The First-Order Theory of Boolean Differentiation

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Abstract. In this contribution we give a first-order axiomatisation of boolean differentiation. More precisely, for each $n \in \mathbb{N}$ and each complete theory T_K of Boolean algebras we will give a complete first-order theory T_n^K of n boolean derivatives where $\bigcap_{i=1}^{n} \ker(\delta_1)$ is a model of the theory T_K. These theories can be obtained by adding just a finite list of axioms to those of T_K. We show that any model of T_K extends uniquely to a model of T_n^K. Moreover, we will also provide a theory of the additive reduct equipped with n boolean derivatives, and we will see that these theories are categorical in every infinite cardinality.
We then show that the theories are indeed the asymptotic theories of the class of the algebras of switching functions equipped with any of the ordinarily used notions of derivative. Furthermore, we see that for the case $n = 1$ they also axiomatise the Fraisse limit of the finite switching functions with a derivative, and we use this fact to deduce quantifier elimination.

1 Introduction

1.1 Our approach

Derivative operations on boolean algebras have been much studied since they were first described as such in the 1950s. An up-to-date textbook focused on the calculus as well as on the numerous applications of Boolean diiferential operations is [12], while a concise systematic treatment of the calculus can be found in Chapter 10 of [11]. However, while algebraic and numeric aspects of differentiation on boolean algebras has been widely studied, and various fields of application have been explored, to the best of our knowledge there has been no investigation of the first-order theory of boolean differentiation to date.

Over the last decades, however, two areas of model theory that are pertinent to this have been explored in great depth:

Firstly, there is the model theory of difference fields, which are fields equipped with an automorphism. This has been developed extensively, using cutting-edge model-theoretic analysis such as the calculus of simple theories, and has found deep applications in number theory and algebraic dynamics (see [1] for an introduction). We will see later that, in fact, this setting of difference algebra is more aligned to our setting than the differential algebra that it is often compared with.

Secondly, there has been an upsurge in research on the connection between infinite models and their finite , with conference volumes such as [4] dedicated to the topic and the monograph [2] summing up a whole line of research. This is particularly relevant here since most of the application interest lies in differentiation on finite rather than infinite boolean algebras, while the power of model theoretic methods will be felt on the infinite level.

However, any application presupposes a good understanding of the first-order theory, and that is what we are presenting in this contribution. While we are inspired by the work on difference fields, our setting is quite different, since firstly we are entirely concerned with characteristic 2 (since $x + x = 0$ for the symmetric difference in a boolean algebra) and secondly the automorphisms we study are involutions rather than free. We will see that this combination will allow us to use a very small set of axioms compared to the axiomatisations of algebraically closed fields with an automorphism in [1]. This remains true even when we move towards several derivations, while the situation for several commuting automorphisms of fields is rather complicated.

However, one model-theoretic advantage of the field setting over that of boolean algebras is that algebraically closed fields are *uncountably categorical*, while boolean algebras are *unstable*. These classifications, which we will discuss briefly in Section 2 below, mean that the most powerful tools of contemporary model theory, those from stability theory, do not apply to boolean algebras. Therefore, we will also present the first-order theory of the reduct to a language that contains purely the symmetric difference and the derivation(s), and we will show that the theory of boolean differentiation considered in this language is in fact *totally categorical*, a very strong model-theoretic property that means that it has a unique model up to isomorphism in every infinite cardinality. This would then allow the direct applications of methods from [2], say, to our structures.

1.2 Outline

In the section following this introduction, we will be giving an overview of the terms and the results from model theory that we will be using in this paper.

In the main section, we will introduce boolean differentiation and specifically our framework for derivatives. We will provide the first-order axiomatisations for the full language and the additive reduct and prove their completeness.

In Section 4 we will discuss the relationship to finite algebras of logic functions equipped with derivatives, and prove elimination of quantifiers for the theories with a single derivation.

In the final section we will discuss connections to the existing literature on boolean differentiation. We will also highlight possible consequeces of our results and pont out some other putative areas for further research.

2 Some model theory

2.1 Classical model theory

In this section we will rehearse the elements of classical model theory that we will need in the course of the paper. However, we will assume familiarity with the basic principles of first-order logic, such as its syntax and semantics, as well as fundamental concepts such as completeness of a theory and isomorphism of structures, which should be explained in any first textbook on logic.

Due to their traditional connection to propositional logic, boolean algebras were among the first algebraic structures whose model theory was studied. In this work, we will refer to two classical complete theories of boolean algebras: infinite atomic and infinite atomless boolean algebras.

Proposition 1. *The following classes of boolean algebras are axiomatisable by a complete first-order theory:*

1. *The theory of infinite atomless boolean algebras*
2. *The theory of infinite atomic boolean algebras*

We will continue with some additional definitions:

Definition 2. A first-order theory is called *categorical* in a cardinal κ if all its models of cardinality κ are isomorphic.

It is well-known that the theory of infinite atomless boolean algebras is ω-categorical, while the theory of infinite atomic boolean algebras is not.

Categoricity is a central concept in model theory, as it implies both completeness and good model-theoretic behaviour:

Proposition 3. *(Vaught's Test, Theorem 2.2.6 of [8]) If a satisfiable first-order theory with no finite models is categorical in an infinite cardinal κ, then it is complete.*

The most common measure of well-behavedness used in modern model theory is stability and the many variants of this concept, all of which have their root in Saharon Shelah's groundbreaking work on classification theory. We will refer to several steps on this scale, which we will briefly introduce here. We will first need the concept of a type.

Definition 4. Let T be a complete theory, $\mathbb{M}$ a model of T, A a subset of $\mathbb{M}$ and $n \in \mathbb{N}$. Then a *(complete) n-type p of T over A* is a set of formulas with n free variables and parameters in A such that p is satisfiable and for every such formula ϕ, either ϕ or $\neg\phi$ lies in p.

Of special interest are often those models that realise many types, since they encode much of what could possibly happen in models of a theory.

Definition 5. Let T be a complete theory and let $\mathbb{M}$ be a model of T.

Then $\mathbb{M}$ is called *ω-saturated* if all types over finite subsets of $\mathbb{M}$ are reallised in $\mathbb{M}$.

$\mathbb{M}$ is called *saturated* if all types over a subset of $\mathbb{M}$ of smaller cardinality than $\mathbb{M}$ itself is realised in $\mathbb{M}$.

It is a general fact that every theory has ω-saturated models (cf. Theorem 4.3.12 of [8]). Whether saturated models exist is generally dependent on set theory - under the general continuum hypothesis, every theory has saturated models (Corollary 4.3.13 of [8]).

