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On the Cohomology of Modules over Hopf Algebras

BODO PAREIGIS

Mathematisches Institut der Universität München, Munich, Germany

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Let R be a commutative ring. Define an FH-algebra H to be a Hopf algebra and a Frobenius algebra over R with a Frobenius homomorphism ψ such that $\sum_{(h)} h_{(1)} \psi(h_{(2)}) = \psi(h) \cdot 1$ for all $h \in H$. This is essentially the same as to consider finitely generated projective Hopf algebras with antipode. For modules over FH-algebras we develop a cohomology theory which is a generalization of the cohomology of finite groups. It generalizes also the cohomology of finite-dimensional restricted Lie algebras. In particular the following results are shown. The complete homology can be described in terms of the complete cohomology. There is a cup-product for the complete cohomology and some of the theorems for periodic cohomology of finite groups can be generalized. We also prove a duality theorem which expresses the cohomology of the "dual" of an H-module as the "dual" of the cohomology of the module. The last section provides techniques to describe under certain conditions the cohomology of H by the cohomology of sub- and quotient-algebras of H. In particular we have a generalization of the Hochschild–Serre spectral sequence for the cohomology of groups.

1. The cohomology of modules over Hopf algebras as represented in this paper generalizes to a certain degree the cohomology of groups as well as the cohomology of Lie algebras and restricted Lie algebras. In fact if G is a group and A is a G-module, then it is well known that

$$H^n(G, A) \cong \operatorname{ext}^n_{(Z[G], Z)}(Z, A) \qquad \text{for} \quad n \geqslant 0.$$

Furthermore, we have for a (restricted) Lie algebra $\mathfrak{g}$ over a field k with (restricted) universal enveloping algebra $U(\mathfrak{g})$ and a $\mathfrak{g}$-module A isomorphisms $H^n(\mathfrak{g}, A) \cong \operatorname{ext}^n_{(U(\mathfrak{g}), k)}(k, A)$ for $n \geqslant 0$.

In these cases $Z[G]$ and $U(\mathfrak{g})$ are Hopf algebras with antipode over Z and k respectively. A lot about the cohomology of finite groups and finite-dimensional restricted Lie algebras can be derived from the fact that $Z[G]/Z$ and $U(\mathfrak{g})/k$ are Frobenius algebras. So we shall frequently use the fact that a finitely generated projective Hopf algebra H with antipode over a commutative ring R with $\operatorname{pic}(R) = 0$ is a Frobenius algebra [7, Theorem 7] with a Frobenius homomorphism ψ such that $\sum_{(h)} h_{(1)}\psi(h_{(2)}) = \psi(h) \cdot 1$ for all $h \in H$. We shall call H an FH-algebra if H is a Hopf algebra and a Frobenius algebra with a Frobenius homomorphisms with the above mentioned property. Most of the homological content of this paper applies to modules over FH-algebras.

Let H be an FH-algebra over a commutative ring R and A be an H-module. The (co-)homology of H with coefficients in A is defined by $H_n(H, A) := \operatorname{tor}_n^{(H,R)}(R, A)$ and $H^n(H, A) := \operatorname{ext}^n_{(H,R)}(R, A)$, respectively; so we have a generalization of the (co-)homology of finite groups as well as of finite-dimensional restricted Lie algebras. For nonzero Lie algebras $\mathfrak{g}$ over a field k the universal enveloping algebra $U(\mathfrak{g})$ is infinite-dimensional, so it is not a Frobenius algebra and most of the theory developed here does not apply.

For FH-algebras, there are complete resolutions which allow to define the (co-)homology groups also for negative n. We obtain in this paper an expression of the n-th homology group by a $(-n-1)$-th cohomology group. Furthermore we develop a cup-product which has similar properties as in the group case. For some results we need that the FH-algebra under consideration is cocommutative. One result derived by cup-product techniques is a duality theorem which gives an isomorphism

$$H^n(H, \hom(A, B^0)) \cong \hom(H^{-n-1}(H, A), B),$$

where B is an injective R-module and H and A are as above. There are also some results on periodic cohomology which generalize the case of finite groups. In particular we show that an FH-algebra generated by one element as an algebra has periodic cohomology of period 2.

The last section provides techniques to describe under certain conditions the cohomology of H by the cohomology of sub- and quotient algebras of H. In particular we have a generalization of the Hochschild–Serre spectral sequence for the cohomology of groups.

If we restrict ourselves to cocommutative Hopf algebras with antipode, then they are group objects in the category of cocommutative coalgebras. So one may consider the Eilenberg–MacLane cohomology of group objects in this case. The second cohomology group of this theory for example describes the Hopf algebra extensions with antipode which are split as coalgebra extensions. In special cases, the Eilenberg–MacLane cohomology groups $H^n_{EM}(H, M)$ of a Hopf algebra H with antipode with coefficients in a commutative Hopf algebra M with antipode can be expressed with the cohomology groups studied in this paper. If $H = Z[\mathscr{H}]$ and $M = Z[\mathscr{M}]$ for a group $\mathscr{H}$ and an $\mathscr{H}$-module $\mathscr{M}$, then $H^n(H, \mathscr{M}) \cong H^n_{EM}(H, M)$ for $n \geqslant 0$. If $H = U(\mathfrak{h})$, the restricted universal enveloping algebra of a restricted Lie algebra $\mathfrak{h}$, and

$M = U(\mathfrak{m})$ with a commutative restricted $\mathfrak{h}$-Lie algebra $\mathfrak{m}$, then $H^n(H, \mathfrak{m}) \cong H^n_{EM}(H, M)$ for $n \geqslant 3$ [5, V Satz 2 and I Korollar 4.3]. Except from these examples, however, we do not know whether these two cohomology theories are in some sense connected.

2. All rings and algebras are associative with unit element. All modules are unitary modules. R is a commutative ring. All algebras are R-algebras. All unlabelled tensor products and hom's are tensor products and hom's over R.

By [7, Theorem 7] a finitely generated projective Hopf algebra H with antipode S and $P(H^*) \cong R$ is a Frobenius algebra with a Frobenius homomorphism ψ such that $\sum_{(h)} h_{(1)}\psi(h_{(2)}) = \psi(h) \cdot 1$. The condition $P(H^*) \cong R$ holds in particular if $\mathrm{pic}(R) = 0$ [7, Proposition 5]. Conversely a Frobenius algebra H with a Frobenius homomorphism ψ such that $\sum_{(h)} h_{(1)}\psi(h_{(2)}) = \psi(h) \cdot 1$, which is a Hopf algebra, has an antipode [7, Theorem 11]. A Hopf algebra and Frobenius algebra H with a Frobenius homomorphism ψ such that $\sum_{(h)} h_{(1)}\psi(h_{(2)}) = \psi(h) \cdot 1$ will be called an FH-*algebra*.

Let H be an augmented algebra. Let A be an H-module. The (*relative*) *homology* of H with coefficients in A is defined by

$$H_n(H, A) := \mathrm{tor}_n^{(H,R)}(R, A).$$

The (*relative*) *cohomology* of H with coefficients in A is defined by

$$H^n(H, A) := \mathrm{ext}^n_{(H,R)}(R, A).$$

LEMMA 1. *Let H be an* FH-*algebra. Each H-module A has a complete (H, R)-resolution:*

$$\begin{tikzcd}
\mathfrak{A} : \cdots \arrow[r, "\partial_2"] & A_1 \arrow[r, "\partial_1"] & A_0 \arrow[rr, "\partial_0"] \arrow[dr] & & A_{-1} \arrow[r, "\partial_{-1}"] & A_{-2} \arrow[r, "\partial_{-2}"] & \cdots \\
& & & A \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

Proof. Since an H-module is (H, R)-projective if and only if it is (H, R)-injective, we may compose an (H, R)-projective resolution with an (H, R)-injective resolution of A to a complete (H, R)-resolution.

