{\bar{\mathsf{b}}}, \tag{5.5}$$

where we switched to differential form notation for the sake of clarity. The expansion coefficients t^{i} (cf. (3.45)) parametrize the Kähler class moduli space M_{KC} with metric g_{ij}, and z^{a} the complex structure moduli space M_{CS} with metric $g_{\mathsf{a}\bar{\mathsf{b}}}$.

5.2. R–R sector

The most obvious way to proceed for the R–R sector would be to evaluate the corresponding action of (2.27) in four dimensions and then implement the duality relations (2.28) in order to recover the action of $\mathcal{N} = 2$ gauged supergravity. Since handling these duality relations in four dimensions turns out rather demanding, we will, however, pursue a different approach and consider the reduced equations of motion instead. Notice that this has been done for compactifications on $SU(3) \times SU(3)$ structure manifolds in [37], and many of the following technical steps are close to the ones employed in this work.

5.2.1. Type IIA setting

Relation to democratic type IIA supergravity

Starting from (2.27), a first step is to write down the pseudo-action explicitly in terms of poly-form fields and obtain a form similar to (3.35). In doing so, we again neglect all cohomologically trivial expressions and, thus, take into account only those components with zero, two, three, four or six internal indices. Applying the methods presented in section 4 of [47] to evaluate the expressions found in (2.27) and arranging the (now ten-dimensional) $\hat{C}$-fields and R–R fluxes in poly-forms

$$\begin{aligned} \hat{\mathcal{C}}^{(\mathrm{IIA})} &= \hat{C}_1 + \hat{C}_3 + \hat{C}_5 + \hat{C}_7 + \hat{C}_9, \\ \mathcal{G}^{(\mathrm{IIA})} &= G_0 + G_2 + G_4 + G_6, \end{aligned} \tag{5.6}$$

we can define

$$\hat{\mathfrak{G}}^{(\mathrm{IIA})} = e^{-\hat{B}} \mathcal{G}^{(\mathrm{IIA})} + \hat{\mathfrak{D}} \hat{\mathcal{C}}^{(\mathrm{IIA})} = e^{-\hat{B}} \mathcal{G}^{(\mathrm{IIA})} + e^{-\hat{B}} \hat{\mathcal{D}} \left(e^{\hat{B}} \hat{\mathcal{C}}^{(\mathrm{IIA})} \right), \tag{5.7}$$

with the ten-dimensional twisted differential of the general form

$$\hat{\mathcal{D}} = \hat{\mathrm{d}} - H \wedge - F \circ - Q \bullet - R \llcorner, \tag{5.8}$$

to write the complete type IIA R–R pseudo-Lagrangian (2.27) as

$$\star \mathcal{L}_{\mathrm{R\text{–}R}} = -\frac{1}{2} \hat{\mathfrak{G}}^{(\mathrm{IIA})} \wedge \star \hat{\mathfrak{G}}^{(\mathrm{IIA})}. \tag{5.9}$$

Notice that this resembles the R–R sector of democratic type IIA supergravity [46], up to an exchange of signs in the exponential factors and the inclusion of additional background fluxes. Since the action depends on all R–R potentials explicitly, their duality relations (2.28) have to be imposed by hand. For the type IIA case, these are equivalent to

$$\hat{\mathfrak{G}}^{(\mathrm{IIA})} = \lambda \left(\star \hat{\mathfrak{G}}^{(\mathrm{IIA})} \right), \tag{5.10}$$

where λ denotes the involution operator defined in (2.41). Varying the corresponding action of (5.9) with respect to the R–R fields, one obtains the poly-form equation

$$\left(\hat{\mathrm{d}} - \mathrm{d}\hat{B} \wedge + \mathfrak{H} \wedge + \mathfrak{F} \circ + \mathfrak{Q} \bullet + \mathfrak{R} \llcorner \right) \star \hat{\mathfrak{G}}^{(\mathrm{IIA})} = 0. \tag{5.11}$$

Employing the duality relations (5.10), these can be recast to take the form of the Bianchi identities

$$e^{-\hat{B}} \hat{\mathcal{D}} \left(e^{\hat{B}} \hat{\mathfrak{G}}^{(\mathrm{IIA})} \right) = 0, \tag{5.12}$$

where the prefactor of $e^{-\hat{B}}$ was included for later convenience. They are automatically satisfied when imposing nilpotency of the twisted differential by hand, and the nontrivial equations of motion in four dimensions can be obtained by implementation of the duality constraints (5.10).

Reduced equations of motion

In order to evaluate the equations of motion in four dimensions, we next express the appearing objects in a way that the framework of special geometry presented in subsection 3.3.1 can be applied. This can be achieved by switching to the so-called "A-basis"[1] introduced in [46], for which we define

$$e^{\hat{B}}\mathcal{C}^{(\mathrm{IIA})} = \left(\mathsf{C}_1^{\mathsf{I}} + \mathsf{C}_3^{\mathsf{I}}\right)\omega_{\mathsf{I}} + \left(\mathsf{C}_0^{\mathsf{A}} + \mathsf{C}_2^{\mathsf{A}} + \mathsf{C}_4^{\mathsf{A}}\right)\alpha_{\mathsf{A}} - \left(\mathsf{C}_{0\,\mathsf{A}} + \mathsf{C}_{2\,\mathsf{A}} + \mathsf{C}_{4\,\mathsf{A}}\right)\beta^{\mathsf{A}} + \left(\mathsf{C}_{1\,\mathsf{I}} + \mathsf{C}_{3\,\mathsf{I}}\right)\widetilde{\omega}^{\mathsf{I}} \tag{5.13}$$

and

$$G_0 = \mathsf{G}_{\mathrm{flux}\,0}\widetilde{\omega}^0, \quad G_2 = \mathsf{G}^{\mathsf{i}}_{\mathrm{flux}}\omega_{\mathsf{i}}, \quad G_4 = \mathsf{G}_{\mathrm{flux}\,\mathsf{i}}\widetilde{\omega}^{\mathsf{i}}, \quad G_6 = \mathsf{G}^0_{\mathrm{flux}}\,\omega_0, \tag{5.14}$$

where the objects C_n now denote differential n-forms living in four dimensional spacetime. The R–R poly-form (5.7) can then be expressed as

$$\hat{\mathfrak{G}}^{(\mathrm{IIA})} = e^{-\hat{B}}\hat{\mathsf{G}}^{(\mathrm{IIA})} = e^{-\hat{B}}\left(\hat{\mathsf{G}}_0^{(\mathrm{IIA})} + \hat{\mathsf{G}}_2^{(\mathrm{IIA})} + \hat{\mathsf{G}}_4^{(\mathrm{IIA})} + \hat{\mathsf{G}}_6^{(\mathrm{IIA})} + \hat{\mathsf{G}}_8^{(\mathrm{IIA})} + \hat{\mathsf{G}}_{10}^{(\mathrm{IIA})}\right). \tag{5.15}$$

Using the flux matrices (3.65) and the relations (3.66), the appearing poly-forms can be expanded in terms four-dimensional differential form fields,

$$\begin{aligned}
\hat{\mathsf{G}}_0^{(\mathrm{IIA})} &= \mathsf{G}_{0\,0}\widetilde{\omega}^0,\\
\hat{\mathsf{G}}_2^{(\mathrm{IIA})} &= \mathsf{G}_{2\,0}\widetilde{\omega}^0 + \mathsf{G}_0^{\mathsf{i}}\omega_{\mathsf{i}},\\
\hat{\mathsf{G}}_4^{(\mathrm{IIA})} &= \mathsf{G}_{4\,0}\widetilde{\omega}^0 + \mathsf{G}_2^{\mathsf{i}}\wedge\omega_{\mathsf{i}} - \mathsf{G}_1^{\mathsf{A}}\wedge\alpha_{\mathsf{A}} + \mathsf{G}_{1\,\mathsf{A}}\wedge\beta^{\mathsf{A}} + \mathsf{G}_{0\,\mathsf{i}}\widetilde{\omega}^{\mathsf{i}},\\
\hat{\mathsf{G}}_6^{(\mathrm{IIA})} &= \mathsf{G}_4^{\mathsf{i}}\wedge\omega_{\mathsf{i}} - \mathsf{G}_3^{\mathsf{A}}\wedge\alpha_{\mathsf{A}} + \mathsf{G}_{3\,\mathsf{A}}\wedge\beta^{\mathsf{A}} + \mathsf{G}_{2\,\mathsf{i}}\wedge\widetilde{\omega}^{\mathsf{i}} + \mathsf{G}_0^0\wedge\omega_0,\\
\hat{\mathsf{G}}_8^{(\mathrm{IIA})} &= \mathsf{G}_{4\,\mathsf{i}}\wedge\widetilde{\omega}^{\mathsf{i}} + \mathsf{G}_2^0\wedge\omega_0,\\
\hat{\mathsf{G}}_{10}^{(\mathrm{IIA})} &= \mathsf{G}_4^0\wedge\omega_0,
\end{aligned} \tag{5.16}$$

with the expansion coefficients given by

$$\begin{aligned}
\mathsf{G}_0^{\mathbb{I}} &= \mathsf{G}^{\mathbb{I}}_{\mathrm{flux}} + \widetilde{\mathcal{O}}^{\mathbb{I}}{}_{\mathbb{A}}\mathsf{C}_0^{\mathbb{A}},\\
\mathsf{G}_1^{\mathbb{A}} &= \mathrm{d}\mathsf{C}_0^{\mathbb{A}} + \mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}}\mathsf{C}_1^{\mathbb{I}},\\
\mathsf{G}_2^{\mathbb{I}} &= \mathrm{d}\mathsf{C}_1^{\mathbb{I}} + \widetilde{\mathcal{O}}^{\mathbb{I}}{}_{\mathbb{A}}\mathsf{C}_2^{\mathbb{A}},\\
\mathsf{G}_3^{\mathbb{A}} &= \mathrm{d}\mathsf{C}_2^{\mathbb{A}} + \mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}}\mathsf{C}_3^{\mathbb{I}},\\
\mathsf{G}_4^{\mathbb{I}} &= \mathrm{d}\mathsf{C}_3^{\mathbb{I}} + \widetilde{\mathcal{O}}^{\mathbb{I}}{}_{\mathbb{A}}\mathsf{C}_4^{\mathbb{A}}.
\end{aligned} \tag{5.17}$$

This expansion can be used as a starting point to compute the reduced equations of motion descending from (5.12). Substituting the definition (5.15) into (5.12), one obtains in A-basis notation

$$\hat{\mathcal{D}}\hat{\mathsf{G}}^{(\mathrm{IIA})} = 0. \tag{5.18}$$

After separating different components and integrating over CY_3, this gives rise to the four-dimensional equations of motion

[1] The naming was chosen based on the notation used in the original work [46] and will not play any role in the upcoming discussion.

$$
\begin{aligned}
\mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}}\mathsf{G}_0^{\mathbb{I}} &= 0,\\
d\mathsf{G}_0^{\mathbb{I}} - \widetilde{\mathcal{O}}^{\mathbb{I}}{}_{\mathbb{A}}\mathsf{G}_1^{\mathbb{A}} &= 0,\\
d\mathsf{G}_1^{\mathbb{A}} - \mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}}\mathsf{G}_2^{\mathbb{I}} &= 0,\\
d\mathsf{G}_2^{\mathbb{I}} - \widetilde{\mathcal{O}}^{\mathbb{I}}{}_{\mathbb{A}}\mathsf{G}_3^{\mathbb{A}} &= 0,\\
d\mathsf{G}_3^{\mathbb{A}} - \mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}}\mathsf{G}_4^{\mathbb{I}} &= 0.
\end{aligned}
\tag{5.19}
$$

Since the Kalb–Ramond field couples with the C-fields, one furthermore has to take into account the (non-trivial) equation of motion obtained by varying the complete ten-dimensional action with respect to $\hat{B}$, which yields an eight-form equation

$$
d\left(e^{-4\phi} \star dB\right) + \frac{1}{2}\left[\hat{\mathfrak{G}}^{(\mathrm{IIA})} \wedge \star\hat{\mathfrak{G}}^{(\mathrm{IIA})}\right]_8 = 0. \tag{5.20}
$$

Reduced duality constraints

Our aim is now to implement the duality constraints (5.10) into the equations of motion (5.19) and (5.20) in an appropriate way in order to recover the $D = 4$ $\mathcal{N} = 2$ gauged supergravity action found in formula (35) of [34]. In particular, we want the fundamental (but not necessarily propagating) degrees of freedom to be given by[2] $2h^{1,2} + 2$ scalars $\hat{Z}^{\mathbb{A}}$, $h^{1,1} + 1$ one-forms A_1^{I}, $2h^{1,2} + 2$ two-forms $B^{\mathbb{A}}$ and the external Kalb–Ramond field B.

