NON-ADDITIVE RING AND MODULE THEORY V.

PROJECTIVE AND COFLAT OBJECTS.

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The notion of a finitely generated projective K-module, where K is a commutative ring can be defined in many different ways. Several of these definitions may be carried over to monoidal categories. For closed symmetric categories some of these generalizations have been known for some time under different names [1, 3, 4, 5, 6, 12] . In [3] it was observed that these notions coincide with the notion of a strongly dualizable space in the monoidal category Stab , the stable homotopy category.

In this paper we want to study the generalization of finitely generated projective A-modules P in monoidal (not necessarily closed or symmetric) categories, where A is a K-algebra. We call such objects P "finite". At the end of the paper we shall also investigate a slightly stronger notion of "finitely generated projective A-objects".

One principal obstacle for this generalization is that $\mathcal{C}$ is not necessarily closed so that the tensor product does not preserve difference cokernels. Although we can define, like in module categories, the tensor product $M \otimes_A N$ "over" a monoid A , we do not have associativity anymore. The first section of this paper shows a way how to deal with this problem.

Section 2 and 3 are devoted to the study of different equivalent definitions of finite objects (see Theorem 3.10). The last section contains miscellaneous material on finite objects and relative projective objects.

1. Let $\mathcal{C}$ be a monoidal category with tensor product $\otimes$ and neutral object I. Let $A,B,C \in \mathcal{C}$ be monoids and $M \in {}_B\mathcal{C}_A$ resp. $N \in {}_A\mathcal{C}_C$ be B-A- resp. A-C-biobjects. As in [9] the tensor product $M \otimes_A N$ is defined to be the difference cokernel

$$M \otimes A \otimes N \rightrightarrows M \otimes N \longrightarrow M \otimes_A N$$

in ${}_B\mathcal{C}_C$, if it exists.

1.1 *Definition*: $P \in {}_A\mathcal{C}_B$ is called B-*coflat over* A if

i) the tensor product $P \otimes_B M$ in ${}_A\mathcal{C}$ exists for all $M \in {}_B\mathcal{C}$

ii) the morphism

$$P \otimes_B (M \otimes X) \longrightarrow (P \otimes_B M) \otimes X$$

induced by α is an isomorphism for all $X \in \mathcal{C}$ and $M \in {}_B\mathcal{C}$.

Let $P \in {}_A\mathcal{C}_B$. P is B-coflat over A iff the functor $P \otimes_B : {}_B\mathcal{C} \longrightarrow {}_A\mathcal{C}$ is a $\mathcal{C}$-functor with ξ: $P \otimes_B (M \otimes X) \longrightarrow (P \otimes_B M) \otimes X$ induced by α [10,Thm.4.2]. If $P \in {}_I\mathcal{C}_B = \mathcal{C}_B$ is B-coflat over I , we simply say that P is B-coflat.

If $P \in {}_A\mathcal{C}$ and $Q \in \mathcal{C}_A$ both are A-coflat then the diagram

$$\begin{array}{ccc}
(X \otimes Q) \otimes_A (P \otimes Y) & \longrightarrow & X \otimes (Q \otimes_A (P \otimes Y)) \\
\downarrow & & \wr\!\wr \\
((X \otimes Q) \otimes_A P) \otimes Y \cong (X \otimes (Q \otimes_A P)) \otimes Y & \cong & X \otimes ((Q \otimes_A P) \otimes Y)
\end{array}$$

is commutative since it is induced by the coherence diagram of α in $\mathcal{C}$, but the two arrows need not be isomorphisms.

1.2 *Lemma*: *Let* $P \in {}_A\mathcal{C}_B$ *be* B-*coflat over* A . *Then* $P \otimes_B$ *induces a functor* $P \otimes_B : {}_B\mathcal{C}_C \longrightarrow {}_A\mathcal{C}_C$.

Proof: Follows immediately from the above remarks and [10,4.1] . In particular $P \otimes_B M$ formed in ${}_A\mathcal{C}$ is also a difference cokernel in ${}_A\mathcal{C}_C$.

1.3 *Lemma*: *Let* $P \in {}_A\mathcal{C}_B$ *be* B-*coflat over* A *and* $Q \in {}_B\mathcal{C}_C$ *be* C-*coflat over* B . *Assume that* α *induces an isomorphism* $P \otimes_B (Q \otimes_C M) \cong (P \otimes_B Q) \otimes_C M$ *in* ${}_A\mathcal{C}$ *for all* $M \in {}_C\mathcal{C}$. *Then* $P \otimes_B Q$ *is* C-*coflat over* A .

Proof: $(P \otimes_B Q) \otimes_C (M \otimes X) \cong P \otimes_B (Q \otimes_C (M \otimes X)) \cong P \otimes_B ((Q \otimes_C M) \otimes X) \cong (P \otimes_B (Q \otimes_C M)) \otimes X \cong ((P \otimes_B Q) \otimes_C M) \otimes X$

The additional assumption in Lemma 1.3 and the remark before Lemma 1.2 show that the notion of coflatness alone is not strong enough for all applications. Before we introduce a stronger notion, we will investigate a few instances where the isomorphism $P \otimes_B (Q \otimes_C M) \cong (P \otimes_B Q) \otimes_C M$ exists.

1.4 Lemma: If one of the following three conditions holds:

i) $P \in {}_A\mathcal{C}_B$ is B-coflat over A and $P \otimes_B : {}_B\mathcal{C}_D \longrightarrow {}_A\mathcal{C}_D$ preserves difference cokernels,

ii) $\mathcal{C}$ is a biclosed monoidal cocomplete category,

iii) $P \in {}_A\mathcal{C}_B$ is B-coflat over A and $N \in {}_C\mathcal{C}_D$ is C-coflat over D, then α induces an isomorphism in ${}_A\mathcal{C}_D$

$$P \otimes_B (M \otimes_C N) \cong (P \otimes_B M) \otimes_C N$$

for $M \in {}_B\mathcal{C}_C$, $N \in {}_C\mathcal{C}_D$.

Proof: i) The commutative diagram in $\mathcal{C}$

$$\begin{array}{ccccc} P \otimes_B (M \otimes C \otimes N) & \rightrightarrows & P \otimes_B (M \otimes N) & \longrightarrow & P \otimes_B (M \otimes_C N) \\ \wr\| & & \wr\| & & \wr\| \\ (P \otimes_B M) \otimes C \otimes N & \rightrightarrows & (P \otimes_B M) \otimes N & \longrightarrow & (P \otimes_B M) \otimes_C N \end{array}$$

has two difference cokernels in ${}_A\mathcal{C}_D$ as rows, the first row since $P \otimes_B$ preserves difference cokernels, the second row by definition. The first two vertical isomorphisms are given by definition of B-coflatness. The last arises canonically.

ii) All tensor products $P \otimes_B Q$ exist in $\mathcal{C}$. Since $\otimes X$ and $X \otimes$ have right adjoints, they preserve difference cokernels, hence we have isomorphisms $P \otimes_B (Q \otimes X) \cong (P \otimes_B Q) \otimes X$ and $X \otimes (P \otimes_B Q) \cong (X \otimes P) \otimes_B Q$. By definition P is B-coflat and Q is B-coflat. By 1.2 $P \otimes_B Q$ carries additional left or right structure if $P \in {}_A\mathcal{C}_B$ or $Q \in {}_B\mathcal{C}_C$. Hence any $P \in {}_A\mathcal{C}_B$ is B-coflat over A and A-coflat over B. So we are reduced to iii)

iii) The following commutative diagram has difference cokernels as rows and columns in ${}_A\mathcal{C}_D$

$$\begin{array}{ccccc}
P \otimes B \otimes M \otimes C \otimes N & \rightrightarrows & P \otimes M \otimes C \otimes N & \longrightarrow & (P \otimes_B M) \otimes C \otimes N \\
\downdownarrows & & \downdownarrows & & \downdownarrows \\
P \otimes B \otimes M \otimes N & \rightrightarrows & P \otimes M \otimes N & \longrightarrow & (P \otimes_B M) \otimes N \\
\downarrow & & \downarrow & & \downarrow \\
 & & & & (P \otimes_B M) \otimes_C N \\
 & & & & \wr\wr \\
P \otimes B \otimes (M \otimes_C N) & \rightrightarrows & P \otimes (M \otimes_C N) & \longrightarrow & P \otimes_B (M \otimes_C N)
\end{array}$$

where the required isomorphism arises from the fact that colimits commute with colimits.

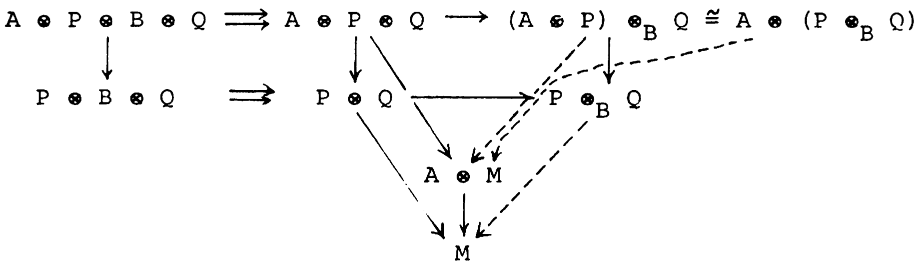
1.5 <u>Definition</u>: $P \in \mathcal{C}_B$ is <u>continuously</u> B-<u>coflat</u> if

i) $X \otimes P \in \mathcal{C}_B$ is B-coflat for all $X \in \mathcal{C}$

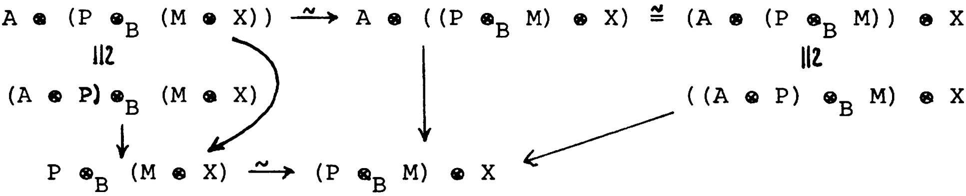
ii) the morphism $(X \otimes P) \otimes_B M \longrightarrow X \otimes (P \otimes_B M)$ induced by α is an isomorphism for all $X \in \mathcal{C}$, $M \in {}_B\mathcal{C}$.

