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MORITA EQUIVALENCE OF MODULE CATEGORIES WITH TENSOR PRODUCTS

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Let k be a commutative ring. If B is a k-bialgebra then the category B-Mod of left B-modules is a monoidal category with the tensor-product $M \otimes_k N$ of two B-modules made again into a B-module by $b(m \otimes n) = \sum b_{(1)}m \otimes b_{(2)}n$, where $\sum b_{(1)} \otimes b_{(2)} = \Delta(b)$ and Δ the diagonal of B. The neutral object of this monoidal structure is k with the trivial action $\varepsilon: B \longrightarrow k$.

Given two k-bialgebras B and B' we want to investigate those equivalences B-Mod $\cong$ B'-Mod which preserve the tensor products in both categories or more precisely those equivalences which are (C-)monoidal functors.

A more general situation would be to study equivalences of module categories A-Mod $\cong$ A'-Mod which preserve arbitrarily given monoidal structures on A-Mod resp. A'-Mod, where A and A' are k-algebras. If A and A' are commutative then one could consider for example the monoidal structures given by the tensor products over A resp. A' . But in this case any equivalence $A\text{-Mod} \cong A'\text{-Mod}$ would already imply $A \cong A'$, since A and A' were assumed to be commutative.

In [4] we showed that a monoidal structure on A-Mod which is preserved by the underlying functor $\mathcal{U}\colon A\text{-Mod} \longrightarrow k\text{-Mod}$ comes already from a bialgebra structure on A and is essentially the monoidal structure described above for B-Mod.

Thus we shall restrict our attention to (C-)monoidal equivalences $B\text{-Mod} \cong B'\text{-Mod}$ where B and B' are k-bialgebras. Since the same problem may be posed for the categories of comodules $B\text{-Comod} \cong B'\text{-Comod}$ (the bialgebra structure of B induces also here a C-monoidal structure) we shall phrase all statements and proofs in the language of monoidal categories as developed e.g. in [2, 3, 5].

The advantage of this procedure lies primarily in the fact that many statements can be more easily proved in the general categorical form than in the concrete example of comodules over coalgebras. This can be seen in the fact that there is no direct proof for the statement that a C-monoidal underlying functor $\mathcal{U}\colon B\text{-Comod} \longrightarrow k\text{-Mod}$ determines the bialgebra structure of B uniquely up to isomorphisms [4]. Similarly we do not have direct proofs for most of the results of this paper in the special case of comodules and coalgebras.

Let $\mathcal{C}$ be a symmetric monoidal category and let B resp. B' be bimonoids in $\mathcal{C}$. Then ${}_B\mathcal{C}$ is a $\mathcal{C}$-monoidal category and $\mathcal{U}: {}_B\mathcal{C} \longrightarrow \mathcal{C}$ is a $\mathcal{C}$-monoidal functor [4].

A functor $F: \mathcal{D} \longrightarrow E$ between two $\mathcal{C}$-monoidal categories $\mathcal{D}$ and E (with tensor products $\hat{\otimes}$ for $\mathcal{D}$ and $\tilde{\otimes}$ for E) is called a _weakly_ $\mathcal{C}$-_monoidal_ _functor_ if

1) F is a $\mathcal{C}$-functor with natural isomorphism

$\xi: F(M \otimes X) \cong F(M) \otimes X$, $M \in \mathcal{D}$, $X \in \mathcal{C}$

2) F is a weakly monoidal functor with natural transformations

$\delta: F(M \hat{\otimes} N) \longrightarrow F(M) \tilde{\otimes} F(N)$, $M,N \in \mathcal{D}$

$\zeta: F(\hat{I}) \longrightarrow \tilde{I}$

3) the given data are coherent in the sense of [1, 2, 5].

PROPOSITION 1. _Let_ B, B' _be bimonoids in_ $\mathcal{C}$ _and let_ $F: {}_B\mathcal{C} \longrightarrow {}_{B'}\mathcal{C}$ _be a covariant functor. Equivalent are_

a) F _is a weakly_ $\mathcal{C}$-_monoidal functor which preserves difference cokernels of_ $\mathcal{U}$-_contractible pairs._

b) _There is a_ B'-B-_biobject_ P _which is_ B-_coflat and a_ B'-B-_comonoid such that_ $F \cong P \otimes_B$ _is a natural isomorphism._

Before we prove this proposition we have to define a B'-B-comonoid. A comonoid P in $\mathcal{C}$ with $\Delta\colon P \longrightarrow P \otimes P$ and $\varepsilon\colon P \longrightarrow I$ is called a B'-B-comonoid if P is in ${}_{B'}\mathcal{C}_B$ and so are the structure morphisms Δ and ε. Here we use the $\mathcal{C}$-monoidal structures on ${}_{B'}\mathcal{C}$ resp. $\mathcal{C}_B$ (and hence on ${}_{B'}\mathcal{C}_B$) as defined above.

Proof of Proposition 1: Applying Theorem 4.2 of [2] we only have to compare the comonoid structure of P and the weakly monoidal structure of $\mathcal{F}$. Assume first that $\mathcal{F}$ satisfies a) and P is given with $\mathcal{F} \cong P \otimes_B$. Actually $P = \mathcal{F}(B)$ as a left B'-object and since $\mathcal{F}$ is a $\mathcal{C}$-functor we get a right B-structure on P compatible with B'-multiplication by

