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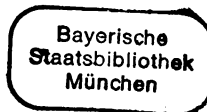
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RÉNYI ALFRÉD, SZELE TIBOR és VARGA OTTÓ

DARÓCZY ZOLTÁN, GYIRES BÉLA, RAPCSÁK ANDRÁS,  
TAMÁSSY LAJOS

KÖZREMŰKÖDÉSÉVEL SZERKESZTI:

BARNA BÉLA



A DEBRECENI TUDOMÁNYEGYETEM MATEMATIKAI INTÉZETE

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# Non-additive Ring and Module Theory, III. Morita Equivalences

*Dedicated to the memory of Andor Kertész*

By B. PAREIGIS (München)

This paper is a continuation of [19], [20]. References are quoted there.

As in ring theory one may ask the question when two categories  ${}_A\mathcal{C}$  and  ${}_B\mathcal{C}$  for monoids  $A$  and  $B$  are equivalent. Now in ring theory we know from the additivity of the equivalences  $\mathcal{F}: {}_A\mathcal{C} \rightarrow {}_B\mathcal{C}$  and  $\mathcal{G}: {}_B\mathcal{C} \rightarrow {}_A\mathcal{C}$  that the natural bijections

$$\text{Hom}_B(\mathcal{F}(M), N) \cong \text{Hom}_A(M, \mathcal{G}(N)) \quad \text{and} \quad \text{Hom}_A(\mathcal{G}(N), M) \cong \text{Hom}_B(N, \mathcal{F}(M))$$

are isomorphisms of abelian groups. So in the general case we only want to consider equivalences such that there are isomorphisms  ${}_B[\mathcal{F}(M), N] \cong {}_A[M, \mathcal{G}(N)]$  and  ${}_A[\mathcal{G}(N), M] \cong {}_B[N, \mathcal{F}(M)]$ . In view of theorem 4.3. this is equivalent to studying equivalences such that  $\mathcal{F}$  and  $\mathcal{G}$  are  $\mathcal{C}$ -functors. The last condition can be studied even in monoidal, non-closed categories.

We call  ${}_A\mathcal{C}$  and  ${}_B\mathcal{C}$   $\mathcal{C}$ -equivalent if there are inverse equivalences  $\mathcal{F}: {}_A\mathcal{C} \rightarrow {}_B\mathcal{C}$  and  $\mathcal{G}: {}_B\mathcal{C} \rightarrow {}_A\mathcal{C}$  such that  $\mathcal{F}$  and  $\mathcal{G}$  are  $\mathcal{C}$ -functors.

Without loss of generality we shall only consider equivalences  $\mathcal{F}$  and  $\mathcal{G}$  together with isomorphisms  $\Phi: \mathcal{F}\mathcal{G} \cong \text{Id}$  and  $\Psi: \mathcal{G}\mathcal{F} \cong \text{Id}$  such that  $\mathcal{F}\Psi = \Phi\mathcal{F}$  and  $\Psi\mathcal{G} = \mathcal{G}\Phi$ . Then  $\Phi$  and  $\Psi$  and their inverses are already adjointness morphisms.

**5.1. Theorem.** *Let  $\mathcal{C}$  be an arbitrary monoidal category. Let  $\mathcal{F}: {}_A\mathcal{C} \rightarrow {}_B\mathcal{C}$  and  $\mathcal{G}: {}_B\mathcal{C} \rightarrow {}_A\mathcal{C}$  be inverse  $\mathcal{C}$ -equivalences. Then there are objects  $P \in {}_A\mathcal{C}_B$  and  $Q \in {}_B\mathcal{C}_A$  such that*

a) *there are natural isomorphisms*

$$\mathcal{F}(M) \cong Q \otimes_A M \cong {}_A[P, M] \quad \text{in} \quad {}_A\mathcal{C},$$

$$\mathcal{G}(N) \cong P \otimes_B N \cong {}_B[Q, N] \quad \text{in} \quad {}_B\mathcal{C},$$

and  $P$   $B$ -coflat and  $Q$   $A$ -coflat.

b) *there are isomorphisms of  $A$ - $A$ -resp.  $B$ - $B$ -bibiobjects*

$$A \cong P \otimes_B Q \quad \text{and} \quad B \cong Q \otimes_A P$$

such that the diagrams

$$\begin{array}{ccc} P \otimes (Q \otimes P) \cong (P \overset{\circ}{\otimes} Q) \otimes P & \longrightarrow & A \otimes P \\ \downarrow & & \downarrow \\ P \otimes B & \longrightarrow & P \end{array}$$

and

$$\begin{array}{ccc} Q \otimes (P \otimes Q) \cong (Q \otimes P) \otimes Q & \longrightarrow & B \otimes Q \\ \downarrow & & \downarrow \\ Q \otimes A & \longrightarrow & Q \end{array}$$

commute,

c) there are isomorphisms

$$\begin{aligned} {}_B[Q, B] &\cong P \text{ in } {}_A\mathcal{C}_B, \\ {}_A[P, A] &\cong Q \text{ in } {}_B\mathcal{C}_A, \end{aligned}$$

d) there are isomorphisms

$$\begin{aligned} {}_B[Q, Q] &\cong A \text{ in } {}_A\mathcal{C}_A \text{ and as monoids,} \\ {}_A[P, P] &\cong B \text{ in } {}_B\mathcal{C}_B \text{ and as monoids.} \end{aligned}$$

PROOF. By the symmetry of the situation we only have to prove one half of the assertions.