The number of types that are realised in a certain model is at the basis of one of a number of equivalent definitions of stability. However, since we will need a different formulation later, we will give that here:

Definition 6. Let T be a complete theory in a countable language.

T is called *stable* if no formula has the order property: that is, there is no model $\mathbb{M}$ of T and formula $\phi(x; y)$ such that for a sequence of pairs of tuples $(a_i; b_i)_{i<\omega}$ in $\mathbb{M}$, $\phi(a_i; b_j)$ holds if and only if $i < j$.

T is called *ω-stable* if there are only countably many types over any countable subset of a model of T

T is called *strongly minimal* if every definable subset of any model of T is either finite or cofinite (i. e. its complement is finite).

These categories of stability are related to another in a strictly descending scale as follows:

Proposition 7. *For complete theories T in a countable language, the following strict implications hold: (i) $\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv), where*

(i)	*T is strongly minimal*
(ii)	*T is categorical in one (equivalently all) uncountable cardinals*
(iii)	*T is ω-stable*
(iv)	*T is stable*

Proof. (i) implies (ii) by Proposition 6.1.12, (ii) implies (iii) by Theorem 5.2.10, (iii) implies (iv) by Proposition 6.2.11 with Theorem 6.2.14, all from [8].

One of the prime reasons for the usefulness of stability theory is its connection to the existence of a good dimension notion on all models of the theory. The most commonly used and strongest dimension notion is known as *Morley Rank (*alongside the associated notion of a *Morley Degree)* and is usually abbreviated as *RM.* While one can find a rigorous introduction of the notion in Chapter 6 of [8], we will here just note the relationship between the existence of a well-defined Morley Rank and the stability hierarchy given above:

Proposition 8. *Let T be a complete theory in a countable language.*

Then T is strongly minimal if and only if every model of T has Morley Rank 1 and Morley Degree 1.

If T is uncountably categorical, every model has finite Morley Rank.

T is ω-stable if and only if (every definable subset of) every model has well-defined Morley Rank.

Note 1. T being just stable is characterised by a different, but less well-behaved rank notion being well-defined.

We will conclude our excursion to stability theory by applying the stability hierarchy to boolean algebras.

Proposition 9. *Let T be a theory that interprets an infinite boolean algebra. Then T is unstable.*

Proof. The canonical order relation of any infinite boolean algebra, given by $a \leq b$ iff $a = a \wedge b$, has the order property in the sense of Definition 6.

Therefore, we will not just study the full theory of the differential boolean calculus, but also its reduct to the additive group of the associated boolean ring. That is an abelian group with $x + x = 0$ for all x, and thus an $\mathbb{F}_2$-vector-space. This reduct is on the opposite end of the stability spectrum:

Proposition 10. *The theory of infinite abelian groups with $x + x = 0$ for all x is strongly minimal.*

Proof. Classical result of model theory, see e. g. Section 4.5 of [5].

We will now continue to those concepts that help to characterise the relationship between finite and infinite structures.

First, we will introduce the concept of a generic theory, specialised to a context appropriate for our investigations:

Definition 11. Let $\mathcal{L}$ be a language and let $(\mathbb{M}_n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{L}$-structures. Let T be a complete $\mathcal{L}$-theory.

T is called the *generic theory* of $(\mathbb{M}_n)_{n\in\mathbb{N}}$ if for all $\varphi \in T$ there is an $N \in \mathbb{N}$ such that $\mathbb{M}_n \models \varphi$ for all $n > N$.

Since all finite boolean algebras are atomic, the generic theory of the cardinality-ascending sequence of finite boolean algebras is the theory of infinite atomic boolean algebras.

A generic theory can be considered as a limit of the individual theories of a sequence of structures.

A different notion which may or may not coincide with a generic theory can be obtained by turning this around and considering instead the first-order theory of the limit of the structures.

The notion of limit used here is the Fraisse limit of structures, for which there are different formalisations in slightly different settings. For our purposes, we will need one that can accommodate functions as well as relations, and we find it in Section 7 of [5].

Definition 12. Let $\mathcal{L}$ be a $\mathbb{M}$ be an $\mathcal{L}$-structure. Then

$\mathbb{M}$ is called *locally finite* if any finitely generated substructure of $\mathbb{M}$ is finite.

A locally finite structure is called *uniformly locally finite* if there is a function $f : \mathbb{N} \to \mathbb{N}$such that the substructure generated by any subset of cardinality n has cardinality at most $f(n)$.

A locally finite $\mathbb{M}$ is called *ultrahomogeneous* if every isomorphism between finite substructures extends to an isomorphism of $\mathbb{M}$.

If $\mathbb{M}$ is countably infinite, ultrahomogeneous and locally finite, it is referred to as a *Fraisse structure.*

Such a Fraisse structure is considered the *Fraisse limit* of the class of its finite substructures.

Proposition 13. *(Theorem 7.1.2 of [5]) A non-empty class of finite strutures $\mathfrak{K}$ is the class of finite substructures of a Fraisse structure (i. e. has a Fraisse limit) if the following are satisfied:*

1. *$\mathfrak{K}$ is closed under isomorphism*
2. *$\mathfrak{K}$ is closed under taking substructures*
3. *$\mathfrak{K}$ contains structures of arbitrarily large cardinalities*
4. *Whenever A and B are in $\mathfrak{K}$, there is a C in $\mathfrak{K}$ such that both A and B can be embedded in C (Joint embedding property)*
5. *Whenever A, B_1 and B_2 are in $\mathfrak{K}$, $f_1 : A \to B_1$ and $f_2 : A \to B_2$, there are a $C \in \mathfrak{K}$ and embeddings $g_1 : B_1 \to C$ and $g_2 : B_2 \to C$ such that $g_1 \circ f_1 = g_2 \circ f_2$. (Amalgamation property)*

Sometimes the generic theory of a class $\mathfrak{K}$ and the theory of the Fraisse limit coincide. For instance, the theory of infinite $\mathbb{F}_2$-vector-spaces is both the generic theory and the theory of the Fraisse limit of the class of finite $\mathbb{F}_2$-vector-spaces. For boolean algebras, however, both notions of limit exist, but they do not coincide: While the generic theory of the class of finite boolean algebras is the theory of infinite atomic boolean algebras, their Fraisse limit is atomless (Classical, see e. g. Example 6.5.25 of [9]).