$\mathcal{F}(B) \otimes B \cong \mathcal{F}(B \otimes B) \xrightarrow{\mathcal{F}(\mu)} \mathcal{F}(B)$ hence $P \otimes B \longrightarrow P$

which makes P a B'-B-biobject. Observe that $B \mathbin{\hat{\otimes}} B$ is in ${}_B\mathcal{C}$ (this object should be distinguished from $B \otimes B$ in ${}_B\mathcal{C}$ where $b\cdot(b' \otimes b'') = bb' \otimes b''$) and that $\Delta\colon B \longrightarrow B \mathbin{\hat{\otimes}} B$ as well as $\varepsilon\colon B \longrightarrow \hat{I}$ are morphisms in ${}_B\mathcal{C}_B$. Hence $\mathcal{F}(\Delta)$ and $\mathcal{F}(\varepsilon)$ are in ${}_{B'}\mathcal{C}$ and induce by the weakly monoidal structure of $\mathcal{F}$ morphisms

$$\Delta\colon P = \mathcal{F}(B) \longrightarrow \mathcal{F}(B \mathbin{\hat{\otimes}} B) \longrightarrow \mathcal{F}(B) \mathbin{\tilde{\otimes}} \mathcal{F}(B) = P \mathbin{\tilde{\otimes}} P$$

$$\varepsilon\colon P = \mathcal{F}(B) \longrightarrow \mathcal{F}(\hat{I}) \longrightarrow \tilde{I} .$$

in ${}_{B'}\mathcal{C}$. It is easy to see from the coherence conditions of weakly $\mathcal{C}$-monoidal functors that (P,Δ,ε) is a comonoid. Since $B \mathbin{\hat{\otimes}} B$ is also in $\mathcal{C}_B$ we get a commutative diagram

$$\begin{tikzcd}
F(B)\otimes B \arrow[d] \arrow[r, phantom, "\cong"] & F(B\otimes B) \arrow[r] \arrow[d] & F(B) \arrow[dd] \\
F(B\hat{\otimes} B)\otimes B \arrow[dddd] \arrow[r, phantom, "\cong"] & F((B\hat{\otimes} B)\otimes B) \arrow[d] & \\
& F((B\otimes B)\hat{\otimes}(B\otimes B)) \arrow[r] \arrow[d] & F(B\hat{\otimes} B) \arrow[dd] \\
& F(B\otimes B)\tilde{\otimes} F(B\otimes B) \arrow[d, phantom, "\cong" rotate=90] \arrow[dr] & \\
(F(B)\tilde{\otimes} F(B))\otimes B \arrow[r] & (F(B)\otimes B)\tilde{\otimes}(F(B)\otimes B) \arrow[r] & F(B)\tilde{\otimes} F(B)
\end{tikzcd}$$

in $_{B'}C$ hence $\Delta: P \longrightarrow P \tilde{\otimes} P$ is also in C_B. The diagram

$$\begin{tikzcd}
F(B)\otimes B \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & F(\hat{I})\otimes B \arrow[rr] \arrow[d, phantom, "\cong" rotate=90] \arrow[ddr] & & \tilde{I}\otimes B \arrow[dd] \\
F(B\otimes B) \arrow[r] \arrow[d] & F(\hat{I}\otimes B) \arrow[d] & & \\
F(B) \arrow[dr] & F(\hat{I}\otimes I) \arrow[r, phantom, "\cong"] \arrow[d, phantom, "\cong" rotate=90] & F(\hat{I})\otimes I \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \tilde{I}\otimes I \arrow[d, phantom, "\cong" rotate=90] \\
& F(\hat{I}) \arrow[r, phantom, "="] & F(\hat{I}) \arrow[r] & \tilde{I}
\end{tikzcd}$$

commutes hence $\varepsilon: P \longrightarrow \tilde{I}$ is in C_B. Thus P is a B'-B-comonoid.

Assume now that P satisfies condition b). Again by Theorem 4.2 of [2] we know that F is a C-functor which preserves difference cokernels of U-contractible pairs. In order to prove that F is weakly monoidal we first prove

LEMMA 2. *Let* B *and* B' *be bimonoids in* C *and* $P \in {}_{B'}C_B$. *Then* ${}_{B'}C(P \otimes_B I, I) \cong {}_{B'}C_B(P,I)$.

<u>Proof</u>: Consider the commutative difference cokernel diagram in ${}_{B'}\mathcal{C}$

$$\begin{tikzcd}
P \otimes B \otimes I \arrow[r, shift left] \arrow[r, shift right] & P \otimes I \arrow[r] & P \otimes_B I \\
P \otimes B \arrow[u, phantom, "\cong" rotate=90] \arrow[r, shift left, "\mathrm{mult}"] \arrow[r, shift right, "1 \otimes \varepsilon"'] & P \arrow[u, phantom, "\cong" rotate=90] \arrow[r, "\omega"] & P \otimes_B I \arrow[u, phantom, "\cong" rotate=90]
\end{tikzcd} .$$

Given $\Psi \in {}_{B'}\mathcal{C}(P \otimes_B I)$ we get a diagram

$$\begin{tikzcd}
P \otimes B \arrow[r, shift left] \arrow[r, shift right] & P \arrow[r, "\omega"] \arrow[dr, "\underline{\Psi}"'] & P \otimes_B I \arrow[d, "\Psi"] \\
& & I
\end{tikzcd} \qquad (*)$$

and $\Psi(pb) = \Psi\omega(pb) = \Psi\omega(p\varepsilon(b)) = \underline{\Psi}(p\varepsilon(b)) = \underline{\Psi}(p)\varepsilon(b) =$ $= \underline{\Psi}(p)b$, furthermore $\underline{\Psi} \in {}_{B'}\mathcal{C}$ since ω and Ψ are. So $\underline{\Psi} \in {}_{B'}\mathcal{C}_B(P,I)$.

Conversely given $\underline{\Psi} \in {}_{B'}\mathcal{C}_B(P,I)$, we get $\underline{\Psi}(pb) =$ $\underline{\Psi}(p)b = \overline{\Psi}(p)\varepsilon(b) = \underline{\Psi}(p\varepsilon(b))$ hence a unique factorization through ω in ${}_B\mathcal{C}$ in the diagram (*).

