Conformal Properties of All-Plus Scattering Amplitudes

Konforme Eigenschaften von Streuamplituden

Submitted by

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Abstract

Symmetry properties of scattering amplitudes often lead to simplifications in their computation and in their final expression. Gluonic scattering amplitudes in pure Yang-Mills theory in the all-plus helicity configuration exhibit remarkable signs of conformal symmetry. At one-loop order they are found to be conformally invariant, but this observation still lacks an explanation. In the present thesis we analyze the conformal properties of all-plus gluon scattering amplitudes at one-loop order and present an attempt to rewrite them in terms of manifestly conformally invariant objects. This might lead to a better understanding of the origin of this symmetry for these amplitudes.

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Chapter 1

Introduction

The Large Hadron Collider is a particle accelerator built by the European Organization for Nuclear Research (CERN) near Geneva, at the French-Swiss border. There, particles are collided at the energy scale of teraelectronvolt, thus producing the highest-energy and most precise measurements of particle collisions in the world. As the LHC prepares to resume its collisions in 2021, in order to interpret the abundant amount of data and compare it to the underlying fundamental theories, it is necessary to have equally precise theoretical predictions of such collisions. In fact, although the Standard Model of particle physics has existed for over 40 years as an established theory for the description of interactions between elementary particles, computing its predictions at high accuracy remains a major challenge. This task is of fundamental importance not only to test its validity, but also to look for possible hints of further extensions of the theory, notably the unification of the three currently described fundamental interactions (electromagnetic, weak and strong interactions) with gravity.

The cornerstone of the Standard Model was put with the development of Quantum Electrodynamics by Feynman, Schwinger, Tomonaga and Dyson in the 40's, which served as a model for the succeeding quantum field theories. At this time, the famous Feynman diagrams were introduced, an intuitive and practical way of computing scattering amplitudes perturbatively. In the 50's, Yang and Mills generalized the notion of gauge invariance present in QED and provided an explicit construction of such a theory for the non-Abelian case. Their aim was to provide a theoretical description of the strong interaction, which was later accomplished by Quantum Chromodynamics. During the 60's the unification of the electromagnetic and the weak interactions, and the later incorporation of the Higgs mechanism, lead to the formulation of the electroweak theory. Together with QCD it forms the Standard Model.

Although Feynman diagrams are a simple and intuitive tool for the computation of scattering amplitudes, their number and complexity grow drastically as the number of external particles and/or the loop order increase. However, the resulting amplitudes often have much simpler final expressions than the intermediate results, suggesting that more efficient methods could simplify the calculation. Significant advances have been made in this sense by Bern, Dixon and Kosower [1], who introduced the unitarity method, and by Britto, Cachazo, Feng and Witten [2], who, using factorization properties, were able to derive a recursion relation for gluon amplitudes at tree-level.

The final amplitude is often simpler than the intermediate results because it is more constrained by symmetries, for instance gauge or Poincaré symmetry. Therefore, one might expect that finding additional symmetries will lead to even simpler expressions. This is the case of the all-plus amplitudes in pure Yang-Mills theory. The known amplitudes for this helicity configuration have very simple expressions. At one-loop order they are observed to be conformally invariant, but it is not understood why; at two-loop we also see unexplained signs of conformal symmetry [3].

In this thesis we study these amplitudes, in particular their behavior under conformal transformations. Imposing conformal symmetry on a theory represents a strong restriction to it, and this leads to a simplification of the possible features such a theory can exhibit. For example, the conformal symmetry of an amplitude reduces dramatically the space of possible terms it may contain, thus simplifying the search for its most general form.

Chapter 2

Tools

Before we proceed to analyze gluon scattering amplitudes, it is convenient to introduce certain useful tools. We begin by introducing Yang-Mills theories, which are the basis for the description of gluon amplitudes. Then we present the spinor helicity formalism, in which gluon amplitudes can be written in a convenient and compact form. Finally, we describe the method of color decomposition, which separates the color degrees of freedom from the kinematic properties of a scattering process.

2.1 Yang-Mills Theory

Yang-Mills theories provide the theoretical foundation of the Standard Model and can be seen as a generalization of QED to a non-Abelian gauge invariant theory [4]. In this procedure we start from a theory of fermions containing a *global* symmetry, in which the symmetry transformation is the same for all space-time points, and upgrade the symmetry to a *local* one, i.e. dependent on the space-time point. Through this construction we end up with a theory of interacting fermions and bosons.

In QED we start from a Lagrangian in terms of the Dirac field $\psi(x)$, $\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$, which contains a *global* $U(1)$ symmetry

$$\psi(x) \to U\psi(x), \tag{2.1}$$

where U is a $U(1)$ transformation, i.e. a phase shift. We can then promote it to a *local* symmetry $U(x)$, by substituting the derivative operator ∂_μ with the covariant derivative $D_\mu = \partial_\mu - iqA_\mu$, where $A_\mu(x)$ is the gauge field introduced to ensure local symmetry and q is the coupling constant of the field. The covariant derivative is defined so that it transforms as

$$D_\mu \to U(x) D_\mu U^\dagger(x), \tag{2.2}$$

which leaves the Lagrangian invariant and implies the transformation rule for the gauge field

$$A_\mu(x) \to U(x) A_\mu(x) U^\dagger(x) + \frac{i}{q} U(x) \partial_\mu(x) U^\dagger(x). \tag{2.3}$$

We can now generalize this procedure to a $SU(N)$ transformation. We begin with the global transformation

$$\psi_i(x) \to U_{ij} \psi_j(x), \tag{2.4}$$

and generalize it to a local one. Any infinitesimal $SU(N)$ transformation can be written in terms of the *generator matrices* T^a:

$$U_{jk}(x) = \delta_{jk} - iq\theta^a(x)(T^a)_{jk}, \tag{2.5}$$

where q is the coupling constant and θ^a are transformation parameters.

The generator matrices are traceless and hermitian, and satisfy the commutator relation

$$[T^a, T^b] = i\sqrt{2} f^{abc} T^c, \tag{2.6}$$

where f^{abc} are the structure constants of the Lie Algebra of the Lie group SU(N). When the structure constants are non-zero, the group is non-Abelian. Being $N \times N$ hermitian traceless matrices, we can find a basis of $N^2 - 1$ such matrices. We can specify it to a diagonal basis such that

$$\mathrm{Tr}(T^a T^b) = \delta^{ab}. \tag{2.7}$$

The structure constants can therefore be rewritten in terms of traces of generators as

$$f^{abc} = -\frac{i}{\sqrt{2}} \mathrm{Tr}(T^a [T^b, T^c]). \tag{2.8}$$

We now introduce the gauge field $A_\mu(x)$ as a traceless hermitian $N \times N$ matrix with an analogous transformation as in 2.3:

$$A_\mu(x) \to U(x) A_\mu(x) U^\dagger(x) + \frac{i}{q} U(x) \partial_\mu U^\dagger(x), \tag{2.9}$$

and define the covariant derivative

$$D_\mu = \partial_\mu - iqA_\mu(x), \tag{2.10}$$

where the $U(x)$ and the $A_\mu(x)$ are no longer understood as scalar objects, but rather as matrices, and the partial derivative ∂_μ is implicitly multiplied by the identity matrix.

Similarly to the Abelian case we define the field strength tensor

$$F_{\mu\nu} = \frac{i}{q}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]. \tag{2.11}$$

We note that the field strength is no longer gauge invariant, so, to ensure gauge invariance in the kinetic term, we write it as

$$\mathcal{L}_{\text{kin}} = -\frac{1}{4}\text{Tr}(F^{\mu\nu}F_{\mu\nu}). \tag{2.12}$$

We also note that the non-vanishing commutator in $F_{\mu\nu}$ implies a self-interacting term for the gauge field in the Lagrangian.

As A_μ was defined as hermitian and traceless, we can write it in terms of the generator matrices

$$A_\mu(x) = A^a_\mu(x)T^a. \tag{2.13}$$

The same is true for the field strength, whose components are then

$$F^c_{\mu\nu} = \partial_\mu A^c_\nu - \partial_\nu A^c_\mu + \sqrt{2}gf^{abc}A^a_\mu A^b_\nu. \tag{2.14}$$

Then we can write

$$\mathcal{L}_{\text{kin}} = -\frac{1}{4}F^c_{\mu\nu}F^{c\mu\nu}, \tag{2.15}$$

and obtain the full Yang-Mills Lagrangian

$$\mathcal{L}_{\text{YM}} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F^c_{\mu\nu}F^{c\mu\nu}. \tag{2.16}$$

Such a theory, containing nonzero structure constants, is called a Yang-Mills theory. One often refers to the *pure* Yang-Mills case, which is the purely gluonic part of a Yang-Mills theory (2.15), excluding all fermions.

An important example of a Yang-Mills theory is QCD, which has the gauge group $SU(3)$ and whose Lagrangian is

$$\mathcal{L}_{QCD} = \bar{\psi}_i(i(\gamma^\mu D_\mu)_{ij} - m\delta_{ij})\psi_j - \frac{1}{4}F^c_{\mu\nu}F^{c\mu\nu}, \tag{2.17}$$

with the gluon field $A^a_\mu(x)$ and the quark field ψ_i.

2.2 Spinor Helicity Formalism

When working with massless particles, it is often useful to think of their helicity properties. Helicity is the projection of the spin of a particle onto its momentum:

$$h := \frac{\mathbf{p} \cdot \mathbf{S}}{|\mathbf{p}|}. \tag{2.18}$$

This quantity is Lorentz-invariant for massless particles, as they move at the speed of light and therefore one cannot find a Lorentz boost which inverts the direction of the momentum.