There is an isomorphism  $\mathcal{F}(M) \cong Q \otimes_A M$  natural in  $M$  by Theorem 4.2 since  $\mathcal{F}$  is a  $\mathcal{C}$ -functor and clearly preserves difference cokernels as an equivalence. By the same theorem we have  $\mathcal{G}(N) \cong {}_B[Q, N]$  natural in  $N$ , since  $\mathcal{G}$  is adjoint to  $\mathcal{F}$ . This proves a).

We have an isomorphism  $A \cong \mathcal{G}\mathcal{F}(A) \cong P \otimes_B (Q \otimes_A A) \cong P \otimes_B Q$  in  ${}_A\mathcal{C}$ . Furthermore we have a commutative diagram

$$\begin{array}{ccccc} A \otimes A & \cong & P \otimes_B (Q \otimes_A (A \otimes A)) & \cong & P \otimes_B (Q \otimes A) \\ \downarrow & & \downarrow & & \downarrow \\ A & \cong & P \otimes_B (Q \otimes_A A) & \cong & P \otimes_B Q \end{array}$$

hence  $A \cong P \otimes_B Q$  as  $A$ - $A$ -bijejects.

The adjunction morphism  $\Psi: \mathcal{G}\mathcal{F} \cong Id$  induces the evaluation morphism  $\Psi': P \otimes_B {}_A[P, M] \cong M$  with  $\Psi'(p \otimes f) = \langle p \rangle f$ . By definition of the isomorphism  $A \cong P \otimes_B Q$  we get a commutative diagram

$$\begin{array}{ccc} P \otimes_B (Q \otimes_A M) \cong (P \otimes_B Q) \otimes_A M \cong A \otimes_A M & & \\ \swarrow & & \searrow \\ P \otimes_B {}_A[P, M] & \xrightarrow{\Psi'} & M \end{array}$$

Hence if the isomorphism  $P \otimes_B Q \cong A$  is described by  $p \otimes_B q \mapsto pq$  and the morphism  $Q \otimes_A M \mapsto {}_A[P, M]$  is given by

$$q \otimes_A m \mapsto \varphi(q \otimes_A m), \text{ we get } (pq)m = \langle p \rangle \varphi(q \otimes_A m).$$

Now if  $M \in {}_A\mathcal{C}_B$  then we get  $\langle p \rangle \varphi(q \otimes_A mb) = (pq)(mb) = ((pq)m)b = (\langle p \rangle \varphi(q \otimes_A m))b = \langle p \rangle (\varphi(q \otimes_A m)b)$ , hence  $Q \otimes_A M$  and  ${}_A[P, M]$  are isomorphic as  $B$ - $B$ -bijejects.

In particular we get  ${}_A[P, A] \cong Q$  as  $B-A$ -bijeobjects and  ${}_A[P, P] \cong Q \otimes_A P \cong B$  as  $B-B$ -bijeobjects.

To prove the monoid isomorphisms we first observe that

$$\begin{array}{ccc} P \otimes (Q \otimes P) \cong (P \otimes Q) \otimes P & \longrightarrow & A \otimes P & \text{and} \\ \downarrow & & \downarrow & \\ P \otimes B & \longrightarrow & P & \\ \\ Q \otimes (P \otimes Q) \cong (Q \otimes P) \otimes Q & \longrightarrow & B \otimes Q & \\ \downarrow & & \downarrow & \\ Q \otimes A & \longrightarrow & Q & \end{array}$$

commute. This follows from  $\mathcal{F} \cong Q \otimes_A$ ,  $\mathcal{G} \cong P \otimes_B$  and from the fact that  $\Phi\mathcal{F}: \mathcal{F}\mathcal{G}\mathcal{F} \rightarrow \mathcal{F}$  and  $\mathcal{F}\Psi: \mathcal{F}\mathcal{G}\mathcal{F} \rightarrow \mathcal{F}$  resp.  $\mathcal{G}\Phi: \mathcal{G}\mathcal{F}\mathcal{G} \rightarrow \mathcal{G}$  and  $\Psi\mathcal{G}: \mathcal{G}\mathcal{F}\mathcal{G} \rightarrow \mathcal{G}$  are equal. So we get

$$\langle p \rangle \varphi(q' \otimes p') \varphi(q'' \otimes p'') = (pq')(p'q'')p'' = p(q'p')(q''p'').$$

Since the isomorphism  ${}_A[P, P] \cong B$  is given by

$${}_A[P, P] \xleftarrow{\varphi} Q \otimes_A P \cong B$$

or  $\varphi(q' \otimes p') \mapsto q'p'$ , the composition  $\varphi(q' \otimes p') \varphi(q'' \otimes p'')$  is mapped to the product  $(q'p')(q''p'')$ . If  $\varphi(q' \otimes p')$  is the identity then  $p(q'p')=p$  for all  $p$ . But  ${}_A[P, P] \rightarrow B$  is an isomorphism hence  $q'p'=1 \in B$ .

**5.2. Corollary:** *The morphisms*

$$P(X) \times {}_A[P, A](Y) \ni (p, f) \mapsto \langle p \rangle f \in A(X \otimes Y)$$

and

$${}_A[P, A](X) \times P(Y) \ni (f, p) \mapsto fp \in {}_A[P, P](X \otimes Y)$$

with  $\langle p' \rangle (fp) := (\langle p' \rangle f)p$  induce isomorphisms

$$P \otimes_B {}_A[P, A] \cong A \quad \text{and} \quad {}_A[P, A] \otimes_A P \cong {}_A[P, P].$$

The analogous assertions hold for  $Q$  and  $B$ .