Let H be an FH-algebra and A an H-module. Let $\mathfrak{R}$ be a complete (H, R)-resolution of R considered as an H-module by the augmentation $\epsilon : H \to R$. The *complete homology* of H with coefficients in A is defined by

$$\hat{H}_n(H, A) := H_n(\mathfrak{R} \otimes_H A).$$

The *complete cohomology* of H with coefficients in A is defined by

$$\hat{H}^n(H, A) := H_n(\hom_H(\mathfrak{R}, A)).$$

By definition we have $\hat{H}_n(H, A) \cong H_n(H, A)$ for $n \geqslant 1$ and $\hat{H}^n(H, A) \cong H^n(H, A)$ for $n \geqslant 1$.

Each H-projective resolution of R is an (H, R)-projective resolution, since all H-projective modules are (H, R)-projective and R-projective. Since R is R-projective, the resolution is R-split, so it is an (H, R)-projective resolution. Consequently $H_n(H, A) \cong \operatorname{tor}_n{}^H(R, A)$ and $H^n(H, A) \cong \operatorname{ext}_H{}^n(R, A)$ for all n.

3. Let ψ be the Frobenius homomorphism of the FH-algebra H. Then ψ is a free generator of H^* as a left H-module and also as a right H-module [3,2.(4)]. So $\psi^*(h)$ defined by $h \circ \psi = \psi \circ \psi^*(h)$ is an algebra automorphism of H, the *Nakayama automorphism* [3].

Let A be a left H-module. Then A^0 is the left H-module with underlying abelian group A and multiplication $h \cdot a = \psi^*(h)a$.

Since H is finitely generated projective, there is a natural isomorphism

$$\hom(\hom(H, R), H) \cong H \otimes H.$$

Let $\sum r_i \otimes l_i$ be the image of the inverse of the Frobenius isomorphism Φ^{-1}. Then we have by [4, Satz 10 and (11)]

$$\sum hr_i \otimes l_i = \sum r_i \otimes l_i h \tag{1}$$

$$\sum r_i h \otimes l_i = \sum r_i \otimes \psi^*(h)\, l_i \tag{2}$$

$$\sum \psi(r_i)\, l_i = 1 = \sum r_i \psi(l_i). \tag{3}$$

Let A be a left H-module and B be an R-module. The homomorphism

$$\operatorname{tr} : \hom(A, B) \ni f \mapsto (A \ni a \mapsto \sum r_i \otimes f(l_i a) \in H \otimes B) \in \hom_H(A, H \otimes B)$$

is called the *trace map*. Clearly $\operatorname{tr}(f) \in \hom_H(A, H \otimes B)$ because of (1).

LEMMA 2. *The trace map* $\operatorname{tr} : \hom(H, R) \to \hom_H(H, H)$ *is an isomorphism.*

Proof. $\operatorname{Hom}(H, R) = H \circ \psi$ is a free H-module.

$$\operatorname{tr}(h \circ \psi)(1) = \sum r_i (h \circ \psi)(l_i) = \sum r_i \psi(l_i h) = h \sum r_i \psi(l_i) = h,$$

where we used (1) and (3). It is sufficient to know $\operatorname{tr}(h \circ \psi)$ on the unit of H. So tr is an isomorphism, actually the inverse of the Frobenius isomorphism.

Let A and B be left H-modules. We denote the map

$$\hom(A, B) \xrightarrow{\text{tr}} \hom_H(A, H \otimes B) \xrightarrow{\hom(A,\text{mult})} \hom_H(A, B)$$

by Tr and call it also the *trace map.*

LEMMA 3. *$f \in \hom_H(A, B)$ is a trace of some $g \in \hom(A, B)$ if and only if there is a factorization of f through $A \to^{\sigma} H \otimes B \to^{\tau} B$ with H-homomorphisms σ and τ.*

Proof. Let $f = \tau\sigma$. $H \otimes B$ is (H, R)-projective [2, 3.1(P1) and also Lemma 5]. So by [4, Satz 11] there is an R-endomorphism ρ of $H \otimes B$ with $\mathrm{Tr}(\rho) = \mathrm{id}_{H\otimes B}$. Since σ and τ are H-homomorphisms, we get $f = \tau\,\mathrm{Tr}(\rho)\sigma = \mathrm{Tr}(\tau\rho\sigma)$. The converse holds by definition of Tr.

COROLLARY 1. *Let $f \in \hom(A, B)$. Then $\hat{H}^n(H, \mathrm{Tr}(f)) = 0$.*

Proof. $\mathrm{Tr}(f)$ can be factored through $H \otimes B$ which is (H, R)-projective. Now for an (H, R)-projective module P all cohomology groups $\hat{H}^n(H, P) = 0$, for $0 \to P \to^{\text{id}} P \to 0$ is a complete resolution of P, and hence $H_n(\hom_H(R, P)) = 0$. Consequently $\hat{H}^n(H, \mathrm{Tr}(f))$ can be factored through zero.

PROPOSITION 1. *$\hat{H}^n(H, A)$ is a module over the center C of H. The ideal I in C generated by $\sum r_i l_i \in C$ annihilates $\hat{H}^n(H, A)$. In particular if $I = C$ then $\hat{H}^n(H, A) = 0$ for all H-modules A. Furthermore if $c \in C$ annihilates A then it annihilates $\hat{H}^n(H, A)$.*

Proof. The first and also the last remark is clear, since multiplication with an element of C defines an H-endomorphism of A. Now $\mathrm{Tr}(\mathrm{id}_A)$ is multiplication by $\sum r_i l_i \in C$ and $\hat{H}^n(H, \mathrm{Tr}(\mathrm{id}_A)) = 0$ by Corollary 1. If $\sum r_i l_i$ is invertible in C then the multiplication by 1 is the zero map, so $\hat{H}^n(H, A) = 0$.

COROLLARY 2. *Let N be the left norm of H with respect to ψ [7]. Then $\epsilon(N) = \epsilon(\sum r_i l_i)$ annihilates $\hat{H}^n(H, A)$.*

Proof. We have

$$\sum r_i \epsilon(l_i) = \sum r_i \psi(l_i N) = N \sum r_i \psi(l_i) = N$$

which implies $\epsilon(\sum r_i l_i) = \epsilon(N)$. Now all homomorphisms $\hom_H(\mathrm{id}_R, \sum r_i l_i)$, $\hom_H(\sum r_i l_i, \mathrm{id}_A)$, $\hom_H(\epsilon(\sum r_i l_i), \mathrm{id}_A)$, and $\hom_H(\mathrm{id}_R, \epsilon(N))$ from $\hom_H(R, A)$ into $\hom_H(R, A)$ are the same, since $\sum r_i l_i \in C$ and R is an

H-module via $\epsilon : H \to R$. So multiplication by $\sum r_i l_i$ and by $\epsilon(N)$ induce the same maps on the cohomology groups $\hat{H}^n(H, A)$.

4. Let H be an FH-algebra. Let A and B be left H-modules. We define a left H-module structure on $A \otimes B$ and $\hom(A, B)$ by

$$h(a \otimes b) = \sum_{(h)} h_{(1)}a \otimes h_{(2)}b$$

and

$$(hf)(a) = \sum_{(h)} h_{(1)} f(S(h_{(2)})a),$$

i.e., we consider $A \otimes B$ and $\hom(A, B)$ as $H \otimes H$-modules and restrict the operation via $\Delta : H \to H \otimes H$.

LEMMA 4. *Let A, B, C be left H-modules. Then*

$$\hom_H(A, \hom(B, C)) \cong \hom_H(A \otimes B, C)$$

is a natural transformation.

Proof. We use the natural transformation

$$\hom(A, \hom(B, C)) \cong \hom(A \otimes B, C)$$

and show that H-homomorphisms correspond to H-homomorphisms. Given $f \in \hom_H(A, \hom(B, C))$. Then

$$\begin{aligned} f'(h(a \otimes b)) &= \sum_{(h)} f(h_{(1)}a)(h_{(2)}b) \\ &= \sum_{(h)} h_{(1)} f(a)(S(h_{(2)})\, h_{(3)}b) \\ &= hf'(a \otimes b) \end{aligned}$$

and for $f' \in \hom_H(A \otimes B, C)$ we get

$$\begin{aligned} f(ha)(b) &= f'(ha \otimes b) \\ &= \sum_{(h)} f'(h_{(1)}a \otimes h_{(2)}S(h_{(3)})b) \\ &= \sum_{(h)} h_{(1)} f'(a \otimes S(h_{(2)})b) \\ &= \sum_{(h)} h_{(1)} f(a)(S(h_{(2)})b) \\ &= (hf(a))(b). \end{aligned}$$

COROLLARY 3. *For H-modules A and B there is a natural isomorphism*

$$\hom_H(R, \hom(A, B)) \cong \hom_H(A, B).$$

Proof. This is a consequence of the isomorphism of H-modules $R \otimes A \cong A$.

