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The Conference on Brauer Groups was originally titled Conference on Brauer and Picard Groups. The present title is more nearly representative of the contents of the conference and these proceedings.

The conference was sponsored by Northwestern University and was held there (Evanston, Illinois) from October 11 to 15, 1975.

The list of participants which follows gives the university of each participants at the time of the conference (Department of Mathematics in each case). Professor Chase had to cancel his attendance but kindly submitted his manuscript for these Proceedings.

Besides the papers published here, the following were read:

R. T. Hoobler	How to construct U. F. D.'s
M. Ojanguren	Generic splitting rings
S. Rosset	Some solvable group rings are domains
D. Haile	Generalization of involution for central simple algebras of order m in the Brauer group
S. A. Amitsur	Cyclic splitting of generic division algebras
G. Szeto	Lifting modules and algebras

CONFERENCE ON BRAUER GROUPS

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NON-ADDITIVE RING AND MODULE THEORY IV

The Brauer Group of a Symmetric Monoidal Category

Bodo Pareigis

In [5],[6] and [7] we introduced general techniques in the theory of a monoidal category, i.e. of a category $\mathcal{C}$ with a bifunctor $\boxtimes: \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$, an object $I \in \mathcal{C}$ and natural isomorphisms $\alpha: A \boxtimes (B \boxtimes C) \cong (A \boxtimes B) \boxtimes C$, $\lambda: I \boxtimes A \cong A$ and $\varrho : A \boxtimes I \cong A$ which are coherent in the sense of [3, VII. 2]. In this paper we want to introduce the notion of a Brauer group of $\mathcal{C}$. For this purpose we are going to assume that $\mathcal{C}$ is symmetric, i.e. that there is a natural isomorphism $\gamma: A \boxtimes B \cong B \boxtimes A$ which is coherent with α, λ and ϱ [3]. One of the main models for such a category $\mathcal{C}$ is, apart from the category of k-modules for a commutative ring k , the dual of the category of C-comodules for a cocommutative coalgebra C . This category is a symmetric monoidal category, but it is not closed.

Another type of monoidal categories, which are not symmetric but which allow the construction of Brauer groups, are for example categories of dimodules over a commutative, cocommutative Hopf algebra [2]. Their general theory will be discussed elsewhere.

In many special cases of symmetric monoidal categories the basic object I turns out to be projective, i.e. the functor $\mathcal{C}(I,-)$ preserve epimorphisms. In the general situation, however, it turns out that ther may be constructed two Brauer groups $\mathcal{B}_1(\mathcal{C})$ and $\mathcal{B}_2(\mathcal{C})$ and a group-homomorphism $\mathcal{B}_2(\mathcal{C}) \longrightarrow \mathcal{B}_1(\mathcal{C})$, which is an isomorphism if I projective

We will construct these two Brauer groups and discuss under which codition for a functor $F: \mathcal{C} \longrightarrow \mathcal{D}$ we get an induced homomorphism $\mathcal{B}_i(F): \mathcal{B}_i(\mathcal{C}) \longrightarrow \mathcal{B}_i(\mathcal{D})$.

Preliminaries

In [7] we proved analogues of the Morita Theorems which will be used in this paper. For the convenience of the reader we will collect the most important definitions and facts of [5], [6] and [7].

If P is an object of $\mathcal{C}$ we denote by $P(X)$ the set $\mathcal{C}(X,P)$ for $X \in \mathcal{C}$. Elements in $P \otimes Q\ (X)$ will often be denoted by $p \otimes q$. If the functor $\mathcal{C}(P \otimes -, Q)$ is representable then the representing object is $[P,Q]$, so that $\mathcal{C}(P \otimes X, Q) \cong \mathcal{C}(X,[P,Q]) = [P,Q](X)$. The "evaluation" $P(X) \times [P,Q](Y) \longrightarrow Q(X \otimes Y)$, induced by the composition of morphisms, is denoted by $P(X) \times [P,Q](Y) \ni (p,f) \longmapsto \langle p \rangle f \in Q(X \otimes Y)$. Thus the "inner morphism sets" $[P,Q]$ operate on P from the right. In [5, Proposition 3.2] we prove that any natural transformation $P(X) \longrightarrow Q(X \otimes Y)$, natural in X, is induced by a uniquely determined element of $[P,Q](Y)$, if $[P,Q]$ exists.

