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BROUWER'S FAN THEOREM AND CONVEXITY

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Abstract. In the framework of Bishop's constructive mathematics we introduce co-convexity as a property of subsets B of $\{0,1\}^*$, the set of finite binary sequences, and prove that co-convex bars are uniform. Moreover, we establish a canonical correspondence between detachable subsets B of $\{0,1\}^*$ and uniformly continuous functions f defined on the unit interval such that B is a bar if and only if the corresponding function f is positive-valued, B is a uniform bar if and only if f has positive infimum, and B is co-convex if and only if f satisfies a weak convexity condition.

§1. Introduction. In their seminal article [7], Julian and Richman established the following correspondence between detachable subsets B of $\{0,1\}^*$ and uniformly continuous functions on the unit interval.

PROPOSITION 1.1. *For every detachable subset B of* $\{0,1\}^*$ *there exists a uniformly continuous function* $f : [0,1] \to [0,\infty[$ *such that*

(i) *B is a bar $\Leftrightarrow$ f is positive-valued,*
(ii) *B is a uniform bar $\Leftrightarrow$, f has positive infimum.*

Conversely, for every uniformly continuous function $f : [0,1] \to [0,\infty[$ *there exists a detachable subset B of* $\{0,1\}^*$ *such that (i) and (ii) hold.*

Consequently, Brouwer's fan theorem for detachable bars, D-FAN, is equivalent to the statement that every uniformly continuous, positive-valued function on $[0,1]$ has positive infimum. On the other hand, in [3, Theorem 1] we have shown that if the function is convex, the fan theorem is no longer required.

THEOREM 1.2. *Suppose that* $f : [0,1] \to]0,\infty[$ *is uniformly continuous and convex. Then f has positive infimum.*

Therefore, the question arises whether there is a constructively valid 'convex' version of the fan theorem. To this end, we define 'co-convexity' as a property of subsets B of $\{0,1\}^*$ and show in Theorem 2.1 that there indeed is such a result. Moreover, in Theorem 3.4, we include the following correspondence

(iii) B is co-convex $\Leftrightarrow$ f is weakly convex

into the list of Proposition 1.1, where weak convexity of functions generalises convexity. The way we achieve our aim shows some similarities with the proofs presented in [2] and [7], but in the crucial parts we need to proceed differently in order to include

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(iii), in particular when deriving the function f with properties (i)–(iii) for some given detachable set B.

The framework of our presentation is Bishop's constructive mathematics [4–6]. This includes the use of choice axioms which are compatible with intuitionistic logic like the axiom of *countable choice*:

Let A be a set and let S be a subset of $\mathbb{N} \times A$. If for each n there exists a in A such that $(n, a) \in S$, then there exists a function $f : \mathbb{N} \to A$ such that $(n, f(n)) \in S$ for each $n \in \mathbb{N}$.

§2. A constructive fan theorem. We write $\{0,1\}^*$ for the set of all finite binary sequences u, v, w. Let $\varnothing$ be the empty sequence and let $\{0,1\}^{\mathbb{N}}$ be the set of all infinite binary sequences α, β, γ. For every u let $|u|$ be the *length* of u, that is $|\varnothing| = 0$ and for $u = (u_0, \dots, u_{n-1})$ we have $|u| = n$. For $v = (v_0, \dots, v_{m-1})$, the *concatenation* $u * v$ of u and v is defined by

$$u * v = (u_0, \dots, u_{n-1}, v_0, \dots, v_{m-1}).$$

The *restriction* $\overline{\alpha}n$ of α to n bits is given by

$$\overline{\alpha}n = (\alpha_0, \dots, \alpha_{n-1}).$$

Thus $|\overline{\alpha}n| = n$ and $\overline{\alpha}0 = \varnothing$. For u with $n \leq |u|$, the *restriction* $\overline{u}n$ is defined analogously. A subset B of $\{0,1\}^*$ is *closed under extension* if $u * v \in B$ for all $u \in B$ and for all v. A sequence α *hits* B if there exists n such that $\overline{\alpha}n \in B$. B is a *bar* if every α hits B. B is a *uniform bar* if there exists N such that for every α there exists $n \leq N$ such that $\overline{\alpha}n \in B$. Often one requires B to be *detachable*, that is for every u the statement $u \in B$ is decidable. Now we are ready to introduce Brouwer's *fan theorem for detachable bars*.

D-FAN : Every detachable bar is a uniform bar.