A very useful consequence of ultrahomogeneity is that the theory of a Fraisse structure will often be ω-categorical and admit quantifier elimination:

Proposition 14. *(Theorem 7.4.1 of [5])Let $\mathbb{M}$ be a uniformly locally finite Fraisse structure. Then the theory of $\mathbb{M}$ is ω-categorical and admits quantifier elimination.*

As both abelian groups with $x + x = 0$ for all x and boolean algebras are uniformly locally finite, the theory of atomless

3 Axiomatising boolean differentiation

3.1 Boolean functions, rings and derivations

The first prerequisite for a study of structures endowed with derivative operations is to recognise the underlying algebraic nature of those structures.

We will formulate this paper entirely in the context of *boolean rings*, which is equivalent to that of boolean algebras.

Definition 15. A *boolean ring* $(\mathbb{B}, +, \cdot, 0, 1)$ is a commutative ring with unit that satisfies the following properties

1. *Idempotency*: For any $x \in \mathbb{B}$, $x \cdot x = x$
2. *Characteristic 2*: For any $x \in \mathbb{B}$, $x + x = 0$

Any boolean algebra can be made into a boolean ring by treating $+$ as the symmetric difference (sometimes written $\oplus$ to avoid ambiguity) and $\cdot$ as the conjunction. Conversely, any boolean ring is a boolean algebra with respect to conjunction defined as $\cdot$, disjunction defined as $x + y + xy$ and negation defined as $x + 1$. See [10] for the details.

This representation suits our purposes very well, since derivations are usually defined using the symmetric difference.

The most used derivations arise in the study of *switching functions*, that is, functions from $\{0,1\}^n \to \{0,1\}$ for an $n \in \mathbb{N}$. We will now formally introduce these derivations:

Definition 16. Let $n \in \mathbb{N}$, and let $f : \{0,1\}^n \to \{0,1\}$.

Then the *derivative of f with respect to the i-th coordinate* $\delta_i(f)$ is given by the function

$$\begin{gathered}\delta_i(f) : \{0,1\}^n \to \{0,1\}\,, \\ \delta_i(f)(a_1, \dots, a_i, \dots, a_n) \coloneqq f(a_1, \dots, a_i, \dots, a_n) + f(a_1, \dots, a_i', \dots, a_n).\end{gathered}$$

The *global derivative* $D(f)$ is given by $D(f)(x) = D(f)(x')$.

These derivatives have been extensively studied, and are the topic of the recent monograph [12]. In that and other work, a generalised notion of derivative that the authors call *vectorial derivative* is also introduced.

Definition 17. Let $n \in \mathbb{N}$, $f : \{0,1\}^n \to \{0,1\}$ and let $S \subseteq \{1, \dots, n\}$. Then the *vectorial derivative of f with respect to S, $\delta_S(f)$,* is given by the function

$$\delta_S(f) : \{0,1\}^n \to \{0,1\}\,,\ \delta_S(f)(a_1, \dots, a_n) \coloneqq f(a_1, \dots, a_n) + f(b_1, \dots, b_n),$$

where $b_i = \begin{cases} a_i' & i \in S \\ a_i & i \notin S \end{cases}$.

Heretofore boolean differentiation was studied mainly as an analogue to real or complex differentiation, and its algebraic properties were usually considered analogues to real or complex *differential* algebra (a remarkable exception to this being [13]).

However, while the above-mentioned derivatives are additive and factor over constants (i. e. function whose derivative is 0), they do not satisfy the Leibniz rule of differentiation, that is, $\delta(xy) = x\delta(y) + y\delta(x) + \delta(x)\delta(y)$ rather than

$\delta(xy) = x\delta(y) + y\delta(x)$, and indeed no possible notion of derivative could satisfy the classical definition of a derivation (cf. [11], Ch. 10).

In this paper, we will instead study boolean differentiation as an analogue of classical *difference* algebra, which studies automorphisms of the real or complex field. This possibility arises from the following observation:

Proposition 18. *In all of the cases above, the map $f \to \sigma(f)$, $\sigma(f)(a_1, \ldots, a_n) := f(b_1, \ldots, b_n)$, is an involution of the boolean ring of functions from $\{0,1\}^n \to \{0,1\}$.*

Proof. We need to show that σ respects addition and multiplication and that $\sigma^2 = \sigma$.

1. $\sigma(f+g)(a_1, \ldots, a_n) = (f+g)(b_1, \ldots, b_n) = f(b_1, \ldots, b_n) + g(b_1, \ldots, b_n) = \sigma(f)(a_1, \ldots, a_n) + \sigma(b)(a_1, \ldots, a_n)$
2. Similarly for multiplication
3. $\sigma^2(f)(a_1, \ldots, a_n) = f(b_1, \ldots, b_n)$, where $b_i = \begin{cases} a_i'' & i \in S \\ a_i & i \notin S \end{cases}$. But as $a_i'' = a_i$, $b_i = a_i$ for all i.

As $\delta(f) = f + \sigma(f)$, $\sigma(f) = f + \delta(f)$ and we can (and will) therefore study derivations and their associated involutions interchangeably.

3.2 A complete axiomatisation

In the light of Proposition 18, we can choose between using a derivation δ or an involution σ in our language, and whether to include the operations of a boolean algebra or the ring operations. For the sake of consistency with the notion of boolean *differentiation*, we will officially present our axiomatisation in the following languages:

Definition 19. For $n \in \mathbb{N}$, let $\mathcal{L}_n$ be the language consisting of the binary operations $+$ and $\cdot$, the constant symbols 0 and 1 and the unary functions $\delta_1, \ldots, \delta_n$.

Let $\mathcal{L}_n^+$ be the reduct of this language, where the conjunction $\cdot$ and the constant 1 are ommitted.

Whenever R is an $\mathcal{L}_n$ or $\mathcal{L}_n^+$ structure, let $\sigma_n := \delta_n + \mathrm{id}$.

In $\mathcal{L}_1$ and $\mathcal{L}_1^+$, we usually write δ and σ for δ_1 and σ_1.

For clarity of exposition, we will begin by providing a complete axiomatisation of the boolean derivative on $\mathcal{L}_n^+$ and then extending it to a complete axiomatisation on $\mathcal{L}_n$.

Definition 20. Let T_1^+ be the following $\mathcal{L}_1^+$ theory:

1. V is an abelian group of characteristic 2, that is, an abelian group with the property that $\forall x(x + x = 0)$.
2. σ is an involution of groups.
3. δ is complete, that is, $\forall y(\delta(y) = 0 \Rightarrow \exists x(\delta(x) = y))$.