Now we return to the proof of Proposition 1. The counit $\varepsilon: P \longrightarrow I$ in ${}_{B'}\mathcal{C}_B$ induces $e: P \otimes_B I \longrightarrow I$ by Lemma 2 and thus

$$\zeta: F(\hat{I}) \cong P \otimes_B \hat{I} \xrightarrow{e} \tilde{I} .$$

Furthermore the diagonal $\Delta: P \longrightarrow P \tilde{\otimes} P$ in ${}_{B'}\mathcal{C}_B$ induces

$$F(M \hat{\otimes} N) \cong P \otimes_B (M \hat{\otimes} N) \longrightarrow (P \tilde{\otimes} P) \otimes_B (M \hat{\otimes} N) \xrightarrow{T}$$
$$(P \otimes_B M) \tilde{\otimes} (P \otimes_B N) \cong F(M) \tilde{\otimes} F(N) .$$

The morphism $T \in {}_{B'}\mathcal{C}$ is defined by

$$T((p \tilde{\otimes} p') \otimes (m \hat{\otimes} n)) = (p \otimes_B m) \tilde{\otimes} (p' \otimes_B n) .$$

Observe that $T((p \tilde{\otimes} p')b \otimes (m \otimes n)) =$
$T((pb_{(1)} \otimes p'b_{(2)}) \otimes (m \otimes n)) =$
$(pb_{(1)} \otimes_B m) \tilde{\otimes} (p'b_{(2)} \otimes_B n) =$
$(p \otimes_B b_{(1)}m) \tilde{\otimes} (p' \otimes b_{(2)}n) =$
$T((p \tilde{\otimes} p') \otimes (b_{(1)}m \hat{\otimes} b_{(2)}n)) =$
$T((p \tilde{\otimes} p') \otimes b(m \hat{\otimes} n))$ hence T is defined on $(P \tilde{\otimes} P) \otimes_B (M \hat{\otimes} N)$. Obviously T is a natural transformation in $M,N \in {}_B\mathcal{C}$, thus $\delta\colon \mathcal{F}(M \hat{\otimes} N) \longrightarrow \mathcal{F}(M) \tilde{\otimes} \mathcal{F}(N)$ is a natural transformation. Using the definition of T it is straightforward to see that the coherence diagrams for a weakly $\mathcal{C}$-monoidal functor (as given in [4] for a $\mathcal{C}$-monoidal functor) commute.

Before we show that the two constructions for going from a) to b) resp. from b) to a) are inverse to each other, we need another Lemma.

LEMMA 3. *Let* $\mathcal{F}$ *be as in Proposition* 1 *represented by* $P \otimes_B$. *Then the diagram*

$$\begin{array}{ccc} P & \xrightarrow{\omega} & P \otimes_B I \\ \| & & \wr\| \\ \mathcal{F}(B) & \xrightarrow{\mathcal{F}(\varepsilon)} & \mathcal{F}(I) \end{array}$$

commutes.

Proof: Since $B \otimes B \otimes I \rightrightarrows B \otimes I \xrightarrow{\varepsilon \otimes 1} I$ is a difference cokernel of a $\mathcal{U}$-contractible pair, it is preserved by $\mathcal{F}$, hence we have a commutative diagram with difference cokernels as rows

$$\begin{tikzcd}
P \otimes B \arrow[r, shift left] \arrow[r, shift right] \arrow[d, equal] & P \arrow[r, "\omega"] \arrow[d, equal] & P \otimes_B I \arrow[ddd, "\cong"] \\
F(B) \otimes B \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\cong"] & F(B) \arrow[d, equal] & \\
F(B \otimes B) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\cong"] & F(B) \arrow[d, "\cong"] \arrow[dr, "F(\varepsilon)"] & \\
F(B \otimes B \otimes I) \arrow[r, shift left] \arrow[r, shift right] & F(B \otimes I) \arrow[r, "F(\varepsilon \otimes 1)"'] & F(I)
\end{tikzcd}$$

Completion of the proof of Proposition 1: First we consider the correspondence between $\varepsilon: P \longrightarrow I$ and $\zeta: F(\hat{I}) \longrightarrow \tilde{I}$. The construction $\varepsilon \longmapsto \zeta \longmapsto \varepsilon'$ gives

$$\varepsilon' = (P = F(B) \longrightarrow F(I) \cong P \otimes_B I \xrightarrow{e} I) =$$
$$(P \xrightarrow{\omega} P \otimes_B I \xrightarrow{e} I) = \varepsilon$$

where we used Lemma 3 and the correspondence between ε and e following Lemma 2. The construction $\zeta \longmapsto \varepsilon \longmapsto \zeta'$ gives

$$\varepsilon = (P = F(B) \longrightarrow F(I) \xrightarrow{\zeta} I) =$$
$$(P \xrightarrow{\omega} P \otimes_B I \cong F(I) \xrightarrow{\zeta} I)$$

by Lemma 3 and hence

$$\zeta' = (F(I) \cong P \otimes_B I \cong F(I) \xrightarrow{\zeta} I) = \zeta \ .$$

Now we consider $\Delta_P \longmapsto \delta \longmapsto \Delta'_P$ and get

$$\Delta'_P = (P = F(B) \longrightarrow F(B \hat{\otimes} B) \cong$$
$$P \otimes_B (B \hat{\otimes} B) \xrightarrow{\Delta \otimes 1} (P \tilde{\otimes} P) \otimes_B (B \hat{\otimes} B)$$
$$\xrightarrow{T} (P \otimes_B B) \tilde{\otimes} (P \otimes_B B) \cong P \tilde{\otimes} P)$$
$$= (P \cong P \otimes_B B \longrightarrow (P \tilde{\otimes} P) \otimes_B (B \hat{\otimes} B) \xrightarrow{T}$$
$$(P \otimes_B B) \tilde{\otimes} (P \otimes_B B) \cong P \tilde{\otimes} P) = \Delta_P$$