We call an object $P \in \mathcal{C}$ finite or finitely generated projective if $[P,I]$ and $[P,P]$ exist and if the morphism $[P,I] \otimes P \longrightarrow [P,P]$ induced by $P(X) \times [P,I](Y) \times P(Z) \ni (p,f,p') \longmapsto \langle p \rangle f p' \in P(X \otimes Y \otimes Z)$ is an isomorphism. For $[P,I] \otimes P \longrightarrow [P,P]$ to be an isomorphism it is necessary and sufficient that there is a "dual basis" $f_o \otimes p_o \in [P,I] \otimes P(I)$ such that $\langle p \rangle f_o p_o = p$ for all $p \in P(X)$ and all $X \in \mathcal{C}$. The difference between finite and finitely generated projective objects, as discussed in [8], does not appear here.

A finite object P is called faithfully projective if the morphism $P \otimes_{[P,P]} [P,I] \longrightarrow I$, induced by the evaluation, is an isomorphism. This is equivalent to the existence of $p_1 \otimes_{[P,P]} f_1 \in P \otimes_{[P,P]} [P,I]\ (I)$ with $\langle p_1 \rangle f_1 = 1 \in I(I)$. If there exists an element $p_1 \otimes f_1 \in P \otimes [P,I]\ (I)$ with $\langle p_1 \rangle f_1 = 1$, then P is called a progenerator. Now $P \otimes [P,I] \longrightarrow P \otimes_{[P,P]} [P,I]$ is an epimorphism; if I is projective, then P is faithfully projective iff P is a progenerator.

Let ${}_A\mathcal{C}$ denote the category of A-objects in $\mathcal{C}$ with A a monoid.

Then a functor ${}_A\mathcal{C} \ni X \longmapsto P \otimes_A X \in {}_B\mathcal{C}$ with ${}_BP_A$ a left B right A biobject is a category equivalence iff ${}_BP$ is faithfully projective and $A \cong {}_B[P,P]$ as has been proved in [7]. For this Morita equivalence all the usual conclusions hold, in particular the centers ${}_A[A,A]_A$ and ${}_B[B,B]_B$ of A resp. B are isomorphic monoids if they exist.

<u>The Brauer group $\mathcal{B}_1(\mathcal{C})$.</u>

Let $\mathcal{C}$ be a symmetric monoidal category. A monoid A in $\mathcal{C}$ is called 1-Azumaya if $\mathcal{C} \ni X \longmapsto A \otimes X \in {}_A\mathcal{C}_A$ is an equivalence of cate - gories. Thus the Morita Theorems, in particular [7, Theorem 5.1] can be applied.

<u>Proposition</u> 1: <u>A monoid</u> A <u>is</u> 1-<u>Azumaya iff</u> $A \in \mathcal{C}$ <u>is faithfully projective and</u>

$$\psi\colon A \otimes A\,(X) \ni a \otimes b \longmapsto (A(Y) \ni c \longmapsto acb \in A(X \otimes Y)) \in [A,A](X)$$

<u>is an isomorphism</u>.

Proof: Let A be faithfully projective and ψ be an isomorphism. Let A^{op} be the monoid on A with inverse multiplication. Then $\psi\colon A^{op} \otimes A \longrightarrow [A,A]$ is an isomorphism of monoids. Thus the categories ${}_A\mathcal{C}_A \cong \mathcal{C}_{[A,A]}$ are equivalent with the functor $M \longmapsto M$. Furthermore $\mathcal{C} \ni X \longmapsto A \otimes X \in \mathcal{C}_{[A,A]}$ is an equivalence by [7, Theorem 5.4] hence $\mathcal{C} \ni X \longmapsto A \otimes X \in {}_A\mathcal{C}_A$ is an equivalence and A is 1-Azumaya.

Conversely if A is 1-Azumaya the morphism ψ which exists for any monoid A induces a commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{C} \\ & {\scriptstyle A\,\otimes -}\searrow & \downarrow{\scriptstyle [A,A]} \\ & & {}_A\mathcal{C}_A \end{array}$$

Hence $A \in C$ is faithfully projective in C by [7, Theorem 5.1] and so ψ must induce a category isomorphism and must even be an isomorphism [7, Theorem 5.1 d)] . Q.E.D.

Recall that the center of a monoid A is the object ${}_A[A,A]_A$, if it exists. ${}_A[A,A]_A$ is the representing object of the functor $C \in X \longmapsto {}_AC_A(A \otimes X, A) \in S$. Observe that for this definition we need the symmetry of the monoidal category C .