In Bishop's constructive mathematics, D-FAN is neither provable nor falsifiable, see [5, Section 3 of Chapter 5]. Define

$$u < v \;:\Leftrightarrow\; |u| = |v| \wedge \exists k < |u| \, (\overline{u}k = \overline{v}k \wedge u_k = 0 \wedge v_k = 1)$$

and

$$u \leq v \;:\Leftrightarrow\; u = v \vee u < v.$$

Note that $u < v$ means that u and v are on the same level and u is to the left of v. A subset B of $\{0,1\}^*$ is *co-convex* if for every α which hits B there exists n such that either

$$\{v \mid v \leq \overline{\alpha}n\} \subseteq B \quad \text{or} \quad \{v \mid \overline{\alpha}n \leq v\} \subseteq B.$$

Note that, for detachable B, co-convexity follows from the convexity of the complement of B, where $C \subseteq \{0,1\}^*$ is *convex* if for all u, v, w we have

$$u \leq v \leq w \wedge u, w \in C \Rightarrow v \in C.$$

Define the *upper closure* B' of B by

$$B' = \{u \mid \exists k \leq |u| \, (\overline{u}k \in B)\}.$$

Note that B is a (uniform) bar if and only if B' is a (uniform) bar. Moreover, if B is detachable then B' is also detachable. Therefore, we may assume that bars are closed under extension.

THEOREM 2.1. *Every co-convex bar is a uniform bar.*

PROOF. Fix a co-convex bar B. Since the upper closure of B is also co-convex, we can assume that B is closed under extension. Define

$$C = \{u \mid \exists n \, \forall w \in \{0,1\}^n \, (u * w \in B)\} .$$

Note that C consists of the set of nodes beyond which B is uniform. Note that $B \subseteq C$ and that C is closed under extension as well. Moreover, B is a uniform bar if and only if there exists n such that $\{0,1\}^n \subseteq C$.

First, we show that

$$\forall u \, \exists i \in \{0,1\} \, (u * i \in C) . \tag{1}$$

Fix u. For

$$\beta = u * 1 * 0 * 0 * 0 * \cdots$$

there exist an l such that either

$$\left\{ v \mid v \leq \overline{\beta} l \right\} \subseteq B$$

or

$$\left\{ v \mid \overline{\beta} l \leq v \right\} \subseteq B.$$

Since B is closed under extension, we can assume that $l > |u|+1$. Let $m = l - |u| - 1$. If $\left\{ v \mid v \leq \overline{\beta} l \right\} \subseteq B$, we can conclude that

$$u * 0 * w \in B$$

for every w of length m, which implies that $u * 0 \in C$. If $\left\{ v \mid \overline{\beta} l \leq v \right\} \subseteq B$, we obtain

$$u * 1 * w \in B$$

for every w of length m, which implies that $u * 1 \in C$. This concludes the proof of (1).

By countable choice, there exists a function $F : \{0,1\}^* \to \{0,1\}$ such that

$$\forall u \, \left(u * F(u) \in C \right) .$$

Define α by

$$\alpha_n = 1 - F(\overline{\alpha} n).$$

Next, we show by induction on n that

$$\forall n \, \forall u \in \{0,1\}^n \, (u \neq \overline{\alpha} n \Rightarrow u \in C) . \tag{2}$$

If $n = 0$, the statement clearly holds, since in this case the statement $u \neq \overline{\alpha} n$ is false. Now fix some n such that (2) holds. Moreover, fix $w \in \{0,1\}^{n+1}$ such that $w \neq \overline{\alpha}(n+1)$.

CASE 1. $\overline{w} n \neq \overline{\alpha} n$. Then $\overline{w} n \in C$ and therefore $w \in C$.

CASE 2. $w = \overline{\alpha} n * (1 - \alpha_n) = \overline{\alpha} n * F(\overline{\alpha} n)$. This implies $w \in C$. So we have established (2).

There exists n such that $\overline{\alpha} n \in B$. Applying (2) to this n, we can conclude that every u of length n is an element of C, thus B is a uniform bar. ⊣

REMARK 2.2.

(a) Note that we do not need to require that the co-convex bar B in Theorem 2.1 be detachable.

(b) If B is detachable, the function F in the proof Theorem 2.1 can be defined directly—without using countable choice—by $F(u) = 0$ if

$$\exists m \left(\forall w \in \{0,1\}^m \left(u * 0 * w \in B\right) \wedge \exists w \in \{0,1\}^m \left(u * 1 * w \notin B\right)\right),$$

and $F(u) = 1$, otherwise.

§3. A correspondence between subsets of $\{0,1\}^*$ and functions on $[0,1]$. We recall a few basic notions of constructive analysis. Fix an inhabited subset S of $\mathbb{R}$. A real number x is a *lower bound* of S if

$$\forall s \in S \, (x \leq s)$$

and the *infimum* of S if it is a lower bound of S and

$$\forall \varepsilon > 0 \, \exists s \in S \, (s < x + \varepsilon).$$

In this case we write $x = \inf S$. We cannot assume that every inhabited set with a lower bound has an infimum. However, under some additional conditions, this is the case. See [6, Corollary 2.1.19] for a proof of the following criterion.