1.6 <u>Lemma</u>: <u>Let</u> $P \in {}_A\mathcal{C}_B$ <u>be</u> <u>continuously</u> B-<u>coflat</u>. <u>Then</u> P <u>is</u> B-<u>coflat</u> <u>over</u> A .

Proof: The following commutative diagram in $\mathcal{C}$ shows the A-structure and universal property of $P \otimes_B Q$ in ${}_A\mathcal{C}$:



The next commutative diagram shows that the isomorphism $P \otimes_B (M \otimes X) \cong (P \otimes_B M) \otimes X$ is in ${}_A\mathcal{C}$:



The exterior diagram commutes since it commutes before tensoring over B and $(A \otimes P) \otimes_B (M \otimes X)$ is a difference cokernel.

If $P \in \mathcal{C}_B$ is continuously B-coflat then $X \otimes P$ is continuously B-coflat for all $X \in \mathcal{C}$. This implies that $(X \otimes P) \otimes_B$ is a $\mathcal{C}$-functor for all $X \in \mathcal{C}$ and that $(f \otimes P) \otimes_B$ is a $\mathcal{C}$-morphism for all $f \in \mathcal{C}$. In particular we get for $Q \in {}_C\mathcal{C}$ that $Q \otimes P$ is B-coflat over C.

1.7 Lemma: $X \otimes A$ *is continuously A-coflat for all* $X \in \mathcal{C}$.

Proof: By the previous remark it is sufficient to prove that A is continuously A-coflat. Since

$$A \otimes A \otimes P \rightrightarrows A \otimes P \longrightarrow P \cong A \otimes_A P$$

is a difference cokernel of a contractible pair [8,2.3] it is preserved by tensoring with X, so that we get an isomorphism of difference cokernels

$$(X \otimes A) \otimes_A P \cong X \otimes (A \otimes_A P)$$

which is functorial in X and induced by α.

Furthermore the morphism

$$\begin{aligned}(X \otimes A) \otimes_A (P \otimes Y) &\cong X \otimes (A \otimes_A (P \otimes Y)) \cong X \otimes ((A \otimes_A P) \otimes Y) \\ &\cong (X \otimes (A \otimes_A P)) \otimes Y \cong ((X \otimes A) \otimes_A P) \otimes Y\end{aligned}$$

is induced by α so that all $X \otimes A$ are A-coflat.

1.8 Lemma: a) *If* $\mathcal{C}$ *is a biclosed monoidal cocomplete category, then all* $P \in \mathcal{C}_B$ *are continuously B-coflat.*

b) *If* $\mathcal{C}$ *is a left-closed monoidal category and* $P \in \mathcal{C}_B$ *is B-coflat. Then* P *is continuously B-coflat.*

c) *If* $\mathcal{C}$ *is a symmetric monoidal category and* $P \in \mathcal{C}_B$ *is B-coflat. Then* P *is continuously B-coflat.*

Proof: a) and b) have already been shown in the proof of Lemma 1.4 (ii).

c) The following diagram shows the inverse of the morphism induced by α^{-1}, since $X \otimes \omega$ is a difference cokernel:

$$\begin{array}{ccccc} P \otimes B \otimes M \otimes X & \rightrightarrows & P \otimes M \otimes X & \xrightarrow{\omega \otimes X} & P \otimes_B M \otimes X \\ \wr\wr & & \wr\wr & & \wr\wr \\ X \otimes (P \otimes B \otimes M) & \rightrightarrows & X \otimes (P \otimes M) & \xrightarrow{X \otimes \omega} & X \otimes (P \otimes_B M) \\ \wr\wr & & \wr\wr & & \wr\wr \\ (X \otimes P) \otimes B \otimes M & \rightrightarrows & (X \otimes P) \otimes M & \longrightarrow & (X \otimes P) \otimes_B M \end{array}$$

1.9 *Proposition: Let* $P \in {}_A\mathcal{C}_B$ *be continuously B-coflat and* $Q \in {}_B\mathcal{C}_C$ *be continuously C-coflat, then* $P \otimes_B Q \in {}_A\mathcal{C}_C$ *is continuously C-coflat and* α *induces an isomorphism in* ${}_A\mathcal{C}$

$$P \otimes_B (Q \otimes_C M) \cong (P \otimes_B Q) \otimes_C M$$

for all $M \in {}_C\mathcal{C}$.

Proof: By 1.6 P is B-coflat over A . We show that

$$\begin{array}{ccccc} P \otimes_B (Q \otimes C \otimes M) & \rightrightarrows & P \otimes_B (Q \otimes M) & \longrightarrow & P \otimes_B (Q \otimes_C M) \\ \wr\wr & & \wr\wr & & \wr\wr \\ (P \otimes_B Q) \otimes C \otimes M & \rightrightarrows & (P \otimes_B Q) \otimes M & \longrightarrow & (P \otimes_B Q) \otimes_C M \end{array}$$

is a commutative diagram of difference cokernels. The first two isomorphisms arise from the fact that P is B-coflat over A . The last line is a difference cokernel by definition. The first row arises from the diagram

$$\begin{array}{ccccc}
P \otimes B \otimes Q \otimes C \otimes M & \rightrightarrows & P \otimes B \otimes Q \otimes M & \longrightarrow & P \otimes B \otimes (Q \otimes_C M) \\
\downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow \\
P \otimes Q \otimes C \otimes M & \rightrightarrows & P \otimes Q \otimes M & \longrightarrow & P \otimes (Q \otimes_C M) \\
\downarrow & & \downarrow & & \downarrow \\
P \otimes_B (Q \otimes C \otimes M) & \rightrightarrows & P \otimes_B (Q \otimes M) & \longrightarrow & P \otimes_B (Q \otimes_C M)
\end{array}$$

where the columns are difference cokernels by definition the first two rows are difference cokernels since Q is continuously C-coflat and the last row is a difference cokernel since the diagram commutes and colimits commute with colimits. Thus we get the claimed isomorphism. Observe that we only used that P is B-coflat over A and Q is continuously C-coflat.

The isomorphisms

$$(P \otimes_B Q) \otimes_C (M \otimes X) \cong P \otimes_B (Q \otimes_C (M \otimes X)) \cong P \otimes_B ((Q \otimes_C M) \otimes X) \cong$$
$$(P \otimes_B (Q \otimes_C M)) \otimes X \cong ((P \otimes_B Q) \otimes_C M) \otimes X$$

show that $P \otimes_B Q$ is C-coflat over A.

From 1.7 we have that also $X \otimes P$ is continuously B-coflat so we get isomorphisms

$$(X \otimes (P \otimes_B Q)) \otimes_C M \cong ((X \otimes P) \otimes_B Q) \otimes_C M \cong (X \otimes P) \otimes_B (Q \otimes_C M)$$
$$\cong X \otimes (P \otimes_B (Q \otimes_C M)) \cong X \otimes ((P \otimes_B Q) \otimes_C M)$$

which are induced by α and show that $P \otimes_B Q$ in C_C (and thus also in ${}_AC_C$) is continuously C-coflat.

Up to now we have considered one-sided conditions of coflatness. We can improve the situation by two-sided conditions.

1.10 Lemma: Let $Q \in {}_BC_C$ be continuously B-coflat and continuously C-coflat. Then for $P \in {}_AC_B$ and $M \in {}_CC_D$ we get an isomorphism in ${}_AC_D$ induced by α

$$P \otimes_B (Q \otimes_C M) \cong (P \otimes_B Q) \otimes_C M .$$

Proof: The proof uses the same diagrams as in the proof of 1.9. The isomorphisms in the first diagram arise from the fact that Q is continuously B-coflat. The rest of the proof is the same as in 1.9.

2. Let A and B be monoids in $\mathcal{C}$.

2.1 <u>Definition</u>: An <u>adjunction</u> $P \underset{\eta}{\overset{\varepsilon}{\rightarrow}} Q$ between A and B consists of two objects $P \in {}_A\mathcal{C}_B$, $Q \in {}_B\mathcal{C}_A$ and two morphisms $\eta: B \longrightarrow Q \otimes_A P$ in ${}_B\mathcal{C}_B$ and $\varepsilon: P \otimes_B Q \longrightarrow A$ in ${}_A\mathcal{C}_A$ such that

i) P is continuously B-coflat, Q is continuously A-coflat

ii) the diagrams

$$\begin{array}{ccccccc} P & & & \xrightarrow{\quad 1_P \quad} & & & P \\ \downarrow\wr & & & & & & \uparrow\wr \\ P \otimes_B B & \xrightarrow{P \otimes \eta} & P \otimes_B (Q \otimes_A P) & \xrightarrow{\sim} & (P \otimes_B Q) \otimes_A P & \xrightarrow{\varepsilon \otimes P} & A \otimes_A P \end{array}$$

and

$$\begin{array}{ccccccc} Q & & & \xrightarrow{\quad 1_Q \quad} & & & Q \\ \downarrow\wr & & & & & & \uparrow\wr \\ B \otimes_B Q & \xrightarrow{\eta \otimes Q} & (Q \otimes_A P) \otimes_B Q & \xrightarrow{\sim} & Q \otimes_A (P \otimes_B Q) & \xrightarrow{P \otimes \varepsilon} & Q \otimes_A A \end{array}$$

commute.

Observe that this definition is unsymmetric in i) with respect to the sides. The associativity isomorphisms result from 1.9.

To understand the above notion one should consider the monoids A, B and the A-B-biobjects with their homomorphisms as members of a (generalized) 2-category MOD with

- 0-cells: the monoids in $\mathcal{C}$
- 1-cells: the A-B-biobjects in $\mathcal{C}$, which are continuously B-coflat
- 2-cells: the morphisms in ${}_A\mathcal{C}_B$.

