Endomorphism Bialgebras of Diagrams and of Non-Commutative Algebras and Spaces

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Bialgebras and Hopf algebras have a very complicated structure. It is not easy to construct explicit examples of such and check all the necessary properties. This gets even more complicated if we have to verify that something like a comodule algebra over a bialgebra is given.

Bialgebras and comodule algebras, however, arise in a very natural way in non-commutative geometry and in representation theory. We want to study some general principles on how to construct such bialgebras and comodule algebras.

The leading idea throughout this paper is the notion of an action as can been seen most clearly in the example of vector spaces. Given a vector space V we can associate with it its endomorphism *algebra* $\mathrm{End}(V)$ that, in turn, defines an action $\mathrm{End}(V) \times V \longrightarrow V$. There is also the general linear *group* $\mathrm{GL}(V)$ that defines an action $\mathrm{GL}(V) \times V \longrightarrow V$. In the case of the endomorphism algebra we are in the pleasant situation that $\mathrm{End}(V)$ is a vector space itself so that we can write the action also as $\mathrm{End}(V) \otimes V \longrightarrow V$. The action of $\mathrm{GL}(V)$ on V can also be described using the tensor product by expanding the group $\mathrm{GL}(V)$ to the group algebra $K(\mathrm{GL}(V))$ to obtain $K(\mathrm{GL}(V)) \otimes V \longrightarrow V$.

We are going to find analogues of $\mathrm{End}(V)$ or $K(\mathrm{GL}(V))$ acting on non-commutative geometric spaces or on certain diagrams. This will lead to bialgebras, Hopf algebras, and comodule algebras.

There are two well-known procedures to obtain bialgebras from endomorphisms of certain objects. In the first section we will construct endomorphism spaces in the category of non-commutative spaces. These endomorphism spaces are described through bialgebras.

In the second section we find (co-)endomorphism coalgebras of certain diagrams of vector spaces, graded vector spaces, differential graded vector spaces, or others. Under additional conditions they again will turn out to be bialgebras.

The objects constructed in the first section will primarily be algebras, whereas in the second section the objects $\mathrm{coend}(\omega)$ will have the natural structure of a coalgebra. Nevertheless we will show in the third section that the constructions of bialgebras from non-commutative spaces and of bialgebras from diagrams of vector spaces, remote as they may seem, are closely related, in fact that the case of an endomorphism space of a non-commutative space is a special case of a coendomorphism bialgebra of a certain diagram. Some other constructions of endomorphism spaces from the literature will also be subsumed under the more general construction of coendomorphism bialgebras of diagrams. We also will find such bialgebras coacting on Lie algebras.

This indicates that a suitable setting of non-commutative geometry might be obtained by considering (monoidal) diagrams of vector spaces (which can be considered as partially defined algebras) as a generalization of affine non-commutative spaces. So the problem of finding non-commutative (non-affine) schemes might be resolved in this direction.

In the last section we show that similar results hold for Hopf algebras acting on non-commutative spaces resp. on diagrams. The universal Hopf algebra coacting on an algebra is usually obtained as the Hopf envelope of the universal bialgebra acting on this algebra. We show that this Hopf algebra can also be obtained as the (co-)endomorphism bialgebra of a specific diagram constructed from the given algebra.

Throughout this paper K shall denote an arbitrary field and $\otimes$ stands for $\otimes_K$.

1 Endomorphisms of Non-Commutative Geometric Spaces

In this section we discuss some simple background on non-commutative spaces and their endomorphisms.

1.1 Affine non-commutative spaces

In algebraic geometry an affine algebraic space is given as a subset of K^n consisting of all points satisfying certain polynomial equalities. A typical example is the unit circle in $\mathbf{R}^2$ given by

$$\mathrm{Circ}(\mathbf{R}) = \{(x,y) \in \mathbf{R}^2 |\ x^2 + y^2 = 1\}.$$

Actually one is interested in circles with arbitrary radius r over any commutative $\mathbf{R}$-algebra B. They are defined in a similar way by

$$\mathrm{Circ}(B) = \{(x,y) \in B^2 |\ x^2 + y^2 - r^2 = 0\}.$$

If for example r^2 is -1 instead of 1, there are no points with coefficients in the field of real numbers but lots of points with coefficients in the field of complex numbers.

Furthermore this defines a functor $\mathrm{Circ} : \mathbf{R}\text{-}\mathrm{Alg}_c \longrightarrow \mathrm{Sets}$ from the category of all commutative $\mathbf{R}$-algebras to the category of sets, since an algebra homomorphism $f : B \longrightarrow B'$ induces a map of the corresponding unit circles $\mathrm{Circ}(f) : \mathrm{Circ}(B) \longrightarrow \mathrm{Circ}(B')$. The coordinates for the points which we consider are taken from algebra B, the *coordinate domain*, and Circ(B) is the set of all points of the space, that is the given manifold or geometric space. One has this set for all choices of coordinate domains B.

In general a functor $\mathcal{X} : K\text{-Alg}_c \longrightarrow \text{Sets}$ with

$$\mathcal{X}(B) := \{(b_1, \ldots, b_n) \in B^n | \, p_1(b_1, \ldots, b_n) = \ldots = p_r(b_1, \ldots, b_n) = 0\} \tag{1}$$

is called an *affine algebraic space*. This functor is represented by the algebra

$$A = K[x_1, \ldots, x_n]/(p_1(x_1, \ldots, x_n), \ldots, p_r(x_1, \ldots, x_n)),$$

hence $\mathcal{X}(B) = K\text{-Alg}_c(A, B)$. So for the circle we have

$$\text{Circ}(B) \cong \mathbf{R}\text{-Alg}_c(\mathbf{R}[x, y]/(x^2 + y^2 - r^2), B).$$

The Yoneda Lemma shows that this representing algebra A is uniquely determined (up to isomorphism) by the functor $\mathcal{X}$. It is usually considered as the "function algebra" of the geometric space under consideration. Indeed there is a map $A \times \mathcal{X}(K) \longrightarrow K$, $(a, f) \mapsto f(a)$, where $f \in \mathcal{X}(K) \cong K\text{-Alg}_c(A, K)$.

All of this has nothing to do with the commutativity of the algebras under consideration. Hence one can use non-commutative K-algebras A and B as well. Certain questions in physics in fact require such algebras. Instead of representing algebras

$$A = K[x_1, \ldots, x_n]/(p_1(x_1, \ldots, x_n), \ldots, p_r(x_1, \ldots, x_n))$$

we take now representing algebras

$$A = K\langle x_1, \ldots, x_n\rangle/(p_1(x_1, \ldots, x_n), \ldots, p_r(x_1, \ldots, x_n))$$

where $K\langle x_1, \ldots, x_n\rangle$ is the algebra of polynomials in non-commuting variables or, equivalently, the tensor algebra of the finite-dimensional vector space with basis $x_1, \ldots, x_n$.

DEFINITION 1.1 (B-points of quantum spaces)
A functor $\mathcal{X} = K\text{-Alg}(A, -)$ *is called an* affine non-commutative space *or a* quantum space *and the elements of* $\mathcal{X}(B) = K\text{-Alg}(A, B)$ *the* B-points *of* $\mathcal{X}$.

As in (1) the B-points of a quantum space are elements of B^n, so that $\mathcal{X}(B)$ is a subset of B^n.

Well known examples of quantum spaces are the *quantum planes* with function algebras [5]

$$\mathcal{O}(A_q^{2|0}) = K\langle x, y\rangle/(xy - q^{-1}yx), \qquad q \in K \setminus \{0\}$$

and [6]

$$\mathcal{O}(A_0^{2|0}) = K\langle x, y\rangle/(xy - yx + y^2).$$

As in (commutative) algebraic geometry one can consider the algebra $A = \mathcal{O}(\mathcal{X}) = K\langle x_1, \ldots, x_n\rangle/(p_1(x_1, \ldots, x_n), \ldots, p_r(x_1, \ldots, x_n))$ (which represents the quantum space $\mathcal{X}$) as the *function algebra* of $\mathcal{X}$ consisting of the (natural) functions from $\mathcal{X}(B)$ to the coordinate algebra $\mathcal{U}(B)$ where $\mathcal{U} : K\text{-Alg} \longrightarrow \text{Sets}$ is the underlying functor. So the elements of A are certain functions from $\mathcal{X}(B)$ to B. We denote the set of all natural transformations or functions from $\mathcal{X}$ to $\mathcal{U}$ by $\mathcal{M}\text{ap}(\mathcal{X}, \mathcal{U})$

LEMMA 1.2 (the function algebra of a space)
Let $\mathcal{X} : K\text{-Alg} \longrightarrow \text{Sets}$ be a representable functor with representing algebra A. Then there is an isomorphism $A \cong \mathcal{M}ap(\mathcal{X},\mathcal{U})$ inducing a natural transformation $A \times \mathcal{X}(B) \longrightarrow B$.

Proof: The underlying functor $\mathcal{U} : K\text{-Alg} \longrightarrow \text{Sets}$ is represented by the algebra $K[x] = K\langle x\rangle$. So by the Yoneda Lemma we get

$$\mathcal{M}\mathrm{ap}(\mathcal{X},\mathcal{U}) \cong \mathcal{M}\mathrm{ap}(K\text{-Alg}(A,-), K\text{-Alg}(K[x],-)) \cong K\text{-Alg}(K[x],A) \cong A.$$

□

COROLLARY 1.3 (the universal property of the function algebra)
Let $\mathcal{X}$ and A be as in Lemma 1.2. If $\varphi : C \times \mathcal{X}(B) \longrightarrow B$ is a natural transformation then there exists a unique map $\tilde{\varphi} : C \longrightarrow A$ such that the following diagram commutes

$$\begin{tikzcd}
C \times \mathcal{X}(B) \arrow[d, "\tilde{\varphi}\times 1"'] \arrow[dr, "\varphi"] & \\
A \times \mathcal{X}(B) \arrow[r] & B.
\end{tikzcd}$$

Proof: This follows from the Yoneda lemma. □

If $\mathcal{X}$ and $\mathcal{Y}$ are quantum spaces, then we will call a natural transformation $f : \mathcal{X} \longrightarrow \mathcal{Y}$ simply a *map of quantum spaces*. So the quantum spaces form a category $\mathcal{Q}$ which is antiequivalent to the category of finitely generated K-algebras K-Alg. The set of maps from $\mathcal{X}$ to $\mathcal{Y}$ will be denoted by $\mathcal{M}\mathrm{ap}(\mathcal{X},\mathcal{Y})$.

If $\mathcal{X}$ is a quantum space, $A = \mathcal{O}(\mathcal{X})$ its function algebra, and $I \subseteq A$ a two-sided ideal of A, then $A \longrightarrow A/I$ is an epimorphism which induces a monomorphism $\iota : \mathcal{X}_I \longrightarrow \mathcal{X}$ for the associated quantum spaces. In particular the B-points $\mathcal{X}_I(B) = K\text{-Alg}(A/I, B)$ can be identified with a certain subset of B-points in $\mathcal{X}(B)$. Conversely every subspace $\mathcal{Y} \subseteq \mathcal{X}$ (i. e. $\mathcal{Y}(B) \subseteq \mathcal{X}(B)$ functorially for all B) is induced by an epimorphism $\mathcal{O}(\mathcal{X}) \longrightarrow \mathcal{O}(\mathcal{Y})$. (Observe, however, that not every epimorphism in the category K-Alg is surjective, e. g. $K[x] \longrightarrow K(x)$ is an epimorphism but not a surjection.)

1.2 The commutative part of a quantum space

The quantum plane $A_q^{2|0}$ defines a functor from the category of algebras to the category of sets. We call its restriction to commutative algebras the commutative part $(A_q^{2|0})_{\mathrm{comm}}$ of the quantum plane. In general the restriction of a quantum space $\mathcal{X}$ to commutative algebras is called the *commutative part* of the quantum space and is denoted by $\mathcal{X}_{\mathrm{comm}}$. The commutative part of a quantum space represented by an algebra A is always an affine algebraic (commutative) scheme, since it is represented by the algebra $A/[A,A]$, where $[A,A]$ denotes the two-sided ideal generated by all commutators $[a,b] = ab - ba$ for $a, b \in A$. In particular, the commutative part $\mathcal{X}_{\mathrm{comm}}$ of a quantum space $\mathcal{X}$ is indeed a subspace of $\mathcal{X}$.

For a commutative algebra B the spaces $\mathcal{X}_{\mathrm{comm}}$ and $\mathcal{X}$ have the same B-points: $\mathcal{X}_{\mathrm{comm}}(B) = \mathcal{X}(B)$.

For the quantum plane $A_q^{2|0}$ and commutative fields B the set of B-points consists exactly of the two coordinate axes in B^2 since $(A_q^{2|0})_{\mathrm{comm}}$ is represented by $K[x,y]/(xy - qyx) \cong K[x,y]/(xy)$ for $q \neq 1$.

1.3 Commuting points

In algebraic geometry any two points $(b_1, \ldots, b_m)$ and $(b'_1, \ldots, b'_n)$ with coefficients in the same coordinate algebra B have the property that their coordinates mutually commute under the multiplication $b_i b'_j = b'_j b_i$ of B since B is commutative. This does not hold any longer for non-commutative algebras B and arbitrary quantum spaces $\mathcal{X}$ and $\mathcal{Y}$.

DEFINITION 1.4 (commuting points)
If $A = \mathcal{O}(\mathcal{X})$ and $A' = \mathcal{O}(\mathcal{Y})$ and if $p : A \longrightarrow B \in \mathcal{X}(B)$ and $p' : A' \longrightarrow B \in \mathcal{Y}(B)$ are two points with coordinates in B, we say that they are commuting points *if for all $a \in A, a' \in A'$ we have $p(a)p'(a') = p'(a')p(a)$, i. e. the images of the algebra homomorphisms p and p' commute elementwise.*

Obviously it is sufficient to require this just for a set of algebra generators a_i of A and a'_j of A'. In particular if A is of the form $A = K\langle x_1, \ldots, x_m\rangle/I$ and A' is of the form $A' = K\langle x_1, \ldots, x_n\rangle/J$ then the B-points are given by $(b_1, \ldots, b_m)$ resp. $(b'_1, \ldots, b'_n)$ with coordinates in B, and the two points commute iff $b_i b'_j = b'_j b_i$ for all i and j.

The set of commuting points

$$(\mathcal{X} \perp \mathcal{Y})(B) = \mathcal{X}(B) \perp \mathcal{Y}(B) := \{(p, q) \in \mathcal{X}(B) \times \mathcal{Y}(B) | \ p \text{ and } q \text{ commute}\}$$

is a subfunctor of $\mathcal{X} \times \mathcal{Y}$ since commutativity of two elements in B is preserved by algebra homomorphisms $f : B \longrightarrow B'$. We call it the *orthogonal product* of the quantum spaces $\mathcal{X}$ and $\mathcal{Y}$.

LEMMA 1.5 (the orthogonal product)
If $\mathcal{X}$ and $\mathcal{Y}$ are quantum spaces with function algebras $A = \mathcal{O}(\mathcal{X})$ and $A' = \mathcal{O}(\mathcal{Y})$ then the orthogonal product $\mathcal{X} \perp \mathcal{Y}$ is a quantum space with (representing) function algebra $\mathcal{O}(\mathcal{X} \perp \mathcal{Y}) = A \otimes A' = \mathcal{O}(\mathcal{X}) \otimes \mathcal{O}(\mathcal{Y})$.