LEMMA 3.1. *Let S be an inhabited set of real numbers which has a lower bound. Assume further that for all $p, q \in \mathbb{Q}$ with $p < q$ either p is a lower bound of S or else there exists $s \in S$ with $s < q$. Then S has an infimum.*

For $X \subseteq \mathbb{R}$, a function $f : X \to \mathbb{R}$ is *weakly increasing* if

$$\forall s, t \in X \left(s < t \;\; \Rightarrow \;\; f(s) \leq f(t)\right),$$

strictly increasing if

$$\forall s, t \in X \left(s < t \;\; \Rightarrow \;\; f(s) < f(t)\right),$$

and *monotone* if either f or $-f$ is weakly increasing.

A subset S of a metric space (X, d) is *totally bounded* if for every $\varepsilon > 0$ there exist $s_1, \ldots, s_n \in S$ such that

$$\forall s \in S \, \exists i \in \{1, \ldots, n\} \left(d(s, s_i) < \varepsilon\right)$$

and *compact* if it is totally bounded and *complete* (i.e., every Cauchy sequence in S has a limit in S). Proofs of the following basic statements can be found in [6, Section 2.2].

LEMMA 3.2. (i) *If S is totally bounded, then for all $x \in X$ the distance*

$$d(x, S) = \inf \{d(x, s) \mid s \in S\}$$

exists and the function $x \mapsto d(x, S)$ is uniformly continuous.

(ii) *Uniformly continuous images of totally bounded sets are totally bounded.*

(iii) *If S is totally bounded and $f : S \to \mathbb{R}$ is uniformly continuous, then*

$$\inf f = \inf \{f(s) \mid s \in S\}$$

exists.

We want to include convexity in the list of Proposition 1.1. To this end, we introduce a suitable convexity condition for functions. Let S be a subset of $\mathbb{R}$. A function $f : S \to \mathbb{R}$ is *weakly convex* if for all $t \in S$ with $f(t) > 0$ there exists $\varepsilon > 0$ such that either

$$\forall s \in S \left(s \leq t \quad \Rightarrow \quad f(s) \geq \varepsilon\right)$$

or

$$\forall s \in S \left(t \leq s \quad \Rightarrow \quad f(s) \geq \varepsilon\right).$$

We want to relate this condition to the usual notions of convexity for functions. Recall that a function $f : [0,1] \to \mathbb{R}$ is *convex* if we have

$$f(\lambda s + (1-\lambda)t) \leq \lambda f(s) + (1-\lambda) f(t)$$

and *quasiconvex* if we have

$$f(\lambda s + (1-\lambda)t) \leq \max\left(f(s), f(t)\right)$$

for all $s, t \in [0,1]$ and all $\lambda \in [0,1]$. Note that convexity implies quasiconvexity.

LEMMA 3.3. *Fix a function $f : [0,1] \to \mathbb{R}$.*

(a) *If f is weakly convex, then the set $\{t \mid f(t) \leq 0\}$ is convex. With classical logic, the reverse implication holds as well, if f is continuous. This illustrates that weak convexity is indeed a convexity property.*

(b) *Monotone functions are weakly convex.*

Now assume that f is uniformly continuous.

(c) *If f is quasiconvex, then it is weakly convex.*

(d) *Let D be a dense subset of $[0,1]$. Then f is weakly convex if and only its restriction to D is weakly convex.*

PROOF. We only show (c). Fix $t \in [0,1]$ and suppose that $f(t) > 0$. By part (iii) of Lemma 3.2, the real numbers

$$\imath = \inf\{f(s) \mid s \in [0,t]\}$$

and

$$\eta = \inf\{f(s) \mid s \in [t,1]\}$$

exist. We either have $0 < \imath$ or $\imath < f(t)$. If $0 < \imath$, we are done. So assume that $\imath < f(t)$. We either have $0 < \eta$ or $\eta < f(t)$. Again, in the first case, we are done. The second case can be ruled out in view of $\imath < f(t)$ and the quasiconvexity of f. ⊣

Now we can state the main theorem.

THEOREM 3.4. *For every detachable subset B of $\{0,1\}^*$ which is closed under extension there exists a uniformly continuous function $f : [0,1] \to \mathbb{R}$ such that*

(a) *B is a bar $\Leftrightarrow$ f is positive-valued,*

(b) *B is a uniform bar $\Leftrightarrow$ $\inf f > 0$,*

(c) *B is co-convex $\Leftrightarrow$ f is weakly convex.*

Conversely, for every uniformly continuous function $f : [0,1] \to \mathbb{R}$ there exists a detachable subset B of $\{0,1\}^*$ which is closed under extension such that (a), (b), and (c) hold.

We split the proof of Theorem 3.4 into two parts.

PART I: CONSTRUCTION OF A FUNCTION f FOR GIVEN B.