We introduce the spinor helicity variables, which have proven to be very convenient for expressing scattering amplitudes involving massless particles. We start by writing the four-momentum p^μ as a matrix:

$$p^\mu \to p^{\dot{\alpha}\alpha} = \bar{\sigma}_\mu^{\dot{\alpha}\alpha} p^\mu = \begin{pmatrix} p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & p^0 - p^3 \end{pmatrix}, \tag{2.19}$$

where $(\bar{\sigma}_\mu)^{\dot{\alpha}\alpha} = (\mathbb{1}, \sigma_i)$ and σ_i are the Pauli-matrices (see Appendix A). We then have

$$p_i^{\dot{\alpha}\alpha} p_j^{\dot{\beta}\beta} \varepsilon_{\alpha\beta} \varepsilon_{\dot{\alpha}\dot{\beta}} = p_i^\mu p_j^\nu \underbrace{\bar{\sigma}_\mu^{\dot{\alpha}\alpha} \bar{\sigma}_\nu^{\dot{\beta}\beta} \varepsilon_{\alpha\beta} \varepsilon_{\dot{\alpha}\dot{\beta}}}_{=2\eta_{\mu\nu}} = 2 p_i \cdot p_j, \tag{2.20}$$

with the Levi-Civita symbol $\varepsilon_{\alpha\beta}$. Therefore we can express the mass-shell condition as:

$$p_i^2 = m^2 \Leftrightarrow p_i \cdot p_i = \frac{1}{2} p_i^{\dot{\alpha}\alpha} p_i^{\dot{\beta}\beta} \varepsilon_{\alpha\beta} \varepsilon_{\dot{\alpha}\dot{\beta}} = \det(p_i^{\dot{\alpha}\alpha}) = m^2. \tag{2.21}$$

Being a 2×2 matrix, $p^{\dot{\alpha}\alpha}$ has at most rank 2 and can be written as $p^{\dot{\alpha}\alpha} = \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}} + \mu^\alpha \tilde{\mu}^{\dot{\alpha}}$. However, $p^2 = 0$ implies $\det(p^{\alpha\dot{\alpha}}) = 0$, so the matrix has rank 1 and can therefore be written as

$$p_i^{\alpha\dot{\alpha}} = \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}}, \tag{2.22}$$

where $\lambda_i^\alpha, \tilde{\lambda}_i^{\dot{\alpha}}$ are the so called helicity spinors. Raising and lowering of the indices are achieved with the Levi-Civita symbol:

$$\lambda_\alpha := \varepsilon_{\alpha\beta} \lambda^\beta, \quad \tilde{\lambda}_{\dot{\alpha}} := \varepsilon_{\dot{\alpha}\dot{\beta}} \tilde{\lambda}^{\dot{\beta}}. \tag{2.23}$$

In this formalism the helicity operator can be written as

$$h = \frac{1}{2} \left(-\lambda^\alpha \frac{\partial}{\partial \lambda^\alpha} + \tilde{\lambda}^{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}^{\dot{\alpha}}} \right), \tag{2.24}$$

and so we see that the helicity spinors are eigenvectors of the helicity operator

$$h\lambda^{\alpha} = -\frac{1}{2}\lambda^{\alpha}, \quad h\tilde{\lambda}^{\dot{\alpha}} = \frac{1}{2}\tilde{\lambda}^{\dot{\alpha}}. \tag{2.25}$$

Thus, helicity spinor variables describe both momentum and helicity of particles.

Since scattering amplitudes are Lorentz-invariant, they can only be constructed in terms of Lorentz-invariant variables. In the present formalism we can define the following Lorentz-invariant objects:

$$\begin{aligned} \langle \lambda_i \lambda_j \rangle &:= \lambda_i^{\alpha} \lambda_{j\alpha} = \varepsilon_{\alpha\beta} \lambda_i^{\alpha} \lambda_j^{\beta} = -\langle \lambda_j \lambda_i \rangle =: \langle ij \rangle, \\ [\tilde{\lambda}_i \tilde{\lambda}_j] &:= \tilde{\lambda}_{i\dot{\alpha}} \tilde{\lambda}_j^{\dot{\alpha}} = -\varepsilon_{\dot{\alpha}\dot{\beta}} \tilde{\lambda}_i^{\dot{\alpha}} \tilde{\lambda}_j^{\dot{\beta}} = -[\lambda_j \lambda_i] =: [ij]. \end{aligned} \tag{2.26}$$

We may write the Mandelstam variables in term of these brackets as:

$$s_{ij} = (p_i + p_j)^2 = p_i^{\alpha\dot{\alpha}} p_{j\alpha\dot{\alpha}} = \langle ij \rangle [ji]. \tag{2.27}$$

Important relations of these brackets are $\langle ii \rangle = [ii] = 0$, which follows from the antisymmetry of the brackets, the Schouten identity:

$$\langle \lambda_i \lambda_j \rangle \lambda_k^{\alpha} + \langle \lambda_j \lambda_k \rangle \lambda_i^{\alpha} + \langle \lambda_k \lambda_i \rangle \lambda_j^{\alpha} = 0, \tag{2.28}$$

and momentum conservation:

$$\sum_{i=1}^{n} p_i^{\mu} = 0 \Leftrightarrow \sum_{i=1}^{n} \langle ai \rangle [ib] = 0, \tag{2.29}$$

for any $a, b = 1, \ldots, n$.

In this formalism, we can find representations of the polarization vectors, which must obey

$$p \cdot \epsilon_{\pm}(p) = 0, \tag{2.30}$$
$$\epsilon_{+}(p) \cdot \epsilon_{-}(p) = -1, \tag{2.31}$$
$$\epsilon_{+}(p) \cdot \epsilon_{+}(p) = \epsilon_{-}(p) \cdot \epsilon_{-}(p) = 0, \tag{2.32}$$
$$(\epsilon_{\pm}^{\mu})^{*} = \epsilon_{\mp}^{\mu}. \tag{2.33}$$

These relations are fulfilled if we write the polarization vectors as

$$\epsilon_{+,i}^{\alpha\dot{\alpha}} = -\sqrt{2}\frac{\tilde{\lambda}_i^{\dot{\alpha}} \mu_i^{\alpha}}{\langle \lambda_i \mu_i \rangle}, \quad \epsilon_{-,i}^{\alpha\dot{\alpha}} = \sqrt{2}\frac{\lambda_i^{\alpha} \tilde{\mu}_i^{\dot{\alpha}}}{[\lambda_i \mu_i]}, \tag{2.34}$$

where μ_i and $\tilde{\mu}_i$ are reference spinors that can be chosen arbitrarily. Indeed one has

$$h\epsilon^{\alpha\dot{\alpha}}_{\pm,i} = (\pm 1)\epsilon^{\alpha\dot{\alpha}}_{\pm,i} \tag{2.35}$$

$$(\epsilon_+)^* = \epsilon_-, \tag{2.36}$$

$$p \cdot \epsilon_\pm = \frac{1}{2}\lambda_\alpha\tilde{\lambda}_{\dot{\alpha}}\epsilon^{\alpha\dot{\alpha}}_{\pm,i} \sim ([\lambda\lambda] \text{ or } \langle\lambda\lambda\rangle) = 0, \tag{2.37}$$

$$\epsilon_+ \cdot \epsilon_- = -\frac{\tilde{\lambda}^{\dot{\alpha}}\mu^\alpha\lambda_\alpha\tilde{\mu}_{\dot{\alpha}}}{\langle\lambda\mu\rangle[\lambda\mu]} = -1, \tag{2.38}$$

$$\epsilon_+ \cdot \epsilon_+ = -\frac{\tilde{\lambda}^{\dot{\alpha}}\mu^\alpha\tilde{\lambda}_{\dot{\alpha}}\mu_\alpha}{\langle\lambda\mu\rangle[\lambda\mu]} = 0. \tag{2.39}$$

2.3 Color Decomposition

When studying scattering amplitudes in QCD it is useful to separate the color degrees of freedom from the kinematic terms of the amplitude. For the case of pure Yang-Mills theory, the color factor of a generic diagram is given by a chain of structure constants f^{abc}, which follows directly from the Feynman rules. We can now write the structure constants in terms of the generator matrices using relation 2.8:

$$f^{abc} = -\frac{i}{\sqrt{2}}\text{Tr}(T^a[T^b, T^c]). \tag{2.40}$$

So the color degrees of freedom are all expressed by a product of traces of generator matrices. Employing the $SU(N)$ identity

$$(T^a)_{i_1}^{j_1}(T^a)_{i_2}^{j_2} = \delta_{i_1}^{j_2}\delta_{i_2}^{j_1} - \frac{1}{N}\delta_{i_1}^{j_1}\delta_{i_2}^{j_2}, \tag{2.41}$$

we can further simplify the color dependence at tree-level so that only single traces of the generator matrices T^a are left, and write the amplitude of a scattering process of n gluons of colors a_i, momenta p_i and helicities h_i in the *color decomposed* form

$$\mathscr{A}_n^{\text{tree}}(\{a_i, h_i, p_i\}) = g^{n-2} \sum_{\sigma \in S_n/Z_n} \text{Tr}(T^{a_{\sigma_1}} \ldots T^{a_{\sigma_n}}) A_n^{\text{tree}}(p_{\sigma_1}, h_{\sigma_1}; \ldots ; p_{\sigma_n}, h_{\sigma_n}), \tag{2.42}$$

where the A_n^{tree} are the *partial* or *color-ordered* amplitudes, containing all the kinematic information of the amplitude. They still have to be computed, but they are simpler than the full amplitudes, as they only receive contributions from one specific cyclic ordering of the external gluons each. Therefore, the poles of the partial amplitudes can only occur in channels of adjacent momenta. In 2.42, S_n is the group of all permutations of n objects, and Z_n is the group of all *cyclic* permutations, so S_n/Z_n are all permutations *modulo* cyclic permutations, in order to eliminate cyclic terms, which would leave the trace invariant. This is done to avoid the repetition of equal traces, which would lead to a redundant representation. As a result, all traces are independent, and each amplitude has to be gauge invariant by itself.