The composition of two 1-cells $P \in {}_A\mathcal{C}_B$ and $Q \in {}_B\mathcal{C}_C$ is then given by $P \otimes_B Q \in {}_A\mathcal{C}_C$ by 1.9. The vertical composition of two 2-cells is the usual composition of morphisms in ${}_A\mathcal{C}_B$. Observe that the composition of 1-cells as given in 1.9 is only quasi associative. In such a 2-category the notion of adjointness is precisely the one given above.

Now consider the 2-category $\mathcal{C}$-CAT of $\mathcal{C}$-(right-) categories whose

0-cells are $\mathcal{C}$-categories
1-cells are $\mathcal{C}$-functors
2-cells are $\mathcal{C}$-morphisms.

There is a 2-functor from MOD to $\mathcal{C}$-CAT which sends a monoid A to the $\mathcal{C}$-category ${}_A\mathcal{C}$, an A-B-biobject P which is continuously B-coflat to the $\mathcal{C}$-functor $P \otimes_B : {}_B\mathcal{C} \longrightarrow {}_A\mathcal{C}$, and an A-B-morphism $f: P \longrightarrow Q$ to the $\mathcal{C}$-morphism $f \otimes_B : P \otimes_B \longrightarrow Q \otimes_B$.

2.2 Proposition: Given $P \in {}_A\mathcal{C}_B$ continuously B-coflat and $Q \in {}_B\mathcal{C}_A$ continuously A-coflat. Then there is a bijection between pairs (ε, η) such that $P \underset{\eta}{\overset{\varepsilon}{\rightleftharpoons}} Q$ is an adjunction between A and B and pairs $(\bar{\varepsilon}, \bar{\eta})$ such that $P \otimes_B \underset{\bar{\eta}}{\overset{\bar{\varepsilon}}{\rightleftharpoons}} Q \otimes_A$ is an adjunction in $\mathcal{C}$-CAT.

Proof: Given $\varepsilon: P \otimes_B Q \longrightarrow A$ and $\eta: B \longrightarrow Q \otimes_A P$ we get $\bar{\varepsilon}$ and $\bar{\eta}$ from the diagrams

$$\begin{array}{ccc} P \otimes_B (Q \otimes_A M) & \xrightarrow{\bar{\varepsilon}} & M \\ \downarrow \wr & & \uparrow \wr \\ (P \otimes_B Q) \otimes_A M & \xrightarrow{\varepsilon \otimes_A M} & A \otimes_A M \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{\bar{\eta}} & Q \otimes_A (P \otimes_B M) \\ \downarrow & & \uparrow \wr \\ B \otimes_B M & \xrightarrow{\eta \otimes_B M} & (Q \otimes_A P) \otimes_B M \end{array}$$

The adjointness diagrams are easy to check. For the converse substitute A resp. B for M to obtain η and ε from the above diagrams [6, Prop. 5].

2.3 Proposition: Given $P \in {}_A\mathcal{C}_B$, $P' \in {}_B\mathcal{C}_C$, $Q' \in {}_C\mathcal{C}_B$, $Q \in {}_B\mathcal{C}_A$, and $\eta\colon B \longrightarrow Q \otimes_A P$, $\varepsilon\colon P \otimes_B Q \longrightarrow A$, $\eta'\colon C \longrightarrow Q' \otimes_B P'$, $\varepsilon'\colon P' \otimes Q' \longrightarrow B$ such that

$$P \underset{\eta}{\overset{\varepsilon}{\longrightarrow\!\!\!\!\!\dashv}} Q \qquad \text{and} \qquad P' \underset{\eta}{\overset{\varepsilon}{\longrightarrow\!\!\!\!\!\dashv}} Q' .$$

Then

$$P \otimes_B P' \underset{\eta\eta'}{\overset{\varepsilon'\varepsilon}{\longrightarrow\!\!\!\!\!\dashv}} Q' \otimes_B Q .$$

Proof: By 1.9 $P \otimes_B P'$ is continuously C-coflat and $Q' \otimes_A Q$ is continuously A-coflat. The morphism $\varepsilon'\varepsilon$ and $\eta\eta'$ are defined by

$\eta\eta'\colon C \longrightarrow Q' \otimes_B P' \cong Q' \otimes_B (B \otimes_B P') \longrightarrow$
$Q' \otimes_B ((Q \otimes_A P) \otimes_B P') \cong (Q' \otimes_B Q) \otimes_A (P \otimes_B P'),$
$\varepsilon'\varepsilon\colon (P \otimes_B P') \otimes_C (Q' \otimes_B Q) \cong P \otimes_B ((P' \otimes_C Q') \otimes_B Q \longrightarrow$
$P \otimes_B (B \otimes_B Q) \cong P \otimes_B Q \longrightarrow A$,

where the isomorphisms exist by 1.9 and are induced by coherence morphisms. Hence the diagrams

$$P \otimes_B P' \longrightarrow P \otimes_B P'$$
$$\downarrow 2 \qquad\qquad \uparrow 2$$
$$(P \otimes_B P') \otimes_C C \longrightarrow (P \otimes_B P') \otimes_C ((Q' \otimes_B Q) \otimes_A (P \otimes_B P')) \cong$$
$$\cong ((P \otimes_B P') \otimes_C (Q' \otimes_B Q)) \otimes_A (P \otimes_B P') \longrightarrow$$
$$\longrightarrow A \otimes_A (P \otimes_B P')$$

and

$$Q' \otimes_B Q \longrightarrow Q' \otimes_B Q$$
$$\downarrow 2 \qquad\qquad \uparrow 2$$
$$C \otimes_C (Q' \otimes_B Q) \longrightarrow ((Q' \otimes_B Q) \otimes_A (P \otimes_B P')) \otimes_C (Q' \otimes_B Q) \cong$$
$$\cong (Q' \otimes_B Q) \otimes_A ((P \otimes_B P') \otimes_C (Q' \otimes_B Q)) \longrightarrow$$
$$\longrightarrow (Q' \otimes_B Q) \otimes_A A$$

commute and we have $P \otimes_B P' \underset{\eta\eta'}{\overset{\varepsilon'\varepsilon}{\longrightarrow\!\!\!\!\!\dashv}} Q' \otimes_B Q$.

2.4 Lemma: Let $C \longrightarrow B$ be a monoid homomorphism in $\mathcal{C}$ such that $B \in \mathcal{C}_C$ is continuously C-coflat. Then $B \underset{\eta}{\overset{\varepsilon}{\longrightarrow\!\!\!\!\!\dashv}} B$ with $\eta\colon C \longrightarrow B \cong B \otimes_B B$ and $\varepsilon\colon B \otimes_C B \longrightarrow B \otimes_B B \cong B$.

Proof: By 1.7 B is also continuously B-coflat. With the given morphisms η and ε it is easy to check the adjointness.

2.5 Corollary: Let $P \in {}_A\mathcal{C}_B$, $Q \in {}_B\mathcal{C}_A$ be given such that $P \underset{\eta}{\overset{\varepsilon}{\dashv}} Q$. Let $C \longrightarrow B$ be a monoid homomorphism such that B is continuously C-coflat. Then there is an adjunction $P \underset{\eta'}{\overset{\varepsilon'}{\dashv}} Q$ between A and C.

Proof: follows from 2.2 and 2.3 and the isomorphisms $P \cong P \otimes_B B \in {}_A\mathcal{C}_C$ and $Q \cong B \otimes_B Q \in {}_C\mathcal{C}_A$.

Observe that B is always continuously I-coflat. Hence any adjunction $P \underset{\eta}{\overset{\varepsilon}{\dashv}} Q$ between A and B defines also an adjunction $P \underset{\eta'}{\overset{\varepsilon'}{\dashv}} Q$ between A and I.

2.6 Proposition: Let $P \in {}_A\mathcal{C}_B$ be continuously B-coflat and $Q \in \mathcal{C}_A$ be continuously A-coflat. Let $P \underset{\eta}{\overset{\varepsilon}{\dashv}} Q$ be an adjunction between A and I. Then there is a B-structure on Q such that $Q \in {}_B\mathcal{C}_A$, and there is an adjunction $P \underset{\eta'}{\overset{\varepsilon'}{\dashv}} Q$ between A and B.

Proof: The adjunction is defined by $\eta: I \longrightarrow Q \otimes_A P$ in $\mathcal{C}$ and $\varepsilon: P \otimes Q \longrightarrow A$ in ${}_A\mathcal{C}_A$. Thus we have $\eta \in Q \otimes_A P(I)$ and write it as $\eta = q_o \otimes_A p_o$. Then the adjunction diagrams can be expressed by

$$p = (pq_o)p_o \quad \text{and} \quad q = q_o(p_oq)$$

where we write $\varepsilon(p \otimes q) = pq$.

Define a B-structure on Q by $bq := q_o((p_ob)q)$. Then

$$b(qa) = q_o((p_ob)(qa)) = q_o(((p_ob)q)a) = (q_o((p_ob)q))a = (bq)a$$

$$1q = q_o((p_o1)q) = q$$

$$\begin{aligned} b(b'q) &= b(q_o((p_ob')q)) = (bq_o)((p_ob')q) \\ &= (q_o'((p_o'b)q_o))((p_ob')q) \\ &= q_o'((((p_o'b)q_o)(p_ob'))q) = q_o'(((((p_o'b)q_o)p_o)b')q) \\ &= q_o'(((p_o'b)b')q) = q_o'((p_o'(bb'))q) = (bb')q \end{aligned}$$

where we used in the last calculation the definition of $b'q$, the identity $b(qa) = (bq)a$, the associativity of the A-structure on Q, $\varepsilon \in {}_A\mathcal{C}_A$, $P \in {}_A\mathcal{C}_B$, the identity $p = (pq_o)p_o$ and the associativity of the B-structure on P. Thus $Q \in {}_B\mathcal{C}_A$. Observe that $Q \otimes_A P$ is in ${}_B\mathcal{C}_B$ by 1.2 and 1.6. Furthermore we have

$$(pb)q = (((pq_o)p_o)b)q = ((pq_o)(p_ob))q = (pq_o)((p_ob)q) = p(q_o((p_ob)q)) = p(bq)$$

hence ε factors through $\varepsilon': P \otimes_B Q \longrightarrow A$ in ${}_A\mathcal{C}_A$. We also have $bq_o \otimes_A p_o = q_o'((p_o'b)q_o) \otimes_A p_o = q_o' \otimes_A ((p_o'b)q_o)p_o = q_o' \otimes_A p_o'b$ hence a morphism

$$B \ni b \longmapsto bq_o \otimes_A p_o = q_o \otimes_A p_ob \in Q \otimes_A P$$

in ${}_B\mathcal{C}_B$. Finally the two adjunction diagrams hold since we had already $p = (pq_o)p_o$ and $q = q_o(p_oq)$.