Proof: Let $(f, g) \in (\mathcal{X} \perp \mathcal{Y})(B)$ be a pair of commuting points. Then there is a unique homomorphism $h : A \otimes A' \longrightarrow B$ such that

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[dr, "f"'] & A \otimes A' \arrow[d, "h"] & A' \arrow[l, "p'"'] \arrow[dl, "g"] \\ & B & \end{tikzcd}$$

commutes; in fact the map is given by $h(a \otimes a') = f(a)g(a')$. Conversely every algebra homomorphism $h : A \otimes A' \longrightarrow B$ defines a pair of commuting points by the above diagram. The algebra homomorphism $p : A \longrightarrow A \otimes A'$ is defined by $p(a) = a \otimes 1$, and p' is defined similarly. $\square$

This lemma shows that the set of commuting points

$$((b_1, \ldots, b_m), (b'_1, \ldots, b'_n)) = (b_1, \ldots, b_m, b'_1, \ldots, b'_n)$$

with $b_i b'_j - b'_j b_i = 0$ forms again a quantum space. It is now easy to show that the category $\mathcal{Q}$ of quantum spaces is a monoidal category with the orthogonal product $\mathcal{X} \perp \mathcal{Y}$ (in the sense of [3]). The associativity of the orthogonal product arises from the associativity of the tensor product of the function algebras.

The preceding lemma sheds some light on the reason, why we have restricted our considerations to commuting points. There is a general credo that the function algebra of a "non-commutative" space should be graded and have polynomial growth that is some kind of a Poincaré-Birkhoff-Witt theorem should hold. But the free product of algebras (which would correspond to the product of the quantum spaces) grows exponentially (with the degree). Some kind of commutation relation among the elements of the function algebra is required and this is given by letting the elements of the two function algebras A and B in the orthogonal product of the quantum spaces commute. This is done by the tensor product.

1.4 The endomorphisms of a quantum space

In the category of quantum spaces we want to find an analogue of the endomorphism algebra $\mathrm{End}(V)$ of a vector space V and of its action on V. Actually we consider a somewhat more general situation of an action $\mathcal{H}(\mathcal{X}, \mathcal{Y}) \perp \mathcal{X} \longrightarrow \mathcal{Y}$ which resembles the action $\mathrm{Hom}(V, W) \otimes V \longrightarrow W$ for vector spaces. The tensor product $\otimes$ of vector spaces will be replaced by the orthogonal product of quantum spaces $\perp$.

DEFINITION 1.6 (the hom of quantum spaces)
Let $\mathcal{X}$ and $\mathcal{Y}$ be a quantum spaces. A (universal) quantum space $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ together with a map $\rho : \mathcal{H}(\mathcal{X}, \mathcal{Y}) \perp \mathcal{X} \longrightarrow \mathcal{Y}$, such that for every quantum space $\mathcal{Z}$ and every map $\alpha : \mathcal{Z} \perp \mathcal{X} \longrightarrow \mathcal{Y}$ there is a unique map $\beta : \mathcal{Z} \longrightarrow \mathcal{H}(\mathcal{X}, \mathcal{Y})$ such that the diagram

$$\begin{tikzcd} \mathcal{Z} \perp \mathcal{X} \arrow[d, "\beta \perp 1"'] \arrow[dr, "\alpha"] & \\ \mathcal{H}(\mathcal{X}, \mathcal{Y}) \perp \mathcal{X} \arrow[r, "\rho"'] & \mathcal{Y} \end{tikzcd}$$

commutes, is called a homomorphism space. $\mathcal{E}_{\mathcal{X}} := \mathcal{H}(\mathcal{X}, \mathcal{X})$ *is called an* endomorphism space.

Apart from the map $\mathcal{H}(\mathcal{X}, \mathcal{Y}) \perp \mathcal{X} \longrightarrow \mathcal{Y}$, which we will regard as a multiplication of $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ on $\mathcal{X}$ (with values in $\mathcal{Y}$), this construction leads to further multiplications.

LEMMA 1.7 (the multiplication of homs)
If there exist homomorphism spaces $\mathcal{H}(\mathcal{X}, \mathcal{Y})$, $\mathcal{H}(\mathcal{Y}, \mathcal{Z})$, and $\mathcal{H}(\mathcal{X}, \mathcal{Z})$ for the quantum spaces $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$, then there is a multiplication $m : \mathcal{H}(\mathcal{Y}, \mathcal{Z}) \perp \mathcal{H}(\mathcal{X}, \mathcal{Y}) \longrightarrow \mathcal{H}(\mathcal{X}, \mathcal{Z})$ with respect to the orthogonal product structure in $\mathcal{Q}$. Furthermore this product is associative and unitary if the necessary homomorphism spaces exist.

Proof: Consider the diagram

$$\begin{tikzcd}
(\mathcal{H}(\mathcal{Y},\mathcal{Z})\perp\mathcal{H}(\mathcal{X},\mathcal{Y}))\perp\mathcal{X} \arrow[r, "\cong"] \arrow[d, "m\perp 1"'] & \mathcal{H}(\mathcal{Y},\mathcal{Z})\perp(\mathcal{H}(\mathcal{X},\mathcal{Y})\perp\mathcal{X}) \arrow[r, "1\perp\rho"] & \mathcal{H}(\mathcal{Y},\mathcal{Z})\perp\mathcal{Y} \arrow[d, "\rho"] \\
\mathcal{H}(\mathcal{X},\mathcal{Z})\perp\mathcal{X} \arrow[rr, "\rho"] & & \mathcal{Z}
\end{tikzcd}$$

which induces a unique homomorphism $m : \mathcal{H}(\mathcal{Y},\mathcal{Z})\perp\mathcal{H}(\mathcal{X},\mathcal{Y}) \longrightarrow \mathcal{H}(\mathcal{X},\mathcal{Z})$, the multiplication. □

COROLLARY 1.8 (the endomorphism space is a monoid)
The endomorphism space $\mathcal{E}_\mathcal{X}$ of a quantum space $\mathcal{X}$ is a monoid (an algebra) in the category $\mathcal{Q}$ w. r. t. the orthogonal product. The space $\mathcal{H}(\mathcal{X},\mathcal{Y})$ is a left $\mathcal{E}_\mathcal{Y}$-space and a right $\mathcal{E}_\mathcal{X}$-space (an $\mathcal{E}_\mathcal{Y},\mathcal{E}_\mathcal{X}$-bimodule) in $\mathcal{Q}$.

Proof: The multiplication is given in Lemma 1.7. To get the unit we consider the one point quantum space $\mathcal{J}(B) = \{u_B\}$ with function algebra K and the diagram

$$\begin{tikzcd}
\mathcal{J}\perp\mathcal{X} \arrow[d, "u\perp 1"'] \arrow[dr, "\cong"] & \\
\mathcal{E}_\mathcal{X}\perp\mathcal{X} \arrow[r, "\rho"'] & \mathcal{X}
\end{tikzcd}$$

which induces a unique homomorphism $u : \mathcal{J} \longrightarrow \mathcal{E}_\mathcal{X}$, the unit for $\mathcal{E}_\mathcal{X}$.
Standard arguments show now that $(\mathcal{E}_\mathcal{X}, m, u)$ forms a monoid in the category of quantum spaces and that the spaces $\mathcal{H}(\mathcal{X},\mathcal{Y})$ are $\mathcal{E}_\mathcal{Y}$- resp. $\mathcal{E}_\mathcal{X}$-spaces. □

We will call a *quantum monoid* a space $\mathcal{E}$ together with $\mathcal{E}\perp\mathcal{E} \longrightarrow \mathcal{E}$ in $\mathcal{Q}$ which is a monoid w.r.t. to the orthogonal product. If $\mathcal{E}$ acts on a quantum space $\mathcal{X}$ by $\mathcal{E}\perp\mathcal{X} \longrightarrow \mathcal{X}$ such that this action is associative and unitary then we call $\mathcal{X}$ an *$\mathcal{E}$-space.*

PROPOSITION 1.9 (the universal properties of the endomorphism space)
The endomorphism space $\mathcal{E}_\mathcal{X}$ and the map $\rho : \mathcal{E}_\mathcal{X}\perp\mathcal{X} \longrightarrow \mathcal{X}$ have the following universal properties:
a) *For every quantum space $\mathcal{Z}$ and map $\alpha : \mathcal{Z}\perp\mathcal{X} \longrightarrow \mathcal{X}$ there is a unique map $\beta : \mathcal{Z} \longrightarrow \mathcal{E}_\mathcal{X}$ such that the diagram*

$$\begin{tikzcd}
\mathcal{Z}\perp\mathcal{X} \arrow[d, "\beta\perp 1"'] \arrow[dr, "\alpha"] & \\
\mathcal{E}_\mathcal{X}\perp\mathcal{X} \arrow[r, "\rho"'] & \mathcal{X}
\end{tikzcd}$$

commutes.
b) *For every quantum monoid $\mathcal{M}$ and map $\alpha : \mathcal{M}\perp\mathcal{X} \longrightarrow \mathcal{X}$ which makes $\mathcal{X}$ an $\mathcal{M}$-space there is a unique monoid map $\beta : \mathcal{M} \longrightarrow \mathcal{E}_\mathcal{X}$ such that the diagram*

$$\begin{tikzcd}
\mathcal{M}\perp\mathcal{X} \arrow[d, "\beta\perp 1"'] \arrow[dr, "\alpha"] & \\
\mathcal{E}_{\mathcal{X}}\perp\mathcal{X} \arrow[r, "\rho"'] & \mathcal{X}
\end{tikzcd}$$

commutes.

Proof: a) is the definition of $\mathcal{E}_{\mathcal{X}}$.
b) We consider the diagram

$$\begin{tikzcd}
\mathcal{M}\perp\mathcal{M}\perp\mathcal{X} \arrow[r, shift left, "\bar{m}\perp 1"] \arrow[r, shift right, "1\perp\alpha"'] \arrow[d, "\beta\perp\beta\perp 1"'] & \mathcal{M}\perp\mathcal{X} \arrow[d, "\beta\perp 1"] \arrow[dr, "\alpha"] & \\
& & \mathcal{X} \\
\mathcal{E}_{\mathcal{X}}\perp\mathcal{E}_{\mathcal{X}}\perp\mathcal{X} \arrow[r, shift left, "m\perp 1"] \arrow[r, shift right, "1\perp\rho"'] & \mathcal{E}_{\mathcal{X}}\perp\mathcal{X} \arrow[ur, "\rho"'] &
\end{tikzcd}$$

The right triangle commutes by definition of β. The lower square commutes since

$$(\beta\perp 1)(1\perp\alpha) = \beta\perp\alpha = \beta\perp\rho(\beta\perp 1) = (1\perp\rho)(\beta\perp\beta\perp 1).$$

The horizontal pairs are equalized since $\mathcal{X}$ is an $\mathcal{M}$-space and an $\mathcal{E}_{\mathcal{X}}$-space. So we get $\rho(\beta\perp 1)(\bar{m}\perp 1) = \rho(m\perp 1)(\beta\perp\beta\perp 1)$. By the universal property a) of $\mathcal{E}_{\mathcal{X}}$, the map β is compatible with the multiplication. Similarly one shows that it is compatible with the unit. $\square$

It is not so clear which homomorphism spaces or endomorphisms spaces exist. Tambara [15] has given a construction for homomorphism spaces $\mathcal{H}(\mathcal{X},\mathcal{Y})$ where $\mathcal{X}$ is represented by a finite dimensional algebra. In 3.2 we shall give another proof of his theorem. There are other examples of such coendomorphism bialgebras e. g. for "quadratic algebras" as considered in [5]. We will reconstruct them in 3.3.

1.5 Bialgebras and comodule algebras

With quantum homomorphism spaces and endomorphism spaces we have already obtained bialgebras and comodule algebras.

Let $\mathcal{X}$ resp. $\mathcal{Y}$ be quantum spaces with function algebras $A = \mathcal{O}(\mathcal{X})$ resp. $B = \mathcal{O}(\mathcal{Y})$. Assume that $\mathcal{E}_{\mathcal{X}}$ and $\mathcal{E}_{\mathcal{Y}}$ exist and have function algebras $E_A = \mathcal{O}(\mathcal{E}_{\mathcal{X}})$ resp. $E_B = \mathcal{O}(\mathcal{E}_{\mathcal{Y}})$. Then the operations $\mathcal{E}_{\mathcal{X}}\perp\mathcal{E}_{\mathcal{X}} \longrightarrow \mathcal{E}_{\mathcal{X}}$, $\mathcal{E}_{\mathcal{X}}\perp\mathcal{X} \longrightarrow \mathcal{X}$, $\mathcal{E}_{\mathcal{Y}}\perp\mathcal{H}(\mathcal{X},\mathcal{Y}) \longrightarrow \mathcal{H}(\mathcal{X},\mathcal{Y})$, and $\mathcal{H}(\mathcal{X},\mathcal{Y})\perp\mathcal{E}_{\mathcal{X}} \longrightarrow \mathcal{H}(\mathcal{X},\mathcal{Y})$ (if the quantum space $\mathcal{H}(\mathcal{X},\mathcal{Y})$ exists) lead to algebra homomorphisms

$$\begin{aligned}
E_A &\longrightarrow E_A \otimes E_A, \\
A &\longrightarrow E_A \otimes A, \\
\mathcal{O}(\mathcal{H}(\mathcal{X},\mathcal{Y})) &\longrightarrow E_B \otimes \mathcal{O}(\mathcal{H}(\mathcal{X},\mathcal{Y})), \\
\mathcal{O}(\mathcal{H}(\mathcal{X},\mathcal{Y})) &\longrightarrow \mathcal{O}(\mathcal{H}(\mathcal{X},\mathcal{Y})) \otimes E_A.
\end{aligned}$$

In particular E_A and E_B are bialgebras, and A and $\mathcal{O}(\mathcal{H}(\mathcal{X},\mathcal{Y}))$ are comodule algebras over E_A, and B and $\mathcal{O}(\mathcal{H}(\mathcal{X},\mathcal{Y}))$ are comodule algebras over E_B. Furthermore the map $\mathcal{H}(\mathcal{X},\mathcal{Y})\perp\mathcal{X} \longrightarrow \mathcal{Y}$ induces an algebra homomorphism $B \longrightarrow \mathcal{O}(\mathcal{H}(\mathcal{X},\mathcal{Y})) \otimes A$. Write

$$a(A,B) := \mathcal{O}(\mathcal{H}(\mathcal{X},\mathcal{Y})).$$

Then we have

$$\mathcal{M}\mathrm{ap}(\mathcal{Z}, \mathcal{H}(\mathcal{X}, \mathcal{Y})) \cong \mathcal{M}\mathrm{ap}(\mathcal{Z} \perp \mathcal{X}, \mathcal{Y})$$

and

$$K\text{-Alg}(a(A, B), C) \cong K\text{-Alg}(B, C \otimes A).$$

and a universal algebra homomorphism $B \longrightarrow a(A, B) \otimes A$. Here we have used the notation of Tambara [15] for the universal algebra.