Up to conventions, the reduced duality constraints can be obtained completely analogous to [37]. Inserting the expansion

$$
e^{-\hat{B}}\hat{\mathsf{G}}^{(\mathrm{IIA})} = e^{-b^{\mathsf{i}}\omega_{\mathsf{i}}}\left(K^{\mathsf{I}}\omega_{\mathsf{I}} + K_{\mathsf{I}}\widetilde{\omega}^{\mathsf{I}} + L^{\mathsf{A}}\alpha_{\mathsf{A}} - L_{\mathsf{A}}\beta^{\mathsf{A}}\right) \tag{5.21}
$$

into (5.10), one obtains

$$
\begin{aligned}
K^{\mathsf{I}}\omega_{\mathsf{I}} + K_{\mathsf{I}}\widetilde{\omega}^{\mathsf{I}} + L^{\mathsf{A}}\alpha_{\mathsf{A}} - L_{\mathsf{A}}\beta^{\mathsf{A}} = &-\star\lambda\left(K^{\mathsf{I}}\right)\star_b\omega_{\mathsf{I}} - \star\lambda\left(K_{\mathsf{I}}\right)\star_b\widetilde{\omega}^{\mathsf{I}} - \star\lambda\left(L^{\mathsf{A}}\right)\star_b\alpha_{\mathsf{A}}\\
&+\star\lambda\left(L_{\mathsf{A}}\right)\star_b\beta^{\mathsf{A}}.
\end{aligned}
\tag{5.22}
$$

Applying the operators $\int_{CY_3}\langle\widetilde{\omega}^{\mathsf{I}}, \star_b\cdot\rangle$ and $\int_{CY_3}\langle\beta^{\mathsf{A}}, \star_b\cdot\rangle$ to both sides of the equation and using (3.57)–(3.59), one can separate different internal components and obtain the reduced duality constraints

$$
\begin{aligned}
K_{\mathsf{I}} &= -\mathrm{Im}\mathcal{N}_{\mathsf{IJ}}\star\lambda\left(K^{\mathsf{J}}\right) + \mathrm{Re}\mathcal{N}_{\mathsf{IJ}}K^{\mathsf{J}},\\
L_{\mathsf{A}} &= -\mathrm{Im}\mathcal{M}_{\mathsf{AB}}\star\lambda\left(L^{\mathsf{B}}\right) + \mathrm{Re}\mathcal{M}_{\mathsf{AB}}L^{\mathsf{B}}.
\end{aligned}
\tag{5.23}
$$

The K- and L-poly-forms still contain four-dimension differential forms of different degrees. Separating components by hand and performing a Weyl-rescaling (5.4) according to (5.4), we eventually arrive at

$$
\begin{aligned}
\mathsf{G}_{2\,\mathsf{I}} - B\mathsf{G}_{0\,\mathsf{I}} &= \mathrm{Im}\mathcal{N}_{\mathsf{IJ}}\star\left(\mathsf{G}_2^{\mathsf{J}} - B\wedge\mathsf{G}_0^{\mathsf{J}}\right) + \mathrm{Re}\mathcal{N}_{\mathsf{IJ}}\left(\mathsf{G}_2^{\mathsf{J}} - B\wedge\mathsf{G}_0^{\mathsf{J}}\right),\\
\mathsf{G}_4^{\mathbb{I}} - B\wedge\mathsf{G}_2^{\mathbb{I}} + \frac{1}{2}B^2\mathsf{G}_0^{\mathbb{I}} &= -e^{4\phi}\left(S^{-1}\right)^{\mathbb{IJ}}\mathbb{N}_{\mathbb{JK}}\mathsf{G}_0^{\mathbb{K}}\star\mathbf{1}^{(4)},\\
\mathsf{G}_3^{\mathbb{A}} - B\wedge\mathsf{G}_1^{\mathbb{A}} &= e^{2\phi}\left(S^{-1}\right)^{\mathbb{AB}}\mathbb{M}_{\mathbb{BC}}\star\mathsf{G}_1^{\mathbb{C}}.
\end{aligned}
\tag{5.24}
$$

[2] We preliminarily adopt the notation of [34] and identify the correct definitions in the course of the following discussion.

Evaluating the equations of motion – constraints on fluxes

Before implementing the duality constraints, it makes sense to take a closer look at the first line of (5.19). Unlike the remaining equations of motion, the left hand side does not vanish trivially when imposing the nilpotency conditions (3.69). Instead, we are left with an additional constraint, which after integration over CY_3 via $\int_{CY_3}\langle\Sigma_{\mathbb{I}},\cdot\rangle$ reads

$$\mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}}\mathsf{G}^{\mathbb{I}}_{\text{flux}}=0 \tag{5.25}$$

and resembles the conditions found in (37) of [34]. Notice that these arise automatically from the DFT framework and do not have to be imposed by hand.

Evaluating the equations of motion – $\mathsf{C}^{\mathsf{I}}_1$

The simplest equation of motion to derive are those of the one-forms A^{I}_1, which we will be able to identify with the fields $\mathsf{C}^{\mathsf{I}}_1$ at the end of this subsection. In order to get some intuition for the way of proceeding, we will treat this example in more detail. The underlying idea can then easily be transferred to the remaining degrees of freedom.

Many of the technical steps in the following discussion are again very close to the ones employed in [37]. The essential difference is that in the present setting, the expressions (5.17) are completely determined by the DFT action, whereas in the case of $SU(3)\times SU(3)$ manifolds, their structure is governed only by the equations of motion (5.19). This leads to slight redefinitions of the encountered objects, and we will in particular go without additional assumptions regarding the flux matrices (3.65) and the existence of corresponding operators.

Before presenting explicit calculations, it is helpful to motivate our ansatz to derive the desired equations of motion for $\mathsf{C}^{\mathsf{I}}_1$. For this purpose, we take a look at the corresponding expression obtained by varying the action found in [34] with respect to the A^{I}_1,

$$\mathrm{d}\Big(\mathrm{Im}\mathcal{N}_{\mathsf{IJ}}\star\mathsf{F}^{\mathsf{J}}_2+\mathrm{Re}\mathcal{N}_{\mathsf{IJ}}\mathsf{F}^{\mathsf{J}}_2-e_{\mathsf{I}\,\mathbb{A}}B^{\mathbb{A}}-c_{\mathsf{I}}B\Big)=0. \tag{5.26}$$

The first two terms strongly resemble the first line of (5.24), and since $\mathsf{G}_{0\,\mathsf{I}}$ contains only expressions which we expect to appear in the four-dimensional action, a viable ansatz might be to replace $\mathsf{G}_{2\,\mathsf{I}}$ in one of the equations of motion (5.19). Reverting to the expected structure (5.26) of the final equation of motion once more, we see that the most obvious way to do this is by considering the lower-index components of the fourth equation of motion of (5.19). Applying the nilpotency constraint (3.69) of $\mathcal{D}$ and integrating over CY_3 similarly to the previous case, this can be written as

$$\mathrm{d}\mathsf{G}_{2\,\mathsf{I}}-\widetilde{\mathcal{O}}_{\mathsf{I}\,\mathbb{A}}\mathrm{d}\mathsf{C}^{\mathbb{A}}_2=0. \tag{5.27}$$

Using the first line of (5.24) to substitute $\mathsf{G}_{2\,\mathsf{I}}$ yields

$$\mathrm{d}\Big(\mathrm{Im}\mathcal{N}_{\mathsf{IJ}}\star\mathsf{F}^{\mathsf{J}}_2+\mathrm{Re}\mathcal{N}_{\mathsf{IJ}}\mathsf{F}^{\mathsf{J}}_2-\widetilde{\mathcal{O}}_{\mathsf{I}\,\mathbb{A}}\mathsf{C}^{\mathbb{A}}_2+B\wedge\mathsf{G}_{0\,\mathsf{I}}\Big)=0, \tag{5.28}$$

where

$$\mathsf{F}^{\mathsf{I}}_2:=\mathsf{G}^{\mathsf{I}}_2-B\wedge\mathsf{G}^{\mathsf{I}}_0. \tag{5.29}$$

This can be further simplified by pulling out a factor of $B\wedge$ from the definition (5.13) of $\mathsf{C}^{\mathbb{A}}_2$. We do this by employing the alternative expansion

$$
\begin{aligned}
e^{b^{\mathsf{i}}\omega_{\mathsf{i}}}\hat{\mathcal{C}}^{(\mathrm{IIA})} = & \big(\widetilde{\mathsf{C}}_1^{\mathsf{I}} + \widetilde{\mathsf{C}}_3^{\mathsf{I}}\big)\,\omega_{\mathsf{I}} \\
& + \big(\widetilde{\mathsf{C}}_0^{\mathsf{A}} + \widetilde{\mathsf{C}}_2^{\mathsf{A}} + \widetilde{\mathsf{C}}_4^{\mathsf{A}}\big)\,\alpha_{\mathsf{A}} - \big(\widetilde{\mathsf{C}}_{0\,\mathsf{A}} + \widetilde{\mathsf{C}}_{2\,\mathsf{A}} + \widetilde{\mathsf{C}}_{4\,\mathsf{A}}\big)\,\beta^{\mathsf{A}} \\
& + \big(\widetilde{\mathsf{C}}_{1\,\mathsf{I}} + \widetilde{\mathsf{C}}_{3\,\mathsf{I}}\big)\,\widetilde{\omega}^{\mathsf{I}},
\end{aligned}
\tag{5.30}
$$

from which we infer the relation

$$
\mathsf{C}_2^{\mathbb{A}} = \widetilde{\mathsf{C}}_2^{\mathbb{A}} + B \wedge \mathsf{C}_0^{\mathbb{A}}, \tag{5.31}
$$

while the other fields appearing in (5.28) remain unaffected. Inserting the definitions (5.17) for the $\mathsf{G}_{0\,\mathsf{I}}$, we are left with

$$
\mathsf{F}_2^{\mathsf{I}} = \mathrm{d}\mathsf{C}_1^{\mathsf{I}} + \widetilde{\mathcal{O}}^{\mathsf{I}}{}_{\mathbb{A}}\widetilde{\mathsf{C}}_2^{\mathbb{A}} - B \wedge \mathsf{G}^{\mathsf{I}}_{\mathrm{flux}} \tag{5.32}
$$

and the equations of motion

$$
\mathrm{d}\Big(\mathrm{Im}\mathcal{N}_{\mathsf{IJ}} \star \mathsf{F}_2^{\mathsf{J}} + \mathrm{Re}\mathcal{N}_{\mathsf{IJ}}\mathsf{F}_2^{\mathsf{J}} - \widetilde{\mathcal{O}}_{\mathsf{I}\,\mathbb{A}}\widetilde{\mathsf{C}}_2^{\mathbb{A}} + B \wedge \mathsf{G}_{\mathsf{I}\,\mathrm{flux}}\Big) = 0, \tag{5.33}
$$

which, up to sign convention for B, take precisely the form of the corresponding ones obtained from the action of [34] when identifying $A_1^{\mathsf{I}} = \mathsf{C}_1^{\mathsf{I}}$, $B^{\mathbb{A}} = \widetilde{\mathsf{C}}_2^{\mathbb{A}}$, $e_{\mathsf{I}\,\mathbb{A}} = \widetilde{\mathcal{O}}_{\mathsf{I}\,\mathbb{A}}$ and $c_{\mathsf{I}} = \mathsf{G}_{\mathsf{I}\,\mathrm{flux}}$.