At one-loop, we get a color decomposition containing single and double trace structures (we introduce the short-hand notation $\{p_i, h_i\} := i^{h_i}$):

$$A_n^{\text{1-loop}}(\{a_i, h_i, p_i\}) = g^n \Bigg[N \sum_{\sigma \in S_n/Z_n} \text{Tr}(T^{a_{\sigma_1}} \dots T^{a_{\sigma_n}}) A_{n;1}(\sigma_1^{h_{\sigma_1}}, \dots, \sigma_n^{h_{\sigma_n}}) + \sum_{c=2}^{\lfloor n/2 \rfloor + 1} \sum_{\sigma \in S_n/S_{n;c}} \text{Tr}(T^{a_{\sigma_1}} \dots T^{a_{\sigma_{c-1}}}) \text{Tr}(T^{a_{\sigma_c}} \dots T^{a_{\sigma_n}}) A_{n;c}(\sigma_1^{h_{\sigma_1}}, \dots, \sigma_n^{h_{\sigma_n}}) \Bigg], \tag{2.43}$$

where $S_{n;c}$ is the subset of S_n which leaves the corresponding double trace invariant, and $\lfloor n/2 \rfloor$ is the largest integer smaller or equal to $n/2$. The amplitude $A_{n;1}$ is called the *leading-color* amplitude, and the other amplitudes, $A_{n;r}$ for $r \geq 2$ are called *subleading-color* amplitudes.

It is convenient to extend the $SU(N)$ group to $U(N) = SU(N) \times U(1)$ by adding to the $N^2 - 1$ traceless generators the $U(1)$ generator $(T^0)_i^{\,j} = \frac{1}{\sqrt{N}} \delta_i^j$, which is proportional to the identity matrix, since $U(1)$ is Abelian. The N^2 matrices fulfill the completeness relation

$$(T^a)_{i_1}^{\,j_1} (T^a)_{i_2}^{\,j_2} = \delta_{i_1}^{j_2} \delta_{i_2}^{j_1}. \tag{2.44}$$

Since the $U(1)$ generator commutes with all generators, any structure constant containing a $U(1)$ index vanishes. This means that the corresponding coupling term in the field strength (2.14) vanishes, and thus that the $U(1)$ gauge field, often called photon field, is colorless and does not couple to the gluon fields. Therefore, amplitudes in the $U(N)$ Yang-Mills theory containing a $U(1)$ photon must vanish. This implies relations between the different color components. If, for instance, we assume particle 1 to be a photon, we have that $T^{a_1} \sim \mathbb{1}$, and so

$$\mathrm{Tr}(T^{a_1} T^{a_2} \dots T^{a_n}) = \mathrm{Tr}(T^{a_2} T^{a_1} \dots T^{a_n}) = \dots = \mathrm{Tr}(T^{a_2} T^{a_3} \dots T^{a_n}). \quad (2.45)$$

Inserting this into 2.42 then yields

$$0 \stackrel{!}{=} \mathrm{Tr}(T^{a_2} T^{a_3} \dots T^{a_n})(A_n^{\mathrm{tree}}(1,2,\dots,n) + A_n^{\mathrm{tree}}(2,1,\dots,n) + \dots + A_n^{\mathrm{tree}}(2,3,\dots,1,n)) + \mathrm{Tr}\dots \quad (2.46)$$

However, since all traces are independent, each sum of partial amplitudes must vanish. This implies

$$A_n^{\mathrm{tree}}(1,2,\dots,n) + A_n^{\mathrm{tree}}(2,1,\dots,n) + \dots + A_n^{\mathrm{tree}}(2,3,\dots,1,n) = 0. \quad (2.47)$$

This property is often called “photon decoupling”

At higher order, other relations appear. For instance, at one-loop we can write all the subleading-color amplitudes as sums of permutations of the leading-color amplitude [1].

With these tools we can now tackle some scattering processes.

Chapter 3

Tree-Level Methods

The Feynman rules provide a simple and straightforward way of computing scattering amplitudes. However, the increasing complexity and number of Feynman diagrams for increasing number of particles and of loops makes the process very intensive from the computational point of view. The simplicity of the results however often contrasts with the complexity of the intermediate expressions. This justifies the search for more efficient methods. One very powerful tool for computing scattering amplitudes at tree level even for large multiplicity n is the Britto-Cachazo-Feng-Witten (BCFW) recursion. In the following section we will introduce this technique and in section 3.2 we will use it to prove a compact form of the so-called *maximally helicity violating* (MHV) amplitudes inductively. Finally we will present conformal symmetry, an extension of the Poincaré group that can be very powerful in constraining the form of a scattering amplitude, and show the conformal invariance of the MHV amplitudes at tree-level.

3.1 The BCFW Recursion

We restrict ourselves in the following to the pure Yang-Mills case for simplicity. An extension to include massive particles can be found in [5].

The idea underlying the BCFW recursion is to perform a shift of the amplitude in the complex plane and then express the original amplitude in terms of the residues of the shifted one [2]. We begin with an amplitude

$A_n(p_1, h_1; \ldots; p_n, h_n)$ and perform a complex shift of two helicity spinors:

$$\begin{aligned} \lambda_1 \to \hat{\lambda}_1 &= \lambda_1 - z\lambda_n, \\ \tilde{\lambda}_n \to \hat{\tilde{\lambda}}_n &= \tilde{\lambda}_n + z\tilde{\lambda}_1, \end{aligned} \tag{3.1}$$

which implies a deformation of the momenta

$$\hat{p}_1(z) = \lambda_1(\tilde{\lambda_1} - z\tilde{\lambda}_n), \quad \hat{p}_n(z) = (\lambda_n + z\lambda_1)\tilde{\lambda}_n. \tag{3.2}$$

We note that this transformation maintains the on-shell condition for all momenta and momentum conservation, since

$$\hat{p}_1(z) + \hat{p}_n(z) = p_1 + p_n, \tag{3.3}$$

is independent of z. In Yang-Mills theory, the momentum of a propagator in color-ordered amplitudes is always the sum of adjacent momenta, as was discussed in 2.3, and has the form

$$P_{ij} = p_i + \ldots + p_j, \tag{3.4}$$

where the momenta of the particles between i and j in the color-ordering are summed. Therefore, the shifted amplitude $A_n(z)$ only contains poles in z coming from the propagators, which take the form

$$\frac{1}{\hat{P}_i^2(z)} = \frac{1}{(\hat{p}_1(z) + p_2 \ldots + p_{i-1})^2} = \frac{1}{(p_i + \ldots + p_{n-1}\hat{p}_n(z))^2} = \frac{1}{P_i^2 - z\langle n|P_i|1]}, \tag{3.5}$$

where we define $P_i := P_{1,i-1}$, $\langle n|P_i|1] := \lambda_n^{\alpha} P_{i\alpha\dot{\alpha}} \tilde{\lambda}_1^{\dot{\alpha}}$ and use $\hat{P}_i(z) = P_i + z\lambda_n\tilde{\lambda}_1$. Whenever $\hat{p}_1$ and $\hat{p}_n$ are contained in the same region, the propagator is independent of z, and therefore does not contribute with a pole. Hence, we see that the amplitude $A_n(z)$ has simple poles at

$$z_{P_i} = \frac{P_i^2}{\langle n|P_i|1]}. \tag{3.6}$$

According to Cauchy's residue theorem, the contour integral of a function is equal to the sum of all poles contained in the area within the contour. If we send the contour to infinity, we include all possible poles of the function and obtain the relation

$$\oint_C \frac{dz}{2\pi i} \frac{A(z)}{z} = \sum_{z_i} \text{Res}\left(\frac{A(z)}{z}, z_i\right). \tag{3.7}$$

The original amplitude is just the residue at $z = 0$, so

$$A_n = -\sum_{z_{P_i}\neq 0} \text{Res}\left(\frac{A(z)}{z}, z_{P_i}\right) - \text{Res}\left(\frac{A(z)}{z}, \infty\right), \qquad (3.8)$$

where $\text{Res}\left(\frac{A(z)}{z}, \infty\right)$ is the pole at infinity, defined as $\text{Res}(f(z), \infty) = -\oint_C \frac{dz}{2\pi i} f(z)$.