Let A be a monoid in C . Since we have ${}_A[A,A]_A(X) \subseteq A(X)$, the inclusion given by the isomorphism ${}_A[A,A] \cong A$ using the multiplication with A from the right, it is easy to see that $a \in {}_A[A,A]_A(X)$ iff $ab = ba$ for all $b \in A(Y)$ and all $Y \in C$. Since $ab = ba$ for all $a \in \mathrm{Im}(\eta(X): I(X) \longrightarrow A(X))$ and all $b \in A(Y)$, all $X,Y \in C$, we get that $\eta(X)$ maps $I(X)$ into ${}_A[A,A]_A(X)$. If this morphism is an isomorphism then A is called a central monoid.

Let A be 1-Azumaya. Then $[A,A]$ is Morita equivalent to I hence the center I of I coincides with the center of $[A,A]$ via the morphism $\eta: I \longrightarrow [A,A]$ [7, Corollary 6.3] . Thus the morphism

$$I \xrightarrow{\eta} A \xrightarrow{\varphi} A^{op} \otimes A \xrightarrow{\psi} [A,A]$$

is injective. φ is defined by $\varphi(a) = 1 \otimes a$. Hence $I(X)$ is contained in the center ${}_A[A,A]_A(X)$. Now let $a \in A(X)$ such that $ab = ba$ for all $b \in A(Y)$, all $Y \in C$. Then $(1 \otimes a)(b \otimes c) = b \otimes ac = b \otimes ca = (b \otimes c)(1 \otimes a)$ for all $b \otimes c \in A^{op} \otimes A(Y)$. Thus $\varphi(a)$ is in the center of $A^{op} \otimes A$ or $\psi\varphi(a)$ in the center of $[A,A]$, which was I . Furthermore φ is a monomorphism, even a section with retract $\mu: A \otimes A \longrightarrow A$. Thus $a \in \mathrm{Im}(\eta(X): I(X) \longrightarrow A(X))$, so that I is the center of A .

<u>Corollary</u> 2: <u>If</u> A <u>is</u> 1-<u>Azumaya</u> <u>then</u> A <u>is</u> <u>central</u>.

<u>Proposition</u> 3: <u>Let</u> A , B <u>be</u> 1-<u>Azumaya</u> <u>then</u> $A \otimes B$ <u>is</u> 1-<u>Azumaya</u>.

Proof: Let $f_o \otimes a_o$ resp. $g_o \otimes b_o$ be a dual basis for A resp. B. Then $f_o \otimes g_o \otimes a_o \otimes b_o \in [A \otimes B, I] \otimes A \otimes B(I)$ is a dual basis for $A \otimes B$ where we identified $[A,I] \otimes [B,I]$ with $[A \otimes B, I]$. Further-more we have $[A,A] \otimes [B,B] \cong [A \otimes [A,I], [B,B]] \cong [A \otimes B \otimes [A,I],B] \cong [A \otimes B, A \otimes B]$ since A and B are finite [8, Theorem 1.2]. Hence $[A \otimes B, I]$ and $[A \otimes B, A \otimes B]$ exist.

Let $A^e = A^{op} \otimes A$ and $A' \in {}_{A^e}C_I$ be the dual $[A,I]$ of A. With the analogous notation for B we get

$$(A \otimes B) \underset{A^e \otimes B^e}{\otimes} (A' \otimes B') \cong (A \otimes_{A^e} A') \otimes (B \otimes_{B^e} B') \cong I$$

since A, B are faithfully projective. Hence $A \otimes B$ is faithfully projective in C.

Finally since $[A,A] \otimes [B,B] \cong [A \otimes B, A \otimes B]$ we get that $\psi: A \otimes B \otimes A \otimes B \longrightarrow [A \otimes B, A \otimes B]$ is an isomorphism.

<u>Proposition</u> 4: <u>Let</u> P <u>be faithfully projective then</u> $[P,P]$ <u>is</u> 1-<u>Azumaya</u>.

Proof: We know that $[P,I] \otimes P \cong [P,P]$ as $[P,P]$ - $[P,P]$ - objects. Furthermore $C \ni X \longmapsto [P,I] \otimes X \in {}_{[P,P]}C$ and $C \ni X \longmapsto X \otimes P \in C_{[P,P]}$ are equivalences. Hence $C \ni X \longmapsto [P,P] \otimes X \cong [P,I] \otimes X \otimes P \in {}_{[P,P]}C_{[P,P]}$ is an equivalence, since ${}_{[P,P]}C \ni Y \longmapsto Y \otimes P \in {}_{[P,P]}C_{[P,P]}$ is also an equivalence.