LEMMA 5. *Let A and B be left H-modules.*

(a) *If A is R-flat and B H-injective, then* $\hom(A, B)$ *is H-injective.*
(b) *If B is* (H, R)*-injective, then* $\hom(A, B)$ *is* (H, R)*-injective.*
(c) *If A is* (H, R)*-projective, then* $\hom(A, B)$ *is* (H, R)*-injective.*
(d) *If A is* (H, R)*-projective, then* $A \otimes B$ *is* (H, R)*-projective.*

Proof. (a) $\hom_H(- \otimes A, B) \cong \hom_H(-, \hom(A, B))$ is exact.

(b) The functor $\hom_H(- \otimes A, B) \cong \hom_H(-, \hom(A, B))$ maps (H, R)-exact sequences to exact sequences, since $- \otimes A$ maps (H, R)-exact sequences to (H, R)-exact sequences.

(c) We first prove that there is a natural isomorphism

$$\hom_H(A \otimes C, B) \cong \hom_H(C, [\hom(A, B)]),$$

where $[\hom(A, B)]$ is the abelian group $\hom(A, B)$ with the operation $(hf)(a) = \sum_{(h)} h_{(2)} f(S^{-1}(h_{(1)})a)$. Here we use [7, Proposition 6], that S is invertible. In this case $\sum S^{-1}(h_{(2)})\, h_{(1)} = \epsilon(h) = \sum h_{(2)} S^{-1}(h_{(1)})$ holds. Using this, the proof of the isomorphism is similar to the proof of Lemma 4. Now each (H, R)-exact sequence is sent to an exact sequence by

$$\hom_H(A, [\hom(-, B)]) \cong \hom_H(- \otimes A, B) \cong \hom_H(-, \hom(A, B)),$$

since $[\hom(-, B)]$ maps (H, R)-exact sequences to (H, R)-exact sequences.

(d) With the isomorphism $\hom_H(A, \hom(B, -)) \cong \hom_H(A \otimes B, -)$ a similar proof as for (b) and (c) may be given.

It should be noted that Lemma 5 is different from the result [1, X. Proposition 8.1] since the module structures on $A \otimes B$ and $\hom(A, B)$ are quite different.

For explicit computations of resolutions it is often interesting to know the following results. Let A be a left H-module. Then, $H \otimes A$ and $\hom(H, A)$ can carry the H-module structure as described in the beginning of this section as well as the structure $h \cdot (h' \otimes a) = hh' \otimes a$ and $(h \cdot f)(h') = f(h'h)$. Let us denote these modules by $\langle H \otimes A \rangle$ and $\langle \hom(H, A) \rangle$.

LEMMA 6. *There are isomorphisms of left H-modules:*

$$H \otimes A \cong \langle H \otimes A \rangle$$
$$\hom(H, A) \cong \langle \hom(H, A) \rangle.$$

Proof. The first isomorphism α is defined by

$$\alpha(h \otimes a) = \sum_{(h)} h_{(1)} \otimes S(h_{(2)})a.$$

The inverse is defined by $\alpha^{-1}(h \otimes a) = \sum_{(h)} h_{(1)} \otimes h_{(2)}a$. α and α^{-1} are inverses of each other. It is easy to see that α^{-1} is an H-homomorphism; so α is also an H-homomorphism. We define the second isomorphism β by $\beta(f)(h) = \sum_{(h)} h_{(1)} f(S(h_{(2)}))$ and $\beta^{-1}(f)(h) = \sum_{(h)} h_{(2)} f(S^{-1}(h_{(1)}))$. Again, it is easy to see that β and β^{-1} are inverses of each other and that β is an H-homomorphism.

5. LEMMA 7. *For left H-modules A and B the map*

$$\alpha : \hom(A, R) \otimes_H B \ni f \otimes b \mapsto (A \ni a \mapsto \sum \psi^*(r_i) f(l_i a) b \in B) \in \hom_H(A, B^0)$$

is a natural transformation. If A is a finitely generated projective H-module then α is a natural isomorphism.

Proof. First we have to show $\alpha(fh \otimes b) = \alpha(f \otimes hb)$ and $\alpha(f \otimes b)(ha) = \psi^*(h)\, \alpha(f \otimes b)(a)$. Now

$$\begin{aligned} \alpha(fh \otimes b)(a) &= \psi^*(r_i) f(\psi^* \psi^{*-1}(h)\, l_i a) b \\ &= \sum \psi^*(r_i \psi^{*-1}(h)) f(l_i a) b \\ &= \sum \psi^*(r_i) f(l_i a)\, hb \\ &= \alpha(f \otimes hb)(a) \end{aligned} \qquad \text{by (2)}$$

and

$$\begin{aligned} \alpha(f \otimes b)(ha) &= \sum \psi^*(hr_i) f(l_i a) b \\ &= \psi^*(h)\, \alpha(f \otimes b)(a). \end{aligned} \qquad \text{by (1)}$$

Clearly α is a natural transformation. For $A = H$ we have $\psi h \otimes b = \psi \otimes hb$; so every element in $\hom(H, R) \otimes_H B$ can be written in the form $\psi \otimes b$ for some $b \in B$. Now $\alpha(\psi \otimes b)(1) = b$ hence α is an isomorphism for $A = H$. Finally [6, 4.11 Lemma 2] implies the claim of the Lemma.

THEOREM 1. *Let H be an* FH-*algebra and let A be a left H-module. Then*

$$\hat{H}_n(H, A) \cong \hat{H}^{-n-1}(H, A^\circ) \qquad \textit{for all } n.$$

Proof. By [3, Section 6] the H-module R has a complete (H, R)-resolution of finitely generated projective H-modules. Call this resolution $\mathfrak{R}$. Then $\hom(\mathfrak{R}, R)$ is again (H, R)-exact. By Lemma 5.c $\hom(A_n, R)$ is (H, R)-

injective if A_n is projective. So $\hom(\mathfrak{R}, R)$ is an (H, R)-injective and also (H, R)-projective resolution of the right H-module $R^* \cong R$. So

$$\hom(\mathfrak{R}, R) \otimes_H A \cong \hom_H(\mathfrak{R}, A^\circ)$$

by Lemma 7; hence $\hat{H}_n(H, A) \cong \hat{H}^{-n-1}(H, A^\circ)$.

This shows that the complete homology can be described by the complete cohomology so that we have to deal only with complete cohomology.

6. Proposition 2. *The complete cohomology describes module (H, R)-extensions for left H-modules A and B by*

$$\mathrm{ext}^n_{(H,R)}(A, B) \cong \hat{H}^n(H, \hom(A, B)) \qquad \textit{for} \quad n \geqslant 1.$$

Proof. We have to prove

$$\mathrm{ext}^n_{(H,R)}(A, B) \cong \mathrm{ext}^n_{(H,R)}(R, \hom(A, B)).$$

Let $0 \to B \to \mathfrak{B}$ be an (H, R)-injective resolution of B. Then,

$$0 \to \hom(A, B) \to \hom(A, \mathfrak{B})$$

is an (H, R)-injective resolution by Lemma 5.b. So, by

$$\hom_H(A, \mathfrak{B}) \cong \hom_H(R, \hom(A, \mathfrak{B}))$$

as in Corollary 3, we get the required isomorphism for the homology of the complex.

Corollary 4. *The relative global dimension* $\text{gl-dim}(H, R)$ *of H over R is equal to the H-projective dimension* $\text{p-dim}(R)$ *of the H-module R and is either zero or infinite.*

Proof. The projective dimension of R coincides with the (H, R)-projective dimension of R by $\mathrm{ext}_H{}^n(R, A) \cong \mathrm{ext}^n_{(H,R)}(R, A)$. By (4) and $\hom(R, A) \cong A$ as left H-modules we get $\text{gl-dim}(H, R) = \text{p-dim}(R)$. Now [2, 3.1.(IV)] implies the result.