Fix a detachable subset B of $\{0,1\}^*$ which is closed under extension. We can assume that $\varnothing \notin B$. (Otherwise, let f be the constant function $t \mapsto 1$.) First, we define a function $g : [0,1] \to \mathbb{R}$ which satisfies the properties (1) and (2) of Theorem 3.4. Then, we introduce a refined version f of g which satisfies all properties of Theorem 3.4. Define metrics

$$d_1(s,t) = |s-t|, \quad d_2((x_1,x_2),(y_1,y_2)) = |x_1 - y_1| + |x_2 - y_2|$$

on $\mathbb{R}$ and $\mathbb{R}^2$, respectively. The mapping

$$(\alpha, \beta) \mapsto \inf \left\{ 2^{-k} \mid \overline{\alpha}k = \overline{\beta}k \right\}$$

is a compact metric on $\{0,1\}^{\mathbb{N}}$. See [5, Section 1 of Chapter 5] for an introduction to basic properties of this metric space. Let $\kappa : \{0,1\}^{\mathbb{N}} \to [0,1]$ be the standard embedding of Cantor space into the reals as the Cantor set. Then

$$\kappa(\alpha) = 2 \cdot \sum_{k=0}^{\infty} \alpha_k \cdot 3^{-(k+1)},$$

so κ is uniformly continuous. The next lemma immediately follows from the definition of κ.

LEMMA 3.5. *For all α, β and n, we have*

- $\overline{\alpha}n = \overline{\beta}n \ \Rightarrow \ |\kappa(\alpha) - \kappa(\beta)| \leq 3^{-n}$
- $\overline{\alpha}n = \overline{\beta}n \wedge \alpha_n < \beta_n \ \Rightarrow \ \kappa(\alpha) + 3^{-(n+1)} \leq \kappa(\beta)$
- $\overline{\alpha}n \neq \overline{\beta}n \ \Rightarrow \ |\kappa(\alpha) - \kappa(\beta)| \geq 3^{-n}$
- $\overline{\alpha}n < \overline{\beta}n \ \Rightarrow \ \kappa(\alpha) < \kappa(\beta)$.

Now define

$$\eta_B : \{0,1\}^{\mathbb{N}} \to [0,1], \ \alpha \mapsto \inf \left\{ 3^{-k} \mid \overline{\alpha}k \notin B \right\}.$$

LEMMA 3.6. *The function η_B is well-defined—the infimum in the definition of η_B always exists—and uniformly continuous. If $\eta_B(\alpha) > 0$, there exists k such that*

(1) $\overline{\alpha}k \notin B$
(2) $\overline{\alpha}(k+1) \in B$
(3) $\eta_B(\alpha) = 3^{-k}$.

Moreover,

$$\overline{\alpha}n \in B \ \Leftrightarrow \ \eta_B(\alpha) \geq 3^{-n+1} \ \Leftrightarrow \ \eta_B(\alpha) > 3^{-n}$$

for all α and n.

We consider the following, more abstract version of Lemma 3.6.

LEMMA 3.7. *For every weakly increasing function $h : \mathbb{N} \to \{0,1\}$ with $h(0) = 0$ the set*

$$S = \left\{ 3^{-k} \mid h(k) = 0 \right\}$$

has an infimum. If $\inf S > 0$, *there exists k such that*

(1) $h(k) = 0$
(2) $h(k+1) = 1$
(3) $\inf S = 3^{-k}$.

Moreover,

$$h(n) = 1 \quad \Leftrightarrow \quad \inf S \geq 3^{-n+1} \quad \Leftrightarrow \quad \inf S > 3^{-n}$$

for all n.

PROOF. Note that $1 \in S$ and that 0 is a lower bound of S. Fix $p, q \in \mathbb{Q}$ with $p < q$. If $p \leq 0$, p is a lower bound of S. Now assume that $0 < p$. Then there exists k with $3^{-k} < p$. If $h(k) = 0$, there exist $s \in S$ (choose $s = 3^{-k}$) with $s < q$. If $h(k) = 1$, we can compute the minimum s_0 of S. If $p < s_0$, p is a lower bound of S; if $s_0 < q$, there exists $s \in S$ (choose $s = s_0$) with $s < q$.

If $\inf S > 0$, there exists l such that $3^{-l} < \inf S$. Therefore, $h(l) = 1$. Let k be the largest number such that $h(k) = 0$.

Assume that $h(n) = 1$. Let l be the largest natural number with $h(l) = 0$. Then $l \leq n - 1$ and thus $\inf S = 3^{-l} \geq 3^{-n+1}$.

Assume that $\inf S > 3^{-n}$. Then there exists k with (1), (2), and (3). We obtain $k < n$ and therefore $h(n) = 1$.

⊣

Set

$$C = \left\{\kappa(\alpha) \mid \alpha \in \{0,1\}^{\mathbb{N}}\right\}$$

and

$$K = \left\{(\kappa(\alpha), \eta_B(\alpha)) \mid \alpha \in \{0,1\}^{\mathbb{N}}\right\}.$$

LEMMA 3.8. *The sets C and K are compact.*

PROOF. Both sets are uniformly continuous images of the compact set $\{0,1\}^{\mathbb{N}}$ and therefore totally bounded. Suppose that $\kappa(\alpha^n)$ converges to t and $\eta_B(\alpha^n)$ converges to s. By Lemma 3.5, the sequence (α^n) is Cauchy, therefore it converges to a limit α. Then $\kappa(\alpha^n)$ converges to $\kappa(\alpha)$ and $\eta_B(\alpha^n)$ converges to $\eta_B(\alpha)$. Therefore $t = \kappa(\alpha)$ and $s = \eta_B(\alpha)$. Thus we have shown that both C and K are complete. ⊣

In the following, we will use Bishop's lemma, see [4, Chapter 4, Lemma 3.8].