From Definition 1.6 and Lemma 1.7 we get the following

COROLLARY 1.10 (the coendomorphism bialgebra of an algebra)
Let A be a (non-commutative) algebra. Let E_A be an algebra and $\delta : A \longrightarrow E_A \otimes A$ be an algebra homomorphism, such that for every algebra B and algebra homomorphism $\alpha : A \longrightarrow B \otimes A$ there is a unique algebra homomorphism $\beta : E_A \longrightarrow B$ such that

$$\begin{tikzcd} A \arrow[r, "\delta"] \arrow[dr, "\alpha"'] & E_A \otimes A \arrow[d, "\beta \otimes 1"] \\ & B \otimes A \end{tikzcd}$$

commutes. Then E_A represents the endomorphism space of $\mathcal{X} = K\text{-Alg}(A, \text{-})$, E_A is a bialgebra, and A is an E_A-comodule algebra.

Since the category of algebras is dual to the category of quantum space, we call E_A the *coendomorphism bialgebra* of A and its elements *coendomorphisms*. From Proposition 1.9 we obtain immediately

COROLLARY 1.11 (the universal properties of the coendomorphism bialgebra)
The coendomorphism bialgebra E_A of an algebra A and the map $\delta : A \longrightarrow E_A \otimes A$ have the following universal properties:
a) *For every algebra B and algebra homomorphism $\alpha : A \longrightarrow B \otimes A$ there is a unique algebra homomorphism $\beta : E_A \longrightarrow B$ such that the diagram*

$$\begin{tikzcd} A \arrow[r, "\delta"] \arrow[dr, "\alpha"'] & E_A \otimes A \arrow[d, "\beta \otimes 1"] \\ & B \otimes A \end{tikzcd}$$

commutes.
b) *For every bialgebra B and algebra homomorphism $\alpha : A \longrightarrow B \otimes A$ making A into a comodule algebra there is a unique bialgebra homomorphism $\beta : E_A \longrightarrow B$ such that the diagram*

$$\begin{tikzcd} A \arrow[r, "\delta"] \arrow[dr, "\alpha"'] & E_A \otimes A \arrow[d, "\beta \otimes 1"] \\ & B \otimes A \end{tikzcd}$$

commutes. □

2 Coendomorphism Bialgebras of Diagrams

We have seen that, similar to commutative algebraic geometry, non-commutative spaces represented by non-commutative algebras induce non-commutative endomorphism spaces which are represented by bialgebras.

Now we move to a seemingly unrelated subject which is studied in representation theory and Tannaka duality. The principal question here is whether a group, a monoid, or an algebra are completely determined, if all their representations or modules are "known". We certainly have to specify what the term "known" should mean in this context. Certain reasons, mainly the fundamental theorem on the structure of coalgebras and comodules, make it easier to consider comodules rather than modules as representations.

The purpose of this section is to show that for each diagram of finite dimensional vector spaces there is an associated coalgebra which behaves like the dual of an endomorphism algebra. In particular we associate with the trivial diagram that consists of just one finite dimensional vector space V a coalgebra C that is the dual of the endomorphism algebra of this vector space: $C \cong \mathrm{End}(V)^* \cong V \otimes V^*$. If the diagram of vector spaces has additional properties then the associated coalgebra will have additional properties. This construction renders bialgebras and comodule algebras associated with certain diagrams.

If we start with a coalgebra C then we can construct the diagram of all its finite dimensional comodules and comodule homomorphisms, and also the diagram or category of all comodules. Then the coalgebra C can be recovered from the underlying functor $\omega : \mathcal{C}\text{omod-}C \longrightarrow \mathcal{A}$ as the coalgebra associated with this diagram.

Each bialgebra B induces a tensor product in the category of comodules $\mathcal{C}\text{omod-}B$ over B. This bialgebra can also be recovered from the underlying functor as the bialgebra associated with the given diagram. Additional properties of B induce additional features of $\mathcal{C}\text{omod-}B$ and conversely.

2.1 The base category

All the structures considered in this paper are built on underlying vector spaces over a given field. Certain generalizations of our constructions are straightforward whence we will use a general category $\mathcal{A}$ instead of the category of vector spaces.

We assume that $\mathcal{A}$ is an abelian category and a monoidal category with an associative unitary tensor product $\otimes : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ as in [8]. We also assume that the category $\mathcal{A}$ is cocomplete ([7] and [13] p. 23) and that colimits commute with tensor products. Finally we assume that the monoidal category $\mathcal{A}$ is *quasisymmetric* in the sense of [13] or *braided* that is there is a bifunctorial isomorphism $\sigma : X \otimes Y \longrightarrow Y \otimes X$ such that the diagram

$$\begin{tikzcd}
X \otimes (Y \otimes Z) \arrow[r, "\sigma"] \arrow[d, "\alpha"'] & (Y \otimes Z) \otimes X & Y \otimes (Z \otimes X) \arrow[l, "\alpha"'] \\
(X \otimes Y) \otimes Z \arrow[r, "\sigma \otimes 1"] & (Y \otimes X) \otimes Z & Y \otimes (X \otimes Z) \arrow[l, "\alpha"'] \arrow[u, "1 \otimes \sigma"']
\end{tikzcd}$$

and the corresponding diagram for σ^{-1} commute. σ is called a *symmetry* if in addition $\sigma^{-1}_{X,Y} = \sigma_{Y,X}$ holds.

We give a few interesting examples of such categories.

1) The category $\mathcal{V}\text{ec}$ of all vector spaces over K with the usual tensor product is a symmetric

monoidal category and satisfies the conditions for $\mathcal{A}$.
2) The category $\mathcal{C}$omod-H of comodules over a coquasitriangular or braided bialgebra H [13] with the diagonal action of H on the tensor product of comodules over K and with colimits in $\mathcal{V}$ec and the canonical H-comodule structure on these is a quasisymmetric monoidal category and satisfies the conditions for $\mathcal{A}$ since tensor products commute with colimits.
3) The category of $\mathbf{N}$-graded vector spaces is isomorphic to the category $\mathcal{C}$omod-H, with $H = K[\mathbf{N}]$ the monoid algebra over the monoid of integers. Hence the category of graded vector spaces (with the usual graded tensor product) is a symmetric monoidal category which satisfies the conditions for $\mathcal{A}$.
4) The category of chain complexes of vector spaces K-$\mathcal{C}$omp is isomorphic to the category $\mathcal{C}$omod-H of comodules over the Hopf algebra $H = K\langle x, y, y^{-1}\rangle/(xy + yx, x^2)$ with $\Delta(x) = x \otimes 1 + y^{-1} \otimes x$ and $\Delta(y) = y \otimes y$ [10, 11]. H is a cotriangular Hopf algebra ([10] p. 373) and K-$\mathcal{C}$omp is a symmetric monoidal category with the total tensor product of complexes which satisfies the conditions for $\mathcal{A}$.
5) Super-symmetric spaces are defined as the symmetric monoidal category $\mathcal{C}$omod-H of comodules over the Hopf algebra $H = K[\mathbf{Z}/2\mathbf{Z}]$ which is cotriangular [1].

We are going to assume throughout that the category $\mathcal{A}$ is symmetric, so that we can use the full strength of the coherence theorems for monoidal categories. Thus most of the time we will delete all associativity, unity, and symmetry morphisms assuming that our categories are strict symmetric monoidal categories. Most of the given results hold also over quasisymmetric monoidal categories, cf.[4], but it gets quite technical if one wants to check all the details.

We say that an object X of a category $\mathcal{A}$ has a *dual* (X^*,ev) where X^* is an object and ev: $X^* \otimes X \longrightarrow I$ is a morphism in $\mathcal{A}$, if there is a morphism db: $I \longrightarrow X \otimes X^*$ such that

$$(X \xrightarrow{\mathrm{db}\otimes 1} X \otimes X^* \otimes X \xrightarrow{1\otimes \mathrm{ev}} X) = 1_X \ ,$$
$$(X^* \xrightarrow{1\otimes \mathrm{db}} X^* \otimes X \otimes X^* \xrightarrow{\mathrm{ev}\otimes 1} X^*) = 1_{X^*} \ .$$

The category $\mathcal{A}$ is *rigid* if every object of $\mathcal{A}$ has a dual.

The full subcategory of objects in $\mathcal{A}$ having duals in the sense of [12] is denoted by $\mathcal{A}_0$. The category $\mathcal{A}_0$ then is a rigid symmetric monoidal category. Observe that a vector space has a dual in this sense iff it is finite dimensional.

2.2 Cohomomorphisms of diagrams

In this subsection the category $\mathcal{A}$ does not have to be symmetric or quasisymmetric.

We consider diagrams (commutative or not) in $\mathcal{A}$. They are given by the objects at the vertices and the morphisms along the edges. The vertices and the (directed) edges or arrows alone define the shape of the diagram, e. g. a triangle or a square. This shape can be made into a category of its own right [3, 7], a *diagram scheme*, and the concrete diagram can then be considered as a functor, sending the vertices to the objects at the vertices and the arrows to the morphisms of the diagram. So the diagram scheme for commutative triangles

$$\begin{tikzcd} 1 \arrow[r, "\alpha"] \arrow[dr, "\gamma"'] & 2 \arrow[d, "\beta"] \\ & 3 \end{tikzcd}$$

has a total of three objects $\{1, 2, 3\}$ and 6 morphisms $\{\alpha, \beta, \gamma, \mathrm{id}_1, \mathrm{id}_2, \mathrm{id}_3\}$, identities included.

Let $\omega : \mathcal{D} \longrightarrow \mathcal{A}$ be a diagram in $\mathcal{A}$. The category $\mathcal{D}$ and thus the diagram ω is always assumed to be small. We call the diagram *finite* if the functor $\omega : \mathcal{D} \longrightarrow \mathcal{A}$ factors through $\mathcal{A}_0$ that is if all objects of the diagram have duals.

THEOREM 2.1 (the existence of cohom of diagrams)
Let $(\mathcal{D}, \omega)$ and $(\mathcal{D}, \omega')$ be two diagrams in $\mathcal{A}$ over the same diagram scheme and let $(\mathcal{D}, \omega')$ be finite. Then the set of all natural transformations $\mathcal{M}or_f(\omega, M \otimes \omega')$ is representable as a functor in M.

Proof: This has been proved in various special case in [2], [10], [16], and [12]. We follow [12] 2.1.9 and give the explicit construction here since this construction will be used later on. As remarked in [12] 2.1.7 and 2.1.9 there are isomorphisms

$$\begin{aligned} \mathcal{M}\mathrm{or}_f(\omega, M \otimes \omega') &= \int_{X \in \mathcal{D}} \mathrm{Hom}(\omega(X), M \otimes \omega'(X)) && ([3]\ \mathrm{IX}.5(2)) \\ &= \int_{X \in \mathcal{D}} \mathrm{Hom}(\omega(X) \otimes \omega'(X)^*, M) && [\omega'(X) \text{ must have a dual!}] \\ &= \mathrm{Hom}(\int^{X \in \mathcal{D}} \omega(X) \otimes \omega'(X)^*, M) \end{aligned}$$

where the integral means the end resp. coend of the given bifunctors in the sense of [3]. Hence the representing object is

$$\int^{X \in \mathcal{D}} \omega(X) \otimes \omega'(X)^* =$$

$$\mathrm{diffcoker}\Big(\coprod_{f \in \mathrm{Mor}(\mathcal{D})} \omega(\mathrm{dom}(f)) \otimes \omega'(\mathrm{cod}(f))^* \underset{F_D}{\overset{F_C}{\rightrightarrows}} \coprod_{X \in \mathrm{Ob}(\mathcal{D})} \omega(X) \otimes \omega'(X)^* \Big),$$

whence it can be written as a quotient of a certain coproduct. The two morphisms F_C and F_D are composed of morphisms of the form $1 \otimes \omega'(f)^* : \omega(\mathrm{dom}(f)) \otimes \omega'(\mathrm{cod}(f))^* \longrightarrow \omega(\mathrm{dom}(f)) \otimes \omega'(\mathrm{dom}(f))^*$ respectively

$$\omega(f) \otimes 1 : \omega(\mathrm{dom}(f)) \otimes \omega'(\mathrm{cod}(f))^* \longrightarrow \omega(\mathrm{cod}(f)) \otimes \omega'(\mathrm{cod}(f))^*.$$

$\square$

DEFINITION 2.2 (cohomomorphisms)
The representing object of $\mathcal{M}or_f(\omega, M \otimes \omega')$ is written as

$$\mathrm{cohom}(\omega', \omega) :=$$
$$\mathrm{diffcoker}\Big(\coprod_{f \in \mathrm{Mor}(\mathcal{D})} \omega(\mathrm{dom}(f)) \otimes \omega'(\mathrm{cod}(f))^* \underset{F_D}{\overset{F_C}{\rightrightarrows}} \coprod_{X \in \mathrm{Ob}(\mathcal{D})} \omega(X) \otimes \omega'(X)^* \Big).$$

The elements of cohom(ω', ω) *are called* cohomomorphisms. *If $\omega = \omega'$ then the representing object of $\mathcal{M}or_f(\omega, M \otimes \omega)$ is written as* coend(ω), *the set of* coendomorphisms *of ω.*

It is essential for us to describe very explicitly the equivalence relation for the difference cokernel. Let $f : X \longrightarrow Y$ be a morphism in $\mathcal{D}$. It induces homomorphisms $\omega(f) : \omega(X) \longrightarrow \omega(Y)$ and $\omega'(f)^* : \omega'(Y)^* \longrightarrow \omega'(X)^*$. We thus obtain the two components of the maps F_C and F_D

$$\begin{tikzcd}[row sep=small] & \omega(X) \otimes \omega'(X)^* \\ \omega(X) \otimes \omega'(Y)^* \arrow[ur, "1\otimes\omega'(f)^*"] \arrow[dr, "\omega(f)\otimes 1"'] & \\ & \omega(Y) \otimes \omega'(Y)^* \end{tikzcd}$$

which map elements $x \otimes \eta \in \omega(X) \otimes \omega'(Y)^*$ to two equivalent elements $x \otimes \omega'(f)^*(\eta) \sim \omega(f)(x) \otimes \eta$. So $\mathrm{cohom}(\omega', \omega)$ is the direct sum of the objects $\omega(X) \otimes \omega'(X)^*$ for all $X \in \mathcal{D}$ modulo this equivalence relation.

Since a representable functor defines a universal arrow we get

COROLLARY 2.3 (the universal property of cohoms)
Let $(\mathcal{D}, \omega)$ and $(\mathcal{D}, \omega')$ be two diagrams in $\mathcal{A}$ and let $(\mathcal{D}, \omega')$ be finite. Then there is a natural transformation $\delta : \omega \longrightarrow \mathrm{cohom}(\omega', \omega) \otimes \omega'$, such that for each object $M \in \mathcal{A}$ and each natural transformation $\varphi : \omega \longrightarrow M \otimes \omega'$ there is a unique morphism $\tilde{\varphi} : \mathrm{cohom}(\omega', \omega) \longrightarrow M$ such that the diagram

$$\begin{tikzcd} \omega \arrow[r, "\delta"] \arrow[dr, "\varphi"'] & \mathrm{cohom}(\omega', \omega) \otimes \omega' \arrow[d, "\tilde{\varphi} \otimes 1"] \\ & M \otimes \omega' \end{tikzcd}$$

commutes.