PROOF. The first isomorphism, the evaluation morphism, was discussed in the proof of 5.1. The second isomorphism is just given by

$${}_A[P, A] \otimes_A P \xrightarrow{\varphi \otimes P} Q \otimes_A P \xrightarrow{\varphi} {}_A[P, P].$$

We have seen that each  $\mathcal{C}$ -equivalence is induced by some object  $P \in {}_A\mathcal{C}_B$  with the properties of Corollary 5.2. The converse will be proved after a more detailed study of the properties exhibited in Corollary 5.2.

An object  $P \in {}_A\mathcal{C}$  will be called *finite*, if  ${}_A[P, A]$  and  $B := {}_A[P, P]$  exist, if  $P$  is  $B$ -coflat and  ${}_A[P, A]$  is  $A$ -coflat and if the morphism  ${}_A[P, A] \otimes_A P \rightarrow {}_A[P, P]$  induced by  ${}_A[P, A](X) \times P(Y) \ni (f, p) \mapsto fp \in {}_A[P, P](X \otimes Y)$  with  $\langle p' \rangle fp = (\langle p' \rangle f)p$  is an isomorphism.  $P$  will be called *faithfully projective* if it is finite and if the morphism  $P \otimes_B {}_A[P, A] \rightarrow A$  induced by the evaluation is also an isomorphism.

**5.3. Theorem.** Let  $A, B$  be monoids in  $\mathcal{C}$ ,  $P \in {}_A\mathcal{C}_B$   $B$ -coflat and  $Q \in {}_B\mathcal{C}_A$   $A$ -coflat. Given morphisms  $f: P \otimes_B Q \rightarrow A$  in  ${}_A\mathcal{C}_A$  and  $g: Q \otimes_A P \rightarrow B$  in  ${}_B\mathcal{C}_B$  such that the diagrams

$$\begin{array}{ccc}
 P \otimes_B (Q \otimes_A P) \cong (P \otimes_B Q) \otimes_A P & \xrightarrow{f \otimes_A P} & A \otimes_A P \\
 \downarrow P \otimes_B g & & \downarrow \\
 P \otimes_B B & \xrightarrow{\quad \quad \quad} & P \\
 Q \otimes_A (P \otimes_B Q) \cong (Q \otimes_A P) \otimes_B Q & \xrightarrow{g \otimes_B Q} & B \otimes_B Q \\
 \downarrow Q \otimes_A f & & \downarrow \\
 Q \otimes_A A & \xrightarrow{\quad \quad \quad} & Q
 \end{array}$$

commute. Assume that there is  $p_0 \otimes_B q_0 \in P \otimes_B Q(I)$  such that  $p_0 q_0 := f(p_0 \otimes_B q_0) = 1 \in A(I)$ . Then  $f$  is an isomorphism. Assume that in addition there is  $q_1 \otimes_A p_1 \in Q \otimes_A P(I)$  such that  $q_1 p_1 := g(q_1 \otimes_A p_1) = 1 \in B(I)$ . Then  $P \otimes_B: {}_B\mathcal{C} \rightarrow {}_A\mathcal{C}$  and  $Q \otimes_A: {}_A\mathcal{C} \rightarrow {}_B\mathcal{C}$  are inverse  $\mathcal{C}$ -equivalences. In particular  $P \in {}_A\mathcal{C}$  and  $Q \in {}_B\mathcal{C}$  are faithfully projective.

PROOF. Define  $f': A \rightarrow P \otimes_B Q$  by  $f'(a) = ap_0 \otimes_B q_0$ . Then  $ff'(a) = ap_0 q_0 = a$  and  $f'f(p \otimes_B q) = (pq)p_0 \otimes_B q_0 = p(qp_0) \otimes_B q_0 = p \otimes_B (qp_0)q_0 = p \otimes_B q(p_0 q_0) = p \otimes_B q$ . Hence  $f$  is an isomorphism.

Furthermore the functors  $P \otimes_B Q \otimes_A \cong A \otimes_A$  and  $Q \otimes_A P \otimes_B \cong B \otimes_B$  are both isomorphic to the identity-functors on  ${}_A\mathcal{C}$  resp.  ${}_B\mathcal{C}$ , hence they are inverse equivalences. Furthermore  $P \otimes_B$  and  $Q \otimes_A$  are  $\mathcal{C}$ -functors by Theorem 4.2.

**5.4. Theorem.** Let  $P \in {}_A\mathcal{C}$  be faithfully projective. Then  ${}_A[P, -]: {}_A\mathcal{C} \rightarrow {}_A[P, P]\mathcal{C}$  exists and is a  $\mathcal{C}$ -equivalence.

PROOF. By definition  ${}_A[P, A]$  and  ${}_A[P, P] = B$  exist. Furthermore  $P \in {}_A\mathcal{C}_B$ ,  $Q := {}_A[P, A] \in {}_B\mathcal{C}_A$  and the hypotheses of Theorem 5.3. are satisfied by the very definition of  $Q \otimes_A P \rightarrow B$  and  $P \otimes_B Q \rightarrow A$ . So  $Q \otimes_A: {}_A\mathcal{C} \rightarrow {}_B\mathcal{C}$  is a  $\mathcal{C}$ -equivalence. By Theorem 5.1. we get  $Q \otimes_A \cong {}_A[P, -]$ .