Evaluating the equations of motion – $\widetilde{\mathsf{C}}_2^{\mathbb{A}}$

A similar analysis for the fields $B^{\mathbb{A}}$ in [34] implies that a viable strategy is to use lines one and three of the duality constraints (5.24) in order to eliminate the expressions $\mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}}\mathsf{C}_1^{\mathbb{I}}$ and $\mathsf{G}_{2\,\mathsf{I}}$ from the third equation of motion of (5.19). This can be done by first left-multiplying line three of (5.24) with $\widetilde{\mathcal{O}}_{\mathsf{I}\,\mathbb{A}}$, yielding

$$
\widetilde{\mathcal{O}}_{\mathsf{I}\,\mathbb{A}}\mathrm{d}\mathsf{C}_2^{\mathbb{A}} - B \wedge \mathrm{d}(\widetilde{\mathcal{O}}_{\mathsf{I}\,\mathbb{A}}\mathsf{C}_0^{\mathbb{A}}) = e^{2\phi}\widetilde{\mathcal{O}}_{\mathsf{I}\,\mathbb{A}}\left(S^{-1}\right)^{\mathbb{A}\mathbb{B}}\mathbb{M}_{\mathbb{B}\mathbb{C}} \star \mathsf{G}_1^{\mathbb{C}}. \tag{5.34}
$$

Employing the expansion (5.30) and solving for $\mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}}\mathsf{C}_1^{\mathbb{I}}$, we obtain

$$
\mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}}\mathsf{C}_1^{\mathbb{I}} = -\mathcal{O}^{\mathbb{A}}{}_{\mathsf{I}}(\Delta^{-1})^{\mathsf{IJ}}\left(\star\mathrm{d}(\widetilde{\mathcal{O}}_{\mathsf{J}\,\mathbb{B}}\widetilde{\mathsf{C}}_2^{\mathbb{B}}) + \widetilde{\mathcal{O}}_{\mathsf{J}\,\mathbb{B}}\mathsf{C}_0^{\mathbb{B}} \star \mathrm{d}B + e^{2\phi}(\mathcal{O}^T)_{\mathsf{J}}{}^{\mathbb{B}}\mathbb{M}_{\mathbb{B}\mathbb{C}}\mathrm{d}\mathsf{C}_0^{\mathbb{C}}\right), \tag{5.35}
$$

with

$$
\Delta_{\mathsf{IJ}} = e^{2\phi}(\mathcal{O}^T)_{\mathsf{I}}{}^{\mathbb{A}}\mathbb{M}_{\mathbb{A}\mathbb{B}}\mathcal{O}^{\mathbb{B}}{}_{\mathsf{J}}. \tag{5.36}
$$

Starting from line three of (5.19), we separate desired and undesired components to get

$$
\mathrm{d}(\mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}}\mathsf{C}_1^{\mathbb{I}}) - \mathrm{d}(\mathcal{O}^{\mathbb{A}}{}_{\mathsf{I}}\mathsf{C}_1^{\mathsf{I}}) - \mathcal{O}^{\mathbb{A}}{}_{\mathsf{I}}\widetilde{\mathcal{O}}^{\mathsf{I}}{}_{\mathbb{B}}\mathsf{C}_2^{\mathbb{B}} - \mathcal{O}^{\mathbb{A}\,\mathsf{I}}\mathsf{G}_{2\,\mathsf{I}} = 0. \tag{5.37}
$$

The first term can be substituted by (5.35), the third term by the relation

$$
-\Xi_{\mathbb{A}}\mathcal{O}^{\mathbb{A}}{}_{\mathsf{I}}\widetilde{\mathcal{O}}^{\mathsf{I}}{}_{\mathbb{B}}\mathsf{C}_2^{\mathbb{B}} = \Big(\Xi_{\mathbb{A}}\mathcal{O}^{\mathbb{A}\,\mathsf{I}}\widetilde{\mathcal{O}}_{\mathsf{I}\,\mathbb{B}} + \Sigma_{\mathbb{I}} \wedge \mathrm{d}_{\mathrm{int}}\widetilde{\mathcal{O}}^{\mathbb{I}}{}_{\mathbb{B}}\Big)\,\mathsf{C}_2^{\mathbb{B}} \tag{5.38}
$$

derived from (3.69), and the fourth term by the line two of (5.24). Integration over CY_3 then yields after left-multiplication with $S_{\mathbb{A}\mathbb{B}}$,

$$
\begin{aligned}
0 = & -\mathrm{d}\Big[(\widetilde{\mathcal{O}}^T)_{\mathbb{A}\,\mathsf{I}}(\Delta^{-1})^{\mathsf{IJ}}\Big(\star\mathrm{d}(\widetilde{\mathcal{O}}_{\mathsf{J}\,\mathbb{B}}\widetilde{\mathsf{C}}_2^{\mathbb{B}}) + \widetilde{\mathcal{O}}_{\mathsf{J}\,\mathbb{B}}\mathsf{C}_0^{\mathbb{B}} \star \mathrm{d}B + e^{2\phi}(\mathcal{O}^T)_{\mathsf{J}}{}^{\mathbb{B}}\mathbb{M}_{\mathbb{B}\mathbb{C}}\mathrm{d}\mathsf{C}_0^{\mathbb{C}}\Big)\Big] \\
& - \mathrm{d}(\widetilde{\mathcal{O}}^T)_{\mathbb{A}\,\mathsf{I}}\mathsf{C}_1^{\mathsf{I}} + (\widetilde{\mathcal{O}}^T)_{\mathbb{A}}{}^{\mathsf{I}}\Big(\mathrm{Im}\mathcal{N}_{\mathsf{IJ}} \star \mathsf{F}_2^{\mathsf{J}} + \mathrm{Re}\mathcal{N}_{\mathsf{IJ}}\mathsf{F}_2^{\mathsf{J}} + B \wedge \mathsf{G}_{\mathsf{I}\,\mathrm{flux}} - \widetilde{\mathcal{O}}_{\mathsf{I}\,\mathbb{B}}\widetilde{\mathsf{C}}_2^{\mathbb{B}}\Big),
\end{aligned}
\tag{5.39}
$$

revealing that we can identify $\hat{Z}^{\mathbb{A}} = \mathsf{C}_0^{\mathbb{A}}$.

Evaluating the equations of motion – $\mathsf{C}_0^{\mathbb{A}}$

Following the same procedure once more, we implement lines two and three of (5.24) into the fifth equation of motion of (5.19). Simplifying via equations of motion one and three, we obtain after integrating over CY_3

$$\mathrm{d}\left[e^{2\phi}(S^{-1})^{\mathbb{AB}}\mathbb{M}_{\mathbb{BC}}\star\mathsf{G}_1^{\mathbb{C}}\right]+\mathrm{d}B\wedge\mathsf{G}_1^{\mathbb{A}}+e^{4\phi}\mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}}\left(S^{-1}\right)^{\mathbb{IJ}}\mathbb{N}_{\mathbb{JK}}\mathsf{G}_0^{\mathbb{K}}\star\mathbf{1}^{(4)}=0. \tag{5.40}$$

Substituting (5.35) and lowering symplectic indices with $S_{\mathbb{AB}}$, we arrive at

$$\begin{aligned}0=&-\mathrm{d}\left[\widetilde{\Delta}_{\mathbb{AB}}\star\mathrm{d}\mathsf{C}_0^{\mathbb{B}}-e^{2\phi}\mathbb{M}_{\mathbb{AB}}\mathcal{O}^{\mathbb{B}}{}_{\mathsf{I}}(\Delta^{-1})^{\mathsf{IJ}}\left(\mathrm{d}(\widetilde{\mathcal{O}}_{\mathsf{J}\,\mathbb{C}}\widetilde{\mathsf{C}}_2^{\mathbb{C}})+\widetilde{\mathcal{O}}_{\mathsf{J}\,\mathbb{C}}\mathsf{C}_0^{\mathbb{C}}\mathrm{d}B\right)\right]\\&-\mathrm{d}B\wedge\left[S_{\mathbb{AB}}\mathrm{d}\mathsf{C}_0^{\mathbb{B}}-(\widetilde{\mathcal{O}}^T)_{\mathbb{A}\,\mathsf{I}}(\Delta^{-1})^{\mathsf{IJ}}\right.\\&\qquad\left.\cdot\left(\star\mathrm{d}(\widetilde{\mathcal{O}}_{\mathsf{J}\,\mathbb{C}}\widetilde{\mathsf{C}}_2^{\mathbb{C}})+\widetilde{\mathcal{O}}_{\mathsf{J}\,\mathbb{C}}\mathsf{C}_0^{\mathbb{C}}\star\mathrm{d}B+e^{2\phi}(\mathcal{O}^T)_{\mathsf{J}}{}^{\mathbb{C}}\mathbb{M}_{\mathbb{CD}}\mathrm{d}\mathsf{C}_0^{\mathbb{D}}\right)\right]\\&+e^{4\phi}(\widetilde{\mathcal{O}}^T)_{\mathbb{A}}{}^{\mathbb{I}}\mathbb{N}_{\mathbb{IJ}}\left(\mathsf{G}^{\mathbb{J}}_{\text{flux}}+\widetilde{\mathcal{O}}^{\mathbb{J}}{}_{\mathbb{B}}\mathsf{C}_0^{\mathbb{B}}\right)\star\mathbf{1}^{(4)},\end{aligned} \tag{5.41}$$

where

$$\widetilde{\Delta}_{\mathbb{AB}}=e^{2\phi}\left(\mathbb{M}_{\mathbb{AB}}-e^{2\phi}\mathbb{M}_{\mathbb{AC}}\mathcal{O}^{\mathbb{C}}{}_{\mathsf{I}}(\Delta^{-1})^{\mathsf{IJ}}(\mathcal{O}^T)_{\mathsf{J}}{}^{\mathbb{D}}\mathbb{M}_{\mathbb{DB}}\right). \tag{5.42}$$