Proposition 3. *For left H-modules A and B and $H^+ = \ker(\epsilon : H \to R)$ there is a short exact sequence*

$$0 \to \hom_H(A, B) \to \hom(A, B) \to \hom_H(H^+, \hom(A, B))$$

$$\to \mathrm{ext}^1_{(H,R)}(A, B) \to 0$$

and isomorphisms

$$\mathrm{ext}^{n-1}_{(H,R)}(H^+, \hom(A, B)) \cong \mathrm{ext}^n_{(H,R)}(A, B) \qquad \textit{for} \quad n > 1.$$

Proof. The short exact sequence $0 \to H^+ \to H \to R \to 0$ is an (H, R)-exact sequence. So we get an exact sequence

$$0 \to \hom_H(R, \hom(A, B)) \to \hom_H(H, \hom(A, B)) \to \hom_H(H^+, \hom(A, B))$$

$$\to \mathrm{ext}^1_{(H,R)}(R, \hom(A, B)) \longrightarrow \mathrm{ext}^1_{(H,R)}(H, \hom(A, B)),$$

where $\hom_H(R, \hom(A, B)) \cong \hom_H(A, B)$,

$\hom_H(H, \hom(A, B)) \cong \hom(A, B)$, $\mathrm{ext}^1_{(H,R)}(R, \hom(A, B)) \cong \mathrm{ext}^1_{(H,R)}(A, B)$,

and $\mathrm{ext}^1_{(H,R)}(H, \hom(A, B)) = 0$ imply the exact sequence. The isomorphisms are also induced by the exact cohomology sequence by $\mathrm{ext}^n_{(H,R)}(H, \hom(A, B)) = 0$ for $n \geqslant 1$.

PROPOSITION 4. *There is an (H, R)-complete resolution of R by left H-modules where the center of the resolution has the form*:

$$\begin{tikzcd}
\cdots \arrow[r] & \langle H \otimes H^+ \rangle \arrow[r, "m"] & H \arrow[rr, "N"] \arrow[dr, "\epsilon"'] & & H \arrow[r, "m^*"] & \langle H \otimes \hom(H^+, R) \rangle \arrow[r] & \cdots, \\
& & & R \arrow[ur, "\rho"'] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

where N is the right multiplication by the left norm N, $\rho : R \ni r \mapsto rN \in H$, m is the multiplication map, and $m^*(h) = h \sum r_i \otimes (l_i \circ \psi)$.

Proof. Since H^+ is an H-ideal and R is R-free, the sequence $\cdots \to \langle H \otimes H^+ \rangle \to^m H \to^\epsilon R \to 0$ is the beginning of an (H, R)-projective resolution. By dualization we get an (H, R)-exact sequence of left H-modules

$$0 \to \hom(R, R) \to \hom(H, R) \to \hom(\langle H^+ \otimes H \rangle, R) \to \cdots .$$

Now since $H \otimes A \cong \hom(H, A)$ by $h \otimes a \mapsto (h \circ \psi)a$ and $f \mapsto \sum r_i \otimes f(l_i)$, we get isomorphisms of left H-modules $\hom(R, R) \cong R$, $\hom(H, R) \cong H$, and $\hom(\langle H^+ \otimes H \rangle, R) \cong \langle \hom(H, \hom(H^+, R)) \rangle \cong \langle H \otimes \hom(H^+, R) \rangle$. The maps $\hom(\epsilon, R)$ and $\hom(m, R)$ induce the following maps

$$\rho : R \ni r \mapsto rN \in H$$

since $rN \circ \psi = r \circ \epsilon$ and

$$m^* : H \ni h \mapsto h \sum r_i \otimes (l_i \circ \psi) \in \langle H \otimes \hom(H^+, R) \rangle$$

which is easily checked with the above definitions. So the above sequence defines an (H, R)-injective resolution of R. By the fact that $hN = \epsilon(h)N$ [7, Section 4] we get $N = \rho\epsilon$.

PROPOSITION 5. *Let A be a left H-module. Let $A^H = \{a \in A \mid ha = \epsilon(h)a$ for all $h \in H\}$ and $A_N = \{a \mid Na = 0\}$. Then*

$$\hat{H}^{\circ}(H, A) \cong A^H/NA$$
$$\hat{H}^{-1}(H, A) \cong A_N/\psi^{*-1}(H^+)A.$$

For a left H-module A with $\psi^{*-1}(h)a = \epsilon(h)a$ for all $a \in A$, $h \in H$ we have

$$\hat{H}^{-2}(H, A) \cong (H^+/(H^+)^2) \otimes A.$$

Proof. (1) The relevant part of the complex is

$$\begin{tikzcd}
\cdots \hom_H(H, A) \arrow[r, "\hom(N,A)"] & \hom_H(H, A) \arrow[r, "\hom(m,A)"] & \hom_H(H \otimes H^+, A) \cdots . \\
A \arrow[u, "\wr|"'] \arrow[r, "N"] & A \arrow[u, "\wr|"'] \arrow[ur, "m'"'] &
\end{tikzcd}$$

Now $m'(a)(h \otimes h^+) = hh^+a$. So $\mathrm{Im}(N) = NA$ and

$$\ker(m') = \{a \in A \mid H^+a = 0\} = \{a \in A \mid ha = \epsilon(h)a\} = A^H.$$

This implies $\hat{H}^0(H, A) \cong A^H/NA$.

(2) By Theorem 1, we have $\hat{H}_0(H, A') \cong \hat{H}^{-1}(H, A)$, where $(A')^0 = A$ as H-modules. Now $\hat{H}_0(H, A')$ is computed from

$$\begin{tikzcd}
\cdots \langle H^+ \otimes H \rangle \otimes_H A' \arrow[r, "m\otimes 1"] & H \otimes_H A' \arrow[r, "N'\otimes 1"] & H \otimes_H A' \cdots \\
H^+ \otimes A' \arrow[u, "\wr|"'] \arrow[r, "m"] & A' \arrow[u, "\wr|"'] \arrow[r, "N'"] & A' \arrow[u, "\wr|"']
\end{tikzcd}$$

where N' is multiplication with the right norm $N' = \psi^*(N)$. Here, we use the resolution of Proposition 4 but with inverted sides. We get $\ker(N') = \{a \in A' \mid \psi^*(N)a = 0\} = \{a \in A \mid Na = 0\} = A_N$. Furthermore $\mathrm{Im}(m) = H^+A' = \psi^{*-1}(H^+)A$.

(3) By [1, p. 184, (4)] we have $\mathrm{tor}_1^{(H,R)}(R, A') \cong (H^+/(H^+)^2) \otimes A'$, since the relative homology coincides with the (absolute) homology in this case. In the tensor product we are not interested in the H-module structure of A so that we get $H^{-2}(H, A) \cong (H^+/(H^+)^2) \otimes A$, where we used Theorem 1.

COROLLARY 5. *Let* $0 \to A \to B \to C \to 0$ *be an exact sequence of left* H*-modules. Then* $0 \to A^H \to B^H \to C^H$ *is exact.*

Proof. In the preceeding proof we saw that $\ker(m') = A^H$. By definition, m' is a natural transformation. So the result follows from the fact that ker is a left exact functor [6, 2.7 Corollary 2].

COROLLARY 6. *For an* FH*-algebra* H *the following are equivalent:*

(a) $\epsilon(N)$ *is invertible.*
(b) $\text{gl-dim}(H, R) = 0$.
(c) $\hat{H}^0(H, R) = 0$.

Proof. If $\epsilon(N)$ is invertible, then by Corollary 2 all cohomology groups $\hat{H}^n(H, A)$ are zero, which means that $\text{gl-dim}(H, R) = 0$ by Proposition 2. In particular, $\hat{H}^0(H, R) = 0$. Now, if $\hat{H}^0(H, R) = R^H/NR = R/NR = 0$, then $NR = \epsilon(N)R = R$ so $\epsilon(N)$ is invertible.

This corollary is a generalization of a well-known theorem of Maschke.