2.7 <u>Corollary</u>: <u>Let</u> $P \underset{\eta}{\overset{\varepsilon}{\dashv}} Q$ <u>be an adjunction between</u> A <u>and</u> B. <u>Then there is an adjunction</u> $P \underset{\eta'}{\overset{\varepsilon'}{\dashv}} Q$ <u>between</u> A <u>and</u> C <u>for every monoid</u> C <u>such that</u> $P \in {}_A\mathcal{C}_C$ <u>is continuously</u> C-<u>coflat</u>.

Proof: By the remark preceeding 2.5 there is an adjunction $P \underset{\bar{\eta}}{\overset{\bar{\varepsilon}}{\dashv}} Q$ between A and I, thus 2.5 can be applied.

3. Recall that we defined ${}_A[M,N] \in \mathcal{C}$ (if it exists) for $M,N \in {}_A\mathcal{C}$ by ${}_A\mathcal{C}(M \otimes X,N) \cong \mathcal{C}(X,{}_A[M,N])$ [9]. Since $\mathcal{C}$ is not symmetric, $M \otimes X$ does not carry a right A-structure if $M \in \mathcal{C}_A$. There is however the functor $\otimes M: \mathcal{C} \longrightarrow \mathcal{C}_A$. We define $[M,N]_A'$ by $\mathcal{C}_A(X \otimes M,N) \cong \mathcal{C}(X,[M,N]_A')$ if it exists.

If ${}_A[M,N]$ exists then the morphism

$$M \otimes {}_A[M,N] \longrightarrow N \quad \text{in } {}_A\mathcal{C}$$

defined by the identity on ${}_A[M,N]$ is called the <u>evaluation</u> and denoted by $m \otimes f \longmapsto \langle m \rangle f$. If ${}_A[M,M \otimes X]$ exists, then there is also the <u>coevaluation</u>

$$X \longrightarrow {}_A[M, M \otimes X] \quad \text{in} \quad C$$

defined by the identity on $M \otimes X$ and written as

$$x \longmapsto (m \longmapsto m \otimes x) \ .$$

There is another interesting morphism if ${}_A[M,A]$ and ${}_A[M,N]$ exist for $M,N \in {}_AC$. It is the morphism

$${}_A[M,A] \otimes_A N \longrightarrow {}_A[M,N]$$

defined by $f \otimes_A n \longmapsto (m \longmapsto (\langle m\rangle f)n)$. The image of $f \otimes_A n$ will be denoted by fn .

3.1 <u>Definition</u>: $P \in {}_AC$ is called <u>finite</u> if

i) ${}_A[P,A]$ and ${}_A[P,P]$ exist

ii) ${}_A[P,A] \in C_A$ is continuously A-coflat

iii) the morphism

$${}_A[P,A] \otimes_A P \longrightarrow {}_A[P,P]$$

is an isomorphism.

If ${}_A[P,A]$ is continuously A-coflat, then it is called the <u>dual</u> of P . By [9,Cor. 3.4] we know that ${}_A[P,P]$ has the structure of a monoid. In particular there is a unit element $1 = \eta: I \longrightarrow {}_A[P,P]$ which is induced by $\varrho: P \otimes I \longrightarrow P$ or the identity $1: P \longrightarrow P$ if we omit ϱ . With the evaluation morphism it acts as identity: $\langle p\rangle 1 = p$ for all $p \in P(X)$. Condition 3) of the definition of a finite object implies that there is an element

$$f_o \otimes_A p_o \in {}_A[P,A] \otimes_A P(I)$$

whose image is $1 \in {}_A[P,P](I)$, i.e. $\langle p\rangle f_o p_o = p$ for all $p \in P(X)$. This element $f_o \otimes p_o$ is called the <u>dual basis</u> of P . It is uniquely determined, since the morphism ${}_A[P,A] \otimes_A P \longrightarrow {}_A[P,P]$ is an isomorphism. Another fact follows also quite easily from 3) namely that

$${}_A[P,A] \otimes_A P(I) \longrightarrow {}_A[P,P](I)$$

is surjective. In general we will call a morphism $M \longrightarrow N$ rationally surjective if $M(I) \longrightarrow N(I)$ is surjective.

3.2 Lemma: Let $P \in {}_AC$ be finite. Then there is an adjunction $P \underset{\eta}{\overset{\varepsilon}{\dashrightarrow}} {}_A[P,A]$ between A and I .

Proof: Let $f_o \otimes_A p_o$ be the dual basis for P . Then we have $\langle p\rangle f_o p_o = p$. For $p \in P(X)$ and $g \in {}_A[P,A](Y)$ we also have $\langle p\rangle(f_o\langle p_o\rangle g) = (\langle p\rangle f_o)(\langle p_o\rangle g) = \langle\langle p\rangle f_o p_o\rangle g = \langle p\rangle g$ hence $f_o\langle p_o\rangle g = g$. These two identities are precisely the adjointness diagrams. By assumption ${}_A[P,A]$ is continuously A-coflat and we need no assumption about P .

3.3 Lemma: Let $P \underset{\eta}{\overset{\varepsilon}{\dashrightarrow}} Q$ be an adjunction between A and I . Then $P \in {}_AC$ is finite.

Proof: We have ${}_AC(P \otimes X,A) \cong C(X,Q \otimes_A A) \cong C(X,Q)$ by 2.2 hence $Q = {}_A[P,A]$ and continuously A-coflat. The evaluation is given by $\varepsilon\colon P \otimes Q \longrightarrow A$. Furthermore ${}_AC(P \otimes X,P) \cong C(X,Q \otimes_A P) \cong C(X,{}_A[P,P])$, hence there is an isomorphism $Q \otimes_A P \cong {}_A[P,P]$ and this isomorphism is induced by the identity on $Q \otimes_A P$ through

$$C(Q \otimes_A P,Q \otimes_A P) \cong {}_AC(P \otimes Q \otimes_A P,P) \cong C(Q \otimes_A P,{}_A[P,P]) .$$

It is an easy exercise to show that this is the isomorphism described in the definition of a finite object. In particular ${}_A[P,P]$ exists and thus $P \in {}_AC$ is finite.
These two lemmas together with 2.6 and 2.3 give that the tensorproduct of two finite objects $P \in C_B$ and $P' \in {}_BC_C$ is again finite: $P \otimes_B P' \in C_C$. [2,II.5.3]

3.4 Corollary: If $P \in {}_AC$ is finite then ${}_A[P,M]$ exists for all $M \in {}_AC$.

Proof: ${}_AC(P \otimes X,M) \cong C(X,Q \otimes_A M)$ hence ${}_A[P,M] \cong Q \otimes_A M$.

3.5 Definition: $P \in {}_AC$ is called reflexive if

i) ${}_A[P,A]$ and $[{}_A[P,A],A]'_A$ exist

ii) the morphism $P \longrightarrow [{}_A[P,A],A]'_A$ defined by $p \longmapsto (f \longmapsto \langle p\rangle f)$ is an isomorphism.

The morphism given in ii) is the morphism corresponding to the identity under the isomorphism

$$ {}_AC(P,[{}_A[P,A],A]'_A) \cong {}_AC_A(P \otimes {}_A[P,A],A) \cong C_A({}_A[P,A],{}_A[P,A]) $$

3.6 Lemma: If $P \in {}_AC$ is finite then it is reflexive.

Proof: The dual ${}_A[P,A]$ exists by definition. To show that $[{}_A[P,A],A]'_A$ exists, we show that the functor $C_A(X \otimes_A [P,A],A)$ is representable by P. Define maps

$$ \Phi: C(X,P) \longrightarrow C_A(X \otimes {}_A[P,A],A) $$

$$ \Psi: C_A(X \otimes {}_A[P,A],A) \longrightarrow C(X,P) $$

by $\Phi(f)(x \otimes h) := \langle f(x)\rangle h$ and $\Psi(g)(x) := g(x \otimes f_o)p_o$ where $f_o \otimes_A p_o$ is the dual basis of P. For the proper definition of Ψ we have to use $g \in C_A$ and ${}_A[P,A]$ continuously A-coflat. It is clear that $\Phi(f) \in C_A$. The compositions of Φ and Ψ are

$$ \Phi\Psi(g)(x \otimes h) = \langle g(x \otimes f_o)p_o\rangle h = g(x \otimes f_o)\langle p_o\rangle h $$
$$ = g(x \otimes f_o(\langle p_o\rangle h)) = g(x \otimes h) $$

since by 3.2 the second adjointness diagram commutes, and

$$ \Psi\Phi(f)(x) = \langle f(x)\rangle f_o p_o = f(x) \ . $$

So Φ and Ψ are inverses of eachother and they are obviously natural transformations in X so that P represents the functor $C_A(X \otimes {}_A[P,A],A)$ and $[{}_A[P,A],A]_A$ exists.

To show that the morphism defined in 2) of the definition of reflexivity is an isomorphism, we exhibit an inverse. For $\varphi \in [{}_A[P,A],A]'_A$ define an element $(\varphi\langle f_o\rangle)p_o \in P$.

Then $\langle p\rangle f_o p_o = p$ and $\langle(\varphi\langle f_o\rangle)p_o\rangle f = (\varphi\langle f_o\rangle)'(\langle p_o\rangle f) = \varphi\langle f_o\langle p_o\rangle f\rangle = \varphi\langle f\rangle$ show that $P \longrightarrow [{}_A[P,A],A]'_A$ is an isomorphism. Note that we write the evaluation $[M,N]'_A \otimes M \longrightarrow N$ as $f \otimes m \longrightarrow f\langle m\rangle$ and that we may change parentheses as above since ${}_A[P,A]$ is continuously A-coflat.