by elementwise computation. To see that also

$$
\begin{tikzcd}
\mathcal{F}(B \otimes_B (M \hat{\otimes} N)) \arrow[dr] \arrow[dddd, "\wr"'] \arrow[ddr, phantom, "\mathrm{(I)}"] & & \mathcal{F}(B) \otimes_B (M \hat{\otimes} N) \arrow[ll, "\sim"'] \arrow[d] \arrow[dl, phantom, "\mathrm{(II)}"] & \\
& \mathcal{F}((B \hat{\otimes} B) \otimes_B (M \hat{\otimes} N)) \arrow[d, "\mathcal{F}(S)"] \arrow[dr, phantom, "\mathrm{(III)}"] & \mathcal{F}(B \hat{\otimes} B) \otimes_B (M \hat{\otimes} N) \arrow[l] \arrow[r] \arrow[d, "S"] \arrow[dr, phantom, "\mathrm{(IV)}"] & (\mathcal{F}(B) \tilde{\otimes} \mathcal{F}(B)) \otimes_B (M \hat{\otimes} N) \arrow[d, "S"'] \\
& \mathcal{F}(((B \hat{\otimes} B) \otimes_B M) \otimes_B N) \arrow[d] & (\mathcal{F}(B \hat{\otimes} B) \otimes_B M) \otimes_B N \arrow[l] \arrow[r] \arrow[d, phantom, "\mathrm{(V)}"] & ((\mathcal{F}(B) \tilde{\otimes} \mathcal{F}(B)) \otimes_B M) \otimes_B N \arrow[d] \\
& \mathcal{F}((B \otimes_B M) \hat{\otimes} (B \otimes_B N)) \arrow[r] \arrow[dl, "\cong"] \arrow[dr, phantom, "\mathrm{(VI)}"] & \mathcal{F}(B \otimes_B M) \tilde{\otimes} \mathcal{F}(B \otimes_B N) \arrow[d, "\wr"] & (\mathcal{F}(B) \otimes_B M) \tilde{\otimes} (\mathcal{F}(B) \otimes_B N) \arrow[l] \\
\mathcal{F}(M \hat{\otimes} N) \arrow[rr, "\delta"] & & \mathcal{F}(M) \tilde{\otimes} \mathcal{F}(N) &
\end{tikzcd}
$$

$$
\begin{tikzcd}
P \otimes B \otimes (P \otimes B \otimes (M \otimes N)) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & P \otimes B \otimes (P \otimes (M \otimes N)) \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & P \otimes B \otimes (P \otimes_B (M \otimes N)) \arrow[d, shift left] \arrow[d, shift right] \\
P \otimes (P \otimes B \otimes (M \otimes N)) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & P \otimes (P \otimes (M \otimes N)) \arrow[r] \arrow[d] & P \otimes (P \otimes_B (M \otimes N)) \arrow[d] \\
P \otimes_B (P \otimes B \otimes (M \otimes N)) \arrow[r, shift left] \arrow[r, shift right] & P \otimes_B (P \otimes (M \otimes N)) \arrow[r] & P \otimes_B (P \otimes_B (M \otimes N))
\end{tikzcd}
$$

FIG. 1

$\delta \longmapsto \Delta_P \longmapsto \delta'$ gives the identity we must show that the diagram given in FIG. 1 commutes.

If this diagram commutes we get

$$\delta' = (F(M \hat{\otimes} N) \cong P \otimes_B (M \hat{\otimes} N) = F(B) \otimes_B (M \hat{\otimes} N) \longrightarrow$$
$$F(B \hat{\otimes} B) \otimes_B (M \hat{\otimes} N) \longrightarrow (F(B) \tilde{\otimes} F(B)) \otimes_B (M \hat{\otimes} N) =$$
$$(P \tilde{\otimes} P) \otimes_B (M \hat{\otimes} N) \xrightarrow{T} (P \otimes_B M) \tilde{\otimes} (P \otimes_B N) \cong$$
$$F(M) \tilde{\otimes} F(N))$$

(which is going around the diagram along the upperside)

$$= (F(M \hat{\otimes} N) \xrightarrow{\delta} F(M) \tilde{\otimes} F(N)) = \delta \ .$$

The commutativity of the diagram (FIG. 1) will be shown in several steps as marked in the diagram. First we investigate the properties of the morphism S . It is given by $S(q \otimes (m \otimes n)) = (q \otimes_B m) \otimes_B n$ for $q \in Q \in {}_{B'}\mathcal{C}_{B \otimes B}$ because $S(qb \otimes (m \otimes n)) =$ $(qb \otimes_B m) \otimes_B n = (q(b_{(1)} \otimes b_{(2)}) \otimes_B m) \otimes_B n =$ $(q \otimes_B b_{(1)}m) \otimes_B b_{(2)}n = S(q \otimes (b_{(1)}m \otimes b_{(2)}n)) =$ $S(q \otimes b(m \otimes n))$. Obviously S is functorial in Q , M , and N and its composition with the morphism $((F(B) \tilde{\otimes} F(B)) \otimes_B M) \otimes_B N \longrightarrow (F(B) \otimes_B M) \tilde{\otimes} (F(B) \otimes_B N)$ gives T as defined earlier. The last morphism is defined by

$$((P \tilde{\otimes} P) \otimes_B M) \otimes N \longrightarrow ((P \otimes_B M) \tilde{\otimes} P) \otimes N \cong$$
$$(P \otimes_B M) \tilde{\otimes} (P \otimes N) \longrightarrow (P \otimes_B M) \tilde{\otimes} (P \otimes_B N) \ .$$

(I) The following diagram commutes as can be verified by elementwise computation:

$$\begin{tikzcd}
B \otimes_B (M \hat{\otimes} N) \arrow[r, "\Delta \otimes 1"] \arrow[dd, "2"] & (B \hat{\otimes} B) \otimes_B (M \hat{\otimes} N) \arrow[d, "S"] \\
 & ((B \hat{\otimes} B) \otimes_B M) \otimes N \arrow[d] \\
M \hat{\otimes} N \arrow[r, phantom, "\cong"] & (B \otimes_B M) \hat{\otimes} (B \otimes_B N) \; .
\end{tikzcd}$$

Applying F gives (I).

(II) The horizontal morphisms are defined on difference cokernels hence they are functorial in the first component.

(III) commutes by using that $F(B \hat{\otimes} B) \otimes_B (M \hat{\otimes} N)$ is a difference cokernel, F is a C-functor and the definition of S, which is essentially a coherence morphism.

(IV) commutes since S is functorial in the first variable.

(V) commutes by coherence before taking cokernels over B so (V) commutes itself.

(VI) commutes since δ is a natural transformation.

PROPOSITION 4. <u>Given bimonoids</u> B <u>and</u> B' <u>in the symmetric monoidal category</u> C . <u>Assume that there are weakly</u> C-<u>monoidal inverse equivalences</u> $F\colon {}_BC \longrightarrow {}_{B'}C$ <u>and</u> $G\colon {}_{B'}C \longrightarrow {}_BC$. <u>Then</u> F <u>and</u> G <u>are</u> C-<u>monoidal functors</u>.