Let us now apply our theorems to the case where the tensor-product in  $\mathcal{C}$  is the (direct) product (example c) of § 1). Furthermore assume that each canonical epimorphism  $M \times N \rightarrow M \times_A N$  induces a surjective map  $M \times N(I) \rightarrow M \times_A N(I)$ . This is for example the case if  $I$  is projective in the category  $\mathcal{C}$ . We say that  $M \times N \rightarrow M \times_A N$  is *rationally surjective*. Assume that  ${}_A\mathcal{C}$  and  ${}_B\mathcal{C}$  are  $\mathcal{C}$ -equivalent by  $P \otimes_B -: {}_B\mathcal{C} \rightarrow {}_A\mathcal{C}$  and  $Q \otimes_A: {}_A\mathcal{C} \rightarrow {}_B\mathcal{C}$ . Then we have surjective maps  $f: P(I) \times Q(I) \cong P \times Q(I) \rightarrow A(I)$  and  $g: Q(I) \times P(I) \cong Q \times P(I) \rightarrow B(I)$  such that  $(pq)p' = p(qp')$  and  $(qp)q' = q(pq')$  if  $f(p, q) = pq$  and  $g(q, p) = qp$ . Let  $p_i \in P(I)$ ,  $q_i \in Q(I)$ ,  $i = 0, 1$  be chosen such that  $p_0 q_0 = 1 \in A(I)$ ,  $q_1 p_1 = 1 \in B(I)$ .

Let us now assume that each element in  $A(I)$  which has a left inverse has also a right inverse. We wish to show  $A \cong B$  as monoids. First we show  $p_1 q_1 = 1 \in A$ . By definition we have  $(p_0 q_1)(p_1 q_0) = p_0 (q_1 p_1) q_0 = p_0 q_0 = 1 \in A(I)$ , hence  $(p_1 q_0)(p_0 q_1) = 1$ . Furthermore we have  $p_0 q_1 p_1 = p_0$ . This implies  $p_1 q_1 = 1 \cdot p_1 q_1 = (p_1 q_0 p_0 q_1) p_1 q_1 = p_1 q_0 (p_0 q_1 p_1) q_1 = p_1 q_0 p_0 q_1 = 1 \in A(I)$ . Now define morphisms  $P(X) \ni p \mapsto pq_1 \in A(X)$  and  $A(X) \ni a \mapsto ap_1 \in P$ . They are obviously mutually inverse morphisms in  ${}_A\mathcal{C}$ . Hence  $B \cong {}_A[P, P] \cong {}_A[A, A] \cong A$ .

As a special case we get

**5.5. Corollary:** *In the category of sets  $\mathcal{S}$  with the product as monoidal category let  $A$  be a group or a commutative monoid or finite. Then  ${}_A\mathcal{S}$  and  ${}_B\mathcal{S}$  are equivalent iff  $A \cong B$ .*

**PROOF.** In  $\mathcal{S}$  the morphism-sets form an inner hom-functor, so by Theorem 4.3. each equivalence  ${}_A\mathcal{S} \cong {}_B\mathcal{S}$  is an  $\mathcal{S}$ -equivalence. Furthermore  $\{\emptyset\}$  is projective in  $\mathcal{S}$ . If  $A$  is a group or commutative or finite then each element which has a left inverse in  $A(I)$  has also a right inverse. So all conditions of the previous discussion are satisfied. Hence  $A \cong B$ . The converse is trivial.

**The central part of the Morita Theorems**

For this section we will always assume that  $\mathcal{C}$  is a symmetric monoidal category. Let us consider  $\mathcal{C}$ -functors  $\mathcal{U}, \mathcal{V}: {}_A\mathcal{C} \rightarrow {}_B\mathcal{C}$  such that there are  $P, Q \in {}_B\mathcal{C}_A$  with  $\mathcal{C}$ -isomorphisms  $\mathcal{U} \cong P \otimes_A, \mathcal{V} \cong Q \otimes_A$ .

Define a new  $\mathcal{C}$ -functor  $\mathcal{U} \otimes Y$  for  $Y \in \mathcal{C}$  by  $\mathcal{U} \otimes Y(M) := \mathcal{U}(M \otimes Y) \cong \cong P \otimes_A(M \otimes Y)$ . Because of the symmetry of  $\mathcal{C}$  we have  $\mathcal{U} \otimes Y(M) \cong (P \otimes Y) \otimes_A M$  hence  $\mathcal{U} \otimes Y$  indeed is a  $\mathcal{C}$ -functor.

Define  $[\mathcal{U}, \mathcal{V}](Y) := \mathcal{C}\text{-Mor}(\mathcal{U} \otimes Y, \mathcal{V})$  as the set of  $\mathcal{C}$ -morphisms from  $\mathcal{U} \otimes Y$  to  $\mathcal{V}$ . For  $h: Z \rightarrow Y$  in  $\mathcal{C}$  define  $[\mathcal{U}, \mathcal{V}](h): [\mathcal{U}, \mathcal{V}](Y) \rightarrow [\mathcal{U}, \mathcal{V}](Z)$  by  $[\mathcal{U}, \mathcal{V}](h)(\varphi)(M) := (\mathcal{U}(M \otimes Z) \xrightarrow{\mathcal{U}(M \otimes h)} \mathcal{U}(M \otimes Y) \xrightarrow{\varphi(M)} \mathcal{V}(M))$ . Then  $[\mathcal{U}, \mathcal{V}]$  is a contravariant functor from  $\mathcal{C}$  to the category of sets.