3.7 <u>Lemma</u>: <u>Let</u> $P \in {}_A C$ <u>be</u> <u>finite</u>. <u>Then</u> <u>the</u> <u>morphism</u>

$$ {}_A[P,A] \otimes_A P \longrightarrow [{}_A[P,A],\ {}_A[P,A]]'_A $$

<u>defined</u> <u>by</u> $f \otimes_A p \longmapsto (g \longmapsto (p' \longmapsto \langle p'\rangle f\langle p\rangle g))$ <u>is</u> <u>an</u> <u>isomorphism</u>.

Proof: We first remark that the diagram

$$\begin{array}{ccc} & & {}_A[P,P] \\ & \overset{\Phi}{\nearrow} & \downarrow \Lambda \\ {}_A[P,A] \otimes_A P & \underset{\Psi}{\searrow} & \\ & & [{}_A[P,A],{}_A[P,A]]'_A \end{array}$$

commutes where Φ is from the definition of finiteness, Ψ from the Lemma and Λ defined by

$$ {}_A[P,P] \ni h \longmapsto (g \longmapsto (p \longmapsto \langle\langle p\rangle h\rangle g)) \in [{}_A[P,A],{}_A[P,A]]'_A $$

which is just the composition ${}_A[P,P] \otimes {}_A[P,P] \longrightarrow {}_A[P,P]$ transferred by

$$ C_A({}_A[P,P] \otimes {}_A[P,A],{}_A[P,A]) \cong C({}_A[P,P],[{}_A[P,A],{}_A[P,A]]'_A) \ . $$

Λ is an isomorphism since $\varphi \longmapsto (p \longmapsto (\langle p\rangle(\varphi\langle f_o\rangle))p_o)$ is the inverse as can be easily checked. Thus Ψ is an isomorphism.

So we only have to show that $[{}_A[P,A],{}_A[P,A]]'_A$ exists, or that

$$C_A(X \otimes {}_A[P,A], {}_A[P,A]) \cong {}_AC_A(P' \otimes X \otimes {}_A[P,A], A)$$
$$\cong {}_AC(P \otimes X, [{}_A[P,A], A]'_A)$$
$$\cong {}_AC(P \otimes X, P)$$
$$\cong C(X, {}_A[P,P])$$

is representable, which is obvious. The second but last isomorphism uses that P is reflexive.

3.8 Corollary: Let $P \in {}_AC$ be finite. Then the morphism

$$\Lambda: {}_A[P,P] \longrightarrow [{}_A[P,A], {}_A[P,A]]'_A$$

is an isomorphism of monoids.

Proof: Write $\langle p\rangle(\Lambda(h)\langle g\rangle) = \langle\langle p\rangle h\rangle g$, then

$$\langle p\rangle((\Lambda(h)\Lambda(k))\langle g\rangle) = \langle p\rangle(\Lambda(h)\langle\Lambda(k)\langle g\rangle\rangle)$$
$$= \langle\langle\langle p\rangle h\rangle k\rangle g$$
$$= \langle\langle p\rangle hk\rangle g$$
$$= \langle p\rangle(\Lambda(hk)\langle g\rangle)$$

hence $\Lambda(h)\Lambda(k) = \Lambda(hk)$ and

$$\langle p\rangle(\Lambda(1)\langle g\rangle) = \langle\langle p\rangle 1\rangle g = \langle p\rangle g$$

hence $\Lambda(1) = 1$.

If P is finite or equivalently if $P \underset{\eta}{\overset{\varepsilon}{\longrightarrow}} Q$ is an adjunction between A and I then $Q \otimes_A P$ is a monad by using $\varepsilon: P \otimes Q \longrightarrow A$, $\varepsilon(p \otimes q) = pq$ with

$$\mu: Q \otimes_A P \otimes Q \otimes_A P \ni q' \otimes_A p' \otimes q \otimes_A p \longmapsto q' \otimes_A (p'q)p \in Q \otimes_A P$$

$$\eta: I \longrightarrow Q \otimes_A P .$$

Being a monad is nothing else than being a monoid in C with multiplication μ and unit η .

3.9 Corollary: Let $P \in {}_AC$ be finite. Then the morphisms

$$\Phi: {}_A[P,A] \otimes_A P \longrightarrow {}_A[P,P]$$

and

$$\Psi: {}_A[P,A] \otimes_A P \longrightarrow [{}_A[P,A], {}_A[P,A]]'_A$$

are isomorphisms of monoids.

Proof: Because of $\Lambda\Phi = \Psi$ and 3.7, 3.8 it suffices to show that Φ is a monoid homomorphism. The multiplication of the monoid (= monad) ${}_A[P,A] \otimes_A P$ is given by μ in the preceeding remarks. Then

$$\langle p''\rangle\Phi((f' \otimes_A p')\cdot(f \otimes_A p)) = \langle p''\rangle f\langle p'\rangle fp = \langle\langle p''\rangle f'p\rangle fp$$
$$= \langle p''\rangle\Phi(f' \otimes_A p')\Phi(f \otimes_A p)$$

and $\langle p''\rangle\Phi(f_o \otimes_A p_o) = \langle p''\rangle f_o p_o = p''$

hence $\Phi((f' \otimes_A p')\cdot(f \otimes_A p)) = \Phi(f' \otimes_A p')\cdot\Phi(f \otimes_A p)$

$\Phi(f_o \otimes_A p_o) = 1$

where $f_o \otimes_A p_o$ is the dual basis of P .

3.10 *Theorem: Let* $P \in {}_A C$. *The following are equivalent:*

a) P *is finite,*

b) P *has a dual* ${}_A[P,A]$, ${}_A[P,P]$ *exists and the morphism*

$$\Phi: {}_A[P,A] \otimes_A P \longrightarrow {}_A[P,P]$$

is rationally surjective,

c) P *has a dual* ${}_A[P,A]$, ${}_A[P,P]$ *exists and there is an element (dual basis)*

$$f_o \otimes_A p_o \in {}_A[P,A] \otimes_A P(I)$$

such that $\Phi(f_o \otimes_A p_o)$ *is the identity in* ${}_A[P,P]$,

d) P *has a dual* ${}_A[P,A]$, P *is reflexive,* $[{}_A[P,A], {}_A[P,A]]'_A$ *exists, and the morphism*

$$\Psi: {}_A[P,A] \otimes_A P \longrightarrow [{}_A[P,A], {}_A[P,A]]'_A$$

is an isomorphism (of monoids),

e) *there is an adjunction* $P \underset{\eta}{\overset{\varepsilon}{\dashv}} Q$ *between* A *and* I ,

f) *there is a continuously* A-*coflat object* $Q \in C_A$ *and a natural isomorphism*

$${}_A C(P \otimes X, Y) \cong C(X, Q \otimes_A Y)$$

g) P *has a dual* ${}_A[P,A]$ *and there is a morphism* $\eta: I \longrightarrow {}_A[P,A] \otimes_A P$ *such that the diagram*

$$\begin{array}{ccccccc}
P & \xrightarrow{\qquad\qquad\qquad\qquad 1 \qquad\qquad\qquad\qquad} & & & & & P \\
\downarrow 2 & & & & & & \uparrow 2 \\
P \otimes I & \xrightarrow{1 \otimes \eta} & P \otimes ({}_A[P,A] \otimes_A P) & \cong & (P \otimes {}_A[P,A]) \otimes_A P & \xrightarrow{\varepsilon \otimes 1} & A \otimes_A P
\end{array}$$

commutes.

Proof: From the previous paragraphs we know already the implications a) $\Longrightarrow$ b) $\Longrightarrow$ c), a) $\Longrightarrow$ d), a) $\Longleftrightarrow$ e) $\Longleftrightarrow$ f), and a) and e) $\Longrightarrow$ g).

c) $\Longrightarrow$ a): Define a morphism ${}_A[P,P] \longrightarrow {}_A[P,A] \otimes_A P$ by $f \longmapsto f_o \otimes_A \langle p_o \rangle f$. This morphism is well-defined by [9, Thm. 3.2] and because ${}_A[P,A]$ is continuously A-coflat. Then $(\langle p \rangle f_o)(\langle p_o \rangle f) = \langle\langle p \rangle f_o p_o \rangle f = \langle p \rangle f$ and by this equation also $f_o \otimes_A \langle p_o \rangle fp = f_o(\langle p_o \rangle f) \otimes_A p = f \otimes_A p$. Thus the above morphism is inverse to Φ: ${}_A[P,A] \otimes_A P \longrightarrow {}_A[P,P]$.

g) $\Longrightarrow$ e) We only have to show $f_o(\langle p_o \rangle f) = f$ for all $f \in {}_A[P,A](X)$ with $f_o \otimes_A p_o = \eta \in {}_A[P,A](I)$. But the given diagram means $p = \langle p \rangle f_o p_o$, hence $(\langle p \rangle f_o)(\langle p_o \rangle f) = \langle\langle p \rangle f_o p_o \rangle f = \langle p \rangle f$ and thus $f_o(\langle p_o \rangle f) = f$.

d) $\Longrightarrow$ g) Let $f_o \otimes_A p_o \in {}_A[P,A] \otimes_A P(I)$ be the image of the identity under the isomorphism Ψ^{-1}. Then we have for all $p \in P(X)$ and all $g \in {}_A[P,A](Y)$

$$\langle p \rangle g = \langle p \rangle (\Psi(f_o \otimes_A p_o)\langle g \rangle) = (\langle p \rangle f_o)(\langle p_o \rangle g)$$
$$= \langle\langle p \rangle f_o p_o \rangle g .$$

Since P is reflexive the isomorphism $P \cong [{}_A[P,A],A]'_A$ gives $p = \langle p \rangle f_o p_o$. This is the required diagram.

3.11 <u>Corollary</u>: <u>Let</u> $P \in {}_A C$ <u>be continuously</u> A-<u>coflat and finite, then</u> ${}_A[P,A] \in C_A$ <u>is finite</u>.