COROLLARY 7. *Let* A *be a left* H*-module with* $A = A^H$*. Then*

$$\hat{H}_0(H, A) \cong \hat{H}^{-1}(H, A^0) \cong \ker(\epsilon(N) : A \to A).$$

Proof. By Proposition 5 we have $\hat{H}^{-1}(H, A^0) \cong (A^0)_N/\psi^{*-1}(H^+)A^0 = (A^0)_N/H^+A = (A^0)_N = \{a \in A \mid \epsilon\psi^*(N)a = 0\} = \{a \in A \mid \epsilon(N)a = 0\}$, since $\epsilon(N) = \psi(\psi^*(N)N) = \epsilon(\psi^*(N))$.

7. Let $\text{ses}(H, R)$ be the category of all R-split short exact sequences of H-modules with triples of H-homomorphisms as morphisms. We define an (H, R)-connected sequence $\{H^i, E^i\}$ of covariant functors from $\text{mod}(H)$ to $\text{mod}(R)$ like in [8, XII.8]; we restrict ourselves, however, to R-additive functors only. As in [8, XII. Theorem 7.2 and 7.4], one can prove that each R-additive functor from $\text{mod}(H)$ to $\text{mod}(R)$ has a unique R-additive left (and also a right) satellite (up to an isomorphism). We write the right satellite of a functor F as S^1F and the left satellite as S_1F. In the following, we need a slight generalization of [8, XII. Corollary 8.6].

LEMMA 8. *If* $\{H^i, E^i\}$ *is an* (H, R)*-connected sequence with* $H^i \cong S^1H^{i-1}$ *for all (some)* i *and if* $F : \text{mod}(R) \to \text{mod}(R)$ *is a right exact* R*-additive covariant functor, then* $\{FH^i, FE^i\}$ *is an* (H, R)*-connected sequence of functors with* $FH^i \cong S^1(FH^{i-1})$.

Proof. Is essentially dual to the proof of [8, XII. Corollary 8.6].

LEMMA 9. *Let H' and H be* FH-*algebras. Let* $G : \mathrm{mod}(R) \to \mathrm{mod}(R)$ *be an R-additive functor which has an extension* $G^{\#} : \mathrm{mod}(H) \to \mathrm{mod}(H')$ *which maps (H, R)-injective modules into (H', R)-injective modules and such that*

$$\begin{array}{ccc} \mathrm{mod}(H) & \xrightarrow{G^{\#}} & \mathrm{mod}(H') \\ \downarrow{\scriptstyle V} & & \downarrow{\scriptstyle V} \\ \mathrm{mod}(R) & \xrightarrow{G} & \mathrm{mod}(R) \end{array}$$

is commutative where V is the forgetful functor induced by $R \to H$ and $R \to H'$. Then $S^1(LG^{\#}) \cong (S^1L)\, G^{\#}$ for any R-additive functor $L : \mathrm{mod}(H') \to \mathrm{mod}(R)$.

Proof. For a sequence $E = (0 \to E_1 \to E_2 \to E_3 \to 0) \in \mathrm{ses}(H, R)$, the sequences $V(E)$, $GV(E)$, and $VG^{\#}(E)$ are split exact; hence $G^{\#}(E) \in \mathrm{ses}(H', R)$. Now assume that E_2 is (H, R)-injective. Then, $G^{\#}E_2$ is (H', R)-injective; hence $LG^{\#}(E_2) \to LG^{\#}(E_3) \to (S^1L)\, G^{\#}(E_1) \to 0$ is exact. Now we apply [8, XII. Theorem 7.6] to get the result.

LEMMA 10. *Let A and B be left H-modules. Then $\hat{H}^m(H, A) \otimes \hat{H}^n(H, B)$ and $\hat{H}^{m+n}(H, A \otimes B)$ are for fixed n and B (H, R)-connected sequences of functors in A which are right universal. $\hat{H}^{m+n}(H, A \otimes B)$ is also left couniversal.*

Proof. The functor $- \otimes B : \mathrm{mod}(R) \to \mathrm{mod}(R)$ has an extension also denoted by $- \otimes B : \mathrm{mod}(H) \to \mathrm{mod}(H)$ which commutes with the forgetful functor $\mathrm{mod}(H) \to \mathrm{mod}(R)$. Furthermore, it preserves (H, R)-projective modules by Lemma 5 (d). Since a module is (H, R)-projective if and only if it is (H, R)-injective, the functor $- \otimes B$ preserves also (H, R)-injective modules.

Now $\hat{H}^n(H, -)$ is an (H, R)-connected sequence of functors which is right universal and left couniversal for all n, since these functors vanish on (H, R)-projective and also (H, R)-injective modules [8, XII. Corollary 8.5]. So Lemma 9 proves the Lemma for $\hat{H}^{m+n}(H, A \otimes B)$.

The functor $- \otimes \hat{H}^n(H, B) : \mathrm{mod}(R) \to \mathrm{mod}(R)$ is an R-additive right exact functor. So Lemma 8 implies the rest of the proof.

With these means we can prove in a similar way as in [5, III. Satz 3.4] the existence of a cup-product:

THEOREM 2. *Let H be an* FH-*algebra. Let A and B be left H-modules. For fixed integers m_0, n_0 let*

$$\xi : \hat{H}^{m_0}(H, A) \otimes \hat{H}^{n_0}(H, B) \to \hat{H}^{m_0+n_0}(H, A \otimes B)$$

be a natural R-homomorphism. Then there exists exactly one set of R-homomorphisms

$$\varphi^{m,n} : \hat{H}^m(H, A) \otimes \hat{H}^n(H, B) \to \hat{H}^{m+n}(H, A \otimes B)$$

for all m, n and all left H-modules A and B such that

(a) $\varphi^{m_0,n_0} = \xi$,
(b) $\varphi^{m,n}$ *is a natural transformation in A and B*,
(c) $\varphi^{m+1,n}(E_* \otimes 1) = (E \otimes B)_* \varphi^{m,n}$,
(d) $(-1)^n \varphi^{m,n+1}(1 \otimes E_*') = (A \otimes E')_* \varphi^{m,n}$

where E and E′ are in ses(H, R).

We define a natural homomorphism

$$\xi : \hat{H}^0(H, A) \otimes \hat{H}^0(H, B) \to \hat{H}^0(H, A \otimes B)$$

by $A^H/NA \otimes B^H/NB \to (A \otimes B)^H/N(A \otimes B)$ which is induced by the identity $A \otimes B \to A \otimes B$. To show that this is a well-defined homomorphism we first show $A^H \otimes B^H \subseteq (A \otimes B)^H$. Let $a \in A^H$, $b \in B^H$, $h \in H$. Then $h(a \otimes b) = \sum_{(h)} h_{(1)}a \otimes h_{(2)}b = \epsilon(h)(a \otimes b)$; so $a \otimes b \in (A \otimes B)^H$. Furthermore, we show $A^H \otimes NB \subseteq N(A \otimes B)$. Let $a \in A^H$, $b \in B$. Then $a \otimes Nb = \sum_{(N)} \epsilon(N_{(1)})a \otimes N_{(2)}b = \sum_{(N)} N_{(1)}a \otimes N_{(2)}b = N(a \otimes b)$. Obviously the homomorphism ξ is a natural homomorphism. By the preceeding theorem, there is a uniquely defined multiplication $\hat{H}^m(H, A) \otimes \hat{H}^n(H, B) \to \hat{H}^{m+n}(H, A \otimes B)$ which is induced by ξ. This product will be called the *cup-product*.

For $A = R$ we have $\hat{H}^0(H, R) \cong R/NR$. The element corresponding to $1 + NR \in R/NR$ will be denoted simply by $1 \in \hat{H}^0(H, R)$.

LEMMA 11. *If we identify $R \otimes B = B = B \otimes R$, then $1 \cdot b = b = b \cdot 1$ for any $b \in \hat{H}^n(H, B)$.*

Proof. Is the same as the proof of [5, III. Lemma 3.6].
In a similar way one proves that the cup-product is associative.