By [12] 2.1.12 the coendomorphism set $\mathrm{coend}(\omega)$ is a coalgebra. The comultiplication arises from the commutative diagram

$$\begin{tikzcd} \omega \arrow[r, "\delta"] \arrow[d, "\delta"'] & \mathrm{coend}(\omega) \otimes \omega \arrow[d, "\Delta \otimes 1"] \\ \mathrm{coend}(\omega) \otimes \omega \arrow[r, "1 \otimes \delta"] & \mathrm{coend}(\omega) \otimes \mathrm{coend}(\omega) \otimes \omega. \end{tikzcd}$$

Somewhat more generally one shows

PROPOSITION 2.4 (the comultiplication of cohoms)
Let $(\mathcal{D}, \omega), (\mathcal{D}, \omega')$, and $(\mathcal{D}, \omega'')$ be diagrams in $\mathcal{A}$ and let $(\mathcal{D}, \omega')$ and $(\mathcal{D}, \omega'')$ be finite. Then there is a comultiplication

$$\Delta : \mathrm{cohom}(\omega'', \omega) \longrightarrow \mathrm{cohom}(\omega', \omega) \otimes \mathrm{cohom}(\omega'', \omega')$$

which is coassociative and counitary.

Proof: By 2.3 the homomorphism Δ is induced by the following commutative diagram

$$\begin{tikzcd} \omega \arrow[r, "\delta"] \arrow[d, "\delta"'] & \mathrm{cohom}(\omega'', \omega) \otimes \omega'' \arrow[d, "\Delta \otimes 1"] \\ \mathrm{cohom}(\omega', \omega) \otimes \omega' \arrow[r, "1 \otimes \delta"] & \mathrm{cohom}(\omega', \omega) \otimes \mathrm{cohom}(\omega'', \omega') \otimes \omega''. \end{tikzcd}$$

A simple calculation shows that this comultiplication is coassociative and counitary. □

COROLLARY 2.5 (coend(ω) is a coalgebra)
Let $(\mathcal{D},\omega)$ and $(\mathcal{D},\omega')$ be finite diagrams in $\mathcal{A}$. Then $\mathrm{coend}(\omega)$ *and* $\mathrm{coend}(\omega')$ *are coalgebras, all objects $\omega(X)$ resp. $\omega'(X)$ are comodules, and* $\mathrm{cohom}(\omega',\omega)$ *is a right* $\mathrm{coend}(\omega')$*-comodule and a left* $\mathrm{coend}(\omega)$*-comodule.*

Proof: A consequence of the preceding proposition is that $\mathrm{coend}(\omega)$ and $\mathrm{coend}(\omega')$ are coalgebras. The comodule structure on $\omega(X)$ comes from $\delta_X : \omega(X) \longrightarrow \mathrm{coend}(\omega) \otimes \omega(X)$. The other comodule structures are clear. □

What we have found here is in essence the universal coalgebra $C = \mathrm{coend}(\omega)$ such that all vector spaces and all homomorphisms of the diagram are comodules over C resp. comodule homomorphisms. In fact an easy consequence of the last corollary is

COROLLARY 2.6 (the universal coalgebra for a diagram of comodules)
Let $(\mathcal{D},\omega)$ be a finite diagram in $\mathcal{A}$. Then all objects $\omega(X)$ are comodules over $\mathrm{coend}(\omega)$ *and all morphisms $\omega(f)$ are comodule homomorphisms. If D is another coalgebra and all $\omega(X)$ are comodules over D and all $\omega(f)$ are comodule homomorphisms, then there is a unique coalgebra homomorphism $\varphi : \mathrm{coend}(\omega) \longrightarrow D$ such that the diagrams*

$$\begin{tikzcd}
\omega(X) \arrow[r, "\delta(X)"] \arrow[dr, "\varphi"'] & \mathrm{coend}(\omega)\otimes\omega(X) \arrow[d, "\tilde\varphi(X)\otimes 1"] \\
 & D\otimes\omega(X)
\end{tikzcd}$$

commute.

Proof: $\varphi(X) : \omega(X) \longrightarrow D \otimes \omega(X)$ is in fact a natural transformation. Then the existence and uniqueness of a homomorphism of vector spaces $\tilde\varphi : \mathrm{coend}(\omega) \longrightarrow D$ is obvious. The fact that this is a homomorphism of coalgebras follows from the universal property of $C = \mathrm{coend}(\omega)$ by

$$\begin{tikzcd}
\omega \arrow[rr, "\delta"] \arrow[dr, "\delta"] \arrow[dd, "1"'] & & C\otimes\omega \arrow[dr, "\Delta\otimes 1"] \arrow[dd] & \\
 & C\otimes\omega \arrow[rr, "1\otimes\delta", "\tilde\varphi\otimes 1"'] \arrow[dd] & & C\otimes C\otimes\omega \arrow[dd, "\tilde\varphi\otimes\tilde\varphi\otimes 1"] \\
\omega \arrow[rr, "\tilde\varphi\otimes 1", "\varphi"'] \arrow[dr, "\varphi"'] & & D\otimes\omega \arrow[dr, "\Delta\otimes 1"'] & \\
 & D\otimes\omega \arrow[rr, "1\otimes\varphi"'] & & D\otimes D\otimes\omega
\end{tikzcd}$$

where the right side of the cube commutes by the universal property. Similarly one proves that $\tilde{\varphi}$ preserves the counit. $\square$

To describe the coalgebra structure of $\mathrm{coend}(\omega)$ we denote the image of $x \otimes \xi \in \omega(X) \otimes \omega(X)^*$ in $\mathrm{coend}(\omega)$ by $\overline{x \otimes \xi}$. Let furthermore $\sum x_i \otimes \xi_i$ denote the dual basis in $\omega(X) \otimes \omega(X)^*$. Then $\delta : \omega(X) \longrightarrow \mathrm{coend}(\omega) \otimes \omega(X)$ is induced by the map $\omega(X) \otimes \omega(X)^* \longrightarrow \mathrm{coend}(\omega)$ and is given by

$$\delta(x \otimes \xi) = \sum \overline{x \otimes \xi_i} \otimes x_i.$$

The construction of $\Delta : \mathrm{coend}(\omega) \longrightarrow \mathrm{coend}(\omega) \otimes \mathrm{coend}(\omega)$ furnishes

$$\Delta(\overline{x \otimes \xi}) = \sum \overline{x \otimes \xi_i} \otimes \overline{x_i \otimes \xi}.$$

COROLLARY 2.7 (diagram restrictions induce morphisms for the cohoms)
Let $\mathcal{D}$ and $\mathcal{D}'$ be diagram schemes and let $\omega : \mathcal{D}' \longrightarrow \mathcal{A}$ and $\omega' : \mathcal{D}' \longrightarrow \mathcal{A}$ be finite diagrams. Let $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{D}'$ be a functor. Then $\mathcal{F}$ induces a homomorphism $\tilde{\varphi} : \mathrm{cohom}(\omega'\mathcal{F}, \omega\mathcal{F}) \longrightarrow \mathrm{cohom}(\omega', \omega)$ that is compatible with the comultiplication on cohoms as described in 2.4. In particular $\mathcal{F}$ induces a coalgebra homomorphism $\mathrm{coend}(\omega\mathcal{F}) \longrightarrow \mathrm{coend}(\omega)$.

Proof: We obtain a morphism $\tilde{\varphi} : \mathrm{cohom}(\omega''\mathcal{F}, \omega'\mathcal{F}) \longrightarrow \mathrm{cohom}(\omega'', \omega')$ from

$$\begin{tikzcd}
\omega'\mathcal{F} \arrow[r, "\delta"] \arrow[dr, "\delta\mathcal{F}"'] & \mathrm{cohom}(\omega''\mathcal{F}, \omega'\mathcal{F}) \otimes \omega''\mathcal{F} \arrow[d, "\tilde{\varphi} \otimes 1"] \\
 & \mathrm{cohom}(\omega'', \omega') \otimes \omega''\mathcal{F}.
\end{tikzcd}$$

that induces a commutative diagram

$$\begin{tikzcd}
\mathrm{cohom}(\omega''\mathcal{F}, \omega\mathcal{F}) \arrow[r, "\Delta"] \arrow[d, "\tilde{\varphi}"'] & \mathrm{cohom}(\omega'\mathcal{F}, \omega\mathcal{F}) \otimes \mathrm{cohom}(\omega''\mathcal{F}, \omega'\mathcal{F}) \arrow[d, "\tilde{\varphi} \otimes \tilde{\varphi}"] \\
\mathrm{cohom}(\omega'', \omega) \arrow[r, "\Delta"] & \mathrm{cohom}(\omega', \omega) \otimes \mathrm{cohom}(\omega'', \omega')
\end{tikzcd}$$

$\square$

PROPOSITION 2.8 (isomorphic cohoms)
Let $\mathcal{D}$ and $\mathcal{D}'$ be diagram schemes and let $\omega : \mathcal{D}' \longrightarrow \mathcal{A}$ and $\omega' : \mathcal{D}' \longrightarrow \mathcal{A}$ be diagrams such that $(\mathcal{D}', \omega')$ is finite. Let $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{D}'$ be a functor which is bijective an the objects and surjective on the morphism sets. Then the map $\tilde{\varphi} : \mathrm{cohom}(\omega'\mathcal{F}, \omega\mathcal{F}) \longrightarrow \mathrm{cohom}(\omega', \omega)$ is an isomorphism.

Proof: Since by definition the objects $\mathrm{cohom}(\omega'\mathcal{F}, \omega\mathcal{F})$ and $\mathrm{cohom}(\omega', \omega)$ represent the functors $\mathcal{M}\mathrm{or}_f(\omega\mathcal{F}, M \otimes \omega'\mathcal{F})$ resp. $\mathcal{M}\mathrm{or}_f(\omega, M \otimes \omega')$ it suffices to show that these functors are isomorphic. But every natural transformation $\varphi : \omega\mathcal{F} \longrightarrow M \otimes \omega'\mathcal{F}$ makes the diagrams

$$\begin{tikzcd}
\omega\mathcal{F}(X) \arrow[r, "\varphi(X)"] \arrow[d, "\omega\mathcal{F}(f)"'] & M \otimes \omega'\mathcal{F}(X) \arrow[d, "M \otimes \omega'\mathcal{F}(f)"] \\
\omega\mathcal{F}(Y) \arrow[r, "\varphi(Y)"] & M \otimes \omega'\mathcal{F}(Y)
\end{tikzcd}$$

commute for all f in $\mathcal{D}$. Similarly every natural transformation $\psi : \omega \longrightarrow \omega'$ makes the diagrams

$$\begin{tikzcd} \omega(X') \arrow[r, "\psi(X')"] \arrow[d, "\omega(f')"'] & M \otimes \omega'(X') \arrow[d, "M \otimes \omega'(X')"] \\ \omega(Y') \arrow[r, "\psi(Y')"] & M \otimes \omega'(Y') \end{tikzcd}$$

commute for all f' in $\mathcal{D}'$. Since $\mathcal{F}$ is bijective on the objects and surjective on the morphisms, we can identify the natural transformations φ and ψ. $\square$

2.3 Monoidal diagrams and bialgebras

If we consider additional structures like tensor products on the diagrams, we get additional structure for the objects $\mathrm{cohom}(\omega', \omega)$. In particular we find examples of bialgebras and comodule algebras. For this we have to assume now that the category $\mathcal{A}$ is (quasi)symmetric. We first recall some definitions about monoidal categories.

DEFINITION 2.9 (monoidal diagrams)
Let $(\mathcal{D}, \omega)$ be a diagram in $\mathcal{A}$. Assume that $\mathcal{D}$ is a monoidal category and that ω is a monoidal functor. Then we call the diagram $(\mathcal{D}, \omega)$ a monoidal diagram.

DEFINITION 2.10 (monoidal transformations)
Let $(\mathcal{D}, \omega)$ and $(\mathcal{D}, \omega')$ be monoidal diagrams in $\mathcal{A}$. Let $C \in \mathcal{A}$ be an algebra. A natural transformation $\varphi : \omega \longrightarrow C \otimes \omega'$ is called monoidal, *if the diagrams*

$$\begin{tikzcd} \omega(X) \otimes \omega(Y) \arrow[r, "\varphi(X) \otimes \varphi(Y)"] \arrow[d, "\rho"'] & C \otimes C \otimes \omega'(X) \otimes \omega'(Y) \arrow[d, "m \otimes \rho"] \\ \omega(X \otimes Y) \arrow[r, "\varphi(X \otimes Y)"] & C \otimes \omega'(X \otimes Y) \end{tikzcd}$$

and

$$\begin{tikzcd} K \arrow[r, "\cong"] \arrow[d] & K \otimes K \arrow[d] \\ \omega(I) \arrow[r, "\varphi(I)"] & C \otimes \omega'(I) \end{tikzcd}$$

commute.

Let $\omega : \mathcal{D} \longrightarrow \mathcal{A}$ and $\omega' : \mathcal{D}' \longrightarrow \mathcal{A}$ be diagrams in $\mathcal{A}$. We define the tensor product of these diagrams by $(\mathcal{D}, \omega) \otimes (\mathcal{D}', \omega') = (\mathcal{D} \times \mathcal{D}', \omega \otimes \omega')$ where $(\omega \otimes \omega')(X, Y) = \omega(X) \otimes \omega'(Y)$. The tensor product of two diagrams can be viewed as the diagram consisting of all the tensor products of all the objects of the first diagram and all the objects of the second diagram; similarly for the morphisms of the diagrams.

PROPOSITION 2.11 (the tensor product of cohoms)
Let $(\mathcal{D}_1, \omega_1)$, $(\mathcal{D}_2, \omega_2)$, $(\mathcal{D}_1, \omega_1')$, and $(\mathcal{D}_2, \omega_2')$ be diagrams in $\mathcal{A}$ and let $(\mathcal{D}_1, \omega_1')$ and $(\mathcal{D}_2, \omega_2')$ be finite. Then

$$\mathrm{cohom}(\omega_1' \otimes \omega_2', \omega_1 \otimes \omega_2) \cong \mathrm{cohom}(\omega_1', \omega_1) \otimes \mathrm{cohom}(\omega_2', \omega_2).$$

Proof: Similar to the proof in [12] 2. 3. 6 we get

$$\begin{aligned}
\mathrm{cohom}(\omega_1',\omega_1) \quad \otimes \quad & \mathrm{cohom}(\omega_2',\omega_2)\\
& \cong \Big(\int^{X\in\mathcal{D}_1} \omega_1(X)\otimes\omega_1'(X)^*\Big)\otimes\Big(\int^{Y\in\mathcal{D}_2}\omega_2(Y)\otimes\omega_2'(Y)^*\Big)\\
& \cong \int^{X\in\mathcal{D}_1}\int^{Y\in\mathcal{D}_2}(\omega_1(X)\otimes\omega_1'(X)^*\otimes\omega_2(Y)\otimes\omega_2'(Y)^*)\\
& \cong \int^{(X,Y)\in\mathcal{D}_1\times\mathcal{D}_2}(\omega_1(X)\otimes\omega_1'(X)^*\otimes\omega_2(Y)\otimes\omega_2'(Y)^*)\\
& \cong \int^{(X,Y)\in\mathcal{D}_1\times\mathcal{D}_2}(\omega_1\otimes\omega_2)(X,Y)\otimes(\omega_1'\otimes\omega_2')(X,Y)^*\\
& \cong \mathrm{cohom}(\omega_1'\otimes\omega_2',\omega_1\otimes\omega_2).
\end{aligned}$$

This uses the fact that colimits commute with tensor products in $\mathcal{A}$. □

If we look at the representation of coend(ω) in Definition 2.2 then this isomorphism identifies an element $\overline{x\otimes\xi}\otimes\overline{y\otimes\eta}\in\mathrm{cohom}(\omega_1',\omega_1)\otimes\mathrm{cohom}(\omega_2',\omega_2)$ with representative element

$$(x\otimes\xi)\otimes(y\otimes\eta)\in(\omega_1(X)\otimes\omega_1'(X)^*)\otimes(\omega_2(Y)\otimes\omega_2'(Y)^*)$$

with the element $\overline{(x\otimes y)\otimes(\xi\otimes\eta)}\in\mathrm{cohom}(\omega_1'\otimes\omega_2',\omega_1\otimes\omega_2)$.