We will not only show that T_1^+ is complete when restricted to infinite models, but moreover, we will show that it is categorical in every infinite cardinal:

Theorem 21. *T_1^+ is categorical in all infinite cardinals. Its infinite models form a complete ω-stable elementary class.*

The proof of Theorem 21 will go through two Lemmas. First, though a simple observation that we will use throughout and which justifies the formulation of the completeness axiom:

Remark 22. Let V be an abelian group of characteristic 2 and σ an involution of groups. Then $\forall x \in V : \delta(\delta(x)) = 0$.

Proof. $\delta(\delta(x)) = \delta(x+\sigma(x)) = (x+\sigma(x))+\sigma(x+\sigma(x)) = x+\sigma(x)+\sigma(x)+x = 0.$

Lemma 23. *Let V, V' be free finite-dimensional k-modules over a ring k and let $F : V \to V'$ be a linear isomorphism. Let $f : V \to V$ and $f' : V' \to V'$ be linear endomorphisms. Let $\vec{a}$ be a basis for V and M a matrix representing f with respect to $\vec{a}$. Then F is an isomorphism of the structures enriched by a function symbol for f on V and f' on V' iff M is the matrix representation of f' with respect to $\overrightarrow{F(a)}$.*

Proof. It suffices to show that $\forall x(F(f(x)) = f'(F(x)))$. So let $x \in V$ and let $\vec{v}$ be the k-vector representing x with respect to $\vec{a}$. Then $\vec{v}$ is also the k-vector representing $F(x)$ with respect to $\overrightarrow{F(a)}$. Thus

$$F(f(x)) = F(M\vec{v}\vec{a}) = M\vec{v}\overrightarrow{F(a)} = f'(\vec{v}\overrightarrow{F(a)}) = f'(F(\vec{v}\vec{a})) = f'(F(x)).$$

In order to apply Lemma 23 to our structures, we will prove another lemma:

Lemma 24. *Let $(V, +, 0, \delta)$ be a model of T_1^+. Then $(V, +, 0)$ is an $\mathbb{F}_2$ vector space and the following holds:*

1. *$V = \bigoplus_{i=1}^{\kappa} U_i$ for a cardinal κ, where each U_i is a 2-dimensional δ-invariant subspace on which δ can be represented by the matrix $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.*
2. *V has cardinality 2^{2n} for an $n \in \mathbb{N}$ or infinite cardinality.*

Proof. The proof will proceed in steps.

First, as V is an abelian group, being of characteristic 2 is equivalent to being an $\mathbb{F}_2$ vector space.

Let K be the kernel of the group- and thus $\mathbb{F}_2$-vector-space-homomorphism δ. Let $(b_i | i \in I)$ be an $\mathbb{F}_2$-basis for K and let $(a_i | i \in I)$ be such that $\delta(a_i) = b_i$. We claim that $V = \bigoplus_{i \in I} \langle a_i, b_i \rangle$ is a decomposition as required in the statement of the lemma. So, we have to show (a) that $V = \sum_{i \in I} \langle a_i, b_i \rangle$, (b) that the sum is direct and (c) that each$\langle a_i, b_i \rangle$ satisfies the requirements of the lemma.

(a): Let $x \in V$. Then by Remark 22 $\delta(x) \in K$ and thus $\delta(x) = \sum_{j \in J} b_j$. Observe that $\delta(x + \sum_{j \in J} a_j) = \sum_{j \in J} b_j + \sum_{j \in J} b_j = 0$ and thus that $x + \sum_{j \in J} a_j \in K$. But as by definition $K \subseteq \sum_{i \in I} \langle a_i, b_i \rangle$ and $\sum_{j \in J} a_j \in \sum_{i \in I} \langle a_i, b_i \rangle$, we also obtain $x \in \sum_{i \in I} \langle a_i, b_i \rangle$.

(b): We need to show that $\sum_{j \in J} u_j = 0 \Rightarrow u_j = 0$ for all $j \in J$. But by definition $\sum_{j \in J} u_j = \sum_{k \in K} a_k + \sum_{l \in L} b_l$. We see that $\delta(\sum_{k \in K} a_k + \sum_{l \in L} b_l) = \sum_{k \in K} b_k$ and since $(b_i | i \in I)$ is a basis for K, this implies that $K = \emptyset$. Then $\sum_{l \in L} b_l = 0$, which however implies that $L = \emptyset$ by the same argument.

(c): We have already seen that each $\langle a_i, b_i \rangle$ is 2-dimensional, so it remains to show that $\delta(a_i) = b_i$ and that $\delta(b_i) = 0$. But that is just the definition of the a_i and b_i.

This shows the first clause of the Lemma; the second clause follows from the first clause together with additivity of dimension in free sums and the fact that $|V| = 2^{\dim_{\mathbb{F}_2}(V)}$.

Remark 25. The proof shows that one can in fact extend any linearly independent system $\vec{w_i}$ in the kernel together with any $\vec{v_i}$ with $\delta(v_i) = w_i$ into a representation with respect to which the lemma holds.

We can now proceed to prove Theorem 21.

Proof. Let $(V, +, \delta)$ and $(V', +, \delta')$ be two models of T_1^+ of cardinality $\kappa \geq \omega$. Then by Lemma 24, $V = \bigoplus_{i=1}^{\kappa} U_i$ and $V' = \bigoplus_{i=1}^{\kappa} U_i'$ with the properties mentioned there. We define a linear bijection $F : V \to V'$ by defining linear bijections $F_i : U_i \to U_i'$ for each i. Let (a_i, b_i) and (a_i', b_i') be bases for U_i and U_i' repectively for which δ has the matrix representation $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then let $F_i(a_i) = a_i'$ and $F_i(b_i) = b_i'$. Clearly, F_i defines an isomorphism of vector spaces, and by Lemma 23, $F_i(\delta(x)) = \delta'(F(x))$. We will now define $F(x) = F(\sum u_j) := \sum F_j(u_j)$. This is clearly a well-defined linear bijection. It thus only remains to show that $F(\delta(x)) = \delta'(F(x))$:

$$F(\delta(x)) = \sum F_j(\delta(u_j)) = \sum \delta'(F_j(u_j)) = \delta'(\sum F_j(u_j)) = \delta'(F(x)).$$

Therefore, T_1^+ is categorical in all infinite cardinals. By the discussion in Section 2, this implies that T_1^+ is both complete and ω-stable (since it is uncountably categorical).

This categoricity result unlocks powerful model-theoretic tools for boolean differential groups, which we will briefly discuss in the final section. Here we will now adapt our axiomatisation to give a complete first-order theory of boolean differentiation which takes full account of the ring structure.