LEMMA 12. *Let H be a cocommutative* FH-*algebra. Let A and B be H-modules. If we identify the H-modules $A \otimes B$ and $B \otimes A$ and if $a \in \hat{H}^r(H, A)$ and $b \in \hat{H}^s(H, B)$, then $a \cdot b = (-1)^{rs} b \cdot a$.*

Proof. The morphisms $\hat{H}^r(H, A) \otimes \hat{H}^s(H, B) \xrightarrow{\varphi^{r,s}} \hat{H}^{r+s}(H, A \otimes B)$ and $\hat{H}^r(H, A) \otimes \hat{H}^s(H, B) \cong \hat{H}^s(H, B) \otimes \hat{H}^r(H, A) \xrightarrow{(-1)^{rs}\varphi^{s,r}} \hat{H}^{r+s}(H, A \otimes B)$ both fulfill the conditions of Theorem 2 as can be easily checked. So by the uniqueness they have to coincide.

8. Proposition 6. *Let H be an* FH-*algebra and A be a left H-module. Let B be an R-injective R-module, which will be viewed as trivial H-module via $\epsilon : H \to R$. Let $N \in H$ be cocommutative. Then the cup-product*

$$\hat{H}^0(H, \hom(A, B^0)) \otimes \hat{H}^{-1}(H, A) \to \hat{H}^{-1}(H, \hom(A, B^0) \otimes A)$$

and the evaluation

$$\chi : \hom(A, B^0) \otimes A \to B^0$$

define a natural isomorphism

$$\zeta : \hat{H}^0(H, \hom(A, B^0)) \to \hom(\hat{H}^{-1}(H, A), \hat{H}^{-1}(H, B^0)).$$

Proof. First we consider an explicit formula for the cup-product. Let $E = (0 \to C \to^{\lambda} P \to^{\nu} A \to 0)$ be an (H, R)-exact sequence with an (H, R)-projective module P. Then $\hom(A, B^0) \otimes P$ is also (H, R)-projective by Lemma 5 (d). So there is a commutative diagram

$$\begin{array}{ccc} \hat{H}^0(H, \hom(A, B^0)) \otimes \hat{H}^{-1}(H, A) & \xrightarrow{1 \otimes E^*} & \hat{H}^0(H, \hom(A, B^0)) \otimes \hat{H}^0(H, C) \\ \Big\downarrow \varphi^{0,-1} & & \Big\downarrow \varphi^{0,0} \\ \hat{H}^{-1}(H, \hom(A, B^0) \otimes A) & \xrightarrow{\quad \delta \quad} & \hat{H}^0(H, \hom(A, B^0) \otimes C) \end{array}$$

where $\delta = (\hom(A, B^0) \otimes E)_*$ and $1 \otimes E_*$ are isomorphisms in the long exact cohomology sequence. Now

$$\begin{aligned} &\hat{H}^0(H, \hom(A, B^0)) \otimes \hat{H}^{-1}(H, A) \\ &\qquad \cong \hom_H(A, B^0)/N \hom(A, B^0) \otimes A_N/\psi^{*-1}(H^+)A. \end{aligned}$$

Here we use the fact $\hom_H(A, B) = \hom(A, B)^H$, for let $f \in \hom_H(A, B)$. Then,

$$(hf)(a) = \sum_{(h)} h_{(1)} f(S(h_{(2)})a) = f(\epsilon(h)a) = (\epsilon(h)f)(a);$$

so $f \in \hom(A, B)^H$. Let $f \in \hom(A, B)^H$. Then $f(ha) = \sum_{(h)} (\epsilon(h_{(1)})f)(h_{(2)}a) = \sum_{(h)} (h_{(1)}f)(h_{(2)}a) = \sum_{(h)} h_{(1)} f(S(h_{(2)})\, h_{(3)}a) = hf(a)$, so $f \in \hom_H(A, B)$.

Let $f \in \hom_H(A, B^0)$ and $a \in A_N$. Choose $p \in P$ such that $\nu(p) = a$. Then $(1 \otimes E_*)(f \otimes a) = f \otimes Np$ where $Np \in \lambda(C)$. This may be seen by computing the connecting homomorphism E_* using the complete resolution of Proposition 4 for R. By definition of $\varphi^{0,0}$ the representative $f \otimes Np$ is mapped into $f \otimes Np$. On the other hand, $f \otimes a$ is a representative of an element in $\hat{H}^{-1}(H, \hom(A, B^0) \otimes A) \cong (\hom(A, B^0) \otimes A)_N/\psi^{*-1}(H^+)(\hom(A, B^0) \otimes A)$, for $N(f \otimes a) = \sum_{(N)} N_{(1)} f \otimes N_{(2)} a = \sum_{(N)} \epsilon(N_{(1)}) f \otimes N_{(2)} a = f \otimes Na = 0$. So $\delta(f \otimes a) = N(f \otimes p) = f \otimes Np$ with a similar calculation as above.

Since δ and $1 \otimes E_*$ are isomorphisms, this shows $\varphi^{0,-1}(f \otimes a) = f \otimes a$ for the representatives.

ζ defines a homomorphism

$$\mathrm{Hom}_H(A, B^0)/N \,\mathrm{hom}(A, B^0) \to \mathrm{hom}(A_N/\psi^{*-1}(H^+)A, \ker(\epsilon(N) : B \to B)),$$

where we use $\hat{H}^{-1}(H, B^0) \cong \ker(\epsilon(N) : B \to B)$ (Proposition 5). Given $f : A_N \to \ker(\epsilon(N) : B \to B)$ with $f(\psi^{*-1}(H^+)A) = 0$. Then, there is an R-homomorphism $f' : A \to B$ whose restriction to A_N is f, since B is R-injective. We have $f'(\psi^{*-1}(H^+)A) = 0$. We also have

$$\psi^{*-1}(H^+)f'(A) \subseteq \psi^{*-1}(H^+)B^0 = 0.$$

H is a direct sum $R \cdot 1 \oplus \psi^{*-1}(H^+)$, since ψ^* is an algebra automorphism. Let $h = h_1 + h_2$ with respect to this decomposition. Then $f'(ha) = f'(h_1 a) + f'(h_2 a) = h_1 f'(a) = h_1 f'(a) + h_2 f'(a) = hf'(a)$; so $f' \in \mathrm{hom}_H(A,B^0)$, i.e., f' is a representative of an element in $\hat{H}^0(H, \mathrm{hom}(A,B^0))$. Let $a \in A_N$ be a representative for an element in $\hat{H}^{-1}(H, A) \cong A_N/\psi^{*-1}(H^+)A$, then $\zeta(f')(a) = \chi\varphi^{0,-1}(f' \otimes a) = \chi(f' \otimes a) = f'(a) = f(a)$; so $\zeta(f') = f$. Hence, ζ is an epimorphism.

To show that ζ is a monomorphism let $f \in \mathrm{hom}_H(A, B^0)$ be given such that $f(A_N) = 0$. We want to show that $f \in N \,\mathrm{hom}(A, B^0)$. The sequence $0 \to A_N \to A \to^N A$ is exact. Since B^0 is R-injective, the sequence

$$\mathrm{hom}(A, B^0) \to^{N_*} \mathrm{hom}(A, B^0) \to \mathrm{hom}(A_N, B^0) \to 0$$

is exact. Since $f(A_N) = 0$ there is an $f' \in \mathrm{hom}(A, B^0)$ such that $N_*(f') = f$, i.e., $f(a) = f'(Na)$ for all $a \in A$.

We observe that, by [7, Theorem 11], $S(h) = \sum_{(N)} N_{(1)}\psi(hN_{(2)})$; hence, we get

$$S(N) = \sum_{(N)} N_{(1)}\psi(\psi^*(N_{(2)})N) = \sum_{(N)} N_{(1)}\epsilon(\psi^*(N_{(2)})) = \sum_{(N)} \epsilon(\psi^*(N_{(1)}))\, N_{(2)},$$

since N is a cocommutative element. We compute for $f' : A \to B^0$

$$\begin{aligned}(Nf')(a) &= \sum_{(N)} \epsilon(\psi^*(N_{(1)}))f'(N_{(2)})a) \\ &= f'\left(S\left(\sum_{(N)} \epsilon(\psi^*(N_{(1)}))\, N_{(2)}\right) a\right) \\ &= f'(SS(N)a).\end{aligned}$$

Now $SS : H \to H$ is a Hopf algebra automorphism [7, Proposition 6]. So $SS(N)$ is again a left integral. By [7, Theorem 12] there is a unique $r \in R$ with $SS(N) = rN$. With the same proof for $S^{-1}S^{-1}$ we get a unique $r' \in R$ with $S^{-1}S^{-1}(N) = r'N$. Consequently, $rr'N = N$, and $r'rN = N$. Again, by

[7, Theorem 12], $rr' = 1$, and $r'r = 1$. So we get $(r^{-1}Nf')(a) = f'(Na) = f(a)$ for all $a \in A$; so $f = r^{-1}Nf' \in N \hom(A, B^0)$.