Proof: If P is finite then there is an adjunction $P \underset{\eta}{\overset{\varepsilon}{\longrightarrow}} Q$ between A and I. By additional assumption that P is continuously A-coflat we get a completely symmetric situation in the definition of an adjunction (2.1) with respect to the sides and the exchange of P and Q. The symmetric counterpart of 3.10 gives the result.

3.12 <u>Corollary</u>: <u>Let</u> C <u>be a left-closed monoidal category and</u> $P \in {}_A C$. <u>Then the following are equivalent</u>:

a) P <u>is finite</u>

b) ${}_A[P,-]: {}_A C \longrightarrow C$ <u>exists, is a</u> C-<u>functor and is co-continuous</u>.

c) ${}_A[P,-] : {}_A\mathcal{C} \longrightarrow \mathcal{C}$ <u>exists, is a $\mathcal{C}$-functor and preserves difference cokernels of $\mathcal{U}$-contractible pairs</u>.

Proof: a) $\Longrightarrow$ b): We have ${}_A[P,-] \cong Q \otimes_A -$ and $\mathcal{C}(Q \otimes_A M,X) \cong {}_A\mathcal{C}(M,[Q,X])$ by [9, 3.10]. Hence $Q \otimes_A -$ is left adjoint and cocontinuous. Furthermore we have

$${}_A[P,M \otimes X] \cong Q \otimes_A (M \otimes X) \cong (Q \otimes_A M) \otimes X \cong {}_A[P,M] \otimes X .$$

b) $\Longrightarrow$ c) is trivial.
c) $\Longrightarrow$ a) By [10, 4.2] there is a $Q \in \mathcal{C}_A$ which is A-coflat such that ${}_A[P,-] \cong Q \otimes_A -$. By Lemma 1.8.b. Q is continuously A-coflat. The isomorphism just given defines a natural isomorphism

$${}_A\mathcal{C}(P \otimes X,M) \cong \mathcal{C}(X,{}_A[P,M]) \cong \mathcal{C}(X,Q \otimes_A M) .$$

By 3.10 P is finite.

Observe that condition b) of the Corollary is often denoted by "P is an atom" .

We conclude this paragraph by giving some examples of finite objects. If $\mathcal{C}$ = K-Mod with a commutative ring K , then $\mathcal{C}$ is closed symmetric cocomplete hence continuous coflatness holds always (1.8). By 3.10 c) and the dual basis lemma $P \in {}_A\mathcal{C}$ = A-Mod is finite iff it is a finitely generated projective A-module.
If $\mathcal{C}$ = R-Mod-R , the category of R-R-bimodules, then $\mathcal{C}$ is biclosed cocomplete and continuous coflatness holds always. The right-adjoint of $X. \otimes_R -$ is $Hom_R(.X,-)$, the right-adjoint of $- \otimes_R X$ is $Hom_R(X.,-)$. If A is a monoid in $\mathcal{C}$ (a so called R-ring), then ${}_A\mathcal{C}$ is the category A-Mod-R of A-R-bimodules. $P \in {}_A\mathcal{C}$ is finite iff P is a finitely generated projective A-module. If $\mathcal{C}$ = Set , the category of sets, then ${}_A[P,M]$ = A-Set(P,M) , the set of maps from P to M which are compatible with the action

of A. Since $\mathcal{C}$ is closed cocomplete, $P \in {}_A\mathcal{C}$ is finite iff there is an element $(f_o,p_o) \in A\text{-Set}(P,A) \times_A P$ such that $p = f_o(p)\cdot p_o$ for all $p \in P$. (Observe that $M(I)$ and M can be identified in $\mathcal{C}$.) Thus $P = A\cdot p_o$ and $f_o(ap_o) = a\cdot f_o(p_o)$ induces an isomorphism $P \ni ap_o \longmapsto af_o(p_o) \in A\cdot f_o(p_o)$ with $f_o(p_o)$ idempotent in A. Conversely any idempotent $e \in A$ defines a finite object $P := A\cdot e$ with dual basis (f_o,e), where f_o is the imbedding $Ae \hookrightarrow A$. In particular I is the only finite object in $\mathcal{C}$. If $\mathcal{C} = K\text{-Mod}^{op}$, then $\mathcal{C}$ is not closed, but cocomplete. $P \in \mathcal{C}$ is finite iff there is an adjoint Q iff P (and Q) are finite in K-Mod iff P (and Q) are finitely generated projective K-modules.
If $\mathcal{C}$ = Stab, the stable homotopy category, then any compact neighborhood retract of $\mathbb{R}^n$ is finite in $\mathcal{C}$ [3, Thm. 3.1].
If $\mathcal{C} = \partial$-Mod, the category of chain-complexes of K-modules, then the finite objects in $\mathcal{C}$ are precisely the chain complexes $P = (P_n,\partial)$ such that P_n is a finitely generated projective K-module and $P_n \neq 0$ for only a finite number of indices. [3, Prop. 1.6]. In view of [13] ∂-Mod is $\mathcal{C}$-monoidally equivalent to B-Comod with isomorphic underlying functors with the Hopf-algebra B defined by
$B = K\langle x,y,y^{-1}\rangle/(xy + yx, x^2)$ with non-commuting variables x and y and $\Delta(x) = x \otimes 1 + y^{-1} \otimes x$, $\Delta(y) = y \otimes y$, $\varepsilon(x) = 0$, $\varepsilon(y) = 1$, $s(x) = xy$, and $s(y) = y^{-1}$. This result should be compared with 4.6 and 4.7. If $\mathcal{C}$ is a monoidal category with $X \otimes Y := X \times Y$, the categorical product in $\mathcal{C}$, and I the final object in $\mathcal{C}$, then $P \in \mathcal{C}$ is finite iff $P \cong I$.
To see this note that $\mathcal{C}(I,-): \mathcal{C} \longrightarrow$ Sets preserves (tensor-) products, so a dual basis for P must be of the form $(f_o,p_o) \in [P,I] \times P(I)$ hence $p = \langle p\rangle f_o p_o$ implies $P = Ip_o$ and $I \ni x \longmapsto xp_o \in P$ defines an isomorphism. The converse, I is finite, is trivial to see.

4. In this paragraph we want to study various properties of finite objects and adjunctions and compare them with the corresponding classical concepts.

Assume that $P \underset{\eta}{\overset{\varepsilon}{\rightleftharpoons}} Q$ is an adjunction between A and B . Then, as for adjoint functors, we can define the corresponding <u>comonad</u> $C := P \otimes_B Q$ in ${}_A\mathcal{C}_A$. The counit is $\varepsilon : P \otimes_B Q \longrightarrow A$ and the comultiplication is

$$\Delta : P \otimes_B Q \cong P \otimes_B (B \otimes_B Q) \longrightarrow P \otimes_B ((Q \otimes_A P) \otimes_B Q) \cong (P \otimes_B Q) \otimes_A (P \otimes_B Q)$$

where the arrow is induced by η . It is easy to verify that Δ and ε define a coassociative counitary comultiplication. In the category ${}_A\overline{\mathcal{C}}_A$ of continuously right A-coflat objects in ${}_A\mathcal{C}_A$ we can define a tensor-product $M \boxtimes N = M \otimes_A N$ by 1.9. Thus $({}_A\overline{\mathcal{C}}_A , \boxtimes , A)$ becomes a monoidal category and $(P \otimes_B Q , \Delta , \varepsilon)$ is a comonoid in ${}_A\overline{\mathcal{C}}_A$ with $\Delta(p \otimes_B q) = p \otimes_B q_o \otimes_A p_o \otimes_B q$ and dual basis $q_o \otimes_A p_o$.

Observe that for any commutative ring K and any finitely generated K-module P the above construction gives $P \otimes_{\mathbb{Z}} \mathrm{Hom}_K(P,K)$ the structure of a coalgebra in K-Mod-K , but not necessarily in K-Mod , since the two K-structures on $P \otimes_{\mathbb{Z}} \mathrm{Hom}_K(P,K)$ do not coincide in general. In particular $K \otimes_{\mathbb{Z}} K$ becomes a coalgebra in K-Mod-K . If we take, however, $A = I$ then ${}_A\overline{\mathcal{C}}_A = \mathcal{C}$ as monoidal categories and thus $P \otimes_B Q$ forms a comonoid in $\mathcal{C}$ for every B such that P is continuously B-coflat (2.7). If $\mathcal{C}$ = K-Mod then for every finitely generated projective K-module P and every K-subalgebra B of $\mathrm{End}_K(P)$ there is a quotient coalgebra $P \otimes_B \mathrm{Hom}_K(P,K)$ of $P \otimes_K \mathrm{Hom}_K(P,K) \cong \mathrm{Hom}_K(\mathrm{Hom}_K(P,P),K)$. The verification that the last isomorphism is a coalgebra isomorphism is left to the reader.