Definition 26. Let K be a boolean ring, and T_K a complete first-order theory of boolean algebras expressed in the language of boolean rings. Then T_1^K is the following theory in the language $\mathcal{L}_1$:

1. σ is an involution of boolean Rings.
2. $\ker(\delta) \models T_K$.
3. δ is complete, i. e. there is a $z \in V$ such that $\delta(z) = 1$.

Remark 27. We remark that we found it rather surprising that one could obtain a complete axiomatisation by just adding a finite number of axioms to the ones regarding K. This seems to be entirely due to the fact that one can define the ring structure on V from the ring structure on the constants (see below).

We will adopt a different and possibly more straightforward strategy to proving completeness here, extending isomorphisms between kernels to isomorphisms between the models of T_1^K. First, we give a more concrete characterisation of completeness:

Proposition 28. *Let V be a model of T_1^K for a boolean ring K. Then V is a free* $\ker(\delta)$*-algebra on two generators $(1, z)$ and δ is a* $\ker(\delta)$*-algebra-morphism given by $\delta(z) = 1$ and $\delta(1) = 0$.*

Proof. Let z be as in the definition of T_1^K.

(a) $(1, z)$ generate V. Indeed, let $x \in V$ be arbitrary. Then $\delta(x) \in \ker(\delta)$ and $\delta(x) = \delta(\delta(x)z)$ by $\ker(\delta)$-linearity. Thus, $x + \delta(x)z \in \ker(\delta)$ and therefore $x = (x + \delta(x)z) + \delta(x)z$ is the required representation.

(b) $(1, z)$ generate V freely. Indeed, if $a + bz = 0$ for some $a, b \in K$, then $\delta(a + bz) = b = 0$ and thus also $a = 0$.

Now we can prove the extension of isomorphisms.

Proposition 29. *There is a one-to-one correspondence between isomorphism classes of boolean algebras K and isomorphism classes of models of T_1^K.*

Proof. Let K be a boolean algebra and V a free K-algebra on 2 generators. Then by Lemma 23, the condition $\delta(z) = 1$ and $\delta(1) = 0$ uniquely determines V as a K-algebra up to isomorphism.

So let $f : V \to V'$ be an isomorphism of K-algebras respecting δ. We claim that f is in fact an isomorphism of boolean rings. So let $(k_1 + k_2 z)$ and $(k_1' + k_2' z)$ be elements of V. Then

$$\begin{aligned} f((k_1 + k_2 z) \cdot (k_1' + k_2' z)) =& f(k_1 k_1' + (k_2 k_1' + k_1 k_2' + k_2 k_2')z) \\ =& f(k_1)f(k_1') + (f(k_2)f(k_1') + f(k_1)f(k_2') + f(k_2)f(k_2'))f(z) \\ =& f(k_1 + k_2 z)f(k_1' + k_2' z) \end{aligned}$$

Therefore f is actually an isomorphism of boolean rings as required.

It follows from the above proposition that whenever the theory of K is ω-categorical, then so is T_1^K. In particular, when K is an infinite atomless boolean algebra, then T_1^K is ω-categorical and therefore complete. In fact, T_1^K is complete regardless of K, and this can be seen using any of a number of classical model-theoretic techniques.

Theorem 30. *Let T_K be any complete theory of boolean rings. Then the theory T_1^K is complete.*

Proof. We will sketch a proof using ultraproducts (See Section 9.5 of [5] for an introduction), since that most easily generalises to several derivations. Let A and B be models of T_1^K, and let K_A and K_B be their respective kernels. Then $K_A \equiv K_B$ and we want to show that $A \equiv B$ also. By the Keisler-Shelah Theorem, K_A and K_B have isomorphic ultrapowers $U(K_A) \simeq U(K_B)$. Using the same index set and the same ultrafilter, we can take the ultrapowers of $U(A)$ of A and $U(B)$ of B. Then the kernel of $U(A)$ is isomorphic to $U(K_A)$ and the kernel of $U(B)$ is isomorphic to $U(K_B)$. Thus, the kernels are isomorphic to each other and by Proposition 29 $U(A)$ and $U(B)$ are also. Therefore, A and B must have been elementarily equivalent.

We will now extend the characterisations above to several derivatives.

In the following we will use the shorthand $\delta_J^{|J|}$ to mean the $|J|$-fold derivative with respect to all δ_j, $j \in J$; for instance, $\delta^n_{\{1,\dots,n\}}(x) = \delta_1\delta_2\dots\delta_n(x)$ and $\delta^1_{\{j\}} = \delta_j$. (We add the cardinality superscript to avoid confusion with the vectorial derivative from Definition 17)

Definition 31. Let T_n^+ be the following $\mathcal{L}_n^+$ theory:

1. V is an abelian group of characteristic 2, that is, an abelian group with the property that $\forall x(x + x = 0)$
2. $\sigma_1, \dots, \sigma_n$ are commuting involutions of groups.
3. $\{\delta_1, \dots, \delta_n\}$ is complete, that is,

$$\forall y(\delta_1(y) = 0 \wedge \delta_2(y) = 0 \wedge \dots \wedge \delta_n(y) = 0 \Rightarrow \exists x(\delta_1\delta_2\dots\delta_n(x) = y)).$$

We will now provide an analogue to Lemma 24 to prove the categoricity of T_n^+ in each uncountable cardinal.

Lemma 32. *Let $(V, +, 0, \delta_1, \dots, \delta_n)$ be a model of T_n^+. Then $(V, +, 0)$ is an $\mathbb{F}_2$ vector space and the following holds:*

1. *$V = \bigoplus_{i=1}^{\kappa} U_i$ for a cardinal κ, where each U_i is a 2^n-dimensional δ-invariant subspace which has a basis $(a_{i,J} | J \subseteq \{1, \dots, n\})$ such that the following holds: $\{\langle\{a_{i,J}, a_{J\cup\{j\}}\}\rangle \,|\, j \notin J\}$ is a decomposition of U_i in the sense of Lemma 24 with respect to δ_j.*
2. *V has cardinality $2^{2^n m}$ for an $m \in \mathbb{N}$ or infinite cardinality.*

Proof. The proof will proceed in steps.

First, as V is an abelian group, being of characteristic 2 is equivalent to being an $\mathbb{F}_2$ vector space.