**6.1. Theorem.** *There is a natural isomorphism of functors from  $\mathcal{C}$  to the category of sets:*

$$[\mathcal{U}, \mathcal{V}](Y) \cong {}_B\mathcal{C}_A(P \otimes Y, Q).$$

If  ${}_B[P, Q]_A$  exists then  $[\mathcal{U}, \mathcal{V}](Y) \cong {}_B[P, Q]_A(Y)$ .

**PROOF.** Let  $f \in {}_B\mathcal{C}_A(P \otimes Y, Q)$ . Then define  $\varphi \in [\mathcal{U}, \mathcal{V}](Y)$  by

$$\varphi(M) := (\mathcal{U}(M \otimes Y) \cong (P \otimes Y) \otimes_A M \xrightarrow{f \otimes_A M} Q \otimes_A M \cong \mathcal{V}(M)).$$

For  $g \in {}_A\mathcal{C}(M, N)$  we get a commutative diagram

$$\begin{array}{ccccc} \mathcal{U}(M \otimes Y) \cong (P \otimes Y) \otimes_A M & \xrightarrow{f \otimes_A M} & Q \otimes_A M \cong \mathcal{V}(M) & & \\ \downarrow \mathcal{U}(g \otimes Y) & & \downarrow Q \otimes_A g & & \downarrow \mathcal{V}(g) \\ \mathcal{U}(N \otimes Y) \cong (P \otimes Y) \otimes_A N & \xrightarrow{f \otimes_A N} & Q \otimes_A N \cong \mathcal{V}(N) & & \end{array}$$

hence  $\varphi$  is a natural transformation from  $\mathcal{U} \otimes Y$  to  $\mathcal{V}$ . Furthermore the following diagram commutes:

$$\begin{array}{ccccccc} \mathcal{U}((M \otimes X) \otimes Y) \cong (P \otimes Y) \otimes_A (M \otimes X) & \xrightarrow{f \otimes_A (M \otimes X)} & Q \otimes_A (M \otimes X) \cong \mathcal{V}(M \otimes X) & & & & \\ \parallel & & \parallel & & \parallel & & \parallel \\ \mathcal{U}(M \otimes Y) \otimes X \cong ((P \otimes Y) \otimes_A M) \otimes X & \xrightarrow{(f \otimes_A M) \otimes X} & (Q \otimes_A M) \otimes X \cong \mathcal{V}(M) \otimes X & & & & \end{array}$$



Hence  $\varphi$  is a  $\mathcal{C}$ -morphism. This defines a map

$$\Sigma : {}_B\mathcal{C}_A(P \otimes Y, Q) \rightarrow [\mathcal{U}, \mathcal{V}](Y).$$

Conversely let  $\varphi \in [\mathcal{U}, \mathcal{V}](Y)$  and define  $f \in {}_B\mathcal{C}_A(P \otimes Y, Q)$  by  $f := (P \otimes Y \cong P \otimes_A A \otimes Y \cong \mathcal{U}(A \otimes Y) \xrightarrow{\varphi(A)} \mathcal{V}(A) \cong Q \otimes_A A \cong Q)$ . Clearly  $f \in {}_B\mathcal{C}(P \otimes Y, Q)$ . To show that  $f \in {}_B\mathcal{C}_A(P \otimes Y, Q)$  consider the following commutative diagram

$$\begin{array}{ccccccc} P \otimes_A (A \otimes Y) \otimes A & \cong & ((P \otimes Y) \otimes_A A) \otimes A & \cong & \mathcal{U}(A \otimes Y) \otimes A & \xrightarrow{\varphi(A) \otimes A} & \mathcal{V}(A) \otimes A \cong (Q \otimes_A A) \otimes A \\ \parallel & & \parallel & & \parallel & & \parallel & \parallel & \parallel \\ (P \otimes Y) \otimes A & \cong & (P \otimes Y) \otimes_A (A \otimes A) & \cong & \mathcal{U}((A \otimes A) \otimes Y) & \xrightarrow{\varphi(A \otimes A)} & \mathcal{V}(A \otimes A) \cong Q \otimes_A (A \otimes A) \cong Q \otimes A \\ \downarrow \nu_{P \otimes Y} & & \downarrow (P \otimes Y) \otimes_A \mu_A & & \downarrow \mathcal{U}(\mu_A \otimes Y) & & \downarrow \mathcal{V}(\mu_A) & \downarrow Q \otimes_A \mu_A & \downarrow \nu_Q \\ P \otimes Y & \cong & (P \otimes Y) \otimes_A A & \cong & \mathcal{U}(A \otimes Y) & \xrightarrow{\varphi(A)} & \mathcal{V}(A) \cong Q \otimes_A A \cong Q \end{array}$$

where the morphism from  $(P \otimes Y) \otimes A$  to  $Q \otimes A$  along the upper side of the diagram is just  $f \otimes A$  and the morphism from  $P \otimes Y$  to  $Q$  along the lower side is  $f$ . Hence  $f$  is a right  $A$ -morphism. So we have a map

$$\Pi : [\mathcal{U}, \mathcal{V}](Y) \rightarrow {}_B\mathcal{C}_A(P \otimes Y, Q).$$