If $P \in {}_AC$ is finite then many natural transformations should become natural isomorphisms, as the example C = K-Mod suggests. In the general case we get

4.1 Proposition: a) Let $P \in {}_AC$ be finite. Then there are natural isomorphisms for $M \in {}_AC$ and $X \in C$

i) ${}_A[P,M] \cong {}_A[P,A] \otimes_A M$

ii) ${}_AC(P,M) \cong {}_A[P,A] \otimes M(I)$

iii) ${}_A[P,M \otimes X] \cong {}_A[P,M] \otimes X$

iv) ${}_AC(P,M \otimes X) \cong {}_A[P,M] \otimes X(I)$

v) ${}_A[P \otimes X,M] \cong [X,{}_A[P,A] \otimes_A M]$

vi) ${}_AC(P \otimes X,M) \cong C(X,{}_A[P,A] \otimes_A M)$

b) Let $P \in {}_AC$ be finite and continuously A-coflat. Then there are natural isomorphisms for $M \in C_A$, $X \in C$

i) $[{}_A[P,A],M]'_A \cong M \otimes_A P$

ii) $C_A({}_A[P,A],M) \cong M \otimes_A P(I)$

iii) $[X \otimes {}_A[P,A],M]'_A \cong [X,M \otimes_A P]'$

iv) $C_A(X \otimes {}_A[P,A],M) \cong C(X,M \otimes_A P)$

c) Let $P \in {}_AC$ be finite and let C be symmetric. Then there are natural isomorphisms for $M \in {}_AC_B$ and $N \in C_B$

i) $[{}_A[P,M],N]_B \cong {}_A[M,[{}_A[P,A],N]_B]$

ii) $C_B({}_A[P,M],N) \cong {}_AC(M,[{}_A[P,A],N]_B)$

if $[{}_A[P,A],N]_B$ exists.

d) Let $P \in {}_AC$, $P' \in {}_BC$ be finite and let C be closed monoidal symmetric. Then there is a natural isomorphism for $M \in {}_AC$ and $N \in {}_BC$

$${}_A[P,M] \otimes {}_B[P',N] \cong {}_{A \otimes B}[P \otimes P', M \otimes N]$$

Proof: a) By 3.10. f. we get ${}_A[P,M] \cong Q \otimes_A M \cong {}_A[P,A] \otimes M$ and ${}_A[P,M \otimes X] \cong Q \otimes_A (M \otimes X) \cong (Q \otimes_A M) \otimes X \cong {}_A[P,M] \otimes X$. The isomorphisms ii) and iv) follow by applying $C(I,-)$. vi) is again 3.10.f. and v) follows by

$$\mathcal{C}(Y, {}_A[P \otimes X, M]) \cong {}_A\mathcal{C}(P \otimes X \otimes Y, M) \cong \mathcal{C}(X \otimes Y, Q \otimes_A M)$$
$$\cong \mathcal{C}(X \otimes Y, {}_A[P,A] \otimes_A M) \cong \mathcal{C}(Y, [X, {}_A[P,A] \otimes_A M]) .$$

b) By 3.11 we get $[Q,M]'_A \cong M \otimes_A P$ as in a) for the other side, where we use $P \cong [{}_A[P,A], A]'_A$ from 3.10.d. Furthermore $[X \otimes Q, M]'_A \cong [X, M \otimes_A P]'$.

c) For $Q = {}_A[P,A]$ we get

$$[{}_A[P,M], N]_B \cong [Q \otimes_A M, N]_B \cong {}_A[M,[Q,N]]_B \quad \text{by [9, 3.11]}$$
$$\cong {}_A[M,[{}_A[P,A],N]]_B .$$

The second isomorphism arises again by applying $\mathcal{C}(I,-)$.

d) This isomorphism reduces to

$$(Q \otimes_A M) \otimes (Q' \otimes_B N) \cong (Q \otimes Q') \otimes_{A \otimes B} (M \otimes N)$$

with $Q = {}_A[P,A]$ and $Q' = {}_B[P',B]$.

For many purposes one needs a slightly stronger property for $P \in {}_A\mathcal{C}$ than being finite.

4.2 <u>Definition</u>: $P \in {}_A\mathcal{C}$ is called <u>finitely generated projective (over A)</u> if

i) ${}_A[P,A]$ and ${}_A[P,P]$ exist

ii) ${}_A[P,A] \in \mathcal{C}_A$ is continuously A-coflat

iii) there is a <u>strong dual basis</u> $f_o \otimes p_o \in {}_A[P,A] \otimes P(I)$ such that

$(\langle p \rangle f_o)p_o = p$ for all $X \in \mathcal{C}$ and all $p \in P(X)$.

4.3 <u>Lemma</u>: $P \in {}_A\mathcal{C}$ <u>is finitely generated projective over A iff</u> P <u>is finite and the map</u>

$${}_A[P,A] \otimes P(I) \longrightarrow {}_A[P,A] \otimes_A P(I)$$

<u>is surjective</u> (i.e. $\varphi : {}_A[P,A] \otimes P \longrightarrow {}_A[P,A] \otimes_A P$ <u>is rationally surjective</u>).

Proof: If P is finite and φ is rationally surjective then the dual basis $f_o \otimes_A p_o$ for P has a counterimage $f_o \otimes p_o$ and conditions 1), 2), and 3) of the definition are satisfied. If P is finitely generated projective then the image of $f_o \otimes p_o$ is a dual basis for P and P is finite. If $f \otimes_A p \in {}_A[P,A] \otimes_A P(I)$ then $f \otimes_A p = f(\langle p\rangle f_o) \otimes_A p_o$ is the image of $f(\langle p\rangle f_o) \otimes p_o \in {}_A[P,A] \otimes P(I)$, thus φ is rationally surjective.

The same argument as in the above proof shows that

$${}_A[P,A] \otimes P(X) \longrightarrow {}_A[P,A] \otimes_A P(X)$$

is surjective for all finitely generated projective objects $P \in {}_A\mathcal{C}$ and all $X \in \mathcal{C}$. If $I \in \mathcal{C}$ is projective, i.e. for each epimorphism $f: X \longrightarrow Y$ in $\mathcal{C}$ the map $f(I)$ is surjective, then every finite object is finitely generated projective. This is for example the case in the categories of sets, of abelian groups, of K-modules and of Banach spaces. If K is injective as a K-module then this holds also for $(K\text{-Mod})^{op}$.

If $\mathcal{F},\mathcal{G}: {}_A\mathcal{C} \longrightarrow {}_B\mathcal{C}$ are $\mathcal{C}$-functors and if $\mathcal{F}$ and $\mathcal{G}$ preserve difference cokernels of $(\mathcal{U}: {}_A\mathcal{C} \longrightarrow \mathcal{C})$-contractible pairs, if furthermore $\varphi: \mathcal{F} \longrightarrow \mathcal{G}$ is a $\mathcal{C}$-morphism and $\varphi(A): \mathcal{F}(A) \longrightarrow \mathcal{G}(A)$ is an isomorphism then the following diagram commutes

$$\begin{array}{ccc} \mathcal{F}(M) & \cong & \mathcal{F}(A) \otimes_A M \\ \downarrow \varphi(M) & & \downarrow \varphi(A) \otimes_A M \\ \mathcal{G}(M) & \cong & \mathcal{G}(M) \otimes_A M \end{array}$$

by the universal property of difference cokernels. Hence $\varphi(M)$ is an isomorphism for all $M \in {}_A\mathcal{C}$. If we drop the assumption that $\mathcal{F}$ and $\mathcal{G}$ preserve difference cokernels of $\mathcal{U}$-contractible pairs we get the following.

4.4 <u>Proposition</u>: <u>Let</u> $\mathcal{F},\mathcal{G}: {}_A\mathcal{C} \longrightarrow {}_B\mathcal{C}$ <u>be</u> $\mathcal{C}$-<u>functors and</u> $\varphi: \mathcal{F} \longrightarrow \mathcal{G}$ <u>be a</u> $\mathcal{C}$-<u>morphism</u>. <u>Assume that</u>

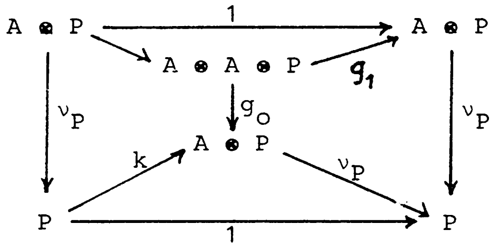
$$\varphi(A): \mathcal{F}(A) \longrightarrow \mathcal{G}(A)$$

<u>is an isomorphism</u>. <u>Then</u>

$\varphi(P): F(P) \longrightarrow G(P)$

<u>is an isomorphism for all finitely generated projective</u> $P \in {}_A C$.

Proof: The following diagram in ${}_A C$ is commutative:



where the morphisms are defined by

$g_o(a \otimes b \otimes p) = ab \otimes p$

$g_1(a \otimes b \otimes p) = a \otimes bp$

$g(a \otimes p) = a \otimes \langle p \rangle f_o \otimes p_o$

$k(p) = \langle p \rangle f_o \otimes p_o$

$\nu_P(a \otimes p) = ap$

and where $f_o \otimes p_o$ is a strong dual basis for P . Both functors preserve this diagram, hence we get difference cokernels in the rows of the following commutative diagram

$$\begin{array}{ccccccc}
F(A) \otimes A \otimes P & \rightrightarrows & F(A) \otimes P & \longrightarrow & F(P) & \cong & F(A) \otimes_A P \\
\wr\!\wr\ \varphi(A) \otimes A \otimes P & & \wr\!\wr\ \varphi(A) \otimes P & & \downarrow \varphi(P) & & \wr\!\wr\ \varphi(A) \otimes_A P \\
G(A) \otimes A \otimes P & \rightrightarrows & G(A) \otimes P & \longrightarrow & G(P) & \cong & G(A) \otimes_A P
\end{array}$$

such that $\varphi(P)$ becomes an isomorphism.

This proposition is a counterpart of the corresponding proposition for additive functors instead of C-functors.

4.5 <u>Proposition</u>: <u>Let</u> $F: {}_A C \longrightarrow {}_B C$ <u>be a</u> C-<u>functor and</u> $F(A) \in {}_B C_A$ <u>be</u> B-<u>finite and continuously</u> A-<u>coflat</u>. <u>Then for every finitely generated projective</u> $P \in {}_A C$ <u>we have</u> $F(P) \in {}_B C$ <u>finite</u>.

Proof: As in the above proof we get a difference cokernel

$$F(A) \otimes A \otimes P \rightrightarrows F(A) \otimes P \longrightarrow F(P) \cong F(A) \otimes_A P$$

in ${}_B\mathcal{C}$. But $F(A) \in {}_B\mathcal{C}_A$ is finite over B and $P \in {}_A\mathcal{C}$ is finite over A, thus by 2.3 and 3.10 we get $F(A) \otimes_A P \cong F(P) \in {}_B\mathcal{C}$ finite.