<u>Proposition</u> 5: <u>Let</u> P, Q <u>be faithfully projective, then</u> $P \otimes Q$ <u>is faithfully projective and</u> $[P,P] \otimes [Q,Q] \cong [P \otimes Q, P \otimes Q]$ <u>as monoids</u>.

Proof: Since P and Q are finite we get for all $X, Y \in C$ that $\varphi: [P,X] \otimes [Q,Y] \ni f \otimes g \longmapsto (p \otimes q \longmapsto \langle p \rangle f \otimes \langle q \rangle g) \in [P \otimes Q, X \otimes Y]$ is an isomorphism and the right side exists. In particular

$[P \otimes Q, I]$ and $[P \otimes Q, P \otimes Q]$ exist. Furthermore we have $<p \otimes q>\varphi(f \otimes g)\varphi(f' \otimes g') = <p>ff' \otimes <q>gg' = <p \otimes q>\varphi(ff' \otimes gg')$ and $\varphi(id_P \otimes id_Q) = id_{P \otimes Q}$, hence $\varphi : [P,P] \otimes [Q,Q] \longrightarrow [P \otimes Q, P \otimes Q]$ is an monoid isomorphism.

If $f_o \otimes p_o$ resp. $g_o \otimes q_o$ are dual bases of P and Q then $\varphi(f_o \otimes g_o) \otimes (p_o \otimes q_o)$ is a dual basis of $P \otimes Q$ for $<p \otimes q>\varphi(f_o \otimes g_o)(p_o \otimes q_o) = <p>f_o p_o \otimes <q>g_o q_o = p \otimes q$. Hence $P \otimes Q$ is finite.

Now let $p_1 \otimes_{[P,P]} f_1$ resp. $q_1 \otimes_{[Q,Q]} g_1$ be elements such that $<p_1>f_1 = 1$ and $<q_1>g_1 = 1$. Then $(p_1 \otimes q_1) \otimes_B \varphi(f_1 \otimes g_1)$ with $B = [P \otimes Q, P \otimes Q]$ has the property $<p_1 \otimes q_1>\varphi(f_1 \otimes g_1) = <p_1>f_1<q_1>g_1 = 1$. Thus $P \otimes Q$ is faithfully projective.

Now we can define the Brauer group $\mathcal{B}_1(\mathcal{C})$ of a symmetric monoidal category $\mathcal{C}$. Let $\mathcal{A}$ be the(illegitimate) set of isomorphism classes of 1-Azumaya monoids A in $\mathcal{C}$. Then we define an equivalence rela - tion on $\mathcal{A}$ by $\bar{A} \sim \bar{B}$ iff there exist faithfully projective $P,Q \in \mathcal{C}$ such that $A \otimes [P,P] \cong B \otimes [Q,Q]$ as monoids. Denote the set of equivalence classes by $\mathcal{B}_1(\mathcal{C})$. $\mathcal{B}_1(\mathcal{C})$ becomes a commutative group in the usual way by $[A][B] = [A \otimes B]$ with unit element $[I]$ and inverse $[A^{op}]$ for $[A]$, where A^{op} is the 1-Azumaya monoid A with inverse multiplication $A \otimes A \xrightarrow{\gamma} A \otimes A \xrightarrow{\mu} A$.

Separable monoids

Let A be a monoid in $\mathcal{C}$. A is called a separable monoid if the multiplication $\mu : A \otimes A \longrightarrow A$ has a splitting $\sigma : A \longrightarrow A \otimes A$ in ${}_A\mathcal{C}_A$ such that $\mu\sigma = id_A$. Observe that $A(X) \ni a \longmapsto a \otimes 1 \in A \otimes A(X)$ is a splitting for μ in ${}_A\mathcal{C}$ but it is no A-right-morphism.

<u>Proposition</u> 6: <u>Let</u> $A \in C$ <u>be a monoid</u>. <u>Equivalent are</u>

a) A <u>is separable</u>

b) <u>There is an element</u> $a \otimes b \in A \otimes A(I)$ <u>such that</u>

i) $\forall c \in A(X): ca \otimes b = a \otimes bc \in A \otimes A\ (X)$,

ii) $ab = 1 \in A(I)$.