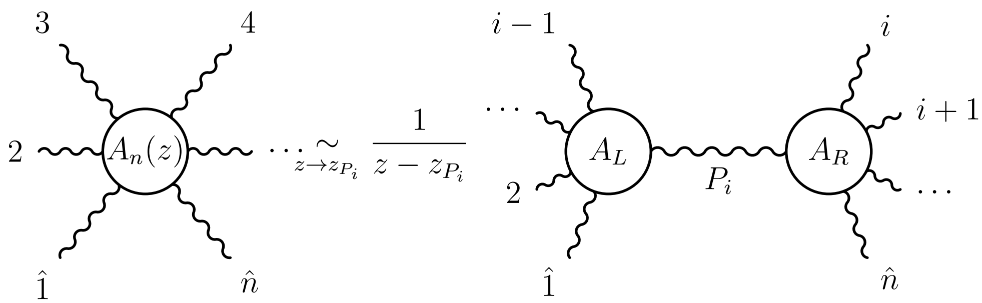

Figure 3.1: Factorization on the pole $z = z_{P_i}$

Sending $z \to z_{P_i}$ corresponds to sending the propagator P_i on-shell, which can be interpreted as splitting the amplitude into two on-shell subamplitudes A_L and A_R connected through an on-shell propagator (see Figure 3.1). Hence, due to factorization properties (for a thorough discussion see [6]), near the poles we have the asymptotic behavior

$$\frac{A_n(z)}{z} \underset{z\to z_{P_i}}{\sim} -\frac{1}{z - z_{P_i}} \sum_h A_L^h(z_{P_i}) \frac{1}{P_i^2} A_R^{-h}(z_{P_i}), \qquad (3.9)$$

where the sum over h is the sum over all possible helicity states of the propagator. The expression for the original amplitude then becomes

$$A_n = \sum_i \sum_h \frac{A_L^h(z_{P_i}) A_R^{-h}(z_{P_i})}{P_i^2} - \text{Res}\left(\frac{A(z)}{z}, \infty\right). \qquad (3.10)$$

In the case of gluons, the possible helicity states are $h = \pm 1$. Note that, since in our convention all gluons are chosen to be outgoing, the helicity of the propagator on the left amplitude is opposite to the one on the right amplitude, hence the minus sign on the helicity state in the right amplitude.

For functions scaling as $f(z) \sim 1/z^2$ for $z \to \infty$, the pole at infinity vanishes, so if $A_n(z)$ scales as $1/z$ we can drop the last term. This is indeed true for the case of the helicity configuration $(+-)$ of the shifted momenta. The z dependence only occurs along the path connecting particles 1 and n. From the Feynman rules, one sees that, along this path, three-gluon vertices contribute at most z, four-gluon vertices contribute 1 and propagators contribute $1/z$. Taking the least favorable case, in which all vertices are three-gluon vertices, due to the fact that between two vertices there is always a propagator, the overall contribution of this path is at most z. Apart from that, there is the contribution from the polarization vectors at legs 1 and n. For our choice of the helicity configuration, we have:

$$\epsilon^{\alpha\dot{\alpha}}_{+,1} = -\sqrt{2}\frac{\tilde{\lambda}^{\dot{\alpha}}_i \mu^{\alpha}_i}{\langle\hat{\lambda}_i \mu_i\rangle} \sim \frac{1}{z}, \tag{3.11}$$

$$\epsilon^{\alpha\dot{\alpha}}_{-,n} = \sqrt{2}\frac{\lambda^{\alpha}_i \tilde{\mu}^{\dot{\alpha}}_i}{[\hat{\lambda}_i \mu_i]} \sim \frac{1}{z}. \tag{3.12}$$

Therefore, considering the contribution from the polarization vectors, we get an overall z-dependence given by $1/z$, thus allowing us to drop the pole at infinity. For the $(++)$ and $(--)$ helicity configurations, one can also show that the shifted amplitude vanishes at $z \to \infty$ [7], whereas for $(-+)$ the amplitude diverges as z^3.

3.2 The MHV Amplitudes

The BCFW recursion owes its power to the fact that it allows us to write large scattering amplitudes in terms of smaller ones recursively, which leads to a description based solely on the simplest types of amplitudes. An example of such an application are the so-called *maximally helicity-violating* (MHV) amplitudes, in which all but two gluons have positive helicity. These are the simplest non-vanishing gluon amplitudes at tree-level, since all-plus amplitudes inevitably contain products of the form $\epsilon_{+,i} \cdot \epsilon_{+,j}$, which necessarily vanish, as can be seen straightforwardly from our construction of the polarization vectors, just as amplitudes with one flipped helicity state, which can be seen by employing a convenient choice of the reference spinors [8].

We can compute, from the Feynman rules, that for three gluons the MHV amplitude for two negative helicity states in gluons i and j is given by

$$A^{\text{MHV}}_3(i^-, j^-) = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}, \tag{3.13}$$

and for the anti-MHV amplitude with two positive helicity states by

$$A_3^{\overline{\text{MHV}}}(i^+, j^+) = -\frac{[ij]^4}{[12][23][31]}. \tag{3.14}$$

It is important to note that momentum conservation implies the vanishing of all Mandelstam variables:

$$p_1^\mu + p_2^\mu + p_3^\mu = 0 \Rightarrow p_1 \cdot p_2 = p_1 \cdot p_3 = p_2 \cdot p_3 = 0, \tag{3.15}$$

or

$$\langle ij\rangle[ji] = 0, \tag{3.16}$$

for any $i, j = 1, 2, 3$. The constraint 3.15, however, admits two distinct solutions in terms of spinors: either $\lambda_1^\alpha \propto \lambda_2^\alpha \propto \lambda_3^\alpha$ (collinear left-handed spinors) or $\tilde{\lambda}_1^{\dot{\alpha}} \propto \tilde{\lambda}_2^{\dot{\alpha}} \propto \tilde{\lambda}_3^{\dot{\alpha}}$ (collinear right-handed spinors). So, since for the MHV case the left-handed spinors are clearly not collinear, the right-handed ones must be, and the opposite is true for the anti-MHV case.

For n-gluons we want to show inductively that

$$A_n^{\text{MHV}}(n^-, 1^-) = \frac{\langle n1\rangle^4}{\langle 12\rangle \ldots \langle n1\rangle}. \tag{3.17}$$

According to the BCFW recursion we can write this amplitude as a sum of split amplitudes. As we have seen, the propagator must lie somewhere between particles 1 and n. Choosing an assignment $(+-)$ to the propagator leaves only one particle with negative helicity on the left side, and such an amplitude is non-vanishing only for three particles. So we only get one contribution from this choice. On the other hand, the assignment $(-+)$ of the propagator gives only one contribution, which is with three particles on the right amplitude, for the same reason. So we are left with two diagrams, which can be seen in Figure 3.2. However, the right amplitude being an $\overline{\text{MHV}}_3$ amplitude implies $\hat{\lambda}_n \propto \lambda_{n-1}$, or $\langle \hat{n} n-1\rangle = \langle n n-1\rangle = 0$, which is a collinearity condition for p_n and p_{n-1}. Since this is not the general case, we can disregard this diagram for a generic configuration of the external momenta.

Assuming that the MHV amplitude for $n-1$ particles is of the form given in 3.17, using the BCFW recursion we construct the amplitude for n gluons.

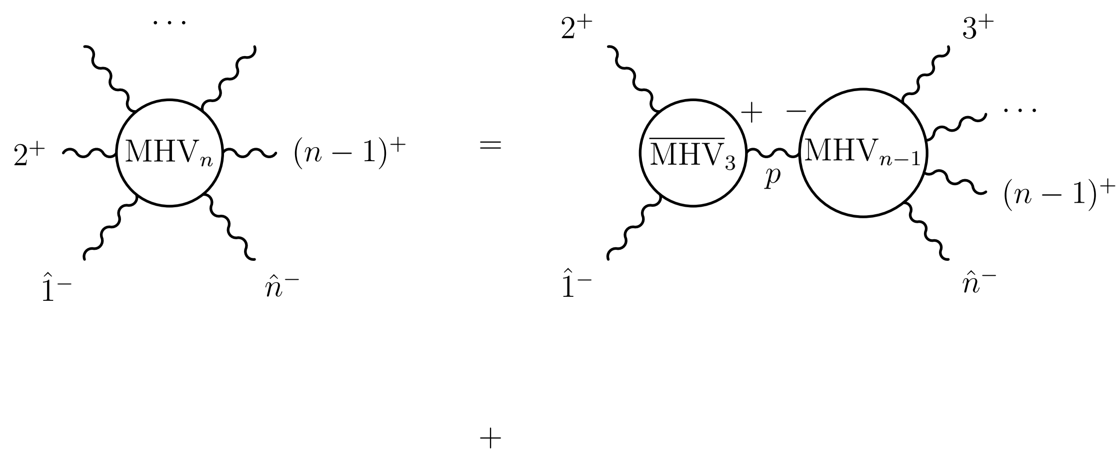

Figure 3.2: Splitting of the MHV amplitude

We have

$$A_L = A_3^{\overline{\text{MHV}}}(\hat{1}^-, 2^+, -\hat{P}^+) = -\frac{[2(-\hat{P})]^3}{[\hat{1}2][(-\hat{P})\hat{1}]}, \tag{3.18}$$

$$A_R = A_{n-1}^{\text{MHV}}(\hat{P}^-, 3^+, \ldots, (n-1)^+, \hat{n}^-) = \frac{\langle \hat{n}\hat{P}\rangle^3}{\langle \hat{P}3\rangle \ldots \langle (n-1)\hat{n}\rangle}, \tag{3.19}$$

where $\hat{P}$ represents the shifted momentum of the propagator. We use the convention that for a negative momentum we have $|-\tilde{\lambda}_P] = -|\tilde{\lambda}_P]$ and $|-\lambda_P\rangle = |\lambda_P\rangle$. Using $\langle \cdot \hat{n}\rangle = \langle \cdot n\rangle$ and $[\hat{1}\cdot] = [1\cdot]$ we have:

$$A = \hat{A}_L \frac{1}{(p_1+p_2)^2}\hat{A}_R = \frac{[2\hat{P}]^3\langle n\hat{P}\rangle^3}{[12][21]\langle 3\hat{P}\rangle[\hat{P}1]\langle 12\rangle\langle 34\rangle \ldots \langle (n-1)n\rangle}. \tag{3.20}$$

Momentum conservation gives:

$$\langle n\hat{P}\rangle[\hat{P}2] = -\langle n\hat{1}\rangle[12] = -\langle n1\rangle[12], \quad \langle 3\hat{P}\rangle[\hat{P}1] = -\langle 32\rangle[21], \tag{3.21}$$

Inserting these relations into eq. 3.20 yields:

$$A = -\frac{[12]^3\langle n1\rangle^3}{[12][21]\langle 32\rangle[21]\langle 12\rangle\langle 34\rangle \ldots \langle (n-1)n\rangle} = \frac{\langle n1\rangle^4}{\langle 12\rangle \ldots \langle (n-1)n\rangle}, \tag{3.22}$$

as we wanted to show.