LEMMA 3.9. *Let A be a compact subset of a metric space X, and x a point of X. Then there exists a point a in A such that $d(x,a) > 0$ entails $d(x,A) > 0$.*

Define

$$g : [0,1] \to [0,\infty[\,,\ t \mapsto d_2((t,0),K).$$

PROPOSITION 3.10. (1) *B is a bar $\Leftrightarrow$ g is positive-valued*
(2) *B is a uniform bar $\Leftrightarrow$ $\inf g > 0$.*

PROOF. Assume that B is a bar. Fix $t \in [0,1]$. In view of Bishop's lemma and the compactness of K, it is sufficient to show that

$$d_2((t,0),(\kappa(\alpha),\eta_B(\alpha))) > 0$$

for each α. This follows from $\eta_B(\alpha) > 0$.

Now assume that g is positive-valued. Fix α. Since

$$d_2((\kappa(\alpha),0),K) = g(\kappa(\alpha)) > 0,$$

we can conclude that

$$d_2((\kappa(\alpha),0),(\kappa(\alpha),\eta_B(\alpha))) > 0.$$

Thus $\eta_B(\alpha)$ is positive which implies that α hits B.

The second equivalence follows from Lemma 3.6 and the fact that $\inf g = \inf \eta_B$. ⊣

Set

$$-C = \{t \in [0,1] \mid d_1(t, C) > 0\}$$

and introduce a new function f by

$$f : [0,1] \to \mathbb{R},\ t \mapsto g(t) - d_1(t, C).$$

The next lemma lists up a few properties of f and g.

LEMMA 3.11. *For all α, n, and t we have*

- $g(\kappa(\alpha)) = f(\kappa(\alpha)) \leq \eta_B(\alpha)$
- $f(\kappa(\alpha)) > 3^{-n} \ \Rightarrow\ \overline{\alpha}n \in B$
- $\overline{\alpha}n \in B \ \Rightarrow\ f(\kappa(\alpha)) \geq 3^{-n}$
- $d_1(t, C) \leq g(t)$.

Next, we clarify how f behaves on $-C$.

LEMMA 3.12. *The set $-C$ is dense in* $[0,1]$. *For every $t \in -C$ there exist unique elements a, a' of C such that*

(a) $t \in \,]a, a'[\, \subseteq -C$.
(b) $d_1(t, C) = \min\left(d_1(t, a), d_1(t, a')\right)$.

Moreover, setting $\gamma = \kappa^{-1}(a)$ and $\gamma' = \kappa^{-1}(a')$, we obtain

(c) $\forall n \left(\overline{\gamma}n \in B \ \wedge\ \overline{\gamma'}n \in B \ \Rightarrow\ f(t) \geq 3^{-n}\right)$
(d) *if $d_1(t, a) < d_1(t, a')$, then*

$$\gamma \text{ hits } B \ \Leftrightarrow\ f(t) > 0 \ \Leftrightarrow\ \inf\{f(s) \mid a \leq s \leq t\} > 0$$

(e) *if $d_1(t, a') < d_1(t, a)$, then*

$$\gamma' \text{ hits } B \ \Leftrightarrow\ f(t) > 0 \ \Leftrightarrow\ \inf\{f(s) \mid t \leq s \leq a'\} > 0.$$

PROOF. Fix $t \in [0,1]$ and $\delta > 0$. If $d_1(t, C) > 0$, then $t \in -C$. Now assume that there exists α such that $d_1(t, \kappa(\alpha)) < \delta/2$. There exists u such that $d_1(\kappa(\alpha), t_u) < \delta/2$ where

$$t_u = \tfrac{1}{2} \cdot \kappa(u * 0 * 1 * 1 * 1 * \cdots) + \tfrac{1}{2} \cdot \kappa(u * 1 * 0 * 0 * 0 * \cdots).$$

Note that $t_u \in -C$ and that $d_1(t, t_u) < \delta$. So $-C$ is dense in $[0,1]$.

Fix $t \in -C$. Since for any α it is decidable whether $\kappa(\alpha) > t$ or $\kappa(\alpha) < t$, the sets $C_{<t} = \{s \in C \mid s < t\}$ and $C_{>t} = \{s \in C \mid s > t\}$ are compact. Let a be the maximum of $C_{<t}$ and let a' be the minimum of $C_{>t}$. Clearly, a and a' fulfil (a) and (b).