<u>Proof</u>: By definition F and G are weakly C-monoidal equivalences if there are C-monoidal isomorphisms $\Psi\colon GF \cong \mathrm{Id}$ and $\Phi\colon FG \cong \mathrm{Id}$. Thus the

diagram

$$\begin{tikzcd}
\mathcal{F}\mathcal{G}(M \tilde{\otimes} N) \arrow[r, "\mathcal{F}(\bar{\delta})"] \arrow[d, "\cong\ \Phi"'] & \mathcal{F}(\mathcal{G}(M) \hat{\otimes} \mathcal{G}(N)) \arrow[r, "\delta"] & \mathcal{F}\mathcal{G}(M) \tilde{\otimes} \mathcal{F}\mathcal{G}(N) \arrow[d, "\cong\ \Phi \otimes \Phi"] \\
M \tilde{\otimes} N \arrow[rr, "\mathrm{id}"] & & M \tilde{\otimes} N
\end{tikzcd}$$

commutes, i.e. there is a right inverse τ for the δ in the diagram with $\delta\tau = \mathrm{id}$. Furthermore $\mathcal{F}(\bar{\delta})$ is a monomorphism for all $M,N \in {}_{B'}\mathcal{C}$. Similarly $\mathcal{G}(\delta)$ is a monomorphism and $\mathcal{G}(\delta)\mathcal{G}(\tau) = \mathrm{id}$ so that $\mathcal{G}(\tau)\mathcal{G}(\delta) = \mathrm{id}$. Now $\mathcal{G}$ is full hence $\tau\delta = \mathrm{id}$ and thus

$$\delta\colon \mathcal{F}(\mathcal{G}(M) \hat{\otimes} \mathcal{G}(N)) \longrightarrow \mathcal{F}\mathcal{G}(M) \tilde{\otimes} \mathcal{F}\mathcal{G}(N)$$

is an isomorphism. Thus in the above diagram $\mathcal{F}(\bar{\delta})$ is an isomorphism. Since $\mathcal{F}$ reflects isomorphisms we get $\bar{\delta}\colon \mathcal{G}(M \tilde{\otimes} N) \longrightarrow \mathcal{G}(M) \hat{\otimes} \mathcal{G}(N)$ an isomorphism. By symmetry also δ is an isomorphism in general.

For the morphism $\zeta\colon \mathcal{F}(\hat{I}) \longrightarrow \tilde{I}$ and $\bar{\zeta}\colon \mathcal{G}(\tilde{I}) \longrightarrow \hat{I}$ we get a commutative diagram

$$\begin{tikzcd}
\mathcal{F}\mathcal{G}(\tilde{I}) \arrow[r, "\mathcal{F}(\bar{\zeta})"] \arrow[d, "\cong\ \Phi"'] & \mathcal{F}(\hat{I}) \arrow[r, "\zeta"] & \tilde{I} \arrow[d, equal] \\
\tilde{I} \arrow[rr, "1"] & & \tilde{I}
\end{tikzcd}$$

hence ζ is a retraction and $\mathcal{F}(\bar{\zeta})$ is a monomorphism. Thus $\mathcal{G}(\zeta)$ is a retraction and a monomorphism, i.e. an isomorphism. So ζ is an isomorphism.

COROLLARY 5. *Let* ${}_B\mathcal{C}$ *and* ${}_{B'}\mathcal{C}$ *be* $\mathcal{C}$-*monoidally equivalent induced by the* B'-B-*progenerator and* B'-B-*coalgebra* P . *Then*

i) <u>there is an isomorphism</u> $P \otimes_B I \cong I$ <u>in</u> $\mathcal{C}_B$ <u>such that the diagram</u>

$$(1)\quad \begin{array}{ccc} P & \xrightarrow{\varepsilon_P} & I \\ \wr\| & & \wr\| \\ P \otimes_B B & \xrightarrow{P \otimes \varepsilon} & P \otimes_B I \end{array}$$

<u>commutes</u>

ii) <u>there is an isomorphism</u> $P \otimes_B (B \hat{\otimes} B) \cong P \otimes P$ <u>in</u> $\mathcal{C}_{B \otimes B}$ <u>such that the diagram</u>

$$(2)\quad \begin{array}{ccc} P & \xrightarrow{\Delta_P} & P \otimes P \\ \wr\| & & \wr\| \\ P \otimes_B B & \xrightarrow{P \otimes \Delta} & P \otimes_B (B \hat{\otimes} B) \end{array}$$

<u>commutes</u>.

<u>Proof</u>: The morphism $P \otimes_B I \cong I$ is just ζ for the $\mathcal{C}$-monoidal functor $P \otimes_B$. The morphism $P \otimes_B (B \otimes B) \cong P \otimes P$ is the isomorphism $P \otimes_B (B \hat{\otimes} B) \cong (P \otimes_B B) \tilde{\otimes} (P \otimes_B B) \cong P \tilde{\otimes} P$ and the diagram commutes by the definition of Δ_P .

In order to show that the conditions in Corollary 5 are also sufficient that ${}_B\mathcal{C}$ and ${}_{B'}\mathcal{C}$ are $\mathcal{C}$-monoidally equivalent let us assume that B is a bimonoid and A is a monoid in the symmetric monoidal category $\mathcal{C}$. Assume further that a $\mathcal{C}$-equivalence between ${}_B\mathcal{C}$ and ${}_A\mathcal{C}$ is given by $F: {}_B\mathcal{C} \longrightarrow {}_A\mathcal{C}$, $G: {}_A\mathcal{C} \longrightarrow {}_B\mathcal{C}$ and $\Phi: FG \cong \mathrm{Id}$, $\Psi: GF \cong \mathrm{Id}$.