Now

$$\begin{aligned} \Pi \Sigma(f) &= (P \otimes Y \cong \mathcal{U}(A \otimes Y) \xrightarrow{\Sigma(f)(A)} \mathcal{V}(A) \cong Q) = \\ &= (P \otimes Y \cong (P \otimes Y) \otimes_A A \xrightarrow{f \otimes_A A} Q \otimes_A A \cong Q) = f \end{aligned}$$

and

$$\begin{aligned} \Sigma \Pi(\varphi)(M) &= (\mathcal{U}(M \otimes Y) \cong (P \otimes Y) \otimes_A M \xrightarrow{\pi(\varphi) \otimes_A M} Q \otimes_A M \cong \mathcal{V}(M)) = \\ &= (\mathcal{U}(M \otimes Y) \cong \mathcal{U}(A \otimes Y) \otimes_A M \xrightarrow{\varphi(A) \otimes_A M} \mathcal{V}(A) \otimes_A M \cong \mathcal{V}(M)) = \varphi(M) \end{aligned}$$

since  $\varphi$  is a  $\mathcal{C}$ -morphism. Hence we have  $[\mathcal{U}, \mathcal{V}](Y) \cong {}_B\mathcal{C}_A(P \otimes Y, Q)$ .

It remains to show that this isomorphism is a natural transformation. Let  $h: Z \rightarrow Y$  be in  $\mathcal{C}$ . Then

$$\begin{array}{ccc} {}_B\mathcal{C}_A(P \otimes Y, Q) & \xrightarrow{\Sigma(Y)} & [\mathcal{U}, \mathcal{V}](Y) \\ \downarrow {}_B\mathcal{C}_A(P \otimes h, Q) & & \downarrow [\mathcal{U}, \mathcal{V}](h) \\ {}_B\mathcal{C}_A(P \otimes Z, Q) & \xrightarrow{\Sigma(Z)} & [\mathcal{U}, \mathcal{V}](Z) \end{array}$$

commutes since for  $f \in {}_B\mathcal{C}_A(P \otimes Y, Q)$  we have

$$\begin{array}{ccc}
 \mathcal{U}(M \otimes Z) \cong (P \otimes Z) \otimes_A M & & \\
 \downarrow \mathcal{U}(M \otimes h) & \downarrow (P \otimes h) \otimes_A M & \searrow (f(P \otimes h)) \otimes_A M \\
 \mathcal{U}(M \otimes Y) \cong (P \otimes Y) \otimes_A M & \xrightarrow{f \otimes_A M} & Q \otimes_A M \cong \mathcal{V}(M)
 \end{array}$$

commutative and thus

$$\begin{aligned}
 & ([\mathcal{U}, \mathcal{V}](h) \circ \Sigma(Y))(f)(M) = \\
 & = (\mathcal{U}(M \otimes Z) \xrightarrow{\mathcal{U}(M \otimes h)} \mathcal{U}(M \otimes Y) \cong (P \otimes Y) \otimes_A M \xrightarrow{f \otimes_A M} Q \otimes_A M \cong \mathcal{V}(M)) = \\
 & = (\mathcal{U}(M \otimes Z) \cong (P \otimes Z) \otimes_A M \xrightarrow{(f(P \otimes h)) \otimes_A M} Q \otimes_A M \cong \mathcal{V}(M)) = \\
 & = (\Sigma(Z) \circ {}_B\mathcal{C}_A(P \otimes h, Q))(f)(M).
 \end{aligned}$$

For this theorem we have two applications. Before we discuss them, we have to introduce the notion of the center of a monoid.

It is clear that  ${}_A\mathcal{C}(A, A) \cong A(I)$  as monoids in the category of sets. The isomorphism is given by

$${}_A\mathcal{C}(A, A) \ni f \mapsto f(1) = f\eta \in A(I)$$

$$A(I) \ni a \mapsto (A(X) \ni b \mapsto ba \in A(X)) \in {}_A\mathcal{C}(A, A).$$

Now those elements  $a \in A(I)$  which commute with all  $b \in A(X)$  for all  $X$  induce in  ${}_A\mathcal{C}(A, A)$  precisely the  $A-A$ -morphisms  ${}_A\mathcal{C}_A(A, A)$ , which then is a commutative monoid. So a possible definition of the center of  $A$  could be  ${}_A\mathcal{C}_A(A, A)$ . But this is only a set, not an object in  $\mathcal{C}$ . A possible generalization to an object in  $\mathcal{C}$  is  ${}_A[A, A]_A$  if this exists. If it does not exist we know at least the functor represented by this object. So we define the *center of  $A$*  as a functor from  $\mathcal{C}$  to  $\mathcal{S}$ , the category of sets, by  $\text{Cent}(A)(X) := {}_A\mathcal{C}_A(A \otimes X, A)$ . If  ${}_A[A, A]_A$  exists we have  $\text{Cent}(A)(X) \cong {}_A[A, A]_A(X)$ .

As in § 3  $\text{Cent}(A)$  can only be defined in a symmetric monoidal category in contrast to  ${}_A\mathcal{C}_A(A, A)$ . In § 2 we showed that  ${}_I\mathcal{C} \cong \mathcal{C}$  and  $\mathcal{C} \cong \mathcal{C}_I$  hence  $\mathcal{C}(I, I) \cong \cong {}_I\mathcal{C}_I(I, I)$  is a commutative monoid [18, Theorem 1] for (possibly nonsymmetric) monoidal categories.