In order to show (c), assume that $\overline{\gamma}n \in B$ and $\overline{\gamma'}n \in B$. Fix α. We show that

$$d_2((t, 0), (\kappa(\alpha), \eta_B(\alpha))) - d_1(t, C) \geq 3^{-n}. \tag{3}$$

First, assume that $\kappa(\alpha) < t$. Then we have

$$d_2((t, 0), (\kappa(\alpha), \eta_B(\alpha))) - d_1(t, C) \geq \kappa(\gamma) - \kappa(\alpha) + \eta_B(\alpha).$$

If $\overline{\alpha}n = \overline{\gamma}n$, then $\overline{\alpha}n \in B$ and we can conclude that $\eta_B(\alpha) \geq 3^{-n+1}$, by Lemma 3.6. On the other hand, Lemma 3.5 implies that $\kappa(\gamma) - \kappa(\alpha) \leq 3^{-n}$. This proves (3). If $\overline{\alpha}n \neq \overline{\gamma}n$, then $\kappa(\gamma) - \kappa(\alpha) \geq 3^{-n}$, by Lemma 3.5. This also proves (3). The case $t < \kappa(\alpha)$ can be treated similarly.

In order to show (d), set $\iota = d_1(t, a') - d_1(t, a)$ and suppose that $\overline{\gamma}n \in B$. Set $\varepsilon = \min(\iota, 3^{-n})$. Fix s with $a \leq s \leq t$. We show that $f(s) \geq \varepsilon$. Note that $d_1(s, C) = s - a$. Fix α. We show that

$$d_2((s, 0), (\kappa(\alpha), \eta_B(\alpha))) - (s - a) \geq \varepsilon.$$

If $a' \leq \kappa(\alpha)$, we obtain

$$d_2((s, 0), (\kappa(\alpha), \eta_B(\alpha))) - (s - a) \geq$$
$$\kappa(\alpha) - s - (s - a) \geq \iota \geq \varepsilon.$$

If $\kappa(\alpha) \leq a$, we obtain

$$d_2((s, 0), (\kappa(\alpha), \eta_B(\alpha))) - (s - a) = s - \kappa(\alpha) + \eta_B(\alpha) - (s - a) =$$
$$\eta_B(\alpha) + a - \kappa(\alpha) \geq 3^{-n} \geq \varepsilon,$$

where $\eta_B(\alpha) + a - \kappa(\alpha) \geq 3^{-n}$ is derived by looking at the cases $\overline{\alpha}n = \overline{\gamma}n$ and $\overline{\alpha}n \neq \overline{\gamma}n$ separately.

Now assume that $f(t) > 0$. We show that γ hits B. If $f(t) > 0$, then $g(t) > t - a$. On the other hand, we have

$$g(t) \leq d_2((t, 0), (a, \eta_B(\gamma))) = t - a + \eta_B(\gamma),$$

so $\eta_B(\gamma) > 0$. By Lemma 3.6, this implies that γ hits B.

The statement (e) is proved analogously to (d). ⊣

The next lemma is very easy to prove, we just formulate it to be able to refer to it.

LEMMA 3.13. *For real numbers $x < y < z$ and $\delta > 0$ there exists a real number y' such that*

- $x < y' < z$
- $d_1(y, y') < \delta$
- $d_1(x, y') < d_1(y', z)$ *or* $d_1(x, y') > d_1(y', z)$.

For a function F defined on $\{0, 1\}^{\mathbb{N}}$, set

$$F(u) = F(u * 0 * 0 * 0 * \cdots). \tag{4}$$

Now we can show that f has all the desired properties.

PROPOSITION 3.14. (a) *B is a bar $\Leftrightarrow$ f is positive-valued*
(b) *B is a uniform bar* $\Leftrightarrow$ $\inf f > 0$
(c) *B is co-convex $\Leftrightarrow$ f is weakly convex.*

PROOF. (a) "⇒". Suppose that B is a bar and fix t. By Proposition 3.10, we obtain $g(t) > 0$. If $d_1(t, C) < g(t)$, then $f(t) > 0$, by the definition of f. If $0 < d_1(t, C)$, we can apply Lemma 3.12 to conclude that $f(t) > 0$.

(a) "⇐". If f is positive-valued, then g is positive-valued as well and Proposition 3.10 implies that B is a bar.

(b) "$\Rightarrow$". If B is a uniform bar, Proposition 3.10 yields

$$\varepsilon := \inf g > 0.$$

Moreover, there exists n such that $\{0,1\}^n \subseteq B$. Fix $\delta > 0$ such that

$$|s-t| < \delta \quad \Rightarrow \quad |f(s) - f(t)| < \varepsilon/2$$

for all s and t. Fix t. If $d_1(t, C) < \delta$, we can conclude that

$$f(t) \geq \varepsilon/2$$

by the choice of ε and δ. If $d_1(t, C) > 0$, Lemma 3.12 and $\{0,1\}^n \subseteq B$ imply that

$$f(t) \geq 3^{-n}.$$

So we have shown that $\inf f \geq \min\left(\varepsilon/2, 3^{-n}\right)$.