Proof: a) $\Rightarrow$ b): Define $a \otimes b: = \sigma(1)$. Then $1 = \mu\sigma(1) = \mu(a \otimes b) = ab$ which is condition (ii). Since σ is an A-A-morphism we have $ca \otimes b = c\sigma(1) = \sigma(c) = \sigma(1)c = a \otimes bc$ for all $a \in A(X)$.

b) $\Rightarrow$ a): Let $\sigma: A(X) \longrightarrow A \otimes A(X)$ be defined by $\sigma(c) = ca \otimes b$. By (i) σ is an A-A-morphism. By (ii) we get $\mu\sigma(c) = \mu(ca \otimes b) = cab = c$, hence $\mu\sigma = id_A$.

Observe that (i) does not depend on a symmetry in C , since $a \otimes b \in A \otimes A(I)$ and $I \otimes X \cong X \otimes I$ even without a symmetry. The element $a \otimes b$ will be called a Casimir element.

Every Casimir element $a \otimes b \in A \otimes A(I)$ induces a map $Tr: C\ (M,N) \ni f \longmapsto (M(X) \ni m \longmapsto af(bm) \in N(X)) \in {}_AC\ (M,N)$ for any two objects $M,N \in {}_AC$. In fact for any $c \in A(Y)$ we have $ca \otimes b = a \otimes bc$ hence $c(af(bm)) = (ca)f(bm) = af(bcm) = af(b(cm))$. This map is called the trace map.

Since the trace map is a natural transformation, natural in X ,

$$Tr: C(M \otimes X,N) \longrightarrow {}_AC(M \otimes X,N) ,$$

we get $Tr: [M,N] \longrightarrow {}_A[M,N]$, if both objects exist.

Since $ab = 1$ we even get that

$${}_AC(M,N) \longrightarrow C(M,N) \xrightarrow{Tr} {}_AC(M,N)$$

is the identity on ${}_AC(M,N)$ since $Tr(f)(m) = af(bm) = abf(m) = f(m)$ and hence $Tr(f) = f$, if $f \in {}_AC(M,N)$. Similarly ${}_A[M,N] \longrightarrow [M,N] \longrightarrow {}_A[M,N]$ is the identity on ${}_A[M,N]$.

If $M,N \in {}_AC_A$ then we clearly get $Tr: C_A(M,N) \longrightarrow {}_AC_A(M,N)$ and ${}_AC_A(M,N) \longrightarrow C_A(M,N) \xrightarrow{Tr} {}_AC_A(M,N)$ is the identity on ${}_AC_A(M,N)$. The same holds for $[M,N]_A$ and ${}_A[M,N]_A$.

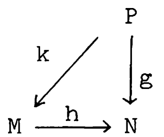
If ${}_A[A,A]_A$ exists then

$${}_A[A,A]_A \longrightarrow [A,A]_A \xrightarrow{Tr} {}_A[A,A]_A$$

is the identity on ${}_A[A,A]_A$. Observe that $[A,A]_A$ exists, since $[A,A]_A \cong A$. Since the last isomorphism is an antiisomorphism of mo - noids, ${}_A[A,A]_A$ is the center of A and ${}_A[A,A]_A \longrightarrow A$ is a mo - noid homomorphism, we get

<u>Proposition</u> 7 [1, Prop. 1.2]: <u>If</u> A <u>is a separable monoid, then the center</u> ${}_A[A,A]_A$ (<u>if it exists</u>) <u>is a "direct summand" of</u> A .

Let $f: A \longrightarrow B$ be monoid homomorphism. $P \in {}_B C$ is called (B,A)-projective if for each commutative diagram



with g,h in ${}_B C$ and k in ${}_A C$ there is a $g' \in {}_B C(P,M)$ with $hg' = g$. The dual notion is that of a (B,A)-injective object [8] .

<u>Proposition</u> 8: <u>Let</u> A <u>be a separable monoid. Then every</u> A-<u>object is</u> (A,I)-<u>projective and</u> (A,I)-<u>injective</u>.

Proof: Let $g \in {}_A C(P,N)$, $h \in {}_A C(M,N)$ and $k \in C(P,M)$ be given such that $hk = g$. Then $g = Tr(g) = Tr(hk) = h\,Tr(k)$ and $Tr(k) \in {}_A C(P,M)$, so that P is (A,I)-projective.Just by reversing the arrows one can prove that each object in ${}_A C$ is (A,I)-injective.