Let  $A$  be a monoid in a symmetric monoidal category  $\mathcal{C}$ . Let  ${}_A Id: {}_A\mathcal{C} \rightarrow {}_A\mathcal{C}$  denote the identity functor. Then  ${}_A Id$  is clearly a  $\mathcal{C}$ -functor and  ${}_A Id \cong A \otimes_A$  as  $\mathcal{C}$ -functors.

**6.2. Theorem.**  $\text{Cent}(A) \cong [{}_A Id, {}_A Id]$ .

PROOF.  $\text{Cent}(A)(X) = {}_A \mathcal{C}_A(A \otimes X, A) \cong [{}_A Id, {}_A Id](X)$ .

The isomorphism of Theorem 6.2. is only an isomorphism of objects in  $\mathcal{C}$ . But there is an additional structure, a multiplication on these functors. If they were representable the representing objects in  $\mathcal{C}$  would be monoids. The multiplicative structure on  ${}_A[A, A]_A$  as it has been studied in §3 is reflected in  ${}_A \mathcal{C}_A(A \otimes X, A)$  by the commutative diagram

$$\begin{array}{ccc} {}_A[A, A]_A(X) \times {}_A[A, A]_A(Y) & \rightarrow & {}_A[A, A]_A(X \otimes Y) \\ \parallel & & \parallel \\ {}_A \mathcal{C}_A(A \otimes X, A) \times {}_A \mathcal{C}_A(A \otimes Y, A) & \rightarrow & {}_A \mathcal{C}_A(A \otimes X \otimes Y, A) \end{array}$$

where the lower map is given by

$$(f, g) \mapsto (A \otimes X \otimes Y \xrightarrow{f \otimes Y} A \otimes Y \xrightarrow{g} A).$$

The unit is described by

$$I(X) \ni f \mapsto (A \otimes X \xrightarrow{A \otimes f} A \otimes I \cong A) \in {}_A \mathcal{C}_A(A \otimes X, A).$$

$[{}_A Id, {}_A Id]$  carries a multiplicative structure via

$$[{}_A Id, {}_A Id](X) \times [{}_A Id, {}_A Id](Y) \xrightarrow{T} [{}_A Id, {}_A Id](X \otimes Y)$$

by 
$$T(\varphi, \psi) = ({}_A Id \otimes X \otimes Y \xrightarrow{\varphi \otimes Y} {}_A Id \otimes Y \xrightarrow{\psi} {}_A Id)$$

and there is a unit

$$I(X) \ni f \mapsto ({}_A Id \otimes X \xrightarrow{{}_A Id \otimes f} {}_A Id \otimes I \cong {}_A Id) \in [{}_A Id, {}_A Id](X).$$

Using the isomorphism of Theorem 6.1. it is easy to see that they are compatible with the multiplication and the unit map. Hence the isomorphism of Theorem 6.2. is a „monoid isomorphism”.

**6.3. Corollary:** *Let  ${}_A \mathcal{C}$  and  ${}_B \mathcal{C}$  be  $\mathcal{C}$ -equivalent. Then  $\text{Cent}(A) \cong \text{Cent}(B)$  as functors from  $\mathcal{C}$  to  $\mathcal{S}$ . If both functors are representable then the two representing objects are isomorphic as commutative monoids in  $\mathcal{C}$ :*

$${}_A[A, A]_A \cong {}_B[B, B]_B.$$

PROOF. We show  $[{}_A Id, {}_A Id] \cong [{}_B Id, {}_B Id]$ . Let  $\mathcal{F}: {}_A \mathcal{C} \rightarrow {}_B \mathcal{C}$  the given  $\mathcal{C}$ -equivalence. First we show

$$[{}_A Id, {}_A Id](X) \cong [\mathcal{F}, \mathcal{F}](Y),$$

or

$$\mathcal{C}\text{-Mor}({}_A Id \otimes Y, {}_A Id) \cong \mathcal{C}\text{-Mor}(\mathcal{F} \otimes Y, \mathcal{F}).$$

Let  $\varphi: {}_A Id \otimes Y \rightarrow {}_A Id$  be a natural transformation. Define  $\mathcal{F} \circ \varphi: \mathcal{F} \otimes Y \rightarrow \mathcal{F}$  by  $\mathcal{F} \circ \varphi(M): \mathcal{F}(M \otimes Y) \rightarrow \mathcal{F}(M)$  as  $\mathcal{F} \circ \varphi(M) = \mathcal{F}(\varphi(M))$ . Since  $\mathcal{F}$  is an equivalence it is clear that  $\varphi \mapsto \mathcal{F} \circ \varphi$  is a bijection between the sets of natural transformations. Now we show that  $\varphi$  is a  $\mathcal{C}$ -morphism iff  $\mathcal{F} \circ \varphi$  is a  $\mathcal{C}$ -morphism.  $\varphi$  is a  $\mathcal{C}$ -morphism iff the diagrams

$$\begin{array}{ccc}
 M \otimes X \otimes Y & \xrightarrow{\varphi(M \otimes X)} & M \otimes X \\
 \parallel & & \parallel \\
 M \otimes Y \otimes X & \xrightarrow{\varphi(M) \otimes X} & M \otimes X
 \end{array}$$