It is now trivial to see that ζ is a natural isomorphism.

THEOREM 3 (Duality-Theorem). *Let H be an* FH-*algebra and A be a left module. Let B be an injective R-module viewed as an H-module via $\epsilon : H \to R$. Let $N \in H$ be cocommutative. Then there are natural isomorphisms*

$$\hat{H}^n(H, \hom(A, B^0)) \cong \hom(\hat{H}^{-n-1}(H, A), B).$$

Proof. By Corollary 2 the R-modules $\hat{H}^{-n-1}(H, A)$ are annihilated by $\epsilon(N)$, so $\hom(\hat{H}^{-n-1}(H, A), B) \cong \hom(\hat{H}^{-n-1}(H, A), \ker(\epsilon(N) : B \to B)) \cong \hom(\hat{H}^{-n-1}(H, A), \hat{H}^{-1}(H, B^0))$. We observe that $\hat{H}^n(H, \hom(-, B^0))$ is a right universal and left couniversal (H, R)-connected sequence of contravariant functors. This is essentially a consequence of Lemma 5. The same holds for the functor $\hom(\hat{H}^{-n-1}(H, -), B)$, since for each (H, R)-injective —and consequently also (H, R)-projective—module A the functor vanishes. So ζ of Proposition 6 and its inverse can uniquely be extended to morphisms of (H, R)-connected sequences of functors and these morphisms are still inverses of each other.

9. We call an FH-algebra *cyclic* if it is generated as an R-algebra by one element X.

LEMMA 13. *Let H be a cyclic* FH-*algebra with generator X. Then $H^+ = H(X - \epsilon(X))$.*

Proof. X and $X - \epsilon(X)$ generate the same R-algebra, so that a cyclic FH-algebra is generated by an element $Y (= X - \epsilon(X))$ with $\epsilon(Y) = 0$. Then each $h \in H$ has the form $h = \alpha_0 + \alpha_1 Y + \cdots + \alpha_n Y^n$, and we get $h \in H^+$ iff $\alpha_0 = 0$ iff $h \in HY$[1].

LEMMA 14. *Let H be a cyclic* FH-*algebra with generator X. Then*

$$\begin{tikzcd}
\cdots \arrow[r, "N"] & H \arrow[r, "X-\epsilon(X)"] & H \arrow[rr, "N"] \arrow[dr, "\epsilon"'] & & H \arrow[r, "X-\epsilon(X)"] & H \arrow[r, "N"] & \cdots \\
& & & R \arrow[ur, "\rho"'] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

is a complete resolution of R as an H-module.

[1] This simplified proof of the lemma was kindly communicated to me by the referee.

Proof. By Proposition 4 we know that $N = \rho\epsilon$. But $\rho\epsilon$ has kernel $H^+ = (X - \epsilon(X))H = \operatorname{Im}(X - \epsilon(X))$. (Observe that H is commutative.) Since $N(X - \epsilon(X)) = (X - \epsilon(X))\, N = 0$, it is sufficient to prove $\ker(X - \epsilon(X)) \subseteq \operatorname{Im}(N)$. Let $a(X - \epsilon(X)) = 0$ for some $a \in H$. Then, for all $h \in H$ we have $ah(X - \epsilon(X)) = 0$ which implies $aH^+ = 0$ and $(a \circ \psi)(H^+) = 0$. This implies $(a \circ \psi)(h) = (a \circ \psi)(\epsilon(h)) = \psi(\epsilon(h)a) = \epsilon(h)\, \psi(a) = \epsilon(h\psi(a)) = \psi(h\psi(a)N) = (\psi(a)\, N \circ \psi)(h)$. Since ψ is a free generator of H^* we get $a = \psi(a)\, N \in \operatorname{Im}(N)$.

We shall call the cohomology $\hat{H}^*(H, -)$ of an FH-algebra H *periodic*, if there are natural isomorphisms $\hat{H}^n(H, -) \cong \hat{H}^{n+q}(H, -)$ for all n. q is called the *period*. The preceeding lemma implies immediately the following corollary.

COROLLARY 8. *Let H be a cyclic* FH-*algebra. Then the cohomology of H is periodic with period* 2. *The cohomology groups are*

$$\hat{H}^{2n}(H, A) \cong \ker(X - \epsilon(X))/\operatorname{Im}(N),$$
$$\hat{H}^{2n+1}(H, A) \cong \ker(N)/\operatorname{Im}(X - \epsilon(X)),$$

where $X - \epsilon(X)$ and N are multiplication of A by $X - \epsilon(X)$ and N respectively.

THEOREM 4. *Let H be a cocommutative* FH-*algebra. Let q be an integer. The following are equivalent*:

(1) $\varphi^{q,-q} : \hat{H}^q(H, R) \otimes \hat{H}^{-q}(H, R) \to \hat{H}^0(H, R)$ *is an isomorphism.*

(2) $\varphi^{q,-q} : \hat{H}^q(H, R) \otimes \hat{H}^{-q}(H, R) \to \hat{H}^0(H, R)$ *is an epimorphism.*

(3) *There is an $x \in \hat{H}^q(H, R)$ and a $y \in \hat{H}^{-q}(H, R)$ such that* $\varphi^{q,-q}(x \otimes y) = 1 \in \hat{H}^0(H, R)$.

(4) *There is an $x \in \hat{H}^q(H, R)$ such that*

$$\hat{H}^n(H, A) \ni a \mapsto \varphi^{n,q}(a \otimes x) \in \hat{H}^{n+q}(H, A)$$

is an isomorphism of (H, R)-connected sequences of functors.

(5) *There is an isomorphism $\varphi^n : \hat{H}^n(H, A) \to \hat{H}^{n+q}(H, A)$ of (H, R)-connected sequences of functors.*

(6) *There is a natural isomorphism $\rho : \hat{H}^n(H, A) \cong \hat{H}^{n+q}(H, A)$ for some n.*

Proof. Trivial implications are (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5), and (5) $\Rightarrow$ (6). Assume that (3) holds. Define $\sigma(a) = ax = \varphi^{n,q}(a \otimes x)$ and $\tau(a) = ay$. Then, $\tau\sigma(a) = axy = a$ and $\sigma\tau(a) = ayx = (-1)^q axy = (-1)^q a$. Hence σ and τ are isomorphisms. By the properties of the cup-product we obtain (4).

Assume that (6) holds. Since $\hat{H}^n(H, -)$ and $\hat{H}^{n+q}(H, -)$ are right universal and left couniversal (H, R)-connected sequences of covariant functors, we get (5).