In [8] we prove that in $(C,\times,E)$, a monoidal category with the pro- duct as tensor-product and E a final object, there are no non-tri - vial finite objects. In Theorem 14 we shall show that $[P,P]$ is a separable monoid for certain finite objects $P \in \mathcal{C}$. So this construction will not produce examples of separable monoids in $(\mathcal{C}, \times, E)$. In fact, there are no non-trivial separable monoids in $\mathcal{C}$ at all.

Proposition 9: *Let A be a separable monoid in the monoidal category* $(\mathcal{C}, \times, E)$. *Then* $A \cong E$ *as monoids.*

Proof: Let (a,b) be the Casimir element for A . Then $(ca,b) = (a,bc)$ for all $c \in A(X)$, hence $ca = a$ and $b = bc$. Here we use $A \times A(X) = A(X) \times A(X)$ and $A(E) \longrightarrow A(X)$ by the unique morphism $X \longrightarrow E$. We also have $1c = c$ and $ab = 1$, hence $c = 1c = abc = ab = 1$ for all $c \in A(X)$, so that $A(X) = \{1\}$ which proves $A \cong E$ .Finally observe that E has a unique monoid structure.

If A and B are monoids in $\mathcal{C}$, then $A \boxtimes B$ is a monoid by $(a_1 \boxtimes b_1) \cdot (a_2 \boxtimes b_2) = a_1a_2 \boxtimes b_1b_2$.

Proposition 10: *Let A and B be separable monoids. Then* $A \boxtimes B$ *is separable.*

Proof: Let $a_1 \boxtimes a_2$ and $b_1 \boxtimes b_2$ be Casimir elements of A resp. B. Then $(a_1 \boxtimes b_1) \boxtimes (a_2 \boxtimes b_2)$ is a Casimir element for $A \boxtimes B$. In fact let $x \boxtimes y \in A \boxtimes B(X)$ then

$$(x \boxtimes y)(a_1 \boxtimes b_1) \boxtimes (a_2 \boxtimes b_2) = (xa_1 \boxtimes yb_1) \boxtimes (a_2 \boxtimes b_2) =$$
$$(a_1 \boxtimes b_1) \boxtimes (a_2x \boxtimes b_2y) = (a_1 \boxtimes b_1) \boxtimes (a_2 \boxtimes b_2)(x \boxtimes y) .$$

Furthermore $(a_1 \boxtimes b_1)(a_2 \boxtimes b_2) = a_1a_2 \boxtimes b_1b_2 = 1 \boxtimes 1$.

Proposition 11: *Let A be a separable monoid with Casimir element* $a \boxtimes b$. *Assume that* ${}_A[A,A]_A$ *exists and that* $I \longrightarrow A$ *is a monomorphism. Then A is central if and only if* $axb \in I(X)$ *for all* $x \in A(X)$.

Proof: Since $ca \otimes b = a \otimes bc$ for all $c \in A(Y)$ we get $c(axb) = (axb)c$ hence $axb \in {}_A[A,A]_A(X)$ for all $x \in A(X)$. If A is central then $axb \in {}_A[A,A]_A(X) = I(X)$ for all $x \in A(X)$. Conversely let $axb \in I(X)$ for all $x \in A(X)$. Let $x \in {}_A[A,A]_A(X)$ then $x = xab = axb \in I(X)$ hence ${}_A[A,A]_A(X) = I(X)$.

The Brauer group $B_2(C)$

A monoid A is called 2-Azumaya if $[A,I]$ and $[A,A]$ exist and $\eta: I \longrightarrow A$ is a monomorphism and if there are elements $a \otimes b \in A \otimes A(I)$ and $c \otimes d \otimes e \in A \otimes A \otimes A(I)$ such that

i) $\forall X \in C \ \ \forall x \in A(X): xa \otimes b = a \otimes bx$,

ii) $ab = 1 \in A(I)$,

iii) $ac \otimes dbe = 1 \otimes 1 \in A \otimes A\ (I)$,

iv) $\forall X \in C \ \ \forall x \in A(X): axb \in I(X)$.

Clearly a 2-Azumaya monoid is a central, separable monoid which follows from i), ii), and iv) by Proposition 6 and 11 . We do not know if the existence of $c \otimes d \otimes e$ with iii) follows from the other conditions.