COROLLARY 2.12 (the universal property of the tensor product of cohoms)
Under the assumptions of Proposition 2.11 there is a natural transformation

$$\delta:\omega_1\otimes\omega_2\longrightarrow\mathrm{cohom}(\omega_1',\omega_1)\otimes\mathrm{cohom}(\omega_2',\omega_2)\otimes\omega_1'\otimes\omega_2',$$

such that for each object $M\in\mathcal{A}$ and each natural transformation $\varphi:\omega_1\otimes\omega_2\longrightarrow M\otimes\omega_1'\otimes\omega_2'$ there is a unique morphism $\tilde\varphi:\mathrm{cohom}(\omega_1',\omega_1)\otimes\mathrm{cohom}(\omega_2',\omega_2)\longrightarrow M$ such that the diagram

$$\begin{tikzcd}
\omega_1\otimes\omega_2 \arrow[r, "\delta"] \arrow[dr, "\varphi"'] & \mathrm{cohom}(\omega_1',\omega_1)\otimes\mathrm{cohom}(\omega_2',\omega_2)\otimes\omega_1'\otimes\omega_2' \arrow[d, "\tilde\varphi\otimes 1"]\\
 & M\otimes\omega_1'\otimes\omega_2'
\end{tikzcd}$$

commutes.

THEOREM 2.13 (cohom is an algebra)
Let $(\mathcal{D},\omega)$ and $(\mathcal{D},\omega')$ be two monoidal diagrams in $\mathcal{A}$ and let $(\mathcal{D},\omega')$ be finite. Then $\mathrm{cohom}(\omega',\omega)$ *is an algebra in $\mathcal{A}$ and $\delta:\omega\longrightarrow\mathrm{cohom}(\omega',\omega)\otimes\omega'$ is monoidal.*

Proof: The multiplication on $\mathrm{cohom}(\omega',\omega)$ results from the commutative diagram

$$\begin{tikzcd}
\omega(X)\otimes\omega(Y) \arrow[r, "\delta\otimes\delta"] \arrow[d] & \mathrm{cohom}(\omega',\omega)\otimes\mathrm{cohom}(\omega',\omega)\otimes\omega'(X)\otimes\omega'(Y) \arrow[d]\\
\omega(X\otimes Y) \arrow[r, "\delta"] & \mathrm{cohom}(\omega',\omega)\otimes\omega'(X\otimes Y).
\end{tikzcd}$$

and from an application of Proposition 2.11.

Let $\mathcal{D}_0 = (\{I\}, \{\mathrm{id}\})$ together with $\omega_0 : \mathcal{D}_0 \longrightarrow \mathcal{A}$ and $\omega_0(I) = K$ be the monoidal "unit object" diagram of the monoidal category $\mathrm{Diag}(\mathcal{A})$. Then $(K \longrightarrow K \otimes K) = (\omega_0 \longrightarrow \mathrm{coend}(\omega_0) \otimes \omega_0)$ is the universal arrow. The unit on $\mathrm{cohom}(\omega', \omega)$ is given by the commutative diagram

$$\begin{tikzcd} K \arrow[r, "\cong"] \arrow[d] & K \otimes K \arrow[d] \\ \omega(I) \arrow[r] & \mathrm{cohom}(\omega', \omega) \otimes \omega'(I). \end{tikzcd}$$

It is easy to verify the algebra laws and the additional claims of the theorem. □

PROPOSITION 2.14 (the product in the algebra cohom)
The product in $\mathrm{cohom}(\omega', \omega)$ *is given by*

$$(\overline{x \otimes \xi}) \cdot (\overline{y \otimes \eta}) = \overline{x \otimes y \otimes \xi \otimes \eta}.$$

Proof: We apply the diagram defining the multiplication of $\mathrm{cohom}(\omega', \omega)$ to an element $\overline{x \otimes y}$ and obtain $\sum (\overline{x \otimes \xi_i}) \cdot (\overline{y \otimes \eta_j}) \otimes x_i \otimes y_j = \sum \overline{x \otimes y \otimes \xi_i \otimes \eta_j} \otimes x_i \otimes y_j$, where for simplicity of notation $\omega(X) \otimes \omega(Y) = \omega(X \otimes Y)$ and $\omega'(X) \otimes \omega'(Y) = \omega'(X \otimes Y)$ are identified and where $\sum x_i \otimes \xi_i \in \omega'(X) \otimes \omega'(X)^*$ and $\sum y_j \otimes \eta_j \in \omega'(Y) \otimes \omega'(Y)^*$ are dual bases. This implies the proposition. □

One easily proves the following corollaries

COROLLARY 2.15 (Δ is an algebra morphism)
Let the assumptions of Proposition2.4 be satisfied and let all diagrams $(\mathcal{D}, \omega)$, $(\mathcal{D}, \omega')$, *and* $(\mathcal{D}, \omega'')$ *be monoidal. Then the morphism*

$$\Delta : \mathrm{cohom}(\omega'', \omega) \longrightarrow \mathrm{cohom}(\omega', \omega) \otimes \mathrm{cohom}(\omega'', \omega')$$

in 2.4 is an algebra homomorphism. □

COROLLARY 2.16 ($\mathrm{coend}(\omega)$ is a bialgebra)
Let $(\mathcal{D}, \omega)$ *and* $(\mathcal{D}, \omega')$ *be two monoidal diagrams in* $\mathcal{A}$ *and let* $(\mathcal{D}, \omega')$ *be finite. Then* $\mathrm{coend}(\omega')$ *and, if* $(\mathcal{D}, \omega)$ *is finite,* $\mathrm{coend}(\omega)$ *are bialgebras and* $\mathrm{cohom}(\omega', \omega)$ *is a right* $\mathrm{coend}(\omega')$*-comodule algebra and a left* $\mathrm{coend}(\omega)$*-comodule algebra.* □

COROLLARY 2.17 (monoidal diagram morphisms induce algebra morphisms for the cohoms)
Let $\mathcal{D}$ *and* $\mathcal{D}'$ *be monoidal diagram schemes and let* $\omega : \mathcal{D}' \longrightarrow \mathcal{A}$ *and* $\omega' : \mathcal{D}' \longrightarrow \mathcal{A}$ *be monoidal diagrams. Let* $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{D}'$ *be a monoidal functor. Then* $\mathcal{F}$ *induces algebra*

homomorphism $f : \mathrm{cohom}(\omega'\mathcal{F}, \omega\mathcal{F}) \longrightarrow \mathrm{cohom}(\omega', \omega)$ *which is compatible with the comultiplication on cohoms as described in 2.4. Furthermore* $\mathcal{F}$ *induces a bialgebra homomorphism* $\mathrm{coend}(\omega'\mathcal{F}) \longrightarrow \mathrm{coend}(\omega')$. $\square$

After having established the structure of an algebra on $\mathrm{cohom}(\omega', \omega)$, we are now interested in algebra homomorphisms from $\mathrm{cohom}(\omega', \omega)$ to other algebras.

COROLLARY 2.18 (the universal property of the algebra cohom)
Let $(\mathcal{D}, \omega)$ *and* $(\mathcal{D}, \omega')$ *be monoidal diagrams in* $\mathcal{A}$ *and let* $(\mathcal{D}, \omega')$ *be finite. Then there is a natural monoidal transformation* $\delta : \omega \longrightarrow \mathrm{cohom}(\omega', \omega) \otimes \omega'$*, such that for each algebra* $C \in \mathcal{A}$ *and each natural monoidal transformation* $\varphi : \omega \longrightarrow C \otimes \omega'$ *there is a unique algebra morphism* $f : \mathrm{cohom}(\omega', \omega) \longrightarrow C$ *such that the diagram*

$$\begin{tikzcd}
\omega \arrow[r, "\delta"] \arrow[dr, "\varphi"'] & \mathrm{cohom}(\omega', \omega) \otimes \omega' \arrow[d, "f \otimes 1"] \\
& C \otimes \omega'
\end{tikzcd}$$

commutes.

Proof: The multiplication of $\mathrm{cohom}(\omega', \omega)$ was given in the proof of Theorem 2.13. Hence we get the following commutative diagram

$$\begin{tikzcd}
\omega(X) \otimes \omega(Y) \arrow[r] \arrow[drr] \arrow[dd] \arrow[ddr, phantom, "2"] & \mathrm{cohom}(\omega', \omega) \otimes \mathrm{cohom}(\omega', \omega) \otimes \omega'(X) \otimes \omega'(Y) \arrow[dd] \arrow[dr] \arrow[d, phantom, "1"] & \\
& & C \otimes C \otimes \omega'(X) \otimes \omega'(Y) \arrow[dd] \arrow[d, phantom, "5"] \\
\omega(X \otimes Y) \arrow[r] \arrow[drr] \arrow[ur, phantom, "3"] & \mathrm{cohom}(\omega', \omega) \otimes \omega'(X \otimes Y) \arrow[dr] \arrow[d, phantom, "4"] & \\
& & C \otimes \omega'(X \otimes Y)
\end{tikzcd}$$

where triangle 1 commutes by Corollary 2.12, square 2 commutes by the definition of the multiplication in Theorem 2.13, square 3 commutes since φ is a natural monoidal transformation, and triangle 4 commutes by the universal property of $\mathrm{cohom}(\omega', \omega)$. Thus by the universal property from Corollary 2.12 the square 5 also commutes so that the universal map $\mathrm{cohom}(\omega', \omega) \longrightarrow C$ is compatible with the multiplication.

We leave it to the reader to check that this map is also compatible with the unit, whence it is an algebra homomorphism. $\square$

We denote by $\mathcal{M}\mathrm{or}_f^{\otimes}(\omega, C \otimes \omega')$ the set of natural monoidal transformations $\varphi : \omega \longrightarrow C \otimes \omega'$. It is easy to see that this is a functor in algebras C. Thus we have proved

COROLLARY 2.19 (cohom as a representing object)
For diagrams $(\mathcal{D}, \omega)$ and $(\mathcal{D}, \omega')$, with $(\mathcal{D}, \omega')$ finite, there is a natural isomorphism

$$\mathcal{M}or_f(\omega, M \otimes \omega') \cong \mathrm{Hom}(\mathrm{cohom}(\omega', \omega), M).$$

If the diagrams are monoidal then there is a natural isomorphism

$$\mathcal{M}or_f^{\otimes}(\omega, C \otimes \omega') \cong K\text{-}\mathrm{Alg}(\mathrm{cohom}(\omega', \omega), C).$$

We end this section with an interesting observation about a particularly large "diagram" namely the underlying functor $\omega : \mathcal{C}\mathrm{omod}\text{-}C \longrightarrow \mathcal{A}$. It says that if all "representations" of a coalgebra resp. bialgebra are known then the coalgebra resp. the bialgebra can be recovered.

THEOREM 2.20 (Tannaka duality)
Let C be a coalgebra. Then $C \cong \mathrm{coend}(\omega)$ as coalgebras where $\omega : \mathcal{C}omod\text{-}C \longrightarrow \mathcal{A}$ is the underlying functor. If C is a bialgebra and ω monoidal, then the coalgebras are isomorphic as bialgebras.

For a proof we refer to [9] Corollary 6. 4. and [10] Theorem 15 and Corollary 16. $\square$

3 Coendomorphism Bialgebras of Quantum Spaces

In this section we will establish the connection between the bialgebras arising as coendomorphism bialgebras of quantum spaces and those arising as coendomorphism bialgebras of diagrams. It will turn out that the construction via diagrams is far more general. So by Theorem 2.1 and Corollary 2.16 we have established the general existence of these bialgebras. And we can apply this to give an explicit description of coendomorphism bialgebras of finite dimensional algebras, of (finite) quadratic algebras, of families of quadratic algebras, and of finite dimensional Lie algebras.

3.1 Generators and relations of the algebra of cohomomorphisms

We will construct monoidal diagrams generated by a given (finite) family of objects, of morphisms, and of commutativity conditions. Then we will calculate the associated coendomorphism bialgebras. Since in a monoidal category there are two compositions, the tensor product and the composition of morphisms, we are constructing a free (partially defined) algebra from the objects, morphisms and relations.

Let $X_1, \ldots, X_n$ be a given set of objects. Then there is a free monoidal category $\mathcal{C}[X_1, \ldots, X_n]$ generated by the $X_1, \ldots, X_n$ constructed in an analogous way as the free monoidal category on one generating object in [3].

Let $X_1, \ldots, X_n$ be a given set of objects and let $\varphi_1, \ldots, \varphi_m$ be *additional* morphisms between the objects of the free monoidal category $\mathcal{C}[X_1, \ldots, X_n]$ generated by $X_1, \ldots, X_n$.

Then there is a free monoidal category $\mathcal{C}[X_1, \ldots, X_n; \varphi_1, \ldots, \varphi_m]$ generated by $X_1, \ldots, X_n$; $\varphi_1, \ldots, \varphi_m$.

If the objects $X_1, \ldots, X_n$ and morphisms $\varphi_1, \ldots, \varphi_m$ are taken in $\mathcal{A}$, then they induce a unique monoidal functor $\omega : \mathcal{C}[X_1, \ldots, X_n; \varphi_1, \ldots, \varphi_m] \longrightarrow \mathcal{A}$.

We indicate how the various free monoidal categories can be obtained. $\mathcal{C}[X_1, \ldots, X_n]$ is generated as follows. The set of objects is given by

(O_1) $X_1, \ldots, X_n$ are objects,

(O_2) I is an object,

(O_3) if Y_1 and Y_2 are objects then $Y_1 \otimes Y_2$ is an object (actually this object should be written as $(Y_1 \otimes Y_2)$ to avoid problems with the explicit associativity conditions),

(O_4) these are all objects.

The set of morphisms is given by

(M_1) for each object there is an identity morphism,

(M_2) for each object Y there are morphisms $\lambda : I \otimes Y \longrightarrow Y$, $\lambda^- : Y \longrightarrow I \otimes Y$, $\rho : Y \otimes I \longrightarrow Y$, and $\rho^- : Y \longrightarrow Y \otimes I$,

(M_3) for any three objects Y_1, Y_2, Y_3 there are morphisms $\alpha : Y_1 \otimes (Y_2 \otimes Y_3) \longrightarrow (Y_1 \otimes Y_2) \otimes Y_3$ and $\alpha^- : (Y_1 \otimes Y_2) \otimes Y_3 \longrightarrow Y_1 \otimes (Y_2 \otimes Y_3)$,

(M_4) if $f : Y_1 \longrightarrow Y_2$ and $g : Y_3 \longrightarrow Y_4$ are morphisms then $f \otimes g : Y_1 \otimes Y_3 \longrightarrow Y_2 \otimes Y_4$ is a morphism,

(M_5) if $f : Y_1 \longrightarrow Y_2$ and $g : Y_2 \longrightarrow Y_3$ are morphisms then $gf : Y_1 \longrightarrow Y_3$ is a morphism,

(M_7) these are all morphisms.