Let (b_i) be a basis for $\bigcap_{j=1}^{n} K_i$, where $K_j := \ker(\delta_j)$. Then choose (a_i) such that $\delta_1\delta_2\dots\delta_n(a_i) = b_i$. Let $a_{i,J} := \delta_J^{|J|}(a_i)$. We claim that this satisfies the requirements, and we will prove this by induction. The case $n = 1$ has been shown in Lemma 24. So assume true for n. It is easy to see that K_{n+1} is a model of T_n^+. Therefore, by the induction hypothesis, $(a_{i,J}|J \subseteq \{1,\dots,n+1\}, n+1 \in J)$ is a basis for K_{n+1} as required. But then by Lemma 24, $(a_{i,J}|J \subseteq \{1,\dots,n\})$ is a basis of V with exactly the properties described in clause 1.

This shows the first clause of the Lemma; the second clause follows from the first clause together with additivity of dimension in free sums and the fact that $|V| = 2^{\dim_{\mathbb{F}_2}(V)}$.

We can now deduce the completeness and indeed the total categoricity of T_n^+ just as we did for T_1^+:

Theorem 33. *T_n^+ is categorical in all infinite cardinals. Its infinite models form a complete ω-stable elementary class.*

Proof. Just as in the proof of Theorem 21, the linear bijection induced by the bases given by Lemma 32 is an $\mathcal{L}_n$-isomorphism by Lemma 23.

We will now finally provide an axiomatisation of the complete theory of several derivations on boolean rings:

Definition 34. Let K be a boolean Ring, and T_K a complete first-order theory of boolean algebras expressed in the language of boolean rings. Then T_n^K is the following theory in the language $\mathcal{L}_n$:

1. $\sigma_1, \dots, \sigma_n$ are commuting involutions of boolean Rings.
2. $\bigcap_{i=1}^{n} \ker(\delta_i) \models T_K$.
3. $\{\delta_1, \dots, \delta_n\}$ is complete, that is,

$$\exists x(\delta_1\delta_2 \dots \delta_n(x) = 1)).$$

The proof will again be preceded by a proposition giving a more concrete representation.

Proposition 35. *Let V be a model of T_n^K for a boolean ring K. Then V is a free $\bigcap_{i=1}^{n} \ker(\delta_i)$-algebra on 2^n generators given by $\{a_J := \delta_J^{|J|}(a)|J \subseteq \{1,\dots,n\}\}$ for any $a \in V$ with $\delta_1\delta_2 \dots \delta_n(a) = 1$.*

Proof. By induction on n. The case $n = 1$ is part of Proposition 28. So assume it true for n and choose any model V of T_{n+1}^K and any $a \in V$ with $\delta_1\delta_2 \dots \delta_{n+1}(a) = 1$.

We will now show that it is a generating system for V. So let $x \in V$. We will proceed by induction on the smallest number m such that the m-fold derivative $\delta_{\{1,\dots,m\}}^m(x) \in \bigcap_{i=1}^{n} \ker(\delta_i)$. If $m = 0$ then $x \in \bigcap_{i=1}^{n} \ker(\delta_i)$ itself. So assume true for m. Then if $\delta_{\{1,\dots,m+1\}}^m x \in \bigcap_{i=1}^{n} \ker(\delta_i)$, $x = (\delta_{\{1,\dots,m+1\}}^m x) \cdot \delta_{\{1,\dots,n\}\setminus\{1,\dots,m+1\}}^{n-(m+1)} a + y$,

$y \coloneqq ((\delta^m_{\{1,\dots,m+1\}} x)\delta^{n-(m+1)}_{\{1,\dots,n\}\setminus\{1,\dots,m+1\}} a + x)$. Here $\delta^m_{\{1,\dots,m+1\}} y = 0$ and thus $\delta^m_{\{1,\dots,m\}} y \in \bigcap_{i=1}^{m+1} \ker(\delta_i)$.

Proposition 36. *Let K be a boolean ring. Then there is exactly one model of T^n_K up to isomorphism with $\bigcap_{i=1}^{n} \ker(\delta_i) = K$.*

Proof. We will prove the theorem by induction on n. The case $n = 1$ is exactly Proposition 29. So assume it true for T^n_K. We will now show it for T^{n+1}_K. Let a be the witness of clause 3. of the definition and let $a_J \coloneqq \delta^{|J|}_J(a)$. Then we claim that the isomorphism of K-modules induced by a_J is an $\mathcal{L}_{n+1}$-isomorphism. By the induction hypothesis, it is an isomorphism of the obvious $\mathcal{L}_n$-structures on $K_i \coloneqq \ker(\delta_i)$ for each derivation δ_i. However, since $\delta_i(a_{\{1,\dots,i-1,i+1,\dots,n\}}) = 1$, another application of Proposition 29 shows that we actually have an isomorphism of boolean rings which also respects δ_i. Since i was arbitrarily chosen, this finishes the proof.

Just as in the proof of Theorem 30, we can now deduce completeness:

Theorem 37. *Let T_K be any complete theory of boolean rings, and let $n \in \mathbb{N}$. Then the theory T^K_n is complete.*

Proof. Verbatim as in the proof of Theorem 30.

4 Relationship to finite models and immediate consequences

In this section we will be connecting the complete theories from Subsection 3.2 with the examples of boolean differentiation studied in the literature.

In particular, we will show that the theories we have introduced can be naturally characterised as the generic or as the limit theories of groups or rings of switching functions equipped with the derivatives introduced in Subsection 3.1. To facilitate notation, we introduce

Definition 38. Let $\mathbb{S}_n$ be the *boolean ring of switching functions in n variables,* that is, the boolean ring made up of all mappings $f : \{0,1\}^n \to \{0,1\}$, eqipped with the ring structure from Subsection 3.1. Let $\mathbb{S}^+_n$ be the additive group reduct of $\mathbb{S}_n$.

Theorem 39. *The theory of infinite models of T^+_1 is the generic theory of the class $\{\mathbb{S}^+_n | n \in \mathbb{N}\}$, where each switching algebra is equipped with any of the derivatives of Subsection 3.1.*

Proof. By the results at the end of Subsection 3.1, each of the structures mentioned is a model of T^+_1. Clearly, $|\mathbb{S}_n| \geq n$ and thus the additional infinity axioms are generically true in the class too. So, the theory of infinite models of T^+_1 is a subset of the generic theory. However, as the theory is complete by Theorem 21, it is the generic theory.

The equivalent result for the theory of boolean rings is obtained in a very similar way; however, one has to choose the boolean algebra that is the generic theory

Theorem 40. *The theory T_1^K, where K is an infinite atomic boolean algebra, is the generic theory of the class $\{\mathbb{S}_n | n \in \mathbb{N}\}$, where each switching algebra is equipped with any of the derivatives of Subsection3.1.*

Proof. By the discussion following Definition 11, the theory of K is the generic theory of the class $\{\mathbb{S}_n | n \in \mathbb{N}\}$ as boolean rings. Since $|\ker(\delta)| = \sqrt{|\mathbb{S}_n|}$ for all derivations mentioned in Subsection 3.1, the theory of $\ker(\delta)$ will indeed be generically T_K. The remainder of the axioms are clear. As T_1^K is complete by Theorem 29, we can conclude that T_1^K is the generic theory.