3.3 Conformal Symmetry

The discussion in this section follows an argument presented in [9].

The Conformal Group is a group of transformations which describes translations, Lorentz boosts, dilation and special conformal transformations. It therefore contains and generalizes the Poincaré group. In general, QFT's are Poincaré invariant, but whenever the theory does not include any dimensionful parameter the symmetry is extended to the conformal group by adding the dilation and the special conformal transformations.

The special conformal transformation is a non-linear transformation which preserves causality. It can be written as a composition of an inversion, a translation and another inversion:

$$x^\mu \to x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2b \cdot x + b^2 x^2}, \tag{3.23}$$

where b^μ is a transformation parameter. The dilation is simply a global rescaling transformation:

$$x^\mu \to x'^\mu = \alpha x^\mu, \tag{3.24}$$

for an arbitrary constant α.

An important property of conformal symmetry for quantum field theories is that, in general, it is only present at tree-level. At loop order, renormalization introduces mass scales that therefore break conformal symmetry.

In configuration space, the generators of the conformal algebra are

$$P_\mu = -i\partial_\mu \quad \text{(Translation)}, \tag{3.25}$$
$$L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) \quad \text{(Lorentz transformations)}, \tag{3.26}$$
$$D = -ix^\mu \partial_\mu \quad \text{(Dilation)}, \tag{3.27}$$
$$K_\mu = -i(2x_\mu x^\nu \partial_\nu - \mathbf{x}^2 \partial_\mu) \quad \text{(SCT)}. \tag{3.28}$$

A thorough introduction to the conformal group can be found in [10]. The generators of the conformal group must obey the commutation relations that define the conformal algebra:

$$[D, P_\mu] = iP_\mu, \tag{3.29}$$
$$[D, K_\mu] = -iK_\mu, \tag{3.30}$$
$$[K_\mu, P_\nu] = 2i(\eta_{\mu\nu}D - L_{\mu\nu}), \tag{3.31}$$
$$[K_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu), \tag{3.32}$$
$$[P_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu), \tag{3.33}$$
$$[L_{\mu\nu}, L_{\rho\sigma}] = i(\eta_{\nu\rho}L_{\mu\sigma} + \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho}). \tag{3.34}$$

Performing a Fourier-transform of the generators, we get the representation of the conformal group in momentum space

$$P^\mu = p^\mu, \tag{3.35}$$
$$L^{\mu\nu} = p^\mu \partial_p^\nu - p^\nu \partial_p^\mu, \tag{3.36}$$
$$D = k \cdot \partial_p, \tag{3.37}$$
$$K^\mu = \frac{1}{2} p^\mu \partial_p^2 - (p \cdot \partial_p)\partial_p^\mu. \tag{3.38}$$

We can then map these operators to the spinor helicity variables (see [8], [9] for further details) and get the operators

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}, \tag{3.39}$$
$$l_{\alpha\beta} = \frac{1}{2}\left(\lambda_\alpha \frac{\partial}{\partial \lambda_\beta} + \lambda_\beta \frac{\partial}{\partial \lambda_\alpha}\right), \tag{3.40}$$
$$\tilde{l}_{\dot{\alpha}\dot{\beta}} = \frac{1}{2}\left(\tilde{\lambda}_{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_{\dot{\beta}}} + \tilde{\lambda}_{\dot{\beta}} \frac{\partial}{\partial \tilde{\lambda}_{\dot{\alpha}}}\right), \tag{3.41}$$
$$d = \frac{1}{2}\left(\lambda_\alpha \frac{\partial}{\partial \lambda_\alpha} + \tilde{\lambda}_{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_{\dot{\alpha}}} + 2\right), \tag{3.42}$$
$$k_{\alpha\dot{\alpha}} = \frac{\partial}{\partial \lambda^\alpha} \frac{\partial}{\partial \tilde{\lambda}^{\dot{\alpha}}}. \tag{3.43}$$

Note that here we defined the conformal generators for a single massless particle. The corresponding generators for an n-particle system are obtained by summing over the n particles, e.g.

$$k_{\alpha\dot{\alpha}} = \sum_{i=1}^{n} \frac{\partial}{\partial \lambda_i^\alpha} \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{\alpha}}}.$$

As an example, we will prove the conformal invariance of MHV amplitudes at tree-level. Without any loss of generality we can consider a specific color-ordered amplitude, e.g.

$$\mathscr{A}_n^{\text{MHV}} = \delta^{(4)}\left(\sum_{i=1}^{n} p_i\right) A_n^{\text{MHV}}, \tag{3.44}$$

where n is the number of external gluons, A_n^{MHV} is the reduced amplitude, which we computed in 3.2, and the overall delta function imposes momentum conservation.

Lorentz invariance is manifest since all indices are contracted, i.e. all spinor brackets are Lorentz invariant, and translational invariance is also fulfilled thanks to the overall delta function. Invariance under dilation is also fulfilled: $d\mathscr{A}_n = (d\delta^{(4)}(P))A_n + \delta^{(4)}(P)dA_n = (-4+4-n+n)A_n = 0$. We want to focus our attention on the special conformal transformation.

Since we know from the form of the reduced amplitude that $\partial A_n^{\text{MHV}}/\partial\tilde{\lambda}_i = 0$, applying a conformal boost on the amplitude gives, with the definition $P^{\alpha\dot{\alpha}} = \sum_i \lambda_i^{\alpha}\tilde{\lambda}_i^{\dot{\alpha}}$:

$$\begin{aligned} k_{\alpha\dot{\alpha}}\mathscr{A}_n^{\text{MHV}} &= \sum_i \frac{\partial}{\partial\lambda_i^{\alpha}}\left(\frac{\partial p^{\beta\dot{\beta}}}{\partial\tilde{\lambda}_i^{\dot{\alpha}}}\left(\frac{\partial}{\partial p^{\beta\dot{\beta}}}\delta^{(4)}(p)\right)A_n^{\text{MHV}}\right) \\ &= \left[\left(n\frac{\partial}{\partial p^{\alpha\dot{\alpha}}} + p^{\beta\dot{\beta}}\frac{\partial}{\partial p^{\beta\dot{\alpha}}}\frac{\partial}{\partial p^{\alpha\dot{\beta}}}\right)\delta^{(4)}(p)\right]A_n^{\text{MHV}} \\ &+ \left(\frac{\partial\delta^{(4)}(p)}{\partial p^{\beta\dot{\alpha}}}\right)\sum_i \lambda_i^{\beta}\frac{\partial}{\partial\lambda_i^{\alpha}}A_n^{\text{MHV}}. \end{aligned} \tag{3.45}$$

One can explicitly check that

$$\lambda_{i\alpha}\frac{\partial}{\partial\lambda_i^{\beta}} = \frac{1}{2}\underbrace{(\lambda_{i\alpha}\frac{\partial}{\partial\lambda_i^{\beta}} + \lambda_{i\beta}\frac{\partial}{\partial\lambda_i^{\alpha}})}_{\sim L_{\alpha\beta}} + \frac{1}{2}\varepsilon_{\alpha\beta}\lambda_i^{\gamma}\frac{\partial}{\partial\lambda_i^{\gamma}}, \tag{3.46}$$

and, since $L_{\alpha\beta}A_n^{\text{MHV}} = 0$, we get that

$$\sum_i \lambda_i^{\beta}\frac{\partial}{\partial\lambda_i^{\alpha}}A_n^{\text{MHV}} = \frac{1}{2}\delta_{\alpha\beta}\sum_i \lambda_i^{\gamma}\frac{\partial}{\partial\lambda_i^{\gamma}}A_n^{\text{MHV}} = \delta_{\alpha\beta}hA_n^{\text{MHV}} = \delta_{\alpha\beta}(4-n)A_n^{\text{MHV}}, \tag{3.47}$$

since the helicity of the amplitude is $(4-n)$. So we get:

$$k_{\alpha\dot{\alpha}}\mathscr{A}_n^{\text{MHV}} = \left[\left(4\frac{\partial}{\partial p^{\alpha\dot{\alpha}}} + p^{\beta\dot{\beta}}\frac{\partial}{\partial p^{\beta\dot{\alpha}}}\frac{\partial}{\partial p^{\alpha\dot{\beta}}}\right)\delta^{(4)}(p)\right] A_n^{\text{MHV}}. \tag{3.48}$$