Theorem 12: Let A be a monoid in C . Equivalent are

a) A is 2-Azumaya.

b) $A \in C$ is a progenerator and the morphism

$\psi: A \otimes A(X) \ni x \otimes y \longmapsto (A(Y)) \ni z \longmapsto xzy \in A(X \otimes Y)) \in [A,A](X)$

is an isomorphism.

c) A is separable and 1-Azumaya.

Proof: Let A be 2-Azumaya. Define $\varphi: A \otimes A \longrightarrow [A,I]$ by $\langle z\rangle(\varphi(x \otimes y)): = axzyb$ where $a \otimes b$ is a Casimir element for A . Then $axzyb \in I(X)$ by iv) hence φ is well-defined. Now

$\varphi(c \boxtimes d) \boxtimes e$ is a dual basis for A since $\langle x\rangle(\varphi(c \boxtimes d)e) = acxdbe = x$ for all $x \in A(X)$. So A is finite.

Now we show that $A \boxtimes [A,I] \longrightarrow I$, the morphism induced by the evaluation, is rationally surjective, i.e. that
$A \boxtimes [A,I]\ (I) \longrightarrow I(I)$ is surjective. We have to find $a_1 \boxtimes f_1 \in A \boxtimes [A,I](I)$ with $\langle a_1\rangle f_1 = 1 \in I(I)$. Take $a_1 = 1$ and $f_1 = (Tr\colon A \longrightarrow {}_A[A,A]_A \cong I)$, the last isomorphism exists in view of Propositions 6 and 11 by the properties i), ii), and iv) of A . Then $\langle 1\rangle f_1 = 1$, hence A is a progenerator.

To show that ψ is an isomorphism we construct the inverse morphism

$$[A,A](X) \ni \sigma \longmapsto \langle ac\rangle\sigma db \boxtimes e \in A \boxtimes A(X) .$$

This morphism is in fact an inverse of ψ since

$$\langle x\rangle\ \psi\ (\langle ac\rangle\sigma db \boxtimes e) = \langle ac\rangle\sigma dbxe = \langle xac\rangle\sigma dbe$$
$$= \langle x1\rangle\sigma 1 = \langle x\rangle\sigma ,$$

hence $\psi\ (\langle ac\rangle\sigma db \boxtimes e) = \sigma$, and

$$xacydb \boxtimes e = x \boxtimes acydbe = x \boxtimes 1y1 = x \boxtimes y .$$

Now assume that b) holds. By Proposition 1 the monoid A is 1-Azumaya. Let $f_o \boxtimes a_o$ be a dual basis for A and $a_1 \boxtimes f_1 \in A \boxtimes [A,I](I)$ such that $\langle a_1\rangle f_1 = 1 \in I(I)$. Let $g_o \in [A,I](I)$ be defined by $\langle x\rangle g_o = \langle xa_1\rangle f_1$. Let $a \boxtimes b \in A \boxtimes A(I)$ be the element which corresponds to $g_o \boxtimes 1 \in [A,I] \boxtimes A(I)$ under the isomorphism $A \boxtimes A \cong [A,A] \cong [A,I] \boxtimes A$. Then we have $ab = a1b = \langle 1\rangle g_o 1 = \langle a_1\rangle f_1 1 = 1 \in A(I)$. Furthermore we have $xayb = x\langle y\rangle g_o = \langle y\rangle g_o x = aybx$ for all $y \in A(Y)$, hence $xa \boxtimes b = a \boxtimes bx$ for all $x \in A(X)$. So $a \boxtimes b$ is a Casimir element for A . Thus c) holds.

Assume that c) holds. By Proposition 1 A is faithfully projective and ψ is an isomorphism. Construct $a_1 \boxtimes f_1 \in A \boxtimes [A,I](I)$ with $\langle a_1\rangle f_1 = 1 \in I(I)$ as in part one of the proof. Then b) holds.

We still have to show that b) and c) imply a). Let $f_o^1 \boxtimes a_o^1$ and

$f_o^2 \otimes a_o^2$ be two copies of the dual basis of A . Let $a_1 \otimes f_1 \in A \otimes [A,I](I)$ with $<a_1>f_1 = 1$ be given. Then define $a \otimes b$ as above corresponding to g_o and $c \otimes d \otimes e :=$ $u \otimes v \otimes xy \in A \otimes A \otimes A\ (I)$, where $u \otimes v \otimes x \otimes y \