The morphisms are subject to the following congruence conditions with respect to the composition and the tensor product

(R_1) the associativity and unitary conditions of composition of morphisms,

(R_2) the associativity and unitary coherence condition for monoidal categories,

(R_3) the conditions that λ and λ^-, ρ and ρ^-, α and α^- are inverses of each other.

The free monoidal category $\mathcal{C}[X_1, \ldots, X_n; \varphi_1, \ldots, \varphi_m]$ for given objects $X_1, \ldots, X_n$ and morphisms $\varphi_1, \ldots, \varphi_m$ (where the $\varphi_1, \ldots, \varphi_m$ are additional new morphisms between objects of $\mathcal{C}[X_1, \ldots, X_n]$) is obtained by adding the following to the list of conditions for generating the set of morphisms

(M_6) $\varphi_1, \ldots, \varphi_m$ are morphisms.

If there are additional commutativity relations $r_1, \ldots, r_k$ for the morphisms expressed by the φ_i, they can be added to the defining congruence relations to define the free monoidal category $\mathcal{C}[X_1, \ldots, X_n; \varphi_1, \ldots, \varphi_m; r_1, \ldots, r_k]$ by

(R_4) $r_1, \ldots, r_k$ are in the congruence relation.

THEOREM 3.1 (the invariance of the coendomorphism bialgebra)
Let $X_1, \ldots, X_n$ be objects in $\mathcal{A}_0$ and let $\varphi_1, \ldots, \varphi_m$ be morphisms in $\mathcal{A}_0$ between tensor products of the objects $X_1, \ldots, X_n, I$. Let $r_1, \ldots, r_k$ be relations between the given morphisms in $\mathcal{A}_0$. Define $\mathcal{D} := \mathcal{C}[X_1, \ldots, X_n; \varphi_1, \ldots, \varphi_m]$ and $\mathcal{D}' := \mathcal{C}[X_1, \ldots, X_n; \varphi_1, \ldots, \varphi_m; r_1, \ldots, r_k]$ and let $\omega : \mathcal{D} \longrightarrow \mathcal{A}$ and $\omega' : \mathcal{D}' \longrightarrow \mathcal{A}$ be the corresponding underlying functors. Then $\mathrm{coend}(\omega) \cong \mathrm{coend}(\omega')$ *as bialgebras.*

Proof: follows immediately from Proposition 2.8 applied to the functor $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{D}'$ that is the identity on the objects and that sends morphisms to their congruence classes so that $\omega = \omega'\mathcal{F}$. □

LEMMA 3.2 (the generating set for cohom)
Let $\mathcal{D} = \mathcal{C}[X_1, \ldots, X_n; \varphi_1, \ldots, \varphi_m]$ be the freely generated monoidal category generated by the objects $X_1, \ldots, X_n$ and the morphisms $\varphi_1, \ldots, \varphi_m$. Let $(\mathcal{D}, \omega)$ and $(\mathcal{D}, \omega')$ be monoidal diagrams and let $(\mathcal{D}, \omega')$ be finite. Then $\mathrm{cohom}(\omega', \omega)$ *is generated as an algebra by the vector spaces $\omega(X_i) \otimes \omega'(X_i)^*$, $i = 1, \ldots, n$.*

Proof: The multiplication in $\mathrm{cohom}(\omega', \omega)$ is given by taking tensor products of the representatives as in Proposition 2.14. □

For objects $X_1, \ldots, X_n$ and morphisms $\varphi_1, \ldots, \varphi_m$ generating a free monoidal category we now get a complete explicit description of the algebra $\mathrm{cohom}(\omega', \omega)$ in terms of generators and relations. This result resembles that of [12] Lemma 2.1.16 and will be central for our further studies.

THEOREM 3.3 (representation of the algebra cohom)
Let $\mathcal{D} = \mathcal{C}[X_1, \ldots, X_n; \varphi_1, \ldots, \varphi_m]$ be the freely generated monoidal category generated by the objects $X_1, \ldots, X_n$ and the morphisms $\varphi_1, \ldots, \varphi_m$. Let $(\mathcal{D}, \omega)$ and $(\mathcal{D}, \omega')$ be monoidal diagrams and let $(\mathcal{D}, \omega')$ be finite. Then

$$\mathrm{cohom}(\omega', \omega) \cong T(\bigoplus_1^n \omega(X_i) \otimes \omega'(X_i)^*)/I$$

where $T(\bigoplus_1^n \omega(X_i) \otimes \omega'(X_i)^*)$ is the (free) tensor algebra generated by the spaces $\omega(X_i) \otimes \omega'(X_i)^*$ and where I is the two-sided ideal generated by the differences of the images of the $\varphi_1, \ldots, \varphi_m$ under the maps

$$\omega(\mathrm{dom}(\varphi_i)) \otimes \omega'(\mathrm{cod}(\varphi_i))^* \underset{F_D}{\overset{F_C}{\rightrightarrows}} \coprod \omega(X) \otimes \omega'(X)^* \cong T(\bigoplus_1^n \omega(X_i) \otimes \omega'(X_i)^*).$$

Proof: The tensor algebra contains all the spaces of the form $\omega(X) \otimes \omega'(X)$, $X \in \mathcal{D}$, since the objects X are iterated tensor products of the generating objects X_i and since the multiplication in $\mathrm{cohom}(\omega', \omega)$ as described in Proposition 2.14 identifies the images of $(\omega(X) \otimes \omega'(X)^*) \cdot ((\omega(Y) \otimes \omega'(Y)^*)$ with $(\omega(X \otimes Y) \otimes \omega'(X \otimes Y)^*)$.

Assume that $(f : X \longrightarrow Y) \in \mathcal{D}$ is a morphism and that for all $x \otimes \eta \in \omega(X) \otimes \omega'(Y)^*$ the elements $x \otimes \omega'(f)^*(\eta) - \omega(f)(x) \otimes \eta$ are in I. Let $Z \in \mathcal{D}$ and apply $f \otimes Z$ to $x \otimes z \otimes \eta \otimes \zeta \in$

$\omega(X \otimes Z) \otimes \omega'(X \otimes Z)^* \cong \omega(X) \otimes \omega'(Y)^* \otimes \omega(Z) \otimes \omega'(Z)^*$. Then we have

$$\begin{aligned}
&x \otimes z \otimes \omega'(f \otimes Z)^*(\eta \otimes \zeta) - \omega(f \otimes Z)(x \otimes z) \otimes \eta \otimes \zeta = \\
&\quad (x \otimes \omega'(f)^*(\eta)) \cdot (z \otimes \zeta) - (\omega(f)(x) \otimes \eta) \cdot (z \otimes \zeta) = \\
&\quad (x \otimes \omega'(f)^*(\eta) - \omega(f(x)) \otimes \eta) \cdot (z \otimes \zeta) \in I.
\end{aligned}$$

Assume now that $(f : X \longrightarrow Y), (g : Y \longrightarrow Z) \in \mathcal{D}$ are morphisms and that for all $x \otimes \eta \in \omega(X) \otimes \omega'(Y)^*$ and all $y \otimes \zeta \in \omega(Y) \otimes \omega'(Z)^*$ the elements $x \otimes \omega'(f)^*(\eta) - \omega(f)(x) \otimes \eta$ and $y \otimes \omega'(f)^*(\zeta) - \omega(f)(y) \otimes \zeta$ are in I. Then we have

$$\begin{aligned}
&x \otimes \omega'(gf)^*(\zeta) - \omega(gf)(x) \otimes \zeta = \\
&\quad \{x \otimes \omega'(f)^*[\omega'(g)^*(\zeta)] - \omega(f)(x) \otimes [\omega'(g)^*(\zeta)]\} + \\
&\quad \{[\omega(f)(x)] \otimes \omega'(g)^*(\zeta) - \omega(g)[\omega(f)(x)] \otimes \zeta\} \in I.
\end{aligned}$$

□

3.2 Finite quantum spaces

Let $\mathcal{D} = \mathcal{C}[X; m, u]$ be the free monoidal category generated by one object X, a multiplication $m : X \otimes X \longrightarrow X$ and a unit $u : I \longrightarrow X$. The objects of $\mathcal{D}$ are n-fold tensor products $X^{\otimes n}$ of X with itself. (Because of coherence theorems we can assume without of loss of generality that we have strict monoidal categories, i.e. that the associativity morphisms and the left and right unit morphisms are the identities.) Observe that we do not require that m is associative or that u acts as a unit, since by Theorem 3.1 any such relations are irrelevant for the construction of the coendomorphism bialgebras of diagrams over $\mathcal{D}$.

Let A be in K-Alg. Let $\omega_A : \mathcal{D} \longrightarrow \mathcal{A}$ be defined by sending X to the object $A \in \mathcal{A}$, and the multiplication and the unit in $\mathcal{D}$ to the multiplication resp. the unit of the algebra A in $\mathcal{A}$. Then ω_A is a monoidal functor. If A is finite dimensional, then the diagram $(\mathcal{D}, \omega_A)$ is finite.

For a finite dimensional algebra $A \in K$-Alg we can construct the coendomorphism bialgebra E_A of A as in subsection 1.5. We also can consider the corresponding diagram $(\mathcal{D}, \omega_A)$ in $\mathcal{A}$ and construct its coendomorphism bialgebra $\mathrm{coend}(\omega_A)$.

THEOREM 3.4 (cohomomorphisms of non-commutative spaces and of diagrams)
Let $\mathcal{X}$ and $\mathcal{Y}$ be quantum spaces with function algebras $A = \mathcal{O}(\mathcal{X})$ and $B = \mathcal{O}(\mathcal{Y})$. Let A be finite dimensional. Then $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ exists with $\mathcal{O}(\mathcal{H}(\mathcal{X}, \mathcal{Y})) = a(A, B) \cong \mathrm{cohom}(\omega_A, \omega_B)$.

Proof: Given a quantum space $\mathcal{Z}$ and a map of quantum spaces $\mathcal{Z} \perp \mathcal{X} \longrightarrow \mathcal{Y}$. Let $C = \mathcal{O}(\mathcal{Z})$. Then the map induces an algebra homomorphism $f : B \longrightarrow C \otimes A$. We construct the associated diagrams $(\mathcal{D}, \omega_A)$ and $(\mathcal{D}, \omega_B)$.

We will show now that there is a bijection between the algebra homomorphisms $f : B \longrightarrow C \otimes A$ and the monoidal natural transformations $\varphi : \omega_B \longrightarrow C \otimes \omega_A$. Given f we get φ by

$$\varphi(X^{\otimes n}) : \omega_B(X^{\otimes n}) = B^{\otimes n} \longrightarrow C^{\otimes n} \otimes A^{\otimes n} \stackrel{m_n \otimes 1}{\longrightarrow} C \otimes A^{\otimes n} = C \otimes \omega_A(X^{\otimes n}),$$

where $m_n : C^{\otimes n} \longrightarrow C$ is the n-fold multiplication. This is a natural transformation since the diagrams

$$\begin{tikzcd}
B \otimes B \arrow[rr, "\varphi(X \otimes X)"] \arrow[dr, "f \otimes f"] \arrow[ddd, "m"'] & & C \otimes A \otimes A \arrow[ddd, "1 \otimes m"] \\
& C \otimes C \otimes A \otimes A \arrow[ur, "m \otimes 1"'] \arrow[d, "m \otimes m"] & \\
& C \otimes A \arrow[dr, "1 \otimes 1"] & \\
B \arrow[ur, "f"] \arrow[rr, "\varphi(X)"'] & & C \otimes A
\end{tikzcd}$$

and

$$\begin{tikzcd}
K \arrow[r, "\varphi(I)"] \arrow[d, "u"'] & C \otimes K \arrow[d, "1 \otimes u"] \\
B \arrow[r, "\varphi(X)"'] & C \otimes A
\end{tikzcd}$$

commute. Furthermore the diagrams

$$\begin{tikzcd}
B^{\otimes r} \otimes B^{\otimes s} \arrow[rr, "\varphi(X^{\otimes r}) \otimes \varphi(X^{\otimes s})"] \arrow[dr] \arrow[ddd] & & C \otimes C \otimes A^{\otimes r} \otimes A^{\otimes s} \arrow[ddd] \\
& C^{\otimes r} \otimes C^{\otimes s} \otimes A^{\otimes r} \otimes A^{\otimes s} \arrow[ur] \arrow[d] & \\
& C^{\otimes(r+s)} \otimes A^{\otimes(r+s)} \arrow[dr] & \\
B^{\otimes(r+s)} \arrow[ur] \arrow[rr, "\varphi(X^{\otimes(r+s)})"'] & & C \otimes A^{\otimes(r+s)}
\end{tikzcd}$$

commute so that $\varphi : \omega_B \longrightarrow C \otimes \omega_A$ is a natural monoidal transformation.

Conversely given a natural transformation $\varphi : \omega_B \longrightarrow C \otimes \omega_A$ we get a morphism $f = \varphi(X) : B \longrightarrow C \otimes A$. The diagrams

$$\begin{tikzcd}
B \otimes B \arrow[r, "f \otimes f"] \arrow[d, "="'] & C \otimes C \otimes A \otimes A \arrow[d, "m \otimes 1"] \\
B \otimes B \arrow[r, "\varphi(X \otimes X)"] \arrow[d, "m"'] & C \otimes A \otimes A \arrow[d, "1 \otimes m"] \\
B \arrow[r, "f"] & C \otimes A
\end{tikzcd}$$

and

$$\begin{tikzcd}
K \arrow[r, "\cong"] \arrow[d, "="'] & K \otimes K \arrow[d] \\
K \arrow[r] \arrow[d, "u"'] & C \otimes K \arrow[d, "1 \otimes u"] \\
B \arrow[r, "f"] & C \otimes A
\end{tikzcd}$$

commute. Hence $f : B \longrightarrow C \otimes A$ is an algebra homomorphism. This defines a natural isomorphism $K\text{-Alg}(B, C \otimes A) \cong \mathcal{M}\mathrm{or}_f^{\otimes}(\omega_B, C \otimes \omega_A)$. If A is finite dimensional, then the left side is represented by $a(A, B)$ and the right side by $\mathrm{cohom}(\omega_A, \omega_B)$ (2.19). $\square$

COROLLARY 3.5 (isomorphic coendomorphism bialgebras)
There is a unique isomorphism $E_A \cong \mathrm{coend}(\omega_A)$ *of bialgebras such that the diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & E_A \otimes A \arrow[d] \\ & \mathrm{coend}(\omega_A) \otimes A \end{tikzcd}$$

commutes.