Multiplying the term $p^{\beta\dot{\beta}}\frac{\partial}{\partial p^{\beta\dot{\alpha}}}\frac{\partial}{\partial p^{\alpha\dot{\beta}}}$ with a test function and integrating by parts yields:

$$\begin{aligned}
&\int d^4p f(p) p^{\beta\dot{\beta}}\frac{\partial}{\partial p^{\beta\dot{\alpha}}}\frac{\partial}{\partial p^{\alpha\dot{\beta}}}\delta^{(4)}(p)\\
=&\int d^4p\left(\left[\frac{\partial}{\partial p^{\beta\dot{\alpha}}}f(p)\right]2\delta_{\alpha\beta} + \left[\frac{\partial}{\partial p^{\alpha\dot{\beta}}}f(p)\right]2\delta_{\dot{\alpha}\dot{\beta}}\right)\delta^{(4)}(p)\\
=&4\int d^4p\left[\frac{\partial}{\partial p^{\alpha\dot{\alpha}}}f(p)\right]\delta^{(4)}(p)\\
=&-4\int d^4p f(p)\frac{\partial}{\partial p^{\alpha\dot{\alpha}}}\delta^{(4)},
\end{aligned} \tag{3.49}$$

from which we see the relation

$$p^{\beta\dot{\beta}}\frac{\partial}{\partial p^{\beta\dot{\alpha}}}\frac{\partial}{\partial p^{\alpha\dot{\beta}}}\delta^{(4)}(p) = -4\frac{\partial}{\partial p^{\alpha\dot{\alpha}}}\delta^{(4)}. \tag{3.50}$$

Inserting this in 3.48 gives: $k_{\alpha\dot{\alpha}}\mathscr{A}_n^{\text{MHV}} = 0$.

Finding conformally invariant objects in momentum space is a non-trivial task, in general, since the generator of special conformal transformations is a second-order operator. Although in configuration space it is easier, since the operator is only of first-order, due to the on-shell condition for the momenta it is not possible to simply Fourier transform these objects to momentum space. This motivates our search for conformally invariant objects in the spinor helicity formalism.

Chapter 4

All-Plus Scattering Amplitudes

The focus of this research project was the study of all-plus helicity amplitudes. As was discussed in section 3.2, they vanish at tree-level. In eq. 2.43 we saw the color decomposition of an amplitude at one-loop, given by a leading-color term and subleading color terms. The leading-color one-loop amplitude for the scattering of n plus-helicity gluons can be written as [11]:

$$A_{n;1} = -\frac{i}{3} \sum_{1 \leq i_1 < i_2 < i_3 < i_4 \leq n} \frac{\langle i_1 i_2 \rangle [i_2 i_3] \langle i_3 i_4 \rangle [i_4 i_1]}{\langle 12 \rangle \langle 23 \rangle \ldots \langle n1 \rangle}. \tag{4.1}$$

Numerically we have checked for n up to 12 that it is conformally invariant. We therefore expect it to be true for arbitrary n and attempt to rewrite it in a manifestly conformally invariant form. This may shed light on the reason why this amplitude is conformally invariant at one-loop, and perhaps indicate mechanisms to construct generic conformally invariant objects. As discussed in section 3.3, the pure Yang-Mills Lagrangian is conformally invariant at classical level, but this symmetry is broken at quantum level by the introduction of the mass scales required by renormalization. It is therefore remarkable to find an example where conformal symmetry survives at loop level, and the reason why this occurs is yet to be understood.

For $n = 4$, the leading-color amplitude takes a simple form:

$$A_{4,\text{all-plus}} \sim \frac{[12][34]}{\langle 12 \rangle \langle 34 \rangle}, \tag{4.2}$$

which, using momentum conservation, can be written as

$$A_{4,\text{all-plus}} \sim \frac{[41]^2}{\langle 23 \rangle^2}. \tag{4.3}$$

Note that this last form is manifestly conformally invariant, as for each particle it depends either on $\tilde{\lambda}_i$ or on λ_i, not on both together. Similarly, for $n = 5$, the amplitude can be written as [3]:

$$A_{5,\text{all-plus}} \sim \sum_{S_5} \left[\frac{[24]^2}{\langle 13 \rangle \langle 35 \rangle \langle 51 \rangle} + 2 \frac{[23]^2}{\langle 14 \rangle \langle 45 \rangle \langle 51 \rangle} \right], \tag{4.4}$$

where S_5 is the group of cyclic permutations of the external legs. Here we note that the amplitude is a linear combination of conformally invariant objects, therefore being itself manifestly conformally invariant.

These observations motivate the definition of the R_n objects:

$$R_n = \frac{[ab]^2}{\underbrace{\langle ij \rangle \langle j \cdot \rangle \ldots \langle \cdot k \rangle}_{n-2 \text{ times}}}, \tag{4.5}$$

where a, b and $i, j, \ldots, k$ are two disjunct sets of particles. We see that the R_n objects are manifestly conformally invariant, as desired. Furthermore, they have the right dimension of an n-point amplitude and have the all-plus helicity configuration. One might therefore naively hope to rewrite the one-loop all-plus amplitude in terms of permutations of these objects for arbitrary multiplicity n, thus making conformal symmetry manifest. However, for $n \geq 6$, these objects do not span a space large enough to contain the leading-color term.

span$\{R_4\}$	1-dimensional space
span$\{R_5\}$	6-dimensional space
span$\{R_6\}$	30-dimensional space
span$\{R_7\}$	126-dimensional space

Table 4.1: Spaces spanned by R_n objects

As shown in [1], all subleading-color terms can be written as sums of permutations of the leading-color amplitude. Therefore, if we can show the conformal invariance of the leading-color amplitude, it also follows directly for the subleading-color ones.

We shift our attention towards the subleading-color amplitudes as they exhibit a simpler form than the leading-color amplitude. The point is that proving the conformal invariance of the leading-color component analytically is very complicated because of the nested summation, whereas for the subleading-color components we have much simpler expressions. Their analysis may therefore allow us to find other conformally invariant objects which will then get us closer to rewriting 4.1 in a manifestly conformally invariant form.

The subleading-color terms are given by [12]:

$$A_{n;2}(1^+;2^+;\ldots;n^+) = -i\sum_{i<j}\frac{[1|ij|1]}{\langle 23\rangle\langle 34\rangle\ldots\langle n2\rangle}, \tag{4.6}$$

$$A_{n;r}(1^+,\ldots,(r-1)^+;r^+,\ldots,n^+) = -2i\frac{(p_1+\ldots+p_{r-1})^2(p_r+\ldots p_n)^2}{\langle 12\rangle\ldots\langle (r-1)1\rangle\langle r(r+1)\rangle\ldots\langle nr\rangle}. \tag{4.7}$$

Indeed we found the subleading terms to be conformally invariant. To see this for the $A_{n;r}$ amplitude we first define new objects:

$$C_n^{A,B} = Z_n^A Z_n^B, \quad A,B\subset \mathbb{N}_n : \{A\cap B=\emptyset, \exists\sigma\in S_n : A\cup B = \sigma\circ\mathbb{N}_n\}, \tag{4.8}$$

where

$$Z_n^{[i_1i_2\ldots i_k]} = \frac{(p_{i_1}+p_{i_2}+\ldots+p_{i_k})^2}{\langle i_1i_2\rangle\langle i_2i_3\rangle\ldots\langle i_{k-1}i_k\rangle\langle i_ki_1\rangle}, \tag{4.9}$$

and

$$\mathbb{N}_n = \{1,2,\ldots,n\}. \tag{4.10}$$

We note that $A_{n;r} = -2iC_n^{[1\ldots r-1][r\ldots n]}$.

Since each Z_n factor contains an independent set of particles, the action of a conformal boost to C_n can be fractorized into two contributions:

$$k_{\alpha\dot\alpha}C_n^{A,B} = \left(k_{\alpha\dot\alpha}Z_n^A\right)Z_n^B + Z_n^A\left(k_{\alpha\dot\alpha}Z_n^B\right). \tag{4.11}$$

We now prove that $C_n^{A,B}$ is conformally invariant.

Because of the on-shell condition $p_i^2=0$, we can write

$$(p_{i_1}+p_{i_2}+\ldots+p_{i_k})^2 = \frac{1}{2}\sum_{a,b=1}^{k}\langle i_ai_b\rangle[i_bi_a]. \tag{4.12}$$

Applying a conformal boost to $Z_n^{[i_1 i_2 \ldots i_k]}$ then yields:

$$
\begin{aligned}
k_{\alpha\dot\alpha}\frac{(p_{i_1}+p_{i_2}+\ldots+p_{i_k})^2}{\langle i_1 i_2\rangle\langle i_2 i_3\rangle\ldots\langle i_k i_1\rangle} &= \frac{1}{\langle i_1 i_2\rangle\langle i_2 i_3\rangle\ldots\langle i_k i_1\rangle}\sum_{a=1}^{k}\left[\sum_{b\neq a}\lambda_{i_b\alpha}\tilde\lambda_{i_b\dot\alpha}\right.\\
&\left.+\sum_{b=1}^{k}\left(-\frac{\langle i_a i_b\rangle}{\langle i_{a-1} i_a\rangle}\lambda_{i_{a-1}\alpha}\tilde\lambda_{i_b\dot\alpha}+\frac{\langle i_a i_b\rangle}{\langle i_a i_{a+1}\rangle}\lambda_{i_{a+1}\alpha}\tilde\lambda_{i_b\dot\alpha}\right)\right],
\end{aligned}
\tag{4.13}
$$

where $a-1$ and $a+1$ are defined modulo k, i.e. $i_0 = i_k, \quad i_{k+1} = i_1$.