Now we define a map $\pi^{0,0}$ by the commutative diagram

$$\begin{array}{ccc} H^0(H, A) \otimes H^0(H, B) & \xrightarrow{\pi^{0,0}} & H^0(H, A \otimes B) \\ \downarrow{\scriptstyle \rho\otimes 1} & & \downarrow{\scriptstyle \rho} \\ H^q(H, A) \otimes H^0(H, A) & \xrightarrow{\varphi^{q,0}} & H^q(H, A \otimes B). \end{array}$$

By Theorem 2, $\pi^{0,0}$ can be extended to a family $\pi^{m,n}$. Now since $\rho^{-1}\varphi^{n+q,m}(\rho \otimes 1)$ are natural transformations with respect to A and B and commute with the connecting homomorphisms, we get $\pi^{m,n} = \rho^{-1}\varphi^{m+q,n}(\rho \otimes 1)$ by Theorem 2. So we get a commutative diagram

$$\begin{array}{ccc} \hat{H}^0(H, R) \otimes \hat{H}^0(H, R) & \xrightarrow{\varphi^{0,0}} & \hat{H}^0(H, R) \\ \downarrow{\scriptstyle \rho^{-1}\otimes 1} & & \downarrow{\scriptstyle \rho^{-1}} \\ \hat{H}^{-q}(H, R) \otimes \hat{H}^0(H, R) & \xrightarrow{\pi^{-q,0}} & \hat{H}^{-q}(H, R) \\ \wr\Vert & & \wr\Vert \\ \hat{H}^0(H, R) \otimes \hat{H}^{-q}(H, R) & \xrightarrow{\pi^{0,-q}} & \hat{H}^{-q}(H, R) \\ \downarrow{\scriptstyle \rho\otimes 1} & & \downarrow{\scriptstyle \rho} \\ \hat{H}^q(H, R) \prod \hat{H}^{-q}(H, R) & \xrightarrow{\varphi^{q,-q}} & \hat{H}^0(H, R), \end{array}$$

where all homomorphisms are isomorphisms in particular $\varphi^{q,-q}$. Hence, (1) holds.

Corollary 9. *Let $2 \neq 0$ in $\hat{H}^0(H, R)$ and let the cohomology of H be periodic with period q. Then q is even.*

Proof. Take $x \in \hat{H}^q(H, R)$, $y \in \hat{H}^{-q}(H, R)$ with $x \cdot y = 1$. Then, $y \cdot x = (-1)^q$. Hence q must be even.

Corollary 10. *Let H' be a Hopf subalgebra of H and let both H and H' be* FH-*algebras. Let H be (H', R)-projective. Let H have periodic cohomology with period q. Then H' has periodic cohomology with period q.*

Proof. Since H is (H', R)-projective, each complete (H, R)-resolution $\mathfrak{X}$ of R is a complete (H', R)-resolution of R. Let A be a left

H-module. Then A is an H'-module. So there is a monomorphism $\hom_H(\mathfrak{X}, A) \to \hom_{H'}(\mathfrak{X}, A)$ which induces a homomorphism

$$i(H, H') : \hat{H}^m(H, A) \to \hat{H}^m(H', A)$$

the *restriction map* which is a natural transformation of (H, R)-connected sequences of functors.

$\hat{H}^*(H, -)$ and $\hat{H}^*(H', -)$ are right universal and left couniversal (H, R)-connected sequences of covariant functors by Lemma 9. By the definition of $i(H, H')$, the cup-product $\varphi^{0,0}$ for $\hat{H}^*(H, R)$, and the cup-product $\varphi'^{0,0}$ for $\hat{H}^*(H', R)$, it is easy to see $i(H, H')\,\varphi^{0,0} = \varphi'^{0,0} i(H, H')$. Hence we get $i(H, H')\,\varphi^{m,n} = \varphi'^{m,n} i(H, H')$. So $i(H, H')$ is a homomorphism with respect to the cup-products, for one can also check $i(H, H')(1) = 1$. Now for $x \in \hat{H}^q(H, R)$ with $x \cdot x^{-1} = 1$ we get that $i(H, H')(x) \in \hat{H}^q(H', R)$ is invertible; hence H' has periodic cohomology with period q.

10. Let H be a Hopf subalgebra with bijective antipode of the Hopf algebra G with bijective antipode. H is called *normal* in G if we have $\sum_{(g)} g_{(1)} h S(g_{(2)}) \in H$ for all $g \in G$, $h \in H$.

LEMMA 15. *Let H be normal in G. Then $GH^+ = H^+G$ and G/GH^+ is uniquely a Hopf algebra with bijective antipode such that $G \to G/GH^+$ is a Hopf algebra homomorphism.*

Proof. Let $gh \in GH^+$. Then $gh = \sum_{(g)} g_{(1)} h S(g_{(2)})\, g_{(3)} \in HG$ by normality. Now, if $h \in H^+$, then we have $\epsilon(\sum_{(g)} g_{(1)} h S(g_{(2)})) = 0$. So we get $gh \in H^+G$. To get the definition of normality symmetric, we show $\sum_{(g)} S(g_{(1)})\, h g_{(2)} \in H$ for all $g \in G$, $h \in H$. We apply S^{-1} to this sum and get $\sum_{(g)} S^{-1}(g_{(2)})\, S^{-1}(h)\, g_{(1)} = \sum_{(S^{-1}g)} S^{-1}(g)_{(1)} S^{-1}(h)\, S(S^{-1}(g)_{(2)}) \in H$ since S is bijective on both H and G. The rest of the proof follows from [9, Theorem 4.3.1], since GH^+ turns out to be a Hopf ideal. We call this Hopf algebra $G//H$.

THEOREM 5. *Let H be normal in G. Let G be H-projective. Then there is a spectral sequence*

$$H^p(G//H, H^q(H, B)) \Rightarrow H^n(G, B).$$

Proof. In Section 2 we saw that $H^n(G, B) \cong \operatorname{ext}_G{}^n(R, B)$. From [1, XVI, Theorem 6.1] we have a spectral sequence

$$\operatorname{ext}^p_{G//H}(A, \operatorname{ext}_H{}^q(R, B)) \Rightarrow \operatorname{ext}_G{}^n(A, B).$$

Replacing A by R gives the result.

COROLLARY 11. *With the assumptions of the preceeding theorem, there is an exact sequence*

$$0 \to H^1(G//H, A^H) \to H^1(G, A) \to H^1(H, A)^G \to H^2(G//H, A^H) \to H^2(G, A).$$

Proof. Apply [1, XV, Theorem 5.12] in the case $n = 1$ to the theorem. Observe that for a $G//H$-module A we have $A^{G//H} = A^G$.

COROLLARY 12. *With the assumptions of the preceeding theorem we have*

(a) *if* gl-dim$(H, R) = 0$ *then* $H^n(G//H, A^H) \cong H^n(G, A)$,

(b) *if* gl-dim$(G//H, R) = 0$ *then* $H^n(H, A)^G \cong H^n(G, A)$.

Proof. Either $H^m(H, A) = 0$ or $H^m(G//H, A) = 0$ for all $m > 0$. So the spectral sequence collapses to the given isomorphisms.

THEOREM 6. *Let H and G be* FH-*algebras and let H be normal in G. Let G be (H, R)-projective and $G//H$ be R-projective. Let $\epsilon(N_H)$ and $\epsilon(N_{G//H})$ be prime to each other,* i.e., *$\epsilon(N_H)$ and $\epsilon(N_{G//H})$ generate R as an R-ideal, where N_H and $N_{G//H}$ are the norms in H and $G//H$ respectively. Then, for all $n > 0$, there is a split exact sequence*

$$0 \to H^n(G//H, A^H) \to^f H^n(G, A) \to^g H^n(H, A)^G \to 0$$

thus giving

$$H^n(G, A) \cong H^n(G//H, A^H) \oplus H^n(H, A)^G.$$

Proof. $H^q(H, A)$ has the annihilator $\epsilon(N_H)$ for $q > 0$. $H^p(G//H, B)$ has the annihilator $\epsilon(N_{G//H})$ for $p > 0$. So $H^p(G//H, H^q(H, A)) = 0$ for $p > 0$ and $q > 0$ since $\epsilon(N_H)$ and $\epsilon(N_{G//H})$ are prime to each other. The nonzero terms of the spectral sequence lie on the edges. The only possibly nonzero differential is $H^{n-1}(H, A)^G \to H^n(G//H, A^H)$, which is also zero since the elements in the image are annihilated by $\epsilon(N_H)$ and $\epsilon(N_{G//H})$. Hence $E_2 = E_\infty$. By [1, XV, Proposition 5.5], for $p = 0$, $k = n$, we get the exact sequence

$$0 \longrightarrow E_n^{n,0} \xrightarrow{f} H^n \xrightarrow{g} E_n^{0,n} \longrightarrow 0,$$

which in our case is the exact sequence of the theorem.

Let $r\epsilon(N_H) + s\epsilon(N_{G//H}) = 1$. Then, the multiplication with $r\epsilon(N_H)$ maps $H^n(G, A)$ into the image of f and leaves the image of f elementwise fixed. So there is a retraction for f which means that the sequence splits.

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