(b) "$\Leftarrow$". If $\inf f > 0$, then $\inf g > 0$, and Proposition 3.10 implies that B is a uniform bar.

(c) "$\Rightarrow$". By part (d) of Lemma 3.3 and Lemma 3.12, it is sufficient to show that the restriction of f to $-C$ is weakly convex. Fix $t \in -C$ and assume that $f(t) > 0$. Choose a, a', γ and γ' according to Lemma 3.12. In view of Lemma 3.13 and the uniform continuity of f, we may assume without loss of generality that either

$$d_1(a,t) < d_1(t,a') \quad \text{or} \quad d_1(a,t) > d_1(t,a').$$

Consider the first case. The second case can be treated analogously. By Lemma 3.12, we obtain

$$\imath = \inf\{f(s) \mid a \leq s \leq t\} > 0.$$

In particular, $f(\kappa(\gamma)) > 0$, so γ hits B. There exists n such that either

$$\{v \mid v \leq \overline{\gamma}n\} \subseteq B \tag{5}$$

or

$$\{v \mid \overline{\gamma}n \leq v\} \subseteq B. \tag{6}$$

Set $\varepsilon = \min\left(\imath, 3^{-n}\right)$. In case (5), we show that

$$\forall s \in -C\,\left(s \leq t \quad \Rightarrow \quad f(s) \geq \varepsilon\right),$$

as follows. Assume that there exists $s \in -C$ with $s \leq t$ such that $f(s) < \varepsilon$. Then, by the definition of $\imath$, we obtain that $s < a$. Applying Lemma 3.12 again, we can choose α and α' such that

$$s \in \left]\kappa(\alpha), \kappa(\alpha')\right[\subseteq -C.$$

Then $\overline{\alpha}n \leq \overline{\alpha'}n \leq \overline{\gamma}n$. Thus both $\overline{\alpha}n$ and $\overline{\alpha'}n$ are in B. This implies $f(s) \geq 3^{-n}$, which is a contradiction. In case (6), a similar argument yields

$$\forall s \in -C\,\left(t \leq s \quad \Rightarrow \quad f(s) \geq \varepsilon\right).$$

(c) "$\Leftarrow$". Assume that f is weakly convex. Fix α and suppose that α hits B. Then Lemma 3.11 implies that $f(\kappa(\alpha)) > 0$. By the weak convexity of f, there exists $\imath > 0$ such that either

$$\forall s\,\left(s \leq \kappa(\alpha) \quad \Rightarrow \quad f(s) \geq \imath\right) \tag{7}$$

or else

$$\forall s\, \big(\kappa(\alpha) \leq s \;\;\Rightarrow\;\; f(s) \geq \iota\big)\,. \tag{8}$$

Fix n large enough such that $\overline{\alpha}n \in B$ and $3^{-n} < \iota$. Assume that (7) holds. Fix v with $v \leq \overline{\alpha}n$. Then $\kappa(v) \leq \kappa(\alpha)$. If $v \notin B$, then, by Lemmas 3.6 and 3.11,

$$f(\kappa(v)) = g(\kappa(v)) \leq \eta_B(v) \leq 3^{-n}.$$

This contradiction shows that

$$\{v \mid v \leq \overline{\alpha}n\} \subseteq B.$$

Now, consider the case (8). Fix v with $\overline{\alpha}n < v$. Then $\kappa(\alpha) \leq \kappa(v)$. If $v \notin B$, then $f(\kappa(v)) \leq 3^{-n}$. This contradiction shows that

$$\{v \mid \overline{\alpha}n \leq v\} \subseteq B. \qquad \dashv$$

PART II: CONSTRUCTION OF A SET B FOR GIVEN f.
Set

$$\kappa' : \{0,1\}^{\mathbb{N}} \to [0,1]\,,\; \alpha \mapsto \sum_{k=0}^{\infty} \alpha_k \cdot 2^{-(k+1)}.$$

One cannot prove that κ' is surjective, since this would imply LLPO. Note, however, that every rational $q \in [0,1]$ is in the range of κ'. Moreover, we make use of the following lemma, see [1, Lemma 1].

LEMMA 3.15. *Let S be a subset of* $[0,1]$ *such that*

$$\forall \alpha\, \exists \varepsilon > 0\, \forall t \in [0,1]\, \big(|t - \kappa'(\alpha)| < \varepsilon \Rightarrow t \in S\big)\,.$$

Then $S = [0,1]$.

The next lemma is a typical application of Lemma 3.15.

LEMMA 3.16. *Fix a uniformly continuous function* $f : [0,1] \to \mathbb{R}$ *and define*

$$F : \{0,1\}^{\mathbb{N}} \to \mathbb{R},\; \alpha \mapsto f(\kappa'(\alpha)).$$

Then

(1) f *is positive-valued* $\Leftrightarrow$ F *is positive-valued,*
(2) $\inf f > 0 \;\Leftrightarrow\; \inf F > 0$.