Evaluating the equations of motion – B

The equations of motion (5.20) of $\hat{B}$ are already non-trivial and only need to be reformulated in a way that the undesired degrees of freedom disappear. We here consider only the relevant part with two external and six internal indices. Using the expansion (5.21) and manually inserting involution operators (2.41), we can use (3.57) and (3.59) to integrate over CY_3, and after another Weyl-rescaling according to (5.4), we arrive at

$$\frac{1}{2}\mathrm{d}\left(e^{-4\phi}\star\mathrm{d}B\right)-\mathsf{G}_0^{\mathsf{I}}\mathsf{G}_{2\,\mathsf{I}}+\mathsf{G}_{0\,\mathsf{I}}\mathsf{G}_2^{\mathsf{I}}+\mathsf{G}_{1\,\mathsf{A}}\wedge\mathsf{G}_1^{\mathsf{A}}=0. \tag{5.43}$$

Substituting the corresponding expressions from (5.17), we eventually find

$$\begin{aligned}0=&\frac{1}{2}\mathrm{d}\left(e^{-4\phi}\star\mathrm{d}B\right)-\mathsf{G}^{\mathsf{I}}_{\text{flux}}\left(\mathrm{Im}\mathcal{N}_{\mathsf{IJ}}\star\mathsf{F}_2^{\mathsf{J}}+\mathrm{Re}\mathcal{N}_{\mathsf{IJ}}\mathsf{F}_2^{\mathsf{J}}\right)+\mathsf{G}_{\mathsf{I}\,\text{flux}}\mathsf{F}_2^{\mathsf{I}}+\frac{1}{2}\mathrm{d}\mathsf{C}_0^{\mathbb{A}}S_{\mathbb{AB}}\mathrm{d}\mathsf{C}_0^{\mathbb{B}}\\&-\mathrm{d}\left[\mathsf{C}_0^{\mathbb{A}}(\widetilde{\mathcal{O}}^T)_{\mathbb{A}\,\mathsf{I}}(\Delta^{-1})^{\mathsf{IJ}}\left(\star\mathrm{d}(\widetilde{\mathcal{O}}_{\mathsf{J}\,\mathbb{B}}\widetilde{\mathsf{C}}_2^{\mathbb{B}})-\widetilde{\mathcal{O}}_{\mathsf{J}\,\mathbb{B}}\mathsf{C}_0^{\mathbb{B}}\star\mathrm{d}B+e^{2\phi}(\mathcal{O}^T)_{\mathsf{J}}{}^{\mathbb{B}}\mathbb{M}_{\mathbb{BC}}\mathrm{d}\mathsf{C}_0^{\mathbb{C}}\right)\right].\end{aligned} \tag{5.44}$$

Reconstructing the action of $D=4$ $\mathcal{N}=2$ gauged supergravity

Taking into account conventions and field identifications, we expect the complete four-dimensional action to take the form

$$\begin{aligned}S_{\mathrm{IIA}}=\int_{M^{1,3}}&\frac{1}{2}R^{(4)}\star\mathbf{1}^{(4)}-\mathrm{d}\phi\wedge\star\mathrm{d}\phi-\frac{e^{-4\phi}}{4}\mathrm{d}B\wedge\star\mathrm{d}B-g_{\mathsf{ij}}\mathrm{d}t^{\mathsf{i}}\wedge\star\mathrm{d}t^{\mathsf{j}}-g_{\mathsf{ab}}\mathrm{d}z^{\mathsf{a}}\wedge\star\mathrm{d}\bar{z}^{\bar{\mathsf{b}}}\\&+\frac{1}{2}\mathrm{Im}\mathcal{N}_{\mathsf{IJ}}\mathsf{F}_2^{\mathsf{I}}\wedge\star\mathsf{F}_2^{\mathsf{J}}+\frac{1}{2}\mathrm{Re}\mathcal{N}_{\mathsf{IJ}}\mathsf{F}_2^{\mathsf{I}}\wedge\mathsf{F}_2^{\mathsf{J}}+\frac{1}{2}\widetilde{\Delta}_{\mathbb{AB}}\mathrm{d}\mathsf{C}_0^{\mathbb{A}}\wedge\star\mathrm{d}\mathsf{C}_0^{\mathbb{B}}\\&+\frac{1}{2}(\Delta^{-1})^{\mathsf{IJ}}\left(\mathrm{d}(\widetilde{\mathcal{O}}_{\mathsf{I}\,\mathbb{A}}\widetilde{\mathsf{C}}_2^{\mathbb{A}})+\widetilde{\mathcal{O}}_{\mathsf{I}\,\mathbb{A}}\mathsf{C}_0^{\mathbb{A}}\mathrm{d}B\right)\wedge\star\left(\mathrm{d}(\widetilde{\mathcal{O}}_{\mathsf{J}\,\mathbb{B}}\widetilde{\mathsf{C}}_2^{\mathbb{B}})+\widetilde{\mathcal{O}}_{\mathsf{J}\,\mathbb{B}}\mathsf{C}_0^{\mathbb{B}}\mathrm{d}B\right)\\&+\left(\mathrm{d}(\widetilde{\mathcal{O}}_{\mathsf{I}\,\mathbb{A}}\widetilde{\mathsf{C}}_2^{\mathbb{A}})+\widetilde{\mathcal{O}}_{\mathsf{I}\,\mathbb{A}}\mathsf{C}_0^{\mathbb{A}}\mathrm{d}B\right)\wedge\left(e^{2\phi}(\Delta^{-1})^{\mathsf{IJ}}(\mathcal{O}^T)_{\mathsf{J}}{}^{\mathbb{B}}\mathbb{M}_{\mathbb{BC}}\mathrm{d}\mathsf{C}_0^{\mathbb{C}}\right)\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{2} \mathrm{d}B \wedge \mathsf{C}_0^{\mathbb{A}} S_{\mathbb{A}\mathbb{B}} \mathrm{d}\mathsf{C}_0^{\mathbb{B}} \\
&- \left(\widetilde{\mathcal{O}}_{\mathsf{I}\,\mathbb{A}} \widetilde{\mathsf{C}}_2^{\mathbb{A}} - \mathsf{G}_{\mathsf{I}\,\mathrm{flux}} B \right) \wedge \left(\mathrm{d}\mathsf{C}_1^{\mathsf{I}} + \frac{1}{2} \widetilde{\mathcal{O}}^{\mathsf{I}}{}_{\mathbb{B}} \widetilde{\mathsf{C}}_2^{\mathbb{B}} - \frac{1}{2} \mathsf{G}^{\mathsf{I}}_{\mathrm{flux}} B \right) + V_{\mathrm{scalar}} \star \mathbf{1}^{(4)},
\end{aligned}
\tag{5.45}
$$

with

$$
\begin{aligned}
V_{\mathrm{scalar}} &= V_{\mathrm{NSNS}} + V_{\mathrm{RR}} \\
&= + \frac{e^{-2\phi}}{2} V^{\mathbb{I}} (\mathcal{O}^T)_{\mathbb{I}}{}^{\mathbb{A}} \mathbb{M}_{\mathbb{A}\mathbb{B}} \mathcal{O}^{\mathbb{B}}{}_{\mathbb{J}} V^{\mathbb{J}} + \frac{e^{-2\phi}}{2} W^{\mathbb{A}} (\widetilde{\mathcal{O}}^T)_{\mathbb{A}}{}^{\mathbb{I}} \mathbb{N}_{\mathbb{I}\mathbb{J}} \widetilde{\mathcal{O}}^{\mathbb{J}}{}_{\mathbb{B}} \overline{W}^{\mathbb{B}} \\
&\quad - \frac{e^{-2\phi}}{4\mathcal{K}} W^{\mathbb{A}} S_{\mathbb{A}\mathbb{C}} \mathcal{O}^{\mathbb{C}}{}_{\mathbb{I}} \left(V^{\mathbb{I}} \overline{V}^{\mathbb{J}} + \overline{V}^{\mathbb{I}} V^{\mathbb{J}} \right) (\mathcal{O}^T)_{\mathbb{J}}{}^{\mathbb{D}} S_{\mathbb{D}\mathbb{B}} \overline{W}^{\mathbb{B}} \\
&\quad + \frac{e^{4\phi}}{2} \left(\mathsf{G}^{\mathbb{I}}_{\mathrm{flux}} + \widetilde{\mathcal{O}}^{\mathbb{I}}{}_{\mathbb{A}} \mathsf{C}_0^{\mathbb{A}} \right) \mathbb{N}_{\mathbb{I}\mathbb{J}} \left(\mathsf{G}^{\mathbb{J}}_{\mathrm{flux}} + \widetilde{\mathcal{O}}^{\mathbb{J}}{}_{\mathbb{B}} \mathsf{C}_0^{\mathbb{B}} \right).
\end{aligned}
\tag{5.46}
$$

One can now verify by direct calculation and use of the relations (3.67) and (5.25) that one indeed obtains the previously derived equations of motion when varying with respect to the corresponding fields. Up to different conventions and additional terms from the remaining sectors, this replicates the structure of (35) from [34].

A similar result was derived for $SU(3) \times SU(3)$ structure manifolds in [37], where the main difference is that the authors used projectors to render the fields $\widetilde{\mathcal{O}}_{\mathsf{I}\,\mathbb{A}} \widetilde{\mathsf{C}}_2^{\mathbb{A}}$ rather than $\widetilde{\mathsf{C}}_2^{\mathbb{A}}$ the fundamental degrees of freedom. This was done in accordance with the fact that $\widetilde{\mathsf{C}}_2^{\mathbb{A}}$ appears as propagating degree of freedom only in conjunction with the fluxes (or charges). Although this is certainly a desirable feature, we intentionally abstain from making any further assumptions regarding CY_3 and the flux matrices (3.65). While this comes with the drawback that $\widetilde{\mathsf{C}}_2^{\mathbb{A}}$ appears explicitly as a fundamental degree of freedom of the action (5.45), an obvious advantage is that one can directly read off the ten-dimensional origin of the four-dimensional fields.

To conclude the discussion of the type IIA setting, let us briefly illustrate how this result relates to the standard formulation of $D = 4$ $\mathcal{N} = 2$ gauged supergravity. As we have remarked at the beginning of this paper, the action constructed in [34] poses an alternative formulation of gauged supergravity in which a subset of the axions is dualized to two-forms. More precisely, the four-dimensional component B of the Kalb–Ramond field appears explicitly, in addition to different combinations of the NS–NS fluxes with the two-form fields $\widetilde{\mathsf{C}}_2^{\mathbb{A}}$. It was shown in [34] that under the assumption that $h^{1,1} \leq h^{1,2}$, the expressions $\widetilde{\mathcal{O}}_{\mathsf{I}\,\mathbb{A}} \widetilde{\mathsf{C}}_2^{\mathbb{A}}$ arise as duals of a subset of axions containing $h^{1,1} + 1$ of the corresponding $h^{1,2} + 1$ scalars of the original formulation. It is precisely the presence of the flux coefficients $q_{\mathsf{A}}{}^{\mathsf{I}}$, $\tilde{q}^{\mathsf{AI}}$ that prevents this dualization procedure from being reversible. Similarly, in the context of [6–8] it was found that the dualization of B to an axion a using Lagrange multipliers does not work out as straightforward when non-vanishing R–R fluxes are considered.