LEMMA 6. <u>Let</u> C <u>be a monoid in</u> $\mathcal{C}$. <u>Then the</u> $\mathcal{C}$-<u>equivalence</u> $F: {}_B\mathcal{C} \longrightarrow {}_A\mathcal{C}$ <u>induces a</u> $\mathcal{C}$-<u>equivalence</u> $F_C: {}_B\mathcal{C}_C \longrightarrow {}_A\mathcal{C}_C$.

Proof: By Lemma 4.1 of [2] the functors F and G induce functors $F_C: {}_B\mathcal{C}_C \longrightarrow {}_A\mathcal{C}_C$ and $G_C: {}_A\mathcal{C}_C \longrightarrow {}_B\mathcal{C}_C$ By the symmetry of $\mathcal{C}$ and the coherence the functors clearly are $\mathcal{C}$-functors. The morphism $\Phi: FG \cong \mathrm{Id}$ induces a $\mathcal{C}$-isomorphism $\Phi_C: F_C G_C \cong \mathrm{Id}_C$ since the diagram

$$\begin{array}{ccccc} FG(M) \otimes C & \cong & FG(M \otimes C) & \longrightarrow & FG(M) \\ \wr\| \ \Phi(M) \otimes C & & \wr\| \ \Phi(M \otimes C) & & \wr\| \ \Phi \\ M \otimes C & = & M \otimes C & \longrightarrow & M \end{array}$$

in ${}_A\mathcal{C}$ commutes. By a symmetric argument we get that F_C is a $\mathcal{C}$-equivalence.

LEMMA 7. *Let* $F: \mathcal{D} \longrightarrow \mathcal{D}'$ *be a* $\mathcal{C}$-*functor between* $\mathcal{C}$-*categories which preserves difference cokernels. Let* C *be a monoid in* $\mathcal{C}$. *Then there is a natural isomorphism of functors in* $\mathcal{D}'$

$$F(M \otimes_C N) \cong F(M) \otimes_C N$$

where $M \in \mathcal{D}_C$ *and* $N \in {}_C\mathcal{C}$.

Proof: By definition $M \otimes C \otimes N \rightrightarrows M \otimes N \longrightarrow M \otimes_C N$ is a difference cokernel in $\mathcal{D}$. Hence the following commutative diagram of difference cokernels

$$\begin{array}{ccccc} F(M \otimes C \otimes N) & \rightrightarrows & F(M \otimes N) & \longrightarrow & F(M \otimes_C N) \\ \wr\| & & \wr\| & & \wr\| \\ F(M) \otimes C \otimes N & \rightrightarrows & F(M) \otimes N & \longrightarrow & F(M) \otimes_C N \end{array}$$

where the last isomorphism clearly is functorial in M and N .

COROLLARY 8. *If* $P \in {}_A\mathcal{C}_B$, $M \in {}_B\mathcal{C}_C$, $N \in {}_C\mathcal{C}_D$ *and if* $P \otimes_B : {}_B\mathcal{C}_D \longrightarrow {}_A\mathcal{C}_D$ *perserves difference cokernels and if* P *is* B-*coflat then*

$$P \otimes_B (M \otimes_C N) \cong (P \otimes_B M) \otimes_C N \quad \textit{in} \quad {}_A\mathcal{C}_D .$$

Proof: This is essentially Lemma 7, except that the diagram in the proof of Lemma 7 is a diagram in ${}_A\mathcal{C}_D$.

LEMMA 9. *Let* $P \in {}_A\mathcal{C}_B$ *be* B-*coflat. Let* $M \in {}_B\mathcal{C}_C$. *Then* $P \otimes_B M$ *defined in* ${}_A\mathcal{C}$ *has a natural* C-*right structure and*

$$P \otimes B \otimes M \rightrightarrows P \otimes M \longrightarrow P \otimes_B M$$

is a difference cokernel in ${}_A\mathcal{C}_C$.

Proof: By the fact that P is B-coflat we have a natural isomorphism $(P \otimes_B M) \otimes C \cong P \otimes_B (M \otimes C)$. From Lemma 4.1 of [2] , we know that $P \otimes_B M \in {}_A\mathcal{C}$ has a right C-structure. Now the diagram

$$\begin{tikzcd}
P \otimes B \otimes M \otimes C \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & P \otimes M \otimes C \arrow[rr] \arrow[d] \arrow[ddr] & & P \otimes_B M \otimes C \arrow[d] \arrow[ddl] \\
P \otimes B \otimes M \arrow[r, shift left] \arrow[r, shift right] & P \otimes M \arrow[rr] \arrow[ddr] & & P \otimes_B M \arrow[ddl] \\
& & N \otimes C \arrow[d] & \\
& & N &
\end{tikzcd}$$

commutes in ${}_A\mathcal{C}$ since $P \otimes_B M \otimes C$ is a difference cokernel. Thus $P \otimes_B M$ is a difference cokernel in ${}_A\mathcal{C}_{\mathcal{C}}$.

THEOREM 10. *Let* A *be a monoid,* B *a bimonoid in the symmetric monoidal category* $\mathcal{C}$. *Let* $\mathcal{F}\colon {}_B\mathcal{C} \longrightarrow {}_A\mathcal{C}$ *be a* $\mathcal{C}$*-functor. The following conditions are equivalent:*

a) *there is a* $\mathcal{C}$*-monoidal structure of* ${}_A\mathcal{C}$ *such that the underlying functor* $(\mathcal{U}\colon {}_A\mathcal{C} \longrightarrow \mathcal{C}, \mathrm{id}, \mathrm{id}, \mathrm{id})$ *is a* $\mathcal{C}$*-monoidal functor and* $\mathcal{F}$ *is a* $\mathcal{C}$*-monoidal equivalence (with suitably chosen natural transformations* δ, ζ, ξ).

b) *there is an* A-B-*progenerator* P, *which is a* B-*comonoid and satisfies the conditions*

i) *there is an isomorphism* $P \otimes_B I \cong I$ *in* $\mathcal{C}_B$ *such that the diagram*

$$(1)\quad \begin{tikzcd} P \arrow[r, "\varepsilon_P"] \arrow[d, phantom, "\cong" rotate=90] & I \arrow[d, phantom, "\cong" rotate=90] \\ P \otimes_B B \arrow[r, "P \otimes \varepsilon"] & P \otimes_B I \end{tikzcd}$$

commutes

ii) *there is an isomorphism* $P \otimes_B (B \hat{\otimes} B) \cong P \otimes P$ *in* $\mathcal{C}_{B \otimes B}$ *such that the diagram*

$$(2)\quad \begin{tikzcd} P \arrow[r, "\Delta_P"] \arrow[d, phantom, "\cong" rotate=90] & P \otimes P \arrow[d, phantom, "\cong" rotate=90] \\ P \otimes_B B \arrow[r, "P \otimes \Delta"] & P \otimes_B (B \hat{\otimes} B) \end{tikzcd}$$

commutes.