In the last sum we added the term $b = a$; as it vanishes, it does not contribute to the sum. Since the summation is cyclic, we can shift the summation index of the last sum by one: $a \to (a-1)$, and invert the summation order

$$
\begin{aligned}
&\sum_a\left[\sum_{b\neq a}\lambda_{i_b\alpha}\tilde\lambda_{i_b\dot\alpha}+\sum_b\left(\frac{\langle i_a i_b\rangle}{\langle i_{a-1} i_a\rangle}\lambda_{i_{a-1}\alpha}\tilde\lambda_{i_b\dot\alpha}+\frac{\langle i_a i_b\rangle}{\langle i_a i_{a+1}\rangle}\lambda_{i_{a+1}\alpha}\tilde\lambda_{i_b\dot\alpha}\right)\right]\\
&=\sum_b\left[\sum_{a\neq b}\lambda_{i_b\alpha}\tilde\lambda_{i_b\dot\alpha}+\sum_a\left(\underbrace{\left(\frac{\langle i_a i_b\rangle}{\langle i_{a-1} i_a\rangle}\lambda_{i_{a-1}\alpha}+\frac{\langle i_{a-1} i_b\rangle}{\langle i_{a-1} i_a\rangle}\lambda_{i_a\alpha}\right)}_{-\frac{\lambda_{i_b\alpha}}{\langle i_{a-1} i_a\rangle}}\tilde\lambda_{i_b\dot\alpha}\right)\right]\\
&=\sum_b\left[\sum_{a\neq b}\lambda_{i_b\alpha}\tilde\lambda_{i_b\dot\alpha}-\sum_a\lambda_{i_b\alpha}\tilde\lambda_{i_b\dot\alpha}\right]\\
&=-\sum_b\lambda_{i_b\alpha}\tilde\lambda_{i_b\dot\alpha}\\
&=-\left(p_{i_1}+p_{i_2}+\ldots+p_{i_k}\right)_{\alpha\dot\alpha},
\end{aligned}
\tag{4.14}
$$

where we used the Schouten identity in the last step. Hence we get

$$
k_{\alpha\dot\alpha}\frac{(p_{i_1}+p_{i_2}+\ldots+p_{i_k})^2}{\langle i_1 i_2\rangle\langle i_2 i_3\rangle\ldots\langle i_k i_1\rangle} = -\frac{(p_{i_1}+p_{i_2}+\ldots+p_{i_k})_{\alpha\dot\alpha}}{\langle i_1 i_2\rangle\langle i_2 i_3\rangle\ldots\langle i_k i_1\rangle}. \tag{4.15}
$$

So, applying a conformal boost to the $C_n^{A,B}$ with $A = \{a_1, a_2, \ldots, a_k\}$ and $B = \{b_1, b_2, \ldots, b_{n-k}\}$ yields:

$$
\begin{aligned}
k_{\alpha\dot\alpha}C_n^{A,B} = &-\frac{(p_{a_1}+\ldots+p_{a_k})_{\alpha\dot\alpha}(p_{b_1}+\ldots+p_{b_{n-k}})^2}{\langle a_1 a_2\rangle\ldots\langle a_n a_1\rangle\langle b_1 b_2\rangle\ldots\langle b_{n-k} b_1\rangle}\\
&-\frac{(p_{b_1}+\ldots+p_{b_{n-k}})_{\alpha\dot\alpha}(p_{a_1}+\ldots+p_{a_k})^2}{\langle a_1 a_2\rangle\ldots\langle a_n a_1\rangle\langle b_1 b_2\rangle\ldots\langle b_{n-k} b_1\rangle} = 0.
\end{aligned}
\tag{4.16}
$$

The vanishing of the sum follows from momentum conservation: $\sum_i p_i = 0$, which implies

$$(p_{a_1}+\ldots+p_{a_k}) = -(p_{b_1}+\ldots+p_{b_{n-k}}), \quad (p_{a_1}+\ldots+p_{a_k})^2 = (p_{b_1}+\ldots+p_{b_{n-k}})^2. \tag{4.17}$$

These new C_n-objects are in fact a generalization of the R_n objects and extend the space spanned by them. Indeed, one has that

$$\begin{aligned} C_n^{[12][3\ldots n]} &= \frac{(p_1+p_2)^2 \overbrace{(p_3+\ldots+p_n)^2}^{(p_1+p_2)^2}}{\langle 12\rangle\langle 21\rangle\langle 34\rangle\ldots\langle (n-1)n\rangle} \\ &= \frac{\langle 21\rangle^2[12]^2}{\langle 12\rangle\langle 21\rangle\langle 34\rangle\ldots\langle (n-1)n\rangle} \\ &= -\frac{[12]^2}{\langle 34\rangle\ldots\langle (n-1)n\rangle}. \end{aligned} \tag{4.18}$$

span$\{C_4\}$	1-dimensional space
span$\{C_5\}$	6-dimensional space
span$\{C_6\}$	35-dimensional space
span$\{C_7\}$	196-dimensional space

Table 4.2: Spaces spanned by C_n objects

For the $A_{n;2}$ amplitude the proof of the conformal invariance follows in a similar way. We have

$$\begin{aligned} A_{n;2} &= -i\sum_{i<j}\frac{[1|ij|1]}{\langle 23\rangle\langle 34\rangle\ldots\langle n2\rangle} \\ &= -i\sum_{i<j}\frac{[1i]\langle ij\rangle[j1]}{\langle 23\rangle\langle 34\rangle\ldots\langle n2\rangle}. \end{aligned} \tag{4.19}$$

Clearly the summation starts at $i=2$, since for $i=1$ the summand vanishes. Applying a conformal boost to this object, we get, with the notation $k_{\alpha\dot{\alpha}}^{[m]} =$

$\frac{\partial^2}{\partial\lambda_m^\alpha\partial\tilde{\lambda}_m^{\dot{\alpha}}}$:

$$k_{\alpha\dot{\alpha}}A_{n;2} = -i\sum_m\sum_i\sum_{j>i}k_{\alpha\dot{\alpha}}^{[m]}\left(\frac{[1i]\langle ij\rangle[j1]}{\langle 23\rangle\langle 34\rangle\ldots\langle n2\rangle}\right)$$
$$= -i\frac{1}{\langle 23\rangle\langle 34\rangle\ldots\langle n2\rangle}\sum_m\sum_i\sum_{j>i}\left[\delta_{im}\left(\lambda_{j\alpha}+\frac{\lambda_{i-1\alpha}\langle ij\rangle}{\langle(i-1)i\rangle}-\frac{\lambda_{i+1\alpha}\langle ij\rangle}{\langle i(i+1)\rangle}\right)\tilde{\lambda}_{1\dot{\alpha}}[j1]\right.$$
$$\left.+\ \delta_{jm}\left(\lambda_{i\alpha}+\frac{\lambda_{j-1\alpha}\langle ji\rangle}{\langle(j-1)j\rangle}-\frac{\lambda_{j+1\alpha}\langle ji\rangle}{\langle j(j+1)\rangle}\right)\tilde{\lambda}_{1\dot{\alpha}}[1i]\right]. \tag{4.20}$$

We now analyze the behavior of each part separately. For the first term we have

$$\sum_m\sum_{i>1}\sum_{j>i}\delta_{im}\left(\lambda_{j\alpha}+\frac{\lambda_{i-1\alpha}\langle ij\rangle}{\langle(i-1)i\rangle}-\frac{\lambda_{i+1\alpha}\langle ij\rangle}{\langle i(i+1)\rangle}\right)\tilde{\lambda}_{1\dot{\alpha}}[j1]$$
$$=\sum_{i>1}\sum_{j>i}\left(\lambda_{j\alpha}+\frac{\lambda_{i-1\alpha}\langle ij\rangle}{\langle(i-1)i\rangle}-\frac{\lambda_{i+1\alpha}\langle ij\rangle}{\langle i(i+1)\rangle}\right)\tilde{\lambda}_{1\dot{\alpha}}[j1]. \tag{4.21}$$