PROOF. In (1), the direction "$\Rightarrow$" is clear. For "$\Leftarrow$", apply Lemma 3.15 to the set

$$S = \{t \in [0,1] \mid f(t) > 0\}\,.$$

The equivalence (2) follows from the density of the image of κ' in $[0,1]$ and the uniform continuity of f. $\dashv$

In the following proposition, we use a similar construction as in [2].

PROPOSITION 3.17. *For every uniformly continuous function*

$$f : [0,1] \to \mathbb{R}$$

there exists a detachable subset B of $\{0,1\}^*$ *which is closed under extension such that*

(a) *B is a bar* $\Leftrightarrow$ *f is positive-valued,*
(b) *B is a uniform bar* $\Leftrightarrow$ $\inf f > 0$,
(c) *B is co-convex* $\Leftrightarrow$ *f is weakly convex.*

PROOF. Since the function

$$F : \{0,1\}^{\mathbb{N}} \to \mathbb{R},\ \alpha \mapsto f(\kappa'(\alpha))$$

is uniformly continuous, there exists a strictly increasing function $M : \mathbb{N} \to \mathbb{N}$ such that

$$|F(\alpha) - F(\overline{\alpha}(M(n)))| < 2^{-n}$$

for all α and n, recalling the convention given in (4). Since M is strictly increasing, for every k the statement

$$\exists n\,(k = M(n))$$

is decidable. Therefore, for every u we can choose $\lambda_u \in \{0,1\}$ such that

$$\lambda_u = 0 \quad \Rightarrow \quad \forall n\,(|u| \neq M(n)) \ \vee\ \exists n\,\left(|u| = M(n) \ \wedge\ F(u) < 2^{-n+2}\right),$$
$$\lambda_u = 1 \quad \Rightarrow \quad \exists n\,\left(|u| = M(n) \ \wedge\ F(u) > 2^{-n+1}\right).$$

The set

$$B = \{u \in \{0,1\}^* \mid \exists l \leq |u|\,(\lambda_{\overline{u}l} = 1)\}$$

is detachable and closed under extension. Note that

$$F(\alpha) \geq 2^{-n+3} \ \Rightarrow\ \overline{\alpha}(M(n)) \in B \tag{9}$$

and

$$\overline{\alpha}(M(n)) \in B \ \Rightarrow\ F(\alpha) \geq 2^{-n} \tag{10}$$

for all α and n. In view of Lemma 3.16, (9) and (10) yield (a) and (b).

In order to show (c), assume that B be co-convex. Moreover, fix $t \in [0,1]$ and assume that $f(t) > 0$. By part (d) of Lemma 3.3, we may assume that t is a rational number, which implies that there exists α such that $\kappa'(\alpha) = t$. Now $F(\alpha) > 0$ implies that α hits B. Therefore, there exists n such that either

$$\{v \mid v \leq \overline{\alpha}n\} \subseteq B$$

or

$$\{v \mid \overline{\alpha}n \leq v\} \subseteq B.$$

In the first case, we show that

$$\inf\{f(s) \mid s \in [0,t]\} \geq \min\left(2^{-n}, F(\alpha)\right). \tag{11}$$

Assume that there exists $s \leq t$ such that $f(s) < 2^{-n}$ and $f(s) < F(\alpha)$. The latter implies that $s < t$. Choose a β with the property that $\kappa'(\beta)$ is close enough to s such that

$$\kappa'(\beta) < \kappa'(\alpha) \tag{12}$$

and

$$F(\beta) = f(\kappa'(\beta)) < 2^{-n}. \tag{13}$$

Now (10) and (13) imply that $\overline{\beta}n \notin B$. On the other hand, (12) implies that $\overline{\beta}n \leq \overline{\alpha}n$ and therefore $\overline{\beta}n \in B$. This is a contradiction, so we have shown (11).

In the case

$$\{v \mid \overline{\alpha}n \leq v\} \subseteq B$$

we can similarly show that

$$\inf\{f(s) \mid s \in [t,1]\} \geq \min\left(2^{-n}, F(\alpha)\right).$$

Now assume that f is weakly convex. Fix an α which hits B. Then there exists n with $\overline{\alpha}(M(n)) \in B$ and (10) implies that $f(\kappa'(\alpha)) > 0$. We choose n large enough such that either

$$\inf\left\{f(t) \mid t \in \left[0, \kappa'(\alpha)\right]\right\} \geq 2^{-n+3}$$

or

$$\inf\left\{f(t) \mid t \in \left[\kappa'(\alpha), 1\right]\right\} \geq 2^{-n+3}.$$

By (9), we obtain

$$\{v \mid v \leq \overline{\alpha}(M(n))\} \subseteq B$$

in the first case and

$$\{v \mid \overline{\alpha}(M(n)) \leq v\} \subseteq B.$$

in the second. Therefore, B is co-convex. ⊣

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