Proof: If the coendomorphism bialgebra E_A exists, then it satisfies the universal property given in Corollary 1.10. $\square$

COROLLARY 3.6 (Tambara [15] Thm.1.1)
Let A, B *be algebras and let* A *be finite dimensional. Then*

$$a(A, B) \cong T(B \otimes A^*)/(xy \otimes \zeta - \sum x \otimes y \otimes \zeta_{(1)} \otimes \zeta_{(2)}, \zeta(1) - 1_A \otimes \zeta | x, y \in B; \zeta \in A^*).$$

Proof: This is an immediate consequence of Theorem 3.3 and Theorem 3.4. In fact the map $m : X \otimes X \longrightarrow X$ induces by the construction in Theorem 2.1 the relations $xy \otimes \zeta \sim \sum x \otimes y \otimes \zeta_{(1)} \otimes \zeta_{(2)}$ for all $x \otimes y \otimes \zeta \in B \otimes B \otimes A^*$. Similarly the map $u : I \longrightarrow X$ induces $\zeta(1) - 1_A \otimes \zeta$ for all $1 \otimes \zeta \in K \otimes A^*$. $\square$

3.3 Quadratic quantum spaces

We now consider quadratic algebras in the sense of Manin [5]. They are $\mathbf{N}$-graded algebras A with $A_0 = K$, A generated by A_1 with (homogeneous) relations generated by $R \subseteq A_1 \otimes A_1$. The quadratic algebra corresponding to (A_1, R) is $T(A_1)/(R)$ where $T(A_1)$ is the tensor algebra generated by A_1. The admissible algebra homomorphisms are those generated by homomorphisms of vector spaces $f : A_1 \longrightarrow B_1$ such that $(f \otimes f)(R_A) \subseteq R_B$. Thus we obtain the category $\mathcal{QA}$ of quadratic algebras. The dual $\mathcal{QQ}$ of this category is the category of quadratic quantum spaces. We will simply denote quadratic algebras by (A, R) where we assume $R \subseteq A \otimes A$.

We consider the free monoidal category $\mathcal{D} = \mathcal{C}[X, Y; \iota]$ where $\iota : Y \longrightarrow X \otimes X$ in $\mathcal{D}$. Then each quadratic algebra (A, R) induces a monoidal functor $\omega_{(A,R)} : \mathcal{D} \longrightarrow \mathcal{A}$ with $\omega(X) = A$, $\omega(Y) = R$, and $\omega(\iota : Y \longrightarrow X \otimes X) = \iota : R \longrightarrow A \otimes A$.

For any two quadratic algebras (A, R) and (B, S), where A and R are finite dimensional, we can construct the universal algebra $\mathrm{cohom}(\omega_{(A,R)}, \omega_{(B,S)})$ satisfying Corollary 2.18. We show that this is the same algebra as the quantum homomorphism space $\mathrm{hom}((A, R), (B, S))$ constructed by Manin [5] 4.4. that has the universal property given in [5] 4. Theorem 5. In particular it is again a quadratic algebra.

THEOREM 3.7 (cohom of quadratic algebras)
Let (A,R) and (B,S) be quadratic algebras with (A,R) finite. Then

$$\mathrm{cohom}(\omega_{(A,R)},\omega_{(B,S)}) \cong (B \otimes A^*, S \otimes R^\perp),$$

where $R^\perp$ is the annihilator of R in $(A \otimes A)^* = A^* \otimes A^*$.

Proof: By Theorem 3.3 the algebra $\mathrm{cohom}(\omega_{(A,R)},\omega_{(B,S)})$ is generated by the vector spaces $B \otimes A^*$ and $S \otimes R^*$. It satisfies the relations generated by the morphism $\iota : Y \longrightarrow X \otimes X$, which induces relations through the diagram

$$\begin{tikzcd} & S \otimes R^* \\ S \otimes A^* \otimes A^* \arrow[ur, "1\otimes\iota^*"] \arrow[dr, "\iota\otimes 1"'] & \\ & B \otimes B \otimes A^* \otimes A^*. \end{tikzcd}$$

Given an element $s \otimes \alpha \otimes \alpha' \in S \otimes A^* \otimes A^*$ we get equivalent elements $s \otimes (\alpha \otimes \alpha')|_R \sim s \otimes \alpha \otimes \alpha'$. Since the map $1 \otimes \iota^*$ is surjective, every element in $S \otimes R^*$ is equivalent to an element in $B \otimes B \otimes A^* \otimes A^*$ so that we can dispose of the generating set $S \otimes R^*$ altogether. Furthermore elements of the form $s \otimes \alpha \otimes \alpha' \in B \otimes B \otimes A^* \otimes A^*$ are equivalent to zero if $s \in S$ and $\alpha \otimes \alpha'$ induces the zero map on R that is if it is in $R^\perp$, so that the set of relations is induced by $S \otimes R^\perp$. □

Given an algebra C in $\mathcal{A}$, we call an algebra homomorphism $f : (A,R) \longrightarrow C \otimes (B,S)$ *quadratic*, if it satisfies $f(A) \subseteq C \otimes B$ and $(m_C \otimes B \otimes B)(f \otimes f)(R) \subseteq C \otimes S$ that is if

$$\begin{tikzcd} R \arrow[rr, "f"] \arrow[d] & & C \otimes S \arrow[d] \\ A \otimes A \arrow[r, "f \otimes f"] & C \otimes C \otimes B \otimes B \arrow[r, "m \otimes 1"] & C \otimes B \otimes B \end{tikzcd}$$

commutes. The set of all quadratic homomorphisms from (A,R) to $C \otimes (B,S)$ is denoted by $K\text{-Alg}^q((A,R), C \otimes (B,S))$. Then one proves as in Theorem 3.4 that $K\text{-Alg}^q((A,R), C \otimes (B,S)) \cong \mathcal{M}\mathrm{or}_f^{\otimes}(\omega_{(A,R)}, C \otimes \omega_{(B,S)})$. Hence we get the following universal property which is different from the one in Manin [5] 4. Theorem 5.

THEOREM 3.8 (universal property of cohom for quadratic algebras)
Let (A,R) and (B,S) be quadratic algebras and let (A,R) be finite. Then there is a quadratic algebra homomorphism $\delta : (A,R) \longrightarrow \mathrm{cohom}(\omega_{(A,R)},\omega_{(B,S)}) \otimes (B,S)$ such that for every algebra C and every quadratic algebra homomorphism $\varphi : (A,R) \longrightarrow C \otimes (B,S)$ there is a unique algebra homomorphism $\tilde{\varphi} : \mathrm{cohom}(\omega_{(A,R)},\omega_{(B,S)}) \longrightarrow C$ such that the diagram

$$\begin{tikzcd} (A,R) \arrow[r, "\delta"] \arrow[dr, "\varphi"'] & \mathrm{cohom}(\omega_{(A,R)},\omega_{(B,S)}) \otimes (B,S) \arrow[d, "\tilde{\varphi} \otimes 1"] \\ & C \otimes (B,S) \end{tikzcd}$$

commutes. □

3.4 Complete quadratic quantum spaces

The most interesting diagram for constructing comodule algebras over bialgebras is defined over the free monoidal category $\mathcal{D} = \mathcal{C}[X,\rho]$ with $\rho : X \otimes X \longrightarrow X \otimes X$. If $\omega : \mathcal{D} \longrightarrow \mathcal{A}$ is a finite diagram over this diagram scheme with $\omega(X) = V$ and $\omega(\rho) = f : V \otimes V \longrightarrow V \otimes V$, then we can define a quadratic algebra $A_{(V,f)} := T(V)/(\mathrm{Im}(f))$. Let us furthermore define $B = B_{(V,f)} := \mathrm{coend}(\omega)$. We say that B is a *bialgebra with R-matrix.*

LEMMA 3.9 (spaces for a bialgebra with R-matrix)
$A_{(V,f)}$ is a $B_{(V,f)}$-comodule algebra.

Proof: Certainly all vector spaces $V^{\otimes n}$ are $B_{(V,f)}$-comodules and f is a comodule homomorphism. Thus $\mathrm{Im}(f) \subseteq V \otimes V$ is a subcomodule. But then one checks easily that $A_{(V,f)} := T(V)/(\mathrm{Im}(f))$ is a comodule as well and that it is in fact a comodule algebra over $B_{(V,f)}$. □

Now the algebra $A_{(V,f)}$ can become very small, in fact degenerate, namely if $f : V \otimes V \longrightarrow V \otimes V$ is bijective. Then $A = K \oplus V$ where the multiplication on V is the zero map. This happens in "most" cases, since for f to be bijective it suffices that $\det(f) \neq 0$. But the next proposition shows that even in the degenerate case we still have room to move.

PROPOSITION 3.10 (change of the R-matrix)
Let $(V, f : V \otimes V \longrightarrow V \otimes V)$ with V finite dimensional be given. Then for every $\lambda \in K$ we have $B_{(V,f-\lambda\cdot\mathrm{id})} = B_{(V,f)}$.

Proof: By Theorem 3.3 we know that $B_{(V,f)} = T(V \otimes V^*)/I$. We look at the relations. The ideal I is generated by elements of the form

$$x \otimes y \otimes f^*(\xi \otimes \eta) - f(x \otimes y) \otimes \xi \otimes \eta$$

with $x, y, \in V, \xi, \eta \in V^*$. But then the same ideal is also generated by elements of the form

$$x \otimes y \otimes (f^* - \lambda \cdot \mathrm{id}^*)(\xi \otimes \eta) - (f - \lambda \cdot \mathrm{id})(x \otimes y) \otimes \xi \otimes \eta,$$

since the λ-terms simply cancel. Thus $B_{(V,f-\lambda\cdot\mathrm{id})} = B_{(V,f)}$. □

COROLLARY 3.11 (the spectrum of quantum spaces for an R-matrix)
Let $(V, f : V \otimes V \longrightarrow V \otimes V)$ with V finite dimensional be given. Then for every $\lambda \in K$ the algebra $A_{(V,f-\lambda\cdot\mathrm{id})}$ is a $B_{(V,f)}$-comodule algebra. It is non-degenerate if and only of λ is an eigenvalue of f.

Proof: Instead of changing f in the definition of $B_{(V,f)}$ by a multiple of the identity, we can as well change it in the definition of $A_{(V,f)}$.

So for $B_{(V,f)}$ we have obtained a one-parameter family $A_{(V,f-\lambda\cdot\mathrm{id})}$ of comodule algebras. Since V is finite dimensional and f has only finitely many eigenvalues, all but finitely many of these comodule algebras are degenerate by Lemma 3.9. □

EXAMPLE 3.12 (two parameter quantum matrices)
Let us take $V = Kx \oplus Ky$ two dimensional and $f : V \otimes V \longrightarrow V \otimes V$ given by the matrix (with $q, p \neq 0$)

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & q^{-1} & 0 \\ 0 & p^{-1} & 1 - q^{-1}p^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The bialgebra generated by this matrix is generated by the elements $a = x \otimes \xi,\ b = x \otimes \eta,\ c = y \otimes \xi,\ d = y \otimes \eta,$ where $\xi,\ \eta$ is the dual basis to $x,\ y$. The relations are

$$ac = q^{-1}ca,\ bd = q^{-1}db,\ ad - q^{-1}cb = da - qbc,\ ab = p^{-1}ba,\ cd = p^{-1}dc,\ ad - p^{-1}bc = da - pcb.$$

From this follows $qbc = pcb$. This is the two parameter version of a quantum matrix bialgebra constructed in [6] Chap. 4, 4.10. The matrix R has two eigenvalues $\lambda_1 = 1$ (of multiplicity three) and $\lambda_2 = -q^{-1}p^{-1}$ (of multiplicity one) which lead to algebras

$$A_1 = K\langle x, y\rangle/(xy - q^{-1}yx)$$

and

$$A_2 = K\langle x, y\rangle/(x^2,\ y^2,\ xy + pyx).$$

These are the quantum plane with parameter q and the dual quantum plane with parameter p.

EXAMPLE 3.13 (two further quantum 2×2-matrices)
In we replace the generating matrix in previous example by

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & q^{-1} & 0 \\ 0 & 1 & 1 - q^{-1} & 0 \\ 0 & -1 & q^{-1} & 1 \end{pmatrix},$$

then the relations for the corresponding bialgebra are given by

$$ac = q^{-1}ca,\ bd = q^{-1}db,\ ad - q^{-1}cb = da - qbc,$$
$$ba = ab + b^2,\ cb + cd + d^2 = da - db + dc,\ ad - b(c - d) = da - (c + d)b.$$

The bialgebra coacts on the algebras

$$A_1 = K\langle x, y\rangle/(xy - q^{-1}yx)$$

and

$$A_2 = K\langle x, y\rangle/(x^2,\ xy + yx,\ xy - y^2).$$

Finally if we take

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{pmatrix},$$

then the bialgebra satisfies the relations

$$ca = ac + c^2,\ bc + bd + d^2 = da + db - dc,\ ad - c(b-d) = da - (b+d)c,$$
$$ba = ab + b^2,\ cb + cd + d^2 = da - db + dc,\ ad - b(c-d) = da - (c+d)b.$$

The nondegenerate algebras it coacts on are

$$A_1 = K\langle x, y\rangle/(yx - xy - y^2)$$

and

$$A_2 = K\langle x, y\rangle/(x^2,\ xy + yx,\ xy - y^2).$$

The endomorphism $f : V \otimes V \longrightarrow V \otimes V$ may have more than two eigenvalues thus inducing more than two non-degenerate comodule algebras. In general the following holds

THEOREM 3.14 (realization of algebras as spectrum of quantum spaces)
Let V be a finite dimensional vector space and let (V, R_i), $i = 1, \ldots, n$ be quadratic algebras. Let $\lambda_1, \ldots, \lambda_n \in K$ be pairwise distinct. Assume that there are subspaces $W_i \subset V_i \subseteq V \otimes V$ such that $V \otimes V = \bigoplus_{i=1}^n V_i$ and $R_i = \bigoplus_{j=1, j\neq i}^k V_j \oplus W_i$. Then there is a homomorphism $f : V \otimes V \longrightarrow V \otimes V$ such that the non-degenerate algebras for B_f are precisely the $(V, R_i) = (V, \mathrm{Im}(f - \lambda_i \cdot \mathrm{id}) = A_{(V, f-\lambda_i \cdot \mathrm{id})}$ for all $i = 1, \ldots, n$.

Proof: We form the Jordan matrix

$$R = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_n \end{pmatrix}$$

with

$$J_i = \begin{pmatrix} \lambda_i & 1 & & & \\ & \ddots & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 0 \\ & & & & \lambda_i \end{pmatrix}$$

having $\dim(V_i)$ entries λ_i and $\dim(W_i)$ entries 1. Then the induced homomorphism $f : V \otimes V \longrightarrow V \otimes V$ with respect to a suitable basis through the W_i and V_i satisfies $R_i = \mathrm{Im}(f - \lambda_i \cdot \mathrm{id})$. □

3.5 Lie algebras

Similar to our considerations about finite quantum spaces, let $\mathcal{D} = \mathcal{C}[X; m]$ be a free monoidal category on an object X with a multiplication $m : X \otimes X \longrightarrow X$. Then every finite dimensional Lie algebra $\mathbf{g}$ induces a diagram $(\mathcal{D}, \omega)$ with $\omega(X) = \mathbf{g}$ and $\omega(m)$ the Lie bracket. Essentially the same arguments as in Lemma 3.9 show that the bialgebra $\mathrm{coend}(\omega)$ makes $\mathbf{g}$ a comodule Lie algebra (the Lie multiplication is a comodule homomorphism) and its universal enveloping algebra a comodule bialgebra.