Before attempting to reconstruct the standard formulation of gauged supergravity, it is important to bear in mind that we did not perform any a posteriori dualizations of four-dimensional fields to obtain (5.45). Instead, the two-forms $\widetilde{\mathsf{C}}_2^{\mathbb{A}}$ descended naturally from the ten-dimensional field $\hat{C}_5$ dual to the "parent" $\hat{C}_3$ of the $\mathsf{C}_0^{\mathbb{A}}$ as well as $B \wedge \hat{C}_3$. In order to obtain a dual formulation, it therefore makes sense to again consider the ten-dimensional equations of motion and assume vanishing coefficients $q_{\mathsf{A}}{}^{\mathsf{I}}$, $\tilde{q}^{\mathsf{AI}}$. This is equivalent to setting

$$
\mathcal{O}^{\mathbb{A}\,\mathsf{I}} = 0, \qquad \widetilde{\mathcal{O}}^{\mathsf{I}}{}_{\mathbb{A}} = 0,
\tag{5.47}
$$

and most of the undesired degrees of freedom found in (5.17) to vanish immediately. One can then proceed differently from the general case by substituting lines one and three of (5.24) into the lower-index components of the fourth equation of motion of (5.19). After integrating over CY_3, this yields the non-trivial equation of motion

$$\begin{aligned} &\mathrm{d}\left(\mathrm{Im}\mathcal{N}_{\mathsf{IJ}} \star \mathsf{F}_2^{\mathsf{J}} + \mathrm{Re}\mathcal{N}_{\mathsf{IJ}}\mathsf{F}_2^{\mathsf{J}}\right) + \left(\mathsf{G}_{\mathsf{I}\,\mathrm{flux}} + \widetilde{\mathcal{O}}_{\mathsf{I}\,\mathbb{A}}\mathsf{C}_0^{\mathbb{A}}\right)\mathrm{d}B \\ &\quad + e^{2\phi}(\mathcal{O}^T)_{\mathsf{I}}{}^{\mathbb{A}}\mathbb{M}_{\mathbb{A}\mathbb{B}} \star \left(\mathrm{d}\mathsf{C}_0^{\mathbb{A}} + \mathcal{O}^{\mathbb{A}}{}_{\mathsf{I}}\mathsf{C}_1^{\mathsf{I}}\right) = 0 \end{aligned} \tag{5.48}$$

with

$$\mathsf{F}_2^{\mathsf{I}} = \mathrm{d}\mathsf{C}_1^{\mathsf{I}} - B \wedge \mathsf{G}_{\mathrm{flux}}^{\mathsf{I}}. \tag{5.49}$$

The first steps for line five of (5.19) and the equation of motion (5.20) of $\hat{B}$ are analogous to the general case. There is no need for a reformulation of the duality constraints in this simplified setting, and they can be evaluated in the forms found in (5.40) and (5.43), respectively. After inserting the duality relations (5.24) once more, it is easy to check that these equations of motion descend from the action

$$\begin{aligned} S_{\mathrm{IIA}} = \int_{M^{1,3}} & \frac{1}{2}R^{(4)} \star \mathbf{1}^{(4)} - \mathrm{d}\phi \wedge \star\mathrm{d}\phi - \frac{e^{-4\phi}}{4}\mathrm{d}B \wedge \star\mathrm{d}B - g_{\mathsf{ij}}\mathrm{d}t^{\mathsf{i}} \wedge \star\mathrm{d}t^{\mathsf{j}} - g_{\mathsf{a}\bar{\mathsf{b}}}\mathrm{d}z^{\mathsf{a}} \wedge \star\mathrm{d}\bar{z}^{\bar{\mathsf{b}}} \\ & + \frac{1}{2}\mathrm{Im}\mathcal{N}_{\mathsf{IJ}}\mathsf{F}_2^{\mathsf{I}} \wedge \star\mathsf{F}_2^{\mathsf{J}} + \frac{1}{2}\mathrm{Re}\mathcal{N}_{\mathsf{IJ}}\mathsf{F}_2^{\mathsf{I}} \wedge \mathsf{F}_2^{\mathsf{J}} + \frac{e^{2\phi}}{2}\mathbb{M}_{\mathbb{A}\mathbb{B}}D\mathsf{C}_0^{\mathbb{A}} \wedge \star D\mathsf{C}_0^{\mathbb{B}} \\ & - \frac{1}{2}\mathrm{d}B \wedge \left[\mathsf{C}_0^{\mathbb{A}}S_{\mathbb{A}\mathbb{B}}D\mathsf{C}_0^{\mathbb{B}} + \left(2\mathsf{G}_{\mathsf{I}\,\mathrm{flux}} + \widetilde{\mathcal{O}}_{\mathsf{I}\,\mathbb{A}}\mathsf{C}_0^{\mathbb{A}}\right)\mathsf{C}_1^{\mathsf{I}}\right] - \frac{1}{2}\mathsf{G}_{\mathsf{I}\,\mathrm{flux}}\mathsf{G}_{\mathrm{flux}}^{\mathsf{I}}B \wedge B \\ & + V_{\mathrm{scalar}} \star \mathbf{1}^{(4)}, \end{aligned} \tag{5.50}$$

where V_{scalar} takes the same form as in (5.46) and we defined the covariant derivative D by

$$D\mathsf{C}_0^{\mathbb{A}} = \mathrm{d}\mathsf{C}_0^{\mathbb{A}} + \mathcal{O}^{\mathbb{A}}{}_{\mathsf{I}}\mathsf{C}_1^{\mathsf{I}}, \tag{5.51}$$

such that the corresponding expression $D\mathsf{C}_0^{\mathbb{A}}$ matches with the field strength $\mathsf{G}_1^{\mathbb{A}}$. Notice that the second term does not appear in (5.45). This is closely related to the dualization procedure described in [34], where the original action contained additional scalars $e_{\mathsf{I}}{}^{\mathbb{A}}Z^{\mathsf{I}}$ orthogonal to the $\hat{Z}^{\mathbb{A}}$, the former of which were then dualized in order to obtain the two-form fields needed to account for the case of non-vanishing geometric and non-geometric fluxes.

From (3.63) and (3.64), we can infer that this setting corresponds to dimensional reduction of type IIA supergravity on CY_3 with non-vanishing F- and R-flux as well as R–R fluxes. The appearance of the non-geometric R-flux is due to the conventions we used for the collective notation (3.42), and one can obtain an analogous expression for non-vanishing F- and H-fluxes by exchanging the roles of the identity $\mathbf{1}^{(6)}$ and the volume form $\star\mathbf{1}^{(6)}$. Again, a similar result was found in [37] and identified as the effective action of compactifications on $SU(3)$ structure manifolds.

Parts of the action (5.50) already resemble the standard formulation of $D=4$ $\mathcal{N}=2$ gauged supergravity. In a final step, we would like to dualize the four-dimensional Kalb–Ramond field B to an axion a. However, since the presence of non-vanishing R–R fluxes gives rise to a mass term for B, the simple recipe for dualization via Lagrange multipliers does not apply. This was already discussed in the context of [6–8] for simpler settings, and we will spare the details here. For the purpose of this paper, it is sufficient to just consider the case

$$\mathsf{G}^{\mathsf{I}}_{\text{flux}} = 0. \tag{5.52}$$

Implementing the axion a as Lagrange multiplier, the standard procedure for dualization (see, e.g. [6] for explicit calculations) then brings us to

$$\begin{aligned} S_{\text{IIA}} = \int_{M^{1,3}} & \frac{1}{2} R^{(4)} \star \mathbf{1}^{(4)} - \mathrm{d}\phi \wedge \star \mathrm{d}\phi - g_{\mathsf{ij}} \mathrm{d}t^{\mathsf{i}} \wedge \star \mathrm{d}t^{\mathsf{j}} - g_{\mathsf{a}\bar{\mathsf{b}}} \mathrm{d}z^{\mathsf{a}} \wedge \star \mathrm{d}\bar{z}^{\bar{\mathsf{b}}} \\ & + \frac{1}{2} \mathrm{Im}\mathcal{N}_{\mathsf{IJ}} \mathsf{F}^{\mathsf{I}}_2 \wedge \star \mathsf{F}^{\mathsf{J}}_2 + \frac{1}{2} \mathrm{Re}\mathcal{N}_{\mathsf{IJ}} \mathsf{F}^{\mathsf{I}}_2 \wedge \mathsf{F}^{\mathsf{J}}_2 + \frac{e^{2\phi}}{2} \mathbb{M}_{\mathbb{AB}} D\mathsf{C}^{\mathbb{A}}_0 \wedge \star D\mathsf{C}^{\mathbb{B}}_0 \\ & - \frac{e^{4\phi}}{4} \left(Da + \mathsf{C}^{\mathbb{A}}_0 S_{\mathbb{AB}} D\mathsf{C}^{\mathbb{B}}_0 \right) \wedge \star \left(Da + \mathsf{C}^{\mathbb{A}}_0 S_{\mathbb{AB}} D\mathsf{C}^{\mathbb{B}}_0 \right) \\ & + V_{\text{scalar}} \star \mathbf{1}^{(4)}, \end{aligned} \tag{5.53}$$

where the covariant derivative of the axion reads

$$Da = \mathrm{d}a - \left(2\mathsf{G}_{\mathsf{I}\,\text{flux}} + \widetilde{\mathcal{O}}_{\mathsf{I}\,\mathbb{A}} \mathsf{C}^{\mathbb{A}}_0 \right) \mathsf{C}^{\mathsf{I}}_1. \tag{5.54}$$

This strongly resembles the well-known form of $D = 4$ $\mathcal{N} = 2$ supergravity, with additional gaugings descending from the non-vanishing NS–NS fluxes. When setting the remaining fluxes to zero, the contributions of $\mathsf{G}_{\mathsf{I}\,\text{flux}}$ as well as the matrices $\mathcal{O}$ and $\widetilde{\mathcal{O}}$ vanish, and one obtains ungauged $D = 4$ $\mathcal{N} = 2$ supergravity as expected.

5.2.2. *Type IIB setting*

The discussion for the type IIB case follows a very similar pattern, and we will only sketch the most important steps here.