The theorems above show that we have indeed given a characterisation of the asymptotic theory of switching functions - so although our results and methods have focused on infinite models, they can be used to study the derivations on arbitrarily large finite switching algebras that have spawned such a large literature.

They also generalise to the theories with several derivations, when one considers derivations that are *linearly independent* in the sense of [12], but we will omit the generalisation of the proofs here for brevity. One example of such linearly independent derivations are the single derivations $\delta_1, \ldots, \delta_n$ on $\mathbb{S}_n$. We have

Theorem 41. *The theory of infinite models of T_n^+ is the generic theory of the class $\{\mathbb{S}_i^+ | i \geq n\}$, where each switching algebra is equipped with the single derivatives δ_1 to δ_n.*

The theory T_n^K, where K is an infinite atomic boolean algebra, is the generic theory of the class $\{\mathbb{S}_i | i \geq n\}$, where each switching algebra is equipped with the single derivatives δ_1 to δ_n.

We will now move on to characterise the theories we have constructed as the complete theories of limit structures. This gives us more information about their model theory and provides a concrete structure into which the finite switching algebras can be uniquely embedded up to isomorphism. We can take the limit over the same classes we have considered above. In the additive case, we will obtain exactly the same theory, as the underlying theory of infinite $\mathbb{F}_2$-vector spaces is both generic and limit theory of the finite $\mathbb{F}_2$-vector spaces. In the full boolean ring case, however, we will have to change the boolean ring K under consideration since the limit structure of finite boolean rings is the countable atomless boolean algebra and not a countable atomic boolean algebra.

Theorem 42. *Let $\mathfrak{C}$ be the class of all substructures of a member of the class $\{\mathbb{S}_i^+ | i \in \mathbb{N}\}$, where each switching algebra is equipped with any of the derivatives of Subsection3.1. Then $\mathfrak{C}$ is Fraisse class and its limit structure is the unique countably infinite model of T_1^+.*

Proof. We will go through the requirements of a Fraisse class one by one.

1. Closure under isomorphisms is clear.
2. Closure under substructure is guaranteed by our definition as being substructures of a certain other class of structures.
3. It contains arbitrarily large structures, as S_n lies in $\mathfrak{C}$.
4. We can always consider the larger of the two indices of the structures that they embed into.
5. Consider the situation of the amalgamation condition. As we can embed B_1 and B_2 into S_i and S_j respectively, we can assume without loss of generality that B_1 and B_2 are in $\{\mathbb{S}_i^+ | i \in \mathbb{N}\}$. Let δ denote the derivations. Without loss of generality, let the index of B_1 be at most the index of B_2. We will first build a basis for A, which we will then extend to bases for B_1 and B_2 in such a way that a natural embedding between the bases defines an embedding from B_1 into B_2. Start with a basis for $\delta(A) \subseteq A$. This can be extended to a basis for $\ker_A(\delta)$, and that in turn to bases $(\vec{k})$ and $(\vec{k'})$ of B_1 and B_2 respectively. By (the proof of) Lemma 24, $\vec{k}$ and $\vec{k'}$ together with any choice of preimages of $\vec{k}$ and $\vec{k'}$ define bases for B_1 and B_2. We can therefore choose the preimages in such a way that the preimage will be chosen from A wherever A contains such a preimage. We argue that the bases $\vec{b}$ and $\vec{b'}$ obtained in that way contain a basis for A. Indeed, by the standard kernel-image decomposition in linear algebra, the dimension of A is equal to the dimension of the image plus the dimension of the kernel, and the number of preimage elements that could be chosen from A is exactly the dimension of the image. So consider the embedding from B_1 into B_2 that is induced by mapping $\vec{b}$ to $\vec{b'}$ in an appropriate way. Then by Lemma 23, this is an isomorphism onto its image, i. e. an embedding, and it respects A as required by clause 5.

Thus, we could also define the theory of infinite models of T_1^+ as the theory of the Fraisse limit of all finite switching algebras equipped with a derivation.

The analysis also yields quantifier elimination as a consequence:

Corollary 43. *The theory of infinite models of T_+^1 has quantifier elimination.*

Proof. The substructure generated by a subset A of a differential group is the group generated by $A \cup \sigma(A)$. Thus, since the group reduct is uniformly locally finite, so is the boolean differential group.

Thus, the result follows from Theorem 42 by 14.

Considering the theory with the full boolean algebra structure, we obtain a representation for T_1^K, where K is the countable atomless boolean algebra.

Theorem 44. *Let $\mathfrak{C}$ be the class of all substructures of a member of the class $\{\mathbb{S}_i | i \in \mathbb{N}\}$, where each switching algebra is equipped with any of the derivatives of Subsection3.1. Then $\mathfrak{C}$ is a Fraisse class and its limit structure is the unique countably infinite model of T_1^K, where K is the countable atomless boolean algebra.*

Proof. As in the proof of Theorem 42, Clauses 1-4 are easily verified. We therefore consider the situation of the amalgamation property, and again we can assume without loss of generality that B_1 and B_2 are in $\{\mathbb{S}_i | i \in \mathbb{N}\}$ and that the index of B_1 is at most the index of B_2. Due to the corresponding property for pure boolean algebras, we can furthermore assume that $\ker(\delta)_{B_1} \subseteq \ker(\delta)_{B_2}$ and that $(\ker(\delta) \cap A)_{B_1} = (\ker(\delta) \cap A)_{B_2}$. By the analysis in Chapter 3 of [12], $\delta(A)$ is itself a lattice of functions. In particular, $\delta(A)$ has a maximum, say $\alpha \in A$. Let $x \in A$ be chosen with $\delta(x) = \alpha$. Choose z_1 and z_2 in B_1 and B_2 respectively s. t. $\delta(z_1) = 1$ and $\delta(z_2) = 1$. We will define an embedding $\iota : B_1 \to B_2$ by setting ι to be the identity on $\ker(\delta)$ and choosing a value of $\iota(z_1)$. If $\delta(\iota(z_1)) = 1$, then ι is an embedding of boolean differntial algebras. So consider $x = \alpha z_1 + a$ in B_1 and $x = \alpha z_2 + b$ in B_2, where $\delta(a) = \delta(b) = 0$. Then we set $\iota(z_1) := z_2 + a + b$. This defines an embedding of boolean differential algebras since $\delta(z_2 + a + b) = 1 + 0 + 0 = 1$. We thus have to show that for all $y \in A$, $\iota(y_{B_1}) = \iota(y_{B_2})$. First, we will see that this holds for x, and we will derive an auxiliary result:

$$
\begin{aligned}
x\alpha &= \alpha z_1 + a\alpha = x + (\alpha + 1)a \\
&= \alpha z_2 + b\alpha = x + (\alpha + 1)b \\
\Rightarrow & (\alpha + 1)a = (\alpha + 1)b \\
\Rightarrow & (\alpha + 1)(a + b) = 0 \\
\Rightarrow & \alpha(a + b) = a + b
\end{aligned}
$$