Proof: By Corollary 5 and by Proposition 6 of [4] we have that a) implies b).

Conversely assume that b) holds. First we observe that the isomorphisms $P \otimes_B I \cong I$ and $P \otimes_B (B \hat{\otimes} B) \cong P \otimes P$ induce a unique A-left-structure on I resp. $P \otimes P$ such that the isomorphisms are in ${}_A\mathcal{C}$. The diagrams (1) and (2) consist of A-morphisms, in particular Δ_P and ε_P are in ${}_A\mathcal{C}$.

Now let $M, N \in {}_B\mathcal{C}$. There is a functorial isomorphism $P \otimes_B (M \hat{\otimes} N) \cong (P \otimes_B (B \hat{\otimes} B)) \otimes_{B \otimes B} (M \otimes N)$ in ${}_A\mathcal{C}$. In fact $P \otimes_B$ defines a $\mathcal{C}$-equivalence between ${}_B\mathcal{C}$ and ${}_A\mathcal{C}$. In particular $P \otimes_B : {}_B\mathcal{C} \longrightarrow {}_A\mathcal{C}$ preserves difference cokernels. By Corollary 8 we get

$P \otimes_B (M \hat{\otimes} N) \cong P \otimes_B ((B \hat{\otimes} B) \otimes_{B \otimes B} (M \otimes N)) \cong$
$(P \otimes_B (B \hat{\otimes} B)) \otimes_{B \otimes B} (M \otimes N)$ in ${}_A\mathcal{C}$.

By (2) we get an isomorphism
$(P \otimes_B (B \hat{\otimes} B)) \otimes_{B \otimes B} (M \otimes N) \cong (P \otimes P) \otimes_{B \otimes B} (M \otimes N)$ in ${}_A\mathcal{C}$, where $(P \otimes P) \otimes_{B \otimes B} (M \otimes N)$ is formed in ${}_{A \otimes A}\mathcal{C}$ by the A-structures on each of the P's .

Furthermore there is an isomorphism
$(P \otimes P) \otimes_{B \otimes B} (M \otimes N) \cong P \otimes_B (P \otimes_B (M \otimes N))$ in $\mathcal{C}$, where the difference cokernels are taken in ${}_{A \otimes A}\mathcal{C}$, resp. $P \otimes_B (M \otimes N)$ is in ${}_{B-A}\mathcal{C}$. The morphism is defined by $(p \otimes p') \otimes (m \otimes n) \longmapsto p \otimes (p' \otimes (m \otimes n))$. One can check that it extends to the difference cokernels by using that in the commutative diagram (FIG. 2) the columns are difference cokernels in ${}_A\mathcal{C}$ and hence in ${}_{A \otimes A}\mathcal{C}$ by Lemma 9 and the last row is a

difference cokernel in ${}_{A \otimes A}\mathcal{C}$ since $P \otimes_B : {}_{B \otimes A}\mathcal{C} \longrightarrow {}_{A \otimes A}\mathcal{C}$ is an equivalence. Now the map $p \otimes (p' \otimes (m \otimes n)) \longmapsto (p \otimes p') \otimes (m \otimes n)$ is in ${}_{A \otimes A}\mathcal{C}$ and can be factored through $P \otimes_B (P \otimes_B (M \otimes N))$. So the $A \otimes A$ - isomorphism $(P \otimes P) \otimes_{B \otimes B} (M \otimes N) \cong P \otimes_B (P \otimes_B (M \otimes N))$ is also an isomorphism in $\mathcal{C}$ and thus induces a unique new A-structure on $P \otimes_B (P \otimes_B (M \otimes N))$.

Finally there is an isomorphism of $A \otimes A$ - objects $P \otimes_B (P \otimes_B (M \otimes N)) \cong P \otimes_B (M \otimes (P \otimes_B N)) \cong (P \otimes_B M) \otimes (P \otimes_B N)$ because $P \otimes_B$ is a $\mathcal{C}$-equivalence and Lemmas 6 and 9 apply. This isomorphism is also in $\mathcal{C}$ and thus defines a unique new A-structure on $(P \otimes_B M) \otimes (P \otimes_B N)$.

Alltogether we have obtained a natural A-isomorphism $P \otimes_B (M \hat{\otimes} N) \cong (P \otimes_B M) \otimes (P \otimes_B N)$.
This isomorphism is induced by the morphism
$p \otimes (m \otimes n) \longmapsto (p_{(1)} \otimes m) \otimes (p_{(2)} \otimes n)$
where $\Delta_P(p) = p_{(1)} \otimes p_{(2)}$.

Let $Q \otimes_A$ be the inverse equivalence of $P \otimes_B$. Then $P \otimes_B Q \otimes_A M \cong M$ for $M \in {}_A\mathcal{C}$ and $(P \otimes_B Q \otimes_A M) \otimes (P \otimes_B Q \otimes_A N) \cong M \otimes N$ thus inducing in a natural way an A-structure on $M \otimes N$ and a bifunctor $M \tilde{\otimes} N$ from ${}_A\mathcal{C} \times {}_A\mathcal{C}$ to ${}_A\mathcal{C}$.

Furthermore by the previous considerations we have a natural isomorphism

$$P \otimes_B (M \hat{\otimes} N) \cong (P \otimes_B M) \tilde{\otimes} (P \otimes_B N)$$

and

$$P \otimes_B \hat{I} \cong \tilde{I}$$

such that $P \otimes_B$ becomes a C-monoidal equivalence if $M \tilde{\otimes} N$ induces a C-monoidal structure on ${}_A C$.