Relation to democratic type IIB supergravity

Our ansatz is again to reformulate the type IIB R–R pseudo-action (2.27) in poly-form notation. The computations are mostly analogous to the type IIA case, and we obtain

$$\star \mathcal{L}^{(\text{IIB})}_{RR} = -\frac{1}{2} \hat{\mathfrak{G}}^{(\text{IIB})} \wedge \star \hat{\mathfrak{G}}^{(\text{IIB})} \tag{5.55}$$

with

$$\hat{\mathfrak{G}}^{(\text{IIB})} = e^{-\hat{B}} \mathcal{G}^{(\text{IIB})} + \hat{\mathfrak{D}} \hat{\mathcal{C}}^{(\text{IIB})} = e^{-\hat{B}} \mathcal{G}^{(\text{IIB})} + e^{-\hat{B}} \hat{\mathcal{D}} \left(e^{\hat{B}} \hat{\mathcal{C}}^{(\text{IIB})} \right), \tag{5.56}$$

and

$$\begin{aligned} \mathcal{G}^{(\text{IIB})} &= G_3, \\ \hat{\mathcal{C}}^{(\text{IIB})} &= \hat{C}_0 + \hat{C}_2 + \hat{C}_4 + \hat{C}_6 + \hat{C}_8. \end{aligned} \tag{5.57}$$

Notice that we consider only the three-form R–R flux since the one- and five-forms appear only in cohomologically trivial expressions on CY_3. The factor $e^{-\hat{B}}$ in front of $\hat{\mathcal{G}}^{(\text{IIB})}$ thus has no effect and is included only for later convenience. The duality constraints (2.28) for the type IIB case can be written as

$$\hat{\mathfrak{G}}^{(\text{IIB})} = -\lambda \left(\star \hat{\mathfrak{G}}^{(\text{IIB})} \right), \tag{5.58}$$

and varying the action with respect to the C-field components yields the equations of motion

$$\left(\mathrm{d} - \mathrm{d}\hat{B}\wedge + \mathfrak{H}\wedge + \mathfrak{F}\circ + \mathfrak{Q}\bullet + \mathfrak{R}_{\llcorner}\right)\star\hat{\mathfrak{G}}^{(\mathrm{IIB})} = 0, \tag{5.59}$$

which are equivalent to the Bianchi identities

$$e^{-\hat{B}}\hat{\mathcal{D}}\left(e^{\hat{B}}\hat{\mathfrak{G}}^{(\mathrm{IIB})}\right) = 0. \tag{5.60}$$

Reduced equations of motion and duality constraints

In order to employ the framework of special geometry, we again rewrite the above expressions in A-basis notation. We define

$$e^{\hat{B}}\mathcal{C}^{(\mathrm{IIB})} = \left(\mathsf{C}_0^{\mathsf{I}} + \mathsf{C}_2^{\mathsf{I}} + \mathsf{C}_4^{\mathsf{I}}\right)\omega_{\mathsf{I}} + \left(\mathsf{C}_1^{\mathsf{A}} + \mathsf{C}_3^{\mathsf{A}}\right)\alpha_{\mathsf{A}} - \left(\mathsf{C}_{1\,\mathsf{A}} + \mathsf{C}_{3\,\mathsf{A}}\right)\beta^{\mathsf{A}} + \left(\mathsf{C}_{0\,\mathsf{I}} + \mathsf{C}_{2\,\mathsf{I}} + \mathsf{C}_{4\,\mathsf{I}}\right)\widetilde{\omega}^{\mathsf{I}} \tag{5.61}$$

and

$$G_3 = -\mathsf{G}^{\mathsf{A}}_{\text{flux}}\alpha_{\mathsf{A}} + \mathsf{G}_{\text{flux}\,\mathsf{A}}\beta^{\mathsf{A}}, \tag{5.62}$$

which can be utilized to reformulate the type IIB R–R poly-form (5.56) as

$$\hat{\mathfrak{G}}^{(\mathrm{IIB})} = e^{-\hat{B}}\hat{\mathsf{G}}^{(\mathrm{IIB})} = e^{-\hat{B}}\left(\hat{\mathsf{G}}_1^{(\mathrm{IIB})} + \hat{\mathsf{G}}_3^{(\mathrm{IIB})} + \hat{\mathsf{G}}_5^{(\mathrm{IIB})} + \hat{\mathsf{G}}_7^{(\mathrm{IIB})} + \hat{\mathsf{G}}_9^{(\mathrm{IIB})}\right). \tag{5.63}$$

Notice that this strongly resembles the corresponding expressions of the type IIA case (cf. (5.13), (5.14) and (5.15)) with exchanged roles of the even and odd cohomology components. We once more employ a shorthand notation

$$\begin{aligned}
\hat{\mathsf{G}}_1^{(\mathrm{IIB})} &= \mathsf{G}_{1\,0}\widetilde{\omega}^0,\\
\hat{\mathsf{G}}_3^{(\mathrm{IIB})} &= \mathsf{G}_{3\,0}\widetilde{\omega}^0 + \mathsf{G}_1^{\mathsf{i}}\omega_{\mathsf{i}} - \mathsf{G}_0^{\mathsf{A}}\wedge\alpha_{\mathsf{A}} + \mathsf{G}_{0\,\mathsf{A}}\wedge\beta^{\mathsf{A}},\\
\hat{\mathsf{G}}_5^{(\mathrm{IIB})} &= \mathsf{G}_3^{\mathsf{i}}\wedge\omega_{\mathsf{i}} - \mathsf{G}_2^{\mathsf{A}}\wedge\alpha_{\mathsf{A}} + \mathsf{G}_{2\,\mathsf{A}}\wedge\beta^{\mathsf{A}} + \mathsf{G}_{1\,\mathsf{i}}\widetilde{\omega}^{\mathsf{i}},\\
\hat{\mathsf{G}}_7^{(\mathrm{IIB})} &= -\mathsf{G}_4^{\mathsf{A}}\wedge\alpha_{\mathsf{A}} + \mathsf{G}_{4\,\mathsf{A}}\wedge\beta^{\mathsf{A}} + \mathsf{G}_{3\,\mathsf{i}}\wedge\widetilde{\omega}^{\mathsf{i}} + \mathsf{G}_1^0\wedge\omega_0,\\
\hat{\mathsf{G}}_9^{(\mathrm{IIB})} &= \mathsf{G}_3^0\wedge\omega_0,
\end{aligned} \tag{5.64}$$

where the expansion coefficients

$$\begin{aligned}
\mathsf{G}_0^{\mathbb{A}} &= \mathsf{G}^{\mathbb{A}}_{\text{flux}} + \mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}}\mathsf{C}_0^{\mathbb{I}},\\
\mathsf{G}_1^{\mathbb{I}} &= \mathrm{d}\mathsf{C}_0^{\mathbb{I}} + \widetilde{\mathcal{O}}^{\mathbb{I}}{}_{\mathbb{A}}\mathsf{C}_1^{\mathbb{A}},\\
\mathsf{G}_2^{\mathbb{A}} &= \mathrm{d}\mathsf{C}_1^{\mathbb{A}} + \mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}}\mathsf{C}_2^{\mathbb{I}},\\
\mathsf{G}_3^{\mathbb{I}} &= \mathrm{d}\mathsf{C}_2^{\mathbb{I}} + \widetilde{\mathcal{O}}^{\mathbb{I}}{}_{\mathbb{A}}\mathsf{C}_3^{\mathbb{A}},\\
\mathsf{G}_4^{\mathbb{A}} &= \mathrm{d}\mathsf{C}_3^{\mathbb{A}} + \mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}}\mathsf{C}_4^{\mathbb{I}}
\end{aligned} \tag{5.65}$$

can be derived by using the flux matrix relations (3.65)–(3.66). The equations of motion (5.60) reduce to

$$\hat{\mathcal{D}}\hat{\mathsf{G}}^{(\mathrm{IIB})} = 0 \tag{5.66}$$

in A-basis notation, giving rise to the set of four-dimensional equations

$$\begin{aligned}
\widetilde{\mathcal{O}}^{\mathbb{I}}{}_{\mathbb{A}}\mathsf{G}_0^{\mathbb{A}} &= 0,\\
\mathrm{d}\mathsf{G}_0^{\mathbb{A}} - \mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}}\mathsf{G}_1^{\mathbb{I}} &= 0,\\
\mathrm{d}\mathsf{G}_1^{\mathbb{I}} - \widetilde{\mathcal{O}}^{\mathbb{I}}{}_{\mathbb{A}}\mathsf{G}_2^{\mathbb{A}} &= 0,\\
\mathrm{d}\mathsf{G}_2^{\mathbb{A}} - \mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}}\mathsf{G}_3^{\mathbb{I}} &= 0,\\
\mathrm{d}\mathsf{G}_3^{\mathbb{I}} - \widetilde{\mathcal{O}}^{\mathbb{I}}{}_{\mathbb{A}}\mathsf{G}_4^{\mathbb{A}} &= 0
\end{aligned} \tag{5.67}$$

after applying the same methods we already used to derive (5.19). The equation of motion for $\hat{B}$ reads after Weyl-rescaling according to (5.4),

$$\mathrm{d}\left(e^{-4\phi}\star\mathrm{d}B\right)+\frac{1}{2}\left[\hat{\mathfrak{G}}^{(\mathrm{IIB})}\wedge\star\hat{\mathfrak{G}}^{(\mathrm{IIB})}\right]_8=0. \tag{5.68}$$

For the duality constraints (5.58), we follow the same pattern as for (5.10) and obtain

$$\begin{aligned}\mathsf{G}_{2\,\mathsf{A}}-B\mathsf{G}_{0\,\mathsf{A}}&=\mathrm{Im}\mathcal{M}_{\mathsf{AB}}\star\left(\mathsf{G}_2^{\mathsf{B}}-B\wedge\mathsf{G}_0^{\mathsf{B}}\right)+\mathrm{Re}\mathcal{M}_{\mathsf{AB}}\left(\mathsf{G}_2^{\mathsf{B}}-B\wedge\mathsf{G}_0^{\mathsf{B}}\right),\\ \mathsf{G}_4^{\mathbb{A}}-B\wedge\mathsf{G}_2^{\mathbb{A}}+\frac{1}{2}B^2\mathsf{G}_0^{\mathbb{A}}&=-e^{4\phi}\left(S^{-1}\right)^{\mathbb{AB}}\mathbb{M}_{\mathbb{BC}}\mathsf{G}_0^{\mathbb{C}}\star\mathbf{1}^{(4)},\\ \mathsf{G}_3^{\mathbb{I}}-B\wedge\mathsf{G}_1^{\mathbb{I}}&=e^{2\phi}\left(S^{-1}\right)^{\mathbb{IJ}}\mathbb{N}_{\mathbb{JK}}\star\mathsf{G}_1^{\mathbb{K}}.\end{aligned} \tag{5.69}$$

Reconstructing the action

As the structural analogies between the two settings suggest, the equations of motion can be evaluated by following the same pattern as in the type IIA case, eventually leading to the effective four-dimensional action

$$\begin{aligned}S_{\mathrm{IIB}}=\int_{M^{1,3}}&\frac{1}{2}R^{(4)}\star\mathbf{1}^{(4)}-\mathrm{d}\phi\wedge\star\mathrm{d}\phi-\frac{e^{-4\phi}}{4}\mathrm{d}B\wedge\star\mathrm{d}B-g_{\mathsf{ij}}\mathrm{d}t^{\mathsf{i}}\wedge\star\mathrm{d}t^{\mathsf{j}}-g_{\mathsf{a}\bar{\mathsf{b}}}\mathrm{d}z^{\mathsf{a}}\wedge\star\mathrm{d}\bar{z}^{\bar{\mathsf{b}}}\\ &+\frac{1}{2}\mathrm{Im}\mathcal{M}_{\mathsf{AB}}\mathsf{F}_2^{\mathsf{A}}\wedge\star\mathsf{F}_2^{\mathsf{B}}+\frac{1}{2}\mathrm{Re}\mathcal{M}_{\mathsf{AB}}\mathsf{F}_2^{\mathsf{A}}\wedge\mathsf{F}_2^{\mathsf{B}}+\frac{1}{2}\widetilde{\Delta}_{\mathbb{IJ}}\mathrm{d}\mathsf{C}_0^{\mathbb{I}}\wedge\star\mathrm{d}\mathsf{C}_0^{\mathbb{J}}\\ &+\frac{1}{2}(\Delta^{-1})^{\mathsf{AB}}\left(\mathrm{d}(\mathcal{O}_{\mathsf{A}\,\mathbb{I}}\widetilde{\mathsf{C}}_2^{\mathbb{I}})+\mathcal{O}_{\mathsf{A}\,\mathbb{I}}\mathsf{C}_0^{\mathbb{I}}\mathrm{d}B\right)\wedge\star\left(\mathrm{d}(\mathcal{O}_{\mathsf{B}\,\mathbb{J}}\widetilde{\mathsf{C}}_2^{\mathbb{J}})+\mathcal{O}_{\mathsf{B}\,\mathbb{J}}\mathsf{C}_0^{\mathbb{J}}\mathrm{d}B\right)\\ &+\left(\mathrm{d}(\mathcal{O}_{\mathsf{A}\,\mathbb{I}}\widetilde{\mathsf{C}}_2^{\mathbb{I}})+\mathcal{O}_{\mathsf{A}\,\mathbb{I}}\mathsf{C}_0^{\mathbb{I}}\mathrm{d}B\right)\wedge\left(e^{2\phi}(\Delta^{-1})^{\mathsf{AB}}(\widetilde{\mathcal{O}}^T)_{\mathsf{B}}{}^{\mathbb{J}}\mathbb{N}_{\mathbb{JK}}\mathrm{d}\mathsf{C}_0^{\mathbb{K}}\right)+\frac{1}{2}\mathrm{d}B\wedge\mathsf{C}_0^{\mathbb{I}}S_{\mathbb{IJ}}\mathrm{d}\mathsf{C}_0^{\mathbb{J}}\\ &-\left(\mathcal{O}_{\mathsf{A}\,\mathbb{I}}\widetilde{\mathsf{C}}_2^{\mathbb{I}}-\mathsf{G}_{\mathsf{A}\,\mathrm{flux}}B\right)\wedge\left(\mathrm{d}\mathsf{C}_1^{\mathsf{A}}+\frac{1}{2}\mathcal{O}^{\mathsf{A}}{}_{\mathbb{J}}\widetilde{\mathsf{C}}_2^{\mathbb{J}}-\frac{1}{2}\mathsf{G}_{\mathrm{flux}}^{\mathsf{A}}B\right)+V_{\mathrm{scalar}}\star\mathbf{1}^{(4)}\end{aligned} \tag{5.70}$$