commute.  $\mathcal{F} \circ \varphi$  is a  $\mathcal{C}$ -morphism iff the outer diagrams of

$$\begin{array}{ccc}
 \mathcal{F}(M \otimes X \otimes Y) & \xrightarrow{\mathcal{F} \circ \varphi(M \otimes X)} & \mathcal{F}(M \otimes X) \\
 \parallel & & \parallel \\
 \mathcal{F}((M \otimes Y) \otimes X) & \xrightarrow{\mathcal{F}(\varphi(M) \otimes X)} & \mathcal{F}(M \otimes X) \\
 \parallel & & \parallel \\
 \mathcal{F}(M \otimes Y) \otimes X & \xrightarrow{\mathcal{F} \circ \varphi(M) \otimes X} & \mathcal{F}(M) \otimes X
 \end{array}$$

commute where the lower part commutes in any case since  $\mathcal{F}$  is a  $\mathcal{C}$ -functor. But the upper part commutes iff the previous diagram commutes. Hence

$$\mathcal{C}\text{-Mor}({}_A Id \otimes Y, {}_A Id) \cong \mathcal{C}\text{-Mor}(\mathcal{F} \otimes Y, \mathcal{F}).$$

Now we show  $[{}_B Id, {}_B Id](Y) \cong [\mathcal{F}, \mathcal{F}](Y)$  or

$$\mathcal{C}\text{-Mor}({}_B Id \otimes Y, {}_B Id) \cong \mathcal{C}\text{-Mor}(\mathcal{F} \otimes Y, \mathcal{F}).$$

It is clear that the correspondence between  $\varphi: {}_B Id \otimes Y \rightarrow {}_B Id$  and  $\varphi \circ \mathcal{F}: \mathcal{F} \otimes Y \rightarrow \mathcal{F}$  with

$$(\varphi \circ \mathcal{F})(M) := (\mathcal{F}(M \otimes Y) \cong \mathcal{F}(M) \otimes Y \xrightarrow{\varphi(\mathcal{F}(M))} \mathcal{F}(M))$$

induces an isomorphism between the sets of natural transformation, since  $\mathcal{F}$  is an equivalence. Furthermore  $\varphi$  is a  $\mathcal{C}$ -morphism iff the diagrams

$$\begin{array}{ccc}
 N \otimes X \otimes Y & \xrightarrow{\varphi(N \otimes X)} & N \otimes X \\
 \parallel & & \parallel \\
 N \otimes Y \otimes X & \xrightarrow{\varphi(N) \otimes X} & N \otimes X
 \end{array}$$

commute. On the other hand  $\varphi \circ \mathcal{F}$  is a  $\mathcal{C}$ -morphism iff the outer diagrams

$$\begin{array}{ccc}
 \mathcal{F}(M \otimes X \otimes Y) & \xrightarrow{\varphi \circ \mathcal{F}(M \otimes X)} & \tilde{\mathcal{F}}(M \otimes X) \\
 \parallel & & \parallel \\
 \mathcal{F}(M \otimes X) \otimes Y & \xrightarrow{\varphi(\tilde{\mathcal{F}}(M \otimes X))} & \tilde{\mathcal{F}}(M \otimes X) \\
 \parallel & & \parallel \\
 \tilde{\mathcal{F}}(M) \otimes Y \otimes X & \xrightarrow{\varphi(\tilde{\mathcal{F}}(M) \otimes X)} & \tilde{\mathcal{F}}(M) \otimes X \\
 \parallel & & \parallel \\
 \tilde{\mathcal{F}}(M \otimes Y) \otimes X & \xrightarrow{\varphi \circ \tilde{\mathcal{F}}(M) \otimes X} & \tilde{\mathcal{F}}(M) \otimes X
 \end{array}$$

commute. The first and third part commute by definition. In the middle part take into account that  $\mathcal{F}$  is a  $\mathcal{C}$ -functor. Then it commutes iff the previous diagram for  $\varphi$  commutes. Hence  $[_B Id, _B Id](Y) \cong [\mathcal{F}, \tilde{\mathcal{F}}](Y) \cong [_A Id, _A Id](Y)$ .

The reader can easily verify that these isomorphisms are natural isomorphisms in  $Y$ . Furthermore they preserve the „multiplication” given by composition of morphisms just before Corollary 6.3. They also preserve the „unit”. Hence

$${}_A[A, A]_A \cong {}_B[B, B]_B$$

as monoids (if they exist) or

$$\text{Cent}(A) \cong \text{Cent}(B)$$

with the multiplicative structure.

**6.4. Corollary:** Let  $\mathcal{U}: {}_A\mathcal{C} \rightarrow \mathcal{C}$  be the underlying functor. Then  $[\mathcal{U}, \mathcal{U}](Y) \cong \cong A^{\text{op}}(Y)$  natural in  $Y \in \mathcal{C}$  and compatible with the multiplication on both sides.

PROOF. By Theorem 6.1. and the fact  $\mathcal{U} \cong A \otimes_A$  we have  $[\mathcal{U}, \mathcal{U}](Y) \cong \cong \mathcal{C}_A(A \otimes Y, A) \cong A^{\text{op}}(Y)$  as left multiplications and these isomorphisms are natural in  $Y$  and compatible with the multiplication.

So we have seen that just from the knowledge of the underlying functor  $\mathcal{U}$  we may regain the monoid  $A$  up to an isomorphism.

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