Again $\mathrm{coend}(\omega)$ has a universal property with respect to its coaction on $\mathbf{g}$. To show this let $\mathbf{g}$ and $\mathbf{g}'$ be Lie algebras. We say that a linear map $f : \mathbf{g} \longrightarrow C \otimes \mathbf{g}'$ is *multiplicative* if the diagram

$$\begin{tikzcd}
\mathbf{g} \otimes \mathbf{g} \arrow[r, "f \otimes f"] \arrow[d, "{[\ ,\ ]}"'] & C \otimes C \otimes \mathbf{g}' \otimes \mathbf{g}' \arrow[d, "{m \otimes [\ ,\ ]}"] \\
\mathbf{g} \arrow[r, "f"] & C \otimes \mathbf{g}'
\end{tikzcd}$$

commutes. Then the set of all multiplicative maps $\mathrm{Mult}(\mathbf{g}, C \otimes \mathbf{g}')$ is certainly a functor in C.

THEOREM 3.15 (the universal bialgebra coacting on a Lie algebra)
Let $\mathbf{g}$ and $\mathbf{g}'$ be Lie algebras and let $\mathbf{g}'$ be finite dimensional. Then $\mathrm{Mult}(\mathbf{g}, C \otimes \mathbf{g}')$ is a representable functor with representing object $\mathrm{cohom}(\omega_{\mathbf{g}}, \omega_{\mathbf{g}'})$.
In particular the multiplicative map $\mathbf{g}' \longrightarrow \mathrm{cohom}(\omega_{\mathbf{g}}, \omega_{\mathbf{g}'}) \otimes \mathbf{g}$ is universal.

Proof: It is analogous to the proof of Theorem 3.4. In particular we get an isomorphism $\mathrm{Mult}(\mathbf{g}, C \otimes \mathbf{g}') \cong \mathcal{M}\mathrm{or}_f^{\otimes}(\omega_{\mathbf{g}}, C \otimes \omega_{\mathbf{g}'})$. □

We compute now a concrete example of a universal bialgebra coacting on a Lie algebra.

EXAMPLE 3.16 (the universal bialgebra coacting on a three-dimensional Lie algebra of upper triangular matrices)
Let $\mathbf{g} = K\{x, y, z\}$ be the three dimensional Lie algebra with basis x, y, z and Lie bracket $[x, y] = 0$ and $[x, z] = [z, y] = z$. Then $\mathbf{g}^*$ is a Lie coalgebra with dual basis ξ, η, ζ and cobracket $\Delta(\xi) = \Delta(\eta) = 0$ and $\Delta(\zeta) = (\xi - \eta) \otimes \zeta - \zeta \otimes (\xi - \eta)$. By Theorem 3.3 the universal bialgebra $\mathrm{coend}(\omega)$ for the diagram induced by $\mathbf{g}$ is generated by the elements of the matrix

$$M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} x \otimes \xi & x \otimes \eta & x \otimes \zeta \\ y \otimes \xi & y \otimes \eta & y \otimes \zeta \\ z \otimes \xi & z \otimes \eta & z \otimes \zeta \end{pmatrix}.$$

The comultiplication is described in the text after Corollary 2.6 and is given by $\Delta(M) = M \otimes M$. With some straightforward computations one gets the relations as

$$(a-b)c = c(a-b),\ (d-e)f = f(d-e),\ (a-b)f = c(d-e),\ f(a-b) = (d-e)c,$$
$$(a-b)i = i(a-b) = 0,\ (d-e)i = i(d-e) = 0,\ g = 0,\ h = 0.$$

EXAMPLE 3.17 (the universal bialgebra coacting on $sl(2)$)
Let $\mathbf{g} = K\{x, y, z\} = sl(2)$ be the three dimensional Lie algebra with basis x, y, z and Lie bracket $[x, y] = z$, $[z, x] = x$ and $[y, z] = y$. Then $\mathbf{g}^*$ is a Lie coalgebra with dual basis ξ, η, ζ and cobracket $\Delta(\xi) = \zeta \otimes \xi - \xi \otimes \zeta$, $\Delta(\eta) = \eta \otimes \zeta - \zeta \otimes \eta$, and $\Delta(\zeta) = \xi \otimes \eta - \eta \otimes \xi$. The universal bialgebra $\mathrm{coend}(\omega)$ for the diagram induced by $\mathbf{g}$ again is generated by the elements of the matrix

$$M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} x \otimes \xi & x \otimes \eta & x \otimes \zeta \\ y \otimes \xi & y \otimes \eta & y \otimes \zeta \\ z \otimes \xi & z \otimes \eta & z \otimes \zeta \end{pmatrix}.$$

The comultiplication is given by $\Delta(M) = M \otimes M$. Here one gets the relations as

$$ab = ba,\ ac = ca,\ bc = cb,\ de = ed,\ df = fd,\ ef = fe,\ gh = hg,\ gi = ig,\ hi = ih,$$
$$a = ai - cg = ia - gc,\ b = ch - bi = hc - ib,\ c = bg - ah = gb - ha,$$
$$d = fg - di = gf - id,\ e = ei - fh = ie - hf,\ f = dh - eg = hd - ge,$$
$$g = cd - af = dc - fa,\ h = bf - ce = fb - ec,\ i = ae - db = ea - bd.$$

4 Automorphisms and Hopf Algebras

In this last short section we want to extend our techniques of using diagrams for determining bialgebras to Hopf algebras.

Hopf algebras arise as function rings of affine algebraic groups which act as group of automorphisms on algebraic varieties. The problem in non-commutative geometry is that the definition of an automorphism group is not that clear. So one defines the *coautomorphism Hopf algebra* of a space X to be the Hopf envelope of the coendomorphism bialgebra of X. The construction of the Hopf envelope H of a bialgebra B was given in a paper of Takeuchi [14].

Hopf algebras also arise as coendomorphism algebras of rigid diagrams [17]. So we first study some properties of rigid monoidal categories and then show that coautomorphism Hopf algebras can also be obtained from diagrams. One of the main theorems in this context is

THEOREM 4.1 (coendomorphism Hopf algebras of rigid diagrams)
(a) Let H be a Hopf algebra. Then the category of right H-comodules that are finite dimensional as vector spaces, is rigid, and the underlying functor $\omega : \mathcal{C}omod\text{-}H \longrightarrow \mathcal{V}ec$ is monoidal and preserves dual objects (up to isomorphism).
(b) Let $(\mathcal{D}, \omega)$ be a finite monoidal diagram and let $\mathcal{D}$ be rigid. Then the coendomorphism bialgebra coend(ω) *is a Hopf algebra.*

Proof: [12] and [16]. □

LEMMA 4.2 (the rigidization of a monoidal category)
Let $\mathcal{D}$ be a small monoidal category. Then there exists a unique (left-)rigidization (up to isomorphism), i.e. a (left-)rigid small monoidal category $\mathcal{D}^*$ and a monoidal functor $\iota : \mathcal{D} \longrightarrow \mathcal{D}^*$ such that for every (left-) rigid small monoidal category $\mathcal{E}$ and monoidal functor $\tau : \mathcal{D} \longrightarrow \mathcal{E}$ there is a unique monoidal functor $\rho : \mathcal{D}^* \longrightarrow \mathcal{E}$ such that

$$\begin{tikzcd} \mathcal{D} \arrow[r, "\iota"] \arrow[dr, "\tau"'] & \mathcal{D}^* \arrow[d, "\rho"] \\ & \mathcal{E} \end{tikzcd}$$

commutes.

Proof: The construction follows essentially the same way as the construction of a free monoidal category over a given (finite) set of objects and morphisms in section 3.1. □

COROLLARY 4.3 (the rigidization of a diagram)
Each finite monoidal diagram $\omega : \mathcal{D} \longrightarrow \mathcal{A}$ has a unique rigidization $\omega^* : \mathcal{D}^* \longrightarrow \mathcal{A}$ such that $\omega^* \iota = \omega$.

Proof: A finite monoidal diagram is by definition a monoidal diagram in the rigid category $\mathcal{A}_0$. □

PROPOSITION 4.4 (extending monoidal transformations to the rigidization)
Given a finite monoidal diagram $(\mathcal{D}, \omega)$ and its (left-)rigidization $(\mathcal{D}^*, \omega^*)$. Given a (left-)Hopf algebra K and a monoidal natural transformation $f : \omega \longrightarrow K \otimes \omega$. Then there is a unique extension $g : \omega^* \longrightarrow K \otimes \omega^*$, a monoidal natural transformation, such that $(g\iota : \omega^*\iota \longrightarrow K \otimes \omega^*\iota) = (f : \omega \longrightarrow K \otimes \omega)$.

Proof: For $X \in \mathcal{D}$ let $X^* \in \mathcal{D}^*$ be its dual. Let $M := \omega(X)$, $M^* := \omega^*(X^*)$ and ev: $M^* \otimes M \longrightarrow K$ be the evaluation. M is a K-comodule. We define a comodule structure on M^*. Since M is finite-dimensional (or more generally has a left dual in $\mathcal{A}$) there is a canonical isomorphism $M^* \otimes K \cong \mathrm{Hom}(M, K)$. Then $\delta(m^*) \in M^* \otimes K$ is defined by

$$\sum m^*_{(0)}(m) \otimes m^*_{(1)} := \sum m^*(m_{(0)}) \otimes S(m_{(1)}).$$

It is tedious but straightforward to check that this is a right comodule structure on M^*. We check the coassociativity:

$$\begin{aligned}
\sum m^*_{(0)(0)}(m) \otimes S(m^*_{(0)(1)}) \otimes S(m^*_{(1)}) &= \sum m^*_{(0)}(m_{(0)}) \otimes m_{(1)} \otimes S(m^*_{(1)}) \\
&= \sum m^*(m_{(0)(0)}) \otimes m_{(1)} \otimes m_{(0)(1)} \\
&= \sum m^*(m_{(0)}) \otimes \tau\Delta(m_{(1)}) \\
&= \sum m^*_{(0)}(m) \otimes \tau\Delta(S(m^*_{(1)})) \\
&= \sum m^*_{(0)}(m) \otimes S(m^*_{(1)}) \otimes S(m^*_{(2)}).
\end{aligned}$$

Then the evaluation ev: $M^* \otimes M \longrightarrow k$ satisfies

$$\sum m^*_{(0)}(m_{(0)}) \otimes m^*_{(1)} m_{(1)} = \sum m^*_{(0)}(m_{(0)}) \otimes S(m_{(1)}) m_{(2)} = m^*(m) \otimes 1_K$$

hence it is a comodule homomorphism.

We need a unique K-comodule structure on M^* such that ev becomes a comodule homomorphism. Let M^* have such a comodule structure with $\delta(m^*) = \sum m^*_{(0)} \otimes m^*_{(1)}$ and let $m \in M$ with $\delta(m) = \sum m_{(0)} \otimes m_{(1)}$ then we get

$$\sum m^*_{(0)}(m_{(0)}) \otimes m^*_{(1)} m_{(1)} = m^*(m) \otimes 1_K$$

from the fact that ev is a comodule homomorphism. Hence we get

$$\begin{aligned}
\sum m^*_{(0)}(m) \otimes m^*_{(1)} &= \sum m^*_{(0)}(m_{(0)}) \otimes m^*_{(1)} \varepsilon(m_{(1)}) \\
&= \sum m^*_{(0)}(m_{(0)}) \otimes m^*_{(1)} m_{(1)} S(m_{(2)}) \\
&= \sum m^*(m_{(0)}) \otimes S(m_{(1)}).
\end{aligned}$$

But this is precisely the induced comodule-structure on M^* by the antipode S of K given above.

For iterated duals and tensor products of objects the K-comodule structure arises from iterating the process given above resp. from using the multiplication of K to give the tensor product a comodule structure. □

THEOREM 4.5 (the Hopf envelope of a coendomorphism bialgebra)
Given a monoidal diagram $(\mathcal{D}, \omega)$ and its rigidization $(\mathcal{D}^*, \omega^*)$. Let $B := \mathrm{coend}(\omega)$, a bialgebra, and $H := \mathrm{coend}(\omega^*)$, a Hopf algebra. Then there is a bialgebra homomorphism $\sigma : B \longrightarrow H$ such that for every Hopf algebra K and every bialgebra homomorphism $f : B \longrightarrow K$ there is a unique bialgebra homomorphism $g : H \longrightarrow K$ such that

$$
\begin{tikzcd}
B \arrow[r, "\sigma"] \arrow[dr, "f"'] & H \arrow[d, "g"] \\
& K
\end{tikzcd}
$$

commutes.

Proof: The monoidal natural transformation $\omega^* \longrightarrow H \otimes \omega^*$ induces a unique homomorphism $\sigma : B \longrightarrow H$ such that

$$
\begin{tikzcd}
\omega = \omega^*\iota \arrow[r] \arrow[dr] & B \otimes \omega \arrow[d, "\sigma \otimes 1"] \\
& H \otimes \omega^*\iota
\end{tikzcd}
$$

The homomorphism $f : B \longrightarrow K$ induces a monoidal natural transformation $(f \otimes 1)\delta : \omega \longrightarrow B \otimes \omega \longrightarrow K \otimes \omega$, which may be extended to $\omega^* \longrightarrow K \otimes \omega^*$. Hence there is a unique $g : H \longrightarrow K$ such that

$$
\begin{tikzcd}
\omega^* \arrow[r] \arrow[dr] & H \otimes \omega^* \arrow[d, "g \otimes 1"] \\
& K \otimes \omega^*
\end{tikzcd}
$$

commutes. Then the following diagrams commute

$$
\begin{tikzcd}
\omega \arrow[r, "\delta"] \arrow[dr] \arrow[ddr] & B \otimes \omega \arrow[d, "\sigma \otimes 1"'] \arrow[dd, bend left, "f \otimes 1"] \\
& H \otimes \omega \arrow[d, "g \otimes 1"'] \\
& K \otimes \omega
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
B \arrow[r, "\sigma"] \arrow[dr, "f"'] & H \arrow[d, "g"] \\
& K.
\end{tikzcd}
$$

$\square$

We close with a few observations and applications of this result.

Every bialgebra B is the coendomorphism bialgebra of the diagram of all finite-dimensional B-comodules by [10]. So the construction with diagrams given above gives the Hopf envelope of an arbitrary bialgebra.

If a finite-dimensional non-commutative algebra A is given then the rigidization of the finite monoidal diagram $\omega_A : \mathcal{D} \longrightarrow \mathcal{A}$ with $\mathcal{D} := \mathcal{C}[X; m, u]$ generated by A as in 3.2 has a coendomorphism bialgebra H which is the Hopf envelope of the coendomorphism bialgebra of A. A similar remark holds for quadratic algebras.

In the category of quantum spaces the endomorphism quantum space $\mathcal{E}$ of a quantum space $\mathcal{X}$ can be restricted to the "automorphism" quantum space $\mathcal{A}$ by using the homomorphism from the representing bialgebra of $\mathcal{E}$ to its Hopf envelope. Thus $\mathcal{A}$ acts also on $\mathcal{X}$. The existence and the construction in the relevant cases can be obtained from associated rigid monoidal diagrams as in the above theorem.

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