with

$$\begin{aligned}V_{\mathrm{scalar}}&=V_{\mathrm{NSNS}}+V_{\mathrm{RR}}\\ &=+\frac{e^{-2\phi}}{2}V^{\mathbb{I}}(\mathcal{O}^T)_{\mathbb{I}}{}^{\mathbb{A}}\mathbb{M}_{\mathbb{AB}}\mathcal{O}^{\mathbb{B}}{}_{\mathbb{J}}V^{\mathbb{J}}+\frac{e^{-2\phi}}{2}W^{\mathbb{A}}(\widetilde{\mathcal{O}}^T)_{\mathbb{A}}{}^{\mathbb{I}}\mathbb{N}_{\mathbb{IJ}}\widetilde{\mathcal{O}}^{\mathbb{J}}{}_{\mathbb{B}}\overline{W}^{\mathbb{B}}\\ &\quad-\frac{e^{-2\phi}}{4\mathcal{K}}W^{\mathbb{A}}S_{\mathbb{AC}}\mathcal{O}^{\mathbb{C}}{}_{\mathbb{I}}\left(V^{\mathbb{I}}\overline{V}^{\mathbb{J}}+\overline{V}^{\mathbb{I}}V^{\mathbb{J}}\right)(\mathcal{O}^T)_{\mathbb{J}}{}^{\mathbb{D}}S_{\mathbb{DB}}\overline{W}^{\mathbb{B}}\\ &\quad+\frac{e^{4\phi}}{2}\left(\mathsf{G}_{\mathrm{flux}}^{\mathbb{A}}+\mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}}\mathsf{C}_0^{\mathbb{I}}\right)\mathbb{M}_{\mathbb{AB}}\left(\mathsf{G}_{\mathrm{flux}}^{\mathbb{B}}+\mathcal{O}^{\mathbb{B}}{}_{\mathbb{J}}\mathsf{C}_0^{\mathbb{J}}\right).\end{aligned} \tag{5.71}$$

Comparing this to (5.45), one can again construct a set of mirror mappings by extending (3.80) to

$$\begin{array}{llll} t^{\mathsf{i}} \leftrightarrow z^{\mathsf{a}}, & & g_{\mathsf{ij}} \leftrightarrow g_{\mathsf{a}\bar{\mathsf{b}}}, & \\ \mathbb{M}_{\mathbb{AB}} \leftrightarrow \mathbb{N}_{\mathbb{IJ}}, & & h^{1,1} \leftrightarrow h^{1,2}, & \\ V^{\mathbb{I}} \leftrightarrow W^{\mathbb{A}}, & & S_{\mathbb{IJ}} \leftrightarrow S_{\mathbb{AB}}, & \\ \mathsf{C}_n^{\mathbb{I}} \leftrightarrow \mathsf{C}_n^{\mathbb{A}}, & & \mathsf{G}_{\mathrm{flux}}^{\mathbb{I}} \leftrightarrow \mathsf{G}_{\mathrm{flux}}^{\mathbb{A}}, & \\ \mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}} \leftrightarrow \widetilde{\mathcal{O}}^{\mathbb{I}}{}_{\mathbb{A}}, & & & \end{array} \tag{5.72}$$

once more confirming preservation of IIA $\leftrightarrow$ IIB Mirror Symmetry in the presence of both geometric and non-geometric fluxes.

6. Conclusion

Let us summarize the results obtain in this work. In section 2 we derived the scalar potential of four-dimensional $\mathcal{N}=2$ gauged supergravity from dimensional reduction of the purely internal type IIA and IIB DFT action on a Calabi–Yau three-fold CY_3. Building upon the elaborations of [33], we extended the discussed setting by relaxing the primitivity constraints and revealing a more general structure of the reformulated DFT action which strongly resembles that of type II supergravities on $SU(3) \times SU(3)$ structure manifolds (cf. [37]).

It was then exemplified through $K3 \times T^2$ (cf. section 3) how the framework can be generalized beyond the Calabi–Yau setting. This was done by utilizing the features of generalized Calabi–Yau and $K3$ structures [19,35] to allow for a special geometric description of the $K3 \times T^2$ moduli space, eventually leading to a scalar potential term resembling that of $\mathcal{N}=4$ gauged supergravity formulated in the $\mathcal{N}=2$ formalism first discussed in [34]. The essential idea here was to exploit the Calabi–Yau property of $K3$ and T^2 to formally construct $K3 \times T^2$ analogues of the structure forms of CY_3,

$$
\begin{aligned}
e^{b_{CY_3}+iJ_{CY_3}} &\longleftrightarrow e^{b_{K3}+iJ_{K3}} \wedge e^{b_{T^2}+iJ_{T^2}}\,, \\
e^{b_{CY_3}} \wedge \Omega_{CY_3} &\longleftrightarrow \left(e^{b_{K3}} \wedge \Omega_{K3}\right) \wedge \left(e^{b_{T^2}} \wedge \Omega_{T^2}\right),
\end{aligned}
\tag{6.1}
$$

where J denotes the Kähler form of the respective manifold and Ω its holomorphic one-, two- or three-form. While the constructed scalar potential shows characteristic features of $\mathcal{N}=4$ gauged supergravity, relating the result to its standard formulation explicitly turned out to be a nontrivial task and will therefore be saved for future work. We expect that the discussion for arbitrary manifolds allowing for a generalized Calabi–Yau structure in the sense of [19,35] follows the same pattern.

Another novel feature of the setting discussed in this paper is its capability of describing generalized dilaton fluxes and non-vanishing trace-terms of the geometric and non-geometric fluxes. While the additional fluxes in the Calabi–Yau setting are set to zero, (cf. section 3.3.2), it is to be expected that they serve as a ten-dimensional origin of the non-unimodular gaugings of $\mathcal{N}=4$ gauged supergravity [51,52] in the $K3 \times T^2$ setting (see also section 4.2.3 of [30] for a brief discussion in the DFT context). Integrating the dilaton flux operators into the twisted differential of DFT did not require including a rescaling charge operator as done in [51], which is in accordance with the result of [37] for $SU(3) \times SU(3)$ structure manifolds.

Finally, in both the CY_3 and the $K3 \times T^2$ setting, a set of mirror mappings relating the results for type IIA and IIB DFT could be read off and featured the characteristic exchange of roles between the Kähler class and complex structure moduli spaces in the former and between the two modular parameters of T^2 in the latter.

In section 5 we reconstructed the full bosonic part of the four-dimensional $\mathcal{N}=2$ gauged supergravity action by including the kinetic terms into the Calabi–Yau setting. Our results replicate the findings of [34] and once more illustrate how simultaneous treatment of all NS–NS and R–R fluxes not only gives rise to gaugings in the effective four-dimensional theory, but also requires a dualization of a subset of the axions in order to account for all fluxes. Turning off half of the fluxes correctly led to the standard formulation of $\mathcal{N}=2$ gauged supergravity, which could be further reduced to its ungauged version when setting the remaining fluxes to zero. The IIA $\leftrightarrow$ IIB mirror mappings constructed in the context of the scalar potential discussion could be straightforwardly generalized to the full action.

Our analysis of the R–R sector strongly resembles that of [37] for $SU(3) \times SU(3)$ manifolds, where the essential difference is that in the discussion of the present paper the field strengths are determined by the DFT action. This leads to a slightly altered formulation of the action in which the ten-dimensional origin of the four-dimensional fields becomes evident. In particular, rather than only the actual propagating fields, the reduced action contains fundamental degrees of freedom which appear in the equations of motion only in conjunction with the flux charges.

It would be interesting to use the procedure elaborated here to derive the remaining four-dimensional gauged supergravities. The next step is to see how the framework can be applied to the full action compactified on $K3 \times T^2$. Since dimensional reduction on Calabi–Yau three-folds leads to a partially dualized formulation of gauged $\mathcal{N} = 2$ supergravity, an important question in this context is whether the action of half-maximal supersymmetric gauged supergravity obtained via $K3 \times T^2$ shows similar properties in the case of non-vanishing non-geometric fluxes. We plan to address these questions in future work by extending the discussion to manifolds with $SU(2)$ structure [61–63]. Other possible directions include extensions of the orientifold setting discussed in [33] or dimensional reduction of heterotic DFT.

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Appendix A. Notation and conventions

A.1. Spacetime geometry and indices

Throughout this paper we make use of various kinds of indices, which are structured as follows:

- We distinguish between serif letters $A, a, \ldots$ denoting spacetime indices and sanserif letters $\mathsf{A}, \mathsf{a}, \ldots$ labeling the coordinates of moduli spaces. We furthermore introduce blackboard typeface capital letters $\mathbb{A}, \mathbb{B}, \ldots, \mathbb{I}, \mathbb{J}, \ldots$ for collective notation summarizing several de Rham cohomology bases, which are specified in subsection 3.3.1 and 4.2.1.
- For spacetime indices, capital letters denote doubled coordinates, and small letters denote normal coordinates.
- For spacetime indices, ten-dimensional indices (including doubled ones) are labeled with a hat symbol, external indices are denoted by small Greek letters and internal indices by checked or normal Latin letters as specified below.

Using this as a guideline, we define the following indices:

- Hatted Latin capital letters $\hat{M}, \hat{N}, \ldots$ and $\hat{A}, \hat{B}, \ldots$ label the curved respectively tangent coordinates of twenty-dimensional doubled spacetime.
- Small hatted letters $\hat{m}, \hat{n}, \ldots$ and $\hat{a}, \hat{b}, \ldots$ label the curved respectively tangent coordinates of ten-dimensional spacetime.
- Small Greek letters $\mu, \nu, \ldots$ and small Latin letters $e, f, \ldots$ label the curved respectively tangent coordinates of four-dimensional external spacetime.