So $\iota(x) = \alpha(z_2 + a + b) + a = \alpha z_2 + b$ as required. So now consider $y \in A$ arbitrary. Then $y = \beta z_1 + c = \beta\alpha z_1 + c = \beta(\alpha z_1 + a) + \beta a + c = \beta x + \beta a + c$. Since ι is the identity on the kernel elements β, a and c and $\iota(x_{B_1}) = \iota(x_{B_2})$ it follows that $\iota(y_{B_1}) = \iota(y_{B_2})$ as required.

Therefore the theory has a Fraisse limit.

Since ultrahomogeneity of the whole structure also implies ultrahomogeneity of the kernel, the kernel must be the countable atomless boolean algebra.

Just as for the additive theory, we can now conclude a quantifier elimination result:

Corollary 45. *Let K be an atomless boolean algebra. Then the theory T_1^K admits quantifier elimination.*

Proof. Just as Corollary 43 follows from Theorem 42

5 Future applications and perspectives

In this section we will briefly discuss the connection between the first-order theory as presented here and Kühnrich's abstract notion of a boolean derivative (see Chapter 10 of [11]). We will then explore potential applications and firections for further research.

While this is to the best of our knowledge the first analysis of the first-order theory of boolean differentiation, there has certainly been some work on a more general framework for the different notions of derivative suggested in the literature. One such framework, which has been proposed by Martin Kühnrich ([6]), is presented in the chapter on boolean differentiation in [11]:

Definition 46. Let B be a boolean ring and let $d : B \to B$. Then d is called a *(Kühnrich) differential operator* if the following hold:

1. For all $x \in B$, $d(d(x)) = 0$.
2. For all $x \in B$, $d(x+1) = dx$.
3. For all $x, y \in B$, $d(xy) = xd(y) + yd(x) + d(x)d(y)$.

We will show that since Kühnrich's axioms do not include any notion of completeness, they are essentially weaker than the theory presented here. In fact, Kühnrich's differential operator has a simple characterisation in terms of involutions:

Proposition 47. *Let B be a boolean ring and let $d : B \to B$. Then d is a* (Kühnrich) differential operator *if and only if $\sigma : B \to B$, $\sigma(x) = x + d(x)$, is an involution of boolean rings.*

Proof. "⇒": We will verify that σ respects addition, multiplication, 0 and 1.

1. d respects addition by Proposition 10.2.1 of [11]. Thus $\sigma(x+y) = x + y + d(x+y) = x + d(x) + y + d(y) = \sigma(x) + \sigma(y)$.
2. $\sigma(xy) = xy + (xd(y) + yd(x) + d(x)d(y)) = (x + d(x))(y + d(y)) = \sigma(x)\sigma(y)$.
3. $d(0) = d(1) = 0$ by Proposition 10.2.1 of [11]. Thus $\sigma(0) = 0 + 0 = 1$ and $\sigma(1) = 0 + 1 = 1$.

"⇐": Let σ be an involution and $d(x) \coloneqq x + \sigma(x)$. We will verify Kühnrich's axioms for d.

1. $$d(d(x)) = d(x)+\sigma(d(x)) = x+\sigma(x)+\sigma(x)+\sigma(\sigma(x)) = x+\sigma(x)+\sigma(x)+x = 0.$$
   .
2. $$d(x+1) = x + 1 + \sigma(x) + \sigma(1) = x + \sigma(x) = d(x).$$
   .
3. $$\begin{aligned} xd(y) + yd(x) + d(x)d(y) =& x(y + \sigma(y)) + y(x + \sigma(x)) + (x + \sigma(x))(y + \sigma(y)) \\ =& xy + x\sigma(y) + xy + y\sigma(x) + xy + x\sigma(y) + y\sigma(x) + \sigma(x)\sigma(y) \\ =& xy + \sigma(x)\sigma(y) = d(xy). \end{aligned}$$

This holds completely analogously for the "boolean differential algebras of order k" that are introduced in Definition 10.2.2 of [11]; they are exactly characterised by k commuting involutions of B.

This characterisation suggests the question of the exact relationship between Kühnrich's operators and the models of the theories introduced here. In particular, it is clear that every substructure of a model of T_1^K (T_n^K) is a differential operator (algebra) in this sense. But does the converse hold? By Corollary 6.5.3 of [5], this is equivalent to the question of whether Kühnrich's axioms axiomatise the universal theory of $T_1^K(T_n^K)$.

A particular interest lies in the connections between the finite structures that are studied in the literature and the complete theories of infinite structures expounded here. For the particular case of the additive reduct, this is especially alluring, since their infinite models form a totally categorical theory. The connection between totally categorical theories and their finite substructures is the subject of a deep model-theoretic analysis around so-called *smoothly approximable structures* as discussed for instance in [7] and [2]. In particular, there is a close connection between the Morley rank of a definable set in the theory of infinite models and the size of the respective definable subset of a finite model.

Of course, stability theory brings a host of inter-related concepts in its own right too, and investigating these notions with respect to additive boolean differentiation would be an important contribution towards bringing the theory of difference algebra in the boolean case to a similar level as the more widely studied difference algebra over fields.

Furthermore, it would be very interesting to extend T_n^K and T_n^+ to countably infinitely many derivations. Then, one would have one single theory encompassing switching functions of arbitrary sizes and their derivatives. Using more sophisticated model-theoretic techniques, one might also be able to extend the stability hierarchy in order to adequately cover this case.

The quantifier elimination results in Section 4 beg the question to what extent they can be extended to the theories with several derivatives. It would also be interesting to consider how quantifier elimination results for other theories of boolean algebras, that might require additional predicates, can be extended to quantifier elimination results for the corresponding theory T_1^K. One example is the theory of infinite atomic boolean algebras, which admits quantifier elimination if one adds predicates for "n atoms lie below x" (see [3] for details and further examples).

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