Now we can exchange the summation order and shift the summation over i for the last term:

$$=\sum_{j>2}\left[\sum_{1<i<j}\left(\lambda_{j\alpha}+\frac{\lambda_{i-1\alpha}\langle ij\rangle}{\langle(i-1)i\rangle}\right)-\sum_{2<i\leq j}\frac{\lambda_{i\alpha}\langle(i-1)j\rangle}{\langle(i-1)i\rangle}\right]\tilde{\lambda}_{1\dot{\alpha}}[j1]$$
$$=\sum_{j>2}\left[\sum_{1<i<j}\left(\underbrace{\lambda_{j\alpha}+\frac{\lambda_{i-1\alpha}\langle ij\rangle}{\langle(i-1)i\rangle}+\frac{\lambda_{i\alpha}\langle j(i-1)\rangle}{\langle(i-1)i\rangle}}_{=0}\right)-\frac{\lambda_{2\alpha}\langle jn\rangle}{\langle n2\rangle}-\frac{\lambda_{j\alpha}\langle(j-1)j\rangle}{\langle(j-1)j\rangle}\right]\tilde{\lambda}_{1\dot{\alpha}}[j1]$$
$$=\sum_{j>2}\left[-\frac{\lambda_{2\alpha}\langle jn\rangle}{\langle n2\rangle}-\lambda_{j\alpha}\right]\tilde{\lambda}_{1\dot{\alpha}}[j1]$$
$$=\frac{\lambda_{2\alpha}\tilde{\lambda}_{1\dot{\alpha}}}{\langle n2\rangle}\overbrace{\sum_{j>2}[1j]\langle jn\rangle}^{=-[12]\langle 2n\rangle}-\tilde{\lambda}_{1\dot{\alpha}}\overbrace{\sum_{j>2}\lambda_{j\alpha}[j1]}^{=\lambda_{2\alpha}[12]}$$
$$=\left(\lambda_{2\alpha}[12]-\lambda_{2\alpha}[12]\right)\tilde{\lambda}_{1\dot{\alpha}}$$
$$=0, \tag{4.22}$$

where in the second step we used the Schouten identity and in the fourth step we used momentum conservation. For the second part the procedure is similar up to the last step:

$$\begin{aligned}
&\sum_m \sum_{i>1} \sum_{j>i} \delta_{jm} \left(\lambda_{i\alpha} + \frac{\lambda_{j-1\alpha}\langle ji\rangle}{\langle (j-1)j\rangle} - \frac{\lambda_{j+1\alpha}\langle ji\rangle}{\langle j(j+1)\rangle} \right) \tilde{\lambda}_{1\dot{\alpha}}[1i] \\
&\ldots = \frac{\lambda_{2\alpha}\tilde{\lambda}_{1\dot{\alpha}}}{\langle n2\rangle} \overbrace{\sum_{i>1}[1i]\langle in\rangle}^{=0} - \tilde{\lambda}_{1\dot{\alpha}} \overbrace{\sum_{i>1}\lambda_{i\alpha}[i1]}^{=0} \\
&= 0.
\end{aligned} \tag{4.23}$$

Therefore we have:

$$k_{\alpha\dot{\alpha}} A_{n;2} = 0. \tag{4.24}$$

Hence, the object $A_{n;2}$ is conformally invariant.

Numerically we have found that for $n = 6$ the extended space of conformally invariant objects contains the leading-color amplitude. However, for $n = 7$ it is already not large enough.

In summary, there are simplified expressions for the leading-color term for $n = 4, 5$, but we did not find more compact expressions for higher particle multiplicity. We checked numerically that the leading-color term is conformally invariant for n up to 12 and therefore expect it to be conformally invariant in general. Since the leading-color term itself has a rather unpractical form for our purposes, we analyzed the subleading terms looking for hints as to how to rewrite the leading-color amplitude in terms of manifestly conformally invariant objects. Indeed we were able to prove the conformal invariance of the subleading-color terms. This analysis lead us to identify new conformally invariant objects, which suffice to rewrite the leading-color component for 6 gluons in a manifestly conformally invariant, although not more compact way. For 7 or more gluons we still have to extend the range of conformally invariant objects at our disposal.

Chapter 5

Conclusion

In this thesis we started by presenting some necessary tools for computing gluon scattering amplitudes: the construction of Yang-Mills theories, the spinor helicity formalism and the color decomposition. The latter allows us to rewrite the amplitudes in such a way that all the kinematic information is separated from the color structure and contained in color-ordered partial amplitudes. These partial amplitudes were the subject of our research.

We further discussed the BCFW recursion relation, which provides a very powerful technique for writing amplitudes at tree-level even for large multiplicity iteratively in terms of lower multiplicity ones, and showed its application for MHV-amplitudes. We then introduced the concept of conformal symmetry and conformal transformations, showed the explicit form of the generators and discussed conformal invariance. In particular, the fact that theories without dimensionful parameters exhibit conformal invariance at tree-level, but in general not at higher order due to the mass scales introduced by renormalization.

In the last chapter we presented the results of our work, regarding gluon amplitudes in the all-plus helicity configuration at one-loop, which surprisingly appear to be conformally invariant. This has been checked numerically for n up to 12, but an analytic proof is still missing. Our goal was to rewrite the leading-color amplitude in terms of manifestly conformally invariant objects, but, due to its involved expression in terms of nested sums, it is very cumbersome to rewrite it directly. For this reason we studied the subleading-color terms and found new conformally invariant objects which, up to $n = 6$, suffice to write the leading-color amplitude, but not for $n \geq 7$. We also proved analytically that the subleading-color amplitudes are conformally invariant for arbitrary n. We expect that a further study of the leading-color amplitude for $n = 7$ might lead to a more general expression in which conformal symmetry becomes manifest for arbitrary n and to an understanding of the mechanism protecting conformal symmetry at one-loop order in the all-plus helicity configuration.

Appendix A

Conventions

$$
\begin{aligned}
\eta_{\mu\nu} &= \text{diag}(+,-,-,-), & p_\mu p^\mu &= p_0^2 - \mathbf{p}^2,\\
\varepsilon^{12} &= \varepsilon_{21} = +1, & \varepsilon^{21} &= \varepsilon_{12} = -1,\\
(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} &= (\mathbb{1}, -\sigma^i), & (\sigma^\mu)_{\alpha\dot{\alpha}} &= \varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}(\bar{\sigma}^\mu)^{\dot{\beta}\beta} = (\mathbb{1}, \sigma),\\
(\bar{\sigma}_\mu)^{\dot{\alpha}\alpha} &= (\mathbb{1}, \sigma^i), & (\sigma_\mu)_{\alpha\dot{\alpha}} &= (\mathbb{1}, -\sigma^i),
\end{aligned}
$$

$$
\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
$$

$$
\begin{aligned}
p^{\alpha\dot{\alpha}} &= \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}}, & p_{\dot{\alpha}\alpha} &= \varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}\lambda^\beta\tilde{\lambda}^{\dot{\beta}}\\
\lambda_\alpha &= \varepsilon_{\alpha\beta}\lambda^\beta, & \tilde{\lambda}_{\dot{\alpha}} &= \varepsilon_{\dot{\alpha}\dot{\beta}}\tilde{\lambda}^{\dot{\beta}}\\
u_+(p) = v_-(p) &= \begin{pmatrix} \lambda_\alpha \\ 0 \end{pmatrix} =: |p\rangle, & u_-(p) = v_+(p) &= \begin{pmatrix} 0 \\ \tilde{\lambda}^{\dot{\alpha}} \end{pmatrix} =: |p],\\
\bar{u}_+(p) = \bar{v}_-(p) &= \begin{pmatrix} 0 & \tilde{\lambda}_{\dot{\alpha}} \end{pmatrix} =: [p|, & \bar{u}_-(p) = \bar{v}_+(p) &= \begin{pmatrix} \lambda^\alpha & 0 \end{pmatrix} =: |p],\\
\langle\lambda_i\lambda_j\rangle &:= \lambda_i^\alpha\lambda_{j\alpha}, & [\tilde{\lambda}_i\tilde{\lambda}_j] &:= \tilde{\lambda}_{i\dot{\alpha}}\tilde{\lambda}_j^{\dot{\alpha}}.
\end{aligned}
$$

Bibliography

[1] Z. Bern, L. Dixon, D. C. Dunbar, and D. A. Kosower, "One-loop n-point gauge theory amplitudes,unitarity and collinear limits," 1994. arXiv: hep-ph/9403226.

[2] R. Britto, F. Cachazo, B. Feng, and E. Witten, "Direct proof of the tree-level scattering amplitude recursion relation in yang-mills theory," vol. 94, 2005. arXiv: hep-th/0501052.

[3] S. Badger, D. Chicherin, T. Gehrmann, G. Heinrich, J. M. Henn, T. Peraro, P. Wasser, Y. Zhang, and S. Zoia, "Analytic form of the full two-loop five-gluon all-plus helicity amplitude," arXiv: 1905.03733 [hep-ph].

[4] C.-N. Yang and R. Mills, "Conservation of isotopic spin and isotopic gauge invariance," vol. 96, no. 1, pp. 191–195, 1954.

[5] S. D. Badger, E. W. N. Glover, V. V. Khozea, and P. Svrček, "Recursion relations for gauge theory amplitudes with massive particles," 2005. arXiv: hep-th/0504159.

[6] L. Dixon, "Calculating scattering amplitudes efficiently," arXiv: hep-ph/9601359.

[7] N. Arkani-Hamed and J. Kaplan, "On tree amplitudes in gauge theory and gravity," 2008. arXiv: 0801.2385 [hep-th].

[8] J. M. Henn and J. C. Plefka, *Scattering Amplitudes in Gauge Theories*. Springer, 2014.

[9] E. Witten, "Perturbative gauge theory as a string theory in twistor space," arXiv: hep-th/0312171.

[10] P. di Francesco, P. Mathieu, and D. Sénéchal, *Conformal Field Theory*. Springer, 1997.

[11] Z. Bern, G. Chalmers, L. Dixon, and D. A. Kosower, “One-loop n gluon amplitudes with maximal helicity violation via collinear limits,” vol. 72, 1994. arXiv: hep-ph/9312333.

[12] D. C. Dunbar, J. H. Godwin, J. M. W. Strong, and W. B. Perkins, “The Full Colour Two-Loop 5-pt All-Plus Helicity Gluon Amplitude Using a 4-D Unitarity Approach.”.