- Checked capital Latin letters $\check{I}, \check{J}, \dots$ and $\check{A}, \check{B}, \dots$ label the curved respectively tangent coordinates of a general twelve-dimensional doubled internal space.
- Checked small Latin letters $\check{i}, \check{j}, \dots$ and $\check{a}, \check{b}, \dots$ label the curved respectively tangent coordinates of a general six-dimensional internal space.
- Coordinates of specific internal manifolds or their components (e.g. CY_3, $K3$ and T^2) are denoted by normal Latin letters specified in the corresponding sections of this paper.
- On CY_3, small Latin letters $a, \bar{a}, b, \bar{b} \dots$ denote complex curved coordinates of six-dimensional internal spacetime. It will be clear from the context whether the letters $a, b, \dots$ without bars denote holomorphic curved coordinates or normal tangent coordinates. On $K3 \times T^2$, $a, \bar{a}, b, \bar{b} \dots$ denote complex curved coordinates of $K3$ and $g, \bar{g}, h, \bar{h} \dots$ those of T^2.
- Moduli space or cohomological indices are specified in the sections where the bases are defined.

A.2. *Tensor formalism and differential forms*

For general tensors, differential forms and related operators, we apply the following conventions:

- The antisymmetrization of a tensor A is defined by

$$A_{[\underline{\hat{m}_1 \hat{m}_2} \dots \underline{\hat{m}_n}]} := \frac{1}{n!} \sum_{\pi \in S_n} (-1)^{\text{sign}(\pi)} A_{\pi(\hat{m}_1)\pi(\hat{m}_2)\dots\pi(\hat{m}_n)}, \tag{A.1}$$

  where S_n denotes the set of permutations of $\{1, 2, \dots n\}$.
- The Levi-Civita tensor $\varepsilon^{\hat{m}_1 \dots \hat{m}_D}$ in D dimensions is defined as the totally antisymmetric tensor with $\varepsilon^{012\dots(D-1)} = 1$ (Lorentzian signature) or $\varepsilon^{123\dots D} = 1$ (Euclidean signature). It satisfies the relations

$$\begin{aligned} \varepsilon^{\hat{m}_1 \dots \hat{m}_D} \varepsilon_{\hat{n}_1 \dots \hat{n}_D} &= D! \delta^{[\underline{\hat{m}_1}}_{\hat{n}_1} \dots \delta^{\underline{\hat{m}_D}]}_{\hat{n}_D} = \delta^{\hat{m}_1 \dots \hat{m}_D}_{\hat{n}_1 \dots \hat{n}_D} \\ \varepsilon^{\hat{m}_1 \dots \hat{m}_p \hat{m}_{p+1} \dots \hat{m}_D} \varepsilon_{\hat{m}_1 \dots \hat{m}_p \hat{n}_{p+1} \dots \hat{n}_D} &= p!\,(D-p)! \delta^{[\underline{\hat{m}_{p+1}}}_{\hat{n}_{p+1}} \dots \delta^{\underline{\hat{m}_D}]}_{\hat{n}_D} = p! \delta^{\hat{m}_{p+1} \dots \hat{m}_D}_{\hat{n}_{p+1} \dots \hat{n}_D} \\ \varepsilon^{\hat{m}_1 \dots \hat{m}_D} \varepsilon_{\hat{m}_1 \dots \hat{m}_D} &= D!. \end{aligned} \tag{A.2}$$

- The components of a differential p-form are defined as

$$\hat{\omega}_p = \frac{1}{p!} \omega_{\hat{m}_1 \dots \hat{m}_p} \mathrm{d}x^{\hat{m}_1} \wedge \dots \wedge dx^{\hat{m}_p}. \tag{A.3}$$

- The exterior product of a p-form $\hat{\omega}_p$ and a q-form $\hat{\chi}_q$ is given by

$$\begin{aligned} \wedge : \Omega^p(\mathcal{M}) \times \Omega^q(\mathcal{M}) &\to \Omega^{p+q}(\mathcal{M}) \\ (\hat{\omega}_p, \hat{\chi}_q) &\mapsto \hat{\omega}_p \wedge \hat{\omega}_q = \frac{(p+q)!}{p!q!} \omega_{[\underline{\hat{m}_1} \dots \underline{\hat{m}_p}} \chi_{\underline{\hat{n}_1} \dots \underline{\hat{n}_q}]} \mathrm{d}x^{\hat{m}_1} \wedge \dots \\ &\qquad \dots \wedge \mathrm{d}x^{\hat{m}_p} \wedge \mathrm{d}x^{\hat{n}_1} \wedge \dots \wedge \mathrm{d}x^{\hat{n}_q}. \end{aligned} \tag{A.4}$$

  In this context, we choose the notation $(\hat{\omega}_p)^n = \overbrace{\hat{\omega}_p \wedge \hat{\omega}_p \wedge \dots \wedge \hat{\omega}_p}^{n \text{ factors}}$ for exterior products of a p-form ω_p with itself.

- The exterior derivative d is given by

$$\begin{aligned} \mathrm{d} : \Omega^{p}(\mathcal{M}) &\to \Omega^{p+1}(\mathcal{M}) \\ \hat{\omega}_p \mapsto \mathrm{d}\hat{\omega}_p &= \frac{1}{p!}\frac{\partial \omega_{\hat{m}_1\ldots\hat{m}_p}}{\partial x^{\hat{n}}}\mathrm{d}x^{\hat{n}} \wedge \mathrm{d}x^{\hat{m}_1} \wedge \ldots \wedge \mathrm{d}x^{\hat{m}_p}. \end{aligned} \tag{A.5}$$

- The Hodge star operator $\star$ is defined by

$$\begin{aligned} \star: \quad \Omega^{p}(\mathcal{M}) &\to \Omega^{D-p}(\mathcal{M}) \\ \hat{\omega}_p \mapsto \star\hat{\omega}_p &= \frac{1}{\sqrt{\hat{g}}\,p!\,(D-p)!}\varepsilon_{\hat{m}_1\ldots\hat{m}_p\hat{m}_{p+1}\ldots\hat{m}_D} g^{\hat{m}_1\hat{n}_1}\ldots g^{\hat{m}_p\hat{n}_p}\omega_{\hat{n}_1\ldots\hat{n}_p}\mathrm{d}^{D-p}x. \end{aligned} \tag{A.6}$$

  In particular, one can define a scalar product of two p-forms $\hat{\omega}_p$ and $\hat{\chi}_p$ by taking the volume form component of

$$\hat{\omega}_p \wedge \star\overline{\hat{\chi}_p} = \frac{\sqrt{\hat{g}}}{p!}\omega_{\hat{m}_1\ldots\hat{m}_p}\overline{\chi_{\hat{n}_1\ldots\hat{n}_p}}g^{\hat{m}_1\hat{n}_1}\ldots g^{\hat{m}_p\hat{n}_p}\mathrm{d}^D x. \tag{A.7}$$

  On D-dimensional Lorentzian manifolds, $\star$ satisfies the bijectivity condition

$$\star\star\hat{\omega}_p = (-1)^{p(d-p)+1}\,\hat{\omega}_p. \tag{A.8}$$

  Using this, one can show that the b-twisted Hodge star operator (2.43) squares to -1,

$$\star_b\star_b = -1. \tag{A.9}$$

  When splitting a differential p-form $\hat{\omega}_p = \eta_{p-n} \wedge \rho_n$ living in M^{10} into two forms $\eta_{p-n} \in \Omega^{p-n}\left(M^{1,3}\right)$ and $\rho_n \in \Omega^n\left(M^6\right)$, the Hodge-star operator splits as

$$\star\hat{\omega}_p = (-1)^{n(p-n)}\star\eta_{p-n}\wedge\star\rho_n. \tag{A.10}$$

  As a consequence, one obtains for the involution operator (2.41)

$$\star\lambda\left(\hat{\omega}_p\right) = \star\lambda\left(\eta_{p-n}\right)\wedge\star\lambda\left(\rho_n\right). \tag{A.11}$$

- For differential poly-forms, we define the projectors $[\cdot]_n$ to give as output the n-form components of the argument.

Appendix B. Complex and Kähler geometry

This appendix provides an overview on geometric properties of Calabi–Yau 3-folds and $K3 \times T^2$ used for the calculations of section 3 and section 4, respectively. Most of the technical steps are based on the notions complex and Kähler geometry, which shall be discussed here.

Both CY_3 and $K3 \times T^2$ are complex manifolds, allowing for a standard complex structure I satisfying

$$\begin{aligned} I^a{}_b &= i\delta^a{}_b, & I^{\bar{a}}{}_{\bar{b}} &= -i\delta^{\bar{a}}{}_{\bar{b}}, \\ I^a{}_{\bar{b}} &= 0, & I^{\bar{a}}{}_b &= 0. \end{aligned} \tag{B.1}$$

Being also Kähler and, thus, Hermitian manifolds, the only non-vanishing components of their metric g are $g_{a\bar{b}} = \overline{g}_{\bar{a}b}$. They are related to the Kähler form J by

$$J_{a\bar{b}} = i g_{a\bar{b}}, \quad J_{\bar{a}b} = -i g_{\bar{a}b} \tag{B.2}$$

and, in real coordinates,

$$J_{ij} = g_{im} I^m{}_j . \tag{B.3}$$

For the holomorphic three-form of CY_3, we employ the normalization

$$\frac{i}{8}\Omega \wedge \star\overline{\Omega} = \frac{1}{3!} J^3 , \tag{B.4}$$

leading to the relations

$$\begin{aligned} \Omega_{abc}\overline{\Omega}_{\bar{a}\bar{b}\bar{c}} g^{c\bar{c}} &= 8\left(g_{a\bar{a}} g_{b\bar{b}} - g_{a\bar{b}} g_{b\bar{a}}\right) , \\ \Omega_{abc}\overline{\Omega}_{\bar{a}\bar{b}\bar{c}} g^{b\bar{b}} g^{c\bar{c}} &= 16 g_{a\bar{a}} , \\ \Omega_{abc}\overline{\Omega}_{\bar{a}\bar{b}\bar{c}} g^{a\bar{a}} g^{b\bar{b}} g^{c\bar{c}} &= 48. \end{aligned} \tag{B.5}$$

The same normalization is applied to holomorphic form $\Omega := \Omega_{K3} \times \Omega_{T^2}$ of $K3 \times T^2$ (with $J := J_{K3} + J_{T^2}$), and one obtains similarly

$$\begin{aligned} \Omega_{gab}\overline{\Omega}_{\bar{g}\bar{a}\bar{b}} g^{g\bar{g}} &= 8\left(g_{a\bar{a}} g_{b\bar{b}} - g_{a\bar{b}} g_{b\bar{a}}\right) , \\ \Omega_{gab}\overline{\Omega}_{\bar{g}\bar{a}\bar{b}} g^{b\bar{b}} &= 8 g_{g\bar{g}} g_{a\bar{a}} , \\ \Omega_{gab}\overline{\Omega}_{\bar{g}\bar{a}\bar{b}} g^{a\bar{a}} g^{b\bar{b}} &= 16 g_{g\bar{g}} , \\ \Omega_{gab}\overline{\Omega}_{\bar{g}\bar{a}\bar{b}} g^{g\bar{g}} g^{a\bar{a}} g^{b\bar{b}} &= 16. \end{aligned} \tag{B.6}$$

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