Following the construction of the A-structure on $M \tilde{\otimes} N$, the coalgebra structure on P and using the coherence in C we get

$$(M \tilde{\otimes} N) \tilde{\otimes} L \cong M \tilde{\otimes} (N \tilde{\otimes} L)$$

induced by the associativity isomorphism in C . Similarly we get

$$\tilde{I} \tilde{\otimes} M \cong M \quad \text{and} \quad M \tilde{\otimes} \tilde{I} \cong M .$$

These isomorphisms are coherent in C , hence also in ${}_A C$, so ${}_A C$ is monoidal and even C-monoidal. This proves also that $(U,\mathrm{id},\mathrm{id},\mathrm{id})$ is a C-monoidal functor. Finally $P \otimes_B$ is a C-monoidal functor since all structural morphisms are defined by the structural coherent morphisms in C . This completes the proof of Theorem 10.

If A and B are bimonoids, then Theorem 10 provides necessary and sufficient conditions, that the categories ${}_A C$ and ${}_B C$ are C-monoidally equivalent. Moreover if the equivalent conditions of Theorem 10 are satisfied we get immediately by Proposition 6 of [4] that A is a bimonoid. Thus Theorem 10 expresses the special Morita equivalence preserving the tensor product without using the tensor product in ${}_A\mathcal{C}$ nor the bialgebra structure of A .

A simple example of a Morita equivalence of module categories with tensor products can be given by using the bialgebra automorphism σ of the bialgebra $k[\mathbb{Z}_3]$, the group algebra over any field k , with $\sigma(\bar{1}) := \bar{2}$ in $\mathbb{Z}_3$.
Since σ is a group autormorphism of $\mathbb{Z}_3$ it is a bialgebra automorphism of $k[\mathbb{Z}_3]$. Let $A = B = k[\mathbb{Z}_3]$ and $P = k[\mathbb{Z}_3]$ as a coalgebra. Then P is a right B-module via the usual multiplication of $k[\mathbb{Z}_3]$ and in fact a B-module coalgebra (i.e. a B-comonoid). By $A \underset{\sigma}{\cong} B \cong \mathrm{End}_B(P)$ P becomes an A-B-progenerator. The canonical isomorphisms $P \otimes_B k \cong k$ and $P \otimes_B (B \hat{\otimes} B) \cong P \otimes P$ satisfy the diagrams (1) and (2) in Theorem 10, so $P \otimes_B$ induces a Morita-equivalence of the module categories B-Mod and A-Mod preserving the tensor products in these categories. This example holds for any group ring with any automorphism on the group.

On the other hand the conditions on P in Theorem 10 b) are quite restrictive. Assume that b) holds and that B is a Hopf monoid in $\mathcal{C}$ with antipode S . Then the morphism $f\colon B \otimes B \longrightarrow B \otimes B$ given by $f(b \otimes b') = b_{(1)} \otimes b_{(2)} b'$ has the inverse $b \otimes b' \longmapsto b_{(1)} \otimes S(b_{(2)})b'$. Now consider

$f: B \otimes B \longrightarrow B \hat{\otimes} B$, then it is clear that f is an isomorphism in ${}_B C$, where $B \otimes B$ carries a B-structure on the left factor and $B \hat{\otimes} B$ is a B-object via the diagonal. Thus the isomorphism $P \otimes_B (B \hat{\otimes} B) \cong P \otimes P$ induces an isomorphism $P \otimes B \cong P \otimes_B (B \otimes B) \cong P \otimes_B (B \hat{\otimes} B) \cong P \otimes P$. In case C = K-Mod with K a field and B (and hence P) finite-dimensional we get $P \cong B$ in K-Mod and similarly $P \cong A$. Thus two finite dimensional Hopf algebras can only be Morita equivalent in the sense of Theorem 10 if they have the same dimension.

A further remark concerning Theorem 10 is in order. The conditions i) and ii) of b) can be expressed differently. To express i) we assume that $\mathcal{U}: {}_A C \longrightarrow C$ preserves epimorphisms. First we consider the morphism $\varepsilon_P \otimes I: P \otimes_B I \longrightarrow I$, which is well-defined since $\varepsilon_P \in C_B$. This morphism satisfies

$$(\varepsilon_P \otimes I)(P \otimes \varepsilon_B)(p \otimes b) = \varepsilon_P(p)\varepsilon_B(b) = \varepsilon_P(p)b = \varepsilon_P(pb),$$

thus the diagram (1) commutes with $\varepsilon_P \otimes I$. Now if i) is satisfied then $P \otimes_B \varepsilon_B$ is an epimorphism in ${}_A C$ and in C hence the isomorphism $P \otimes_B I \cong I$ must be $\varepsilon_P \otimes I$. Thus i) reduces to

i') the morphism $\varepsilon_P \otimes_B I: P \otimes_B I \longrightarrow I$ is an isomorphism.

The diagram (2) is made commutative by the morphism

$\Psi: P \otimes_B (B \hat{\otimes} B) \ni p \otimes b \otimes b' \longmapsto p_{(1)}b \otimes p_{(2)}b' \in P \otimes P$

as can be easily checked. Assume that ii) holds. Then the isomorphism $\varphi: P \otimes_B (B \hat{\otimes} B) \cong P \otimes P$ satisfies $\varphi(p \otimes (1 \otimes 1)) = p_{(1)} \otimes p_{(2)}$ since (2) commutes and $\varphi(p \otimes (b \otimes b')) = p_{(1)}b \otimes p_{(2)}b'$ since φ is a $B \otimes B$ - isomorphism. Thus $\varphi = \Psi$ and ii) reduces to

ii') the morphism

$\Psi: P \otimes_B (B \hat{\otimes} B) \ni p \otimes b \otimes b' \longmapsto p_{(1)}b \otimes p_{(2)}b' \in P \otimes P$

is an isomorphism.

If B is a Hopf monoid then the isomorphism $P \otimes B \cong P \otimes P$ discussed earlier, is the morphism $p \otimes b \longmapsto p_{(1)} \otimes p_{(2)}b$.

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