In [12, Theorem 17] and [3, Corollary 2.4] it is shown that monoidal functors preserve finite objects. We want to investigate a special case of this. For this purpose let $\mathcal{C}$ be a closed symmetric monoidal category and $B \in \mathcal{C}$ a bimonoid. Then it is easy to see that $({}_B\mathcal{C}, \otimes, I)$ is again a monoidal category by diagonal action of B on a tensor product $M \otimes N$ (cf. [13]). To denote this B-structure we will write ${}_\Delta M \otimes N$. It is also clear that the underlying functor $\mathcal{U}: {}_B\mathcal{C} \longrightarrow \mathcal{C}$ is a monoidal functor. Thus we have

4.6 Proposition: Let B be a bimonoid in the closed symmetric monoidal category $\mathcal{C}$. If $P \in {}_B\mathcal{C}$ is finite (with respect to the monoidal structure of ${}_B\mathcal{C}$), then $P \in \mathcal{C}$ is also finite.

Assume further that $\mathcal{C}$ has difference kernels so that ${}_B[{}_\Delta M \otimes B, N]$ exists. Then we have an isomorphism

$$ {}_B\mathcal{C}({}_\Delta M \otimes T, N) \cong {}_B\mathcal{C}(T, {}_B[{}_\Delta M \otimes B, N])$$

which shows that ${}_B\mathcal{C}$ is also closed. The B-structure on ${}_B[{}_\Delta M \otimes B, N]$ is given by $\langle m \otimes b\rangle(b'f) = \langle m \otimes bb'\rangle f$ [4,6.1]. This is indeed again in ${}_\Delta[M \otimes B, N]$ since

$$(b\langle m \otimes b'\rangle)(b''f) = \langle b_{(1)}m \otimes b_{(2)}b'b''\rangle f = b(\langle m \otimes b'\rangle(b''f)) .$$

The isomorphism and its inverse are given by Σ and Γ with

$$\langle m \otimes b\rangle\Sigma(f)(t) := f(m \otimes bt)$$

$$\Gamma(g)(m \otimes t) := \langle m \otimes 1\rangle g(t) .$$

It is easy to verify that Σ and Γ map morphisms of ${}_B C$ into ${}_B C$ and are inverses of eachother. In a similar way we get

$$ {}_B C({}_\Delta T \otimes M, N) \cong {}_B C(T, {}_B[{}_\Delta B \otimes M, N]) $$

so that ${}_B C$ is biclosed (but not symmetric in general).

If H is a Hopf monoid with $S^2 = 1$ (antipode of order 2) in C then ${}_H[{}_\Delta M \otimes H, N]$ can be put into a simpler form. Note that there is an isomorphism in ${}_H C_H$

$$ {}_\Delta M \otimes H \cong M \otimes H $$

where the H-structure on the right of ${}_\Delta M \otimes H$ is multiplication on H from the right, the multiplication on $M \otimes H$ on the right is given by $(m \otimes h) \cdot h' = S(h'_{(1)})m \otimes hh'_{(2)}$ and the H-structure on the left of $M \otimes H$ is multiplication on H from the left. The isomorphism φ and its inverse ψ are given by

$$ \varphi(m \otimes h) := S(h_{(1)})m \otimes h_{(2)} $$
$$ \psi(m \otimes h) := h_{(1)}m \otimes h_{(2)} $$

as is easily verified. Thus we have an isomorphism

$$ {}_H[{}_\Delta M \otimes H, N] \cong {}_H[M \otimes H, N] \cong [M, N] $$

where the last isomorphism τ is defined by

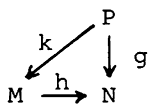
$$ \langle m \rangle (\tau(f)) = \langle m \otimes 1 \rangle f \ , $$
$$ \langle m \otimes h \rangle (\tau^{-1}(g)) = h \cdot \langle m \rangle g \ . $$

Let B again be an arbitrary bimonoid in C. Then the morphism $\varepsilon \colon B \longrightarrow I$ induces a functor $C \longrightarrow {}_B C$ which makes every $X \in C$ into a B-object via ε. This functor is also monoidal so that Theorem 17 of [13] applies again, and we get more examples of finite objects in ${}_B C$.

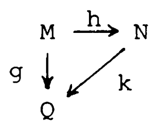
4.7 *Proposition: Let* B *be a bimonoid in the closed symmetric monoidal category* $\mathcal{C}$. *If* $P \in \mathcal{C}$ *is finite, then* $P \in {}_B\mathcal{C}$ *with the trivial* B-*structure induced by* $\varepsilon : B \longrightarrow I$ *is finite*.

In [12] we discussed already the notion of relative or (B,A)-projective (resp. -injective) objects.

4.8 *Definition*: Let $\varphi : A \longrightarrow B$ be a monoid homomorphism in $\mathcal{C}$. A morphism $f: M \longrightarrow N$ in ${}_B\mathcal{C}$ is called a (B,A)-*epimorphisms* if there is a morphism $g: N \longrightarrow M$ in ${}_A\mathcal{C}$ such that $fg = 1_N$. $f \in {}_B\mathcal{C}$ is called a (B,A)-*monomorphism* if there is $g \in {}_A\mathcal{C}$ such that $gf = 1_M$.
$P \in {}_B\mathcal{C}$ is called (B,A)-*projective* if for each commutative diagram



with g,h in ${}_B\mathcal{C}$ and k in ${}_A\mathcal{C}$ there is $g': P \longrightarrow M$ in ${}_B\mathcal{C}$ with $hg' = g$.
$Q \in {}_B\mathcal{C}$ is called (B,A)-*injective* if for each commutative diagram



with g,h in ${}_B\mathcal{C}$ and k in ${}_A\mathcal{C}$ there is $g': N \longrightarrow Q$ in ${}_B\mathcal{C}$ with $g'h = g$.

4.9 *Proposition: Let* $\varphi : A \longrightarrow B$ *be a monoid homomorphism. Consider the following statements for* $P \in {}_B\mathcal{C}$:

a) P *is* (B,A)-*projective*,

b) *for each* (B,A)-*epimorphism* $h: M \longrightarrow N$ *with splitting* $l: N \longrightarrow M$ *in* ${}_A\mathcal{C}$ (*i.e.* $hl = 1_N$) *and each* $g: P \longrightarrow N$ *in* ${}_B\mathcal{C}$ *there is* $g': P \longrightarrow M$ *in* ${}_B\mathcal{C}$ *such that*

$$\begin{array}{ccc} & & P \\ & \overset{g'}{\swarrow} & \downarrow g \\ M & \xrightarrow{h} & N \end{array}$$

commutes,

c) for each (B,A)-epimorphism $h: M \longrightarrow P$ there is a B-morphism $l: P \longrightarrow M$ such that $hl = 1_P$.

Then a) implies b) and b) implies c).

If C has (finite) pull-backs then c) implies a).

Proof: a) $\Longrightarrow$ b): Define $k: P \longrightarrow M$ by $k = lg$. Then apply the definition.

b) $\Longrightarrow$ c): Set $N = P$ and $g = 1_P$.

c) $\Longrightarrow$ a): Let

$$\begin{array}{ccc} M \times_N P & \xrightarrow{q_P} & P \\ {\scriptstyle q_M}\downarrow & & \downarrow {\scriptstyle g} \\ M & \xrightarrow{h} & N \end{array}$$

be a pull-back in C . Then it is also a pull-back in ${}_AC$ and in ${}_BC$ by [9, Cor. 2.4] . Construct $l': P \longrightarrow M \times_N P$ in ${}_AC$ by $q_P l' = 1_P$ and $q_M l' = k$ with $k: P \longrightarrow M$ in ${}_AC$ such that $hk = g$. Then q_P is a (B,A)-epimorphism. Hence there is a B-splitting $h': P \longrightarrow M \times_N P$ with $q_P h' = 1_P$. We get $h(q_M h') = g q_P h' = g$ or with $g' = q_M h'$ in ${}_BC$ we have $hg' = g$.

4.10 Proposition: Let $\varphi: A \longrightarrow B$ be a monoid homomorphism.

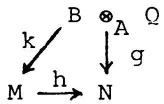
a) If $B \otimes_A Q$ for $Q \in {}_AC$ exists in ${}_BC$, then it is (B,A)-projective.

b) If P is (B,A)-projective and Q is a retract of P in ${}_BC$, then Q is (B,A)-projective.

c) If P is (B,A)-projective and if $B \otimes_A P$ exists in ${}_BC$ then P is a retract of $B \otimes_A P$ in ${}_BC$.

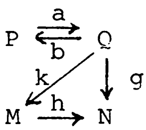
d) If the functor $B \otimes_A -: {}_AC \longrightarrow {}_BC$ exists then for each $M \in {}_BC$ there is a (B,A)-projective P and a (B,A)-epimorphism $P \longrightarrow M$.

Proof: a) Given a commutative diagram



with $k \in {}_A C$ and $g,h \in {}_B C$, define $g'\colon B \otimes_A Q \longrightarrow M$ by $g'(b \otimes_A q) := bk(1 \otimes_A q)$. Then $g' \in {}_B C$ and $hg'(b \otimes_A q) = h(bk(1 \otimes_A q)) = bhk(1 \otimes_A q) = bg(1 \otimes_A q) = g(b \otimes_A q)$ hence $hg' = g$.

b) Consider the diagram



with $a,b,h,g \in {}_B C$ and $k \in {}_A C$ such that $ab = 1_Q$ and $hk = g$. Define $k'\colon P \longrightarrow M$ in ${}_A C$ by $k' = ka$. Then $hk' = ga$ hence there is a $g'\colon P \longrightarrow M$ in ${}_B C$ with $hg' = ga$. Define $g''\colon Q \longrightarrow M$ in ${}_B C$ by $g'' := g'b$. Then $hg'' = hg'b = gab = g$ hence Q is (B,A)-projective.

d) Take $P = B \otimes_A M$, $g\colon B \otimes_A M \longrightarrow M$ given by $g(b \otimes_A m) := bm$ and $k\colon M \longrightarrow B \otimes_A M$ given by $k(m) := 1 \otimes_A m$. Then $g \in {}_B C$, $k \in {}_A C$ and $gk(m) = m$, hence $gk = 1_M$.

c) is a consequence of d) and the (B,A)-projectivity of P .

Dual results may be proved for (B,A)-injective objects replacing the left-adjoint $B \otimes_A -$ of $U\colon {}_B C \longrightarrow {}_A C$ by the right adjoint ${}_A[B,-]$. The details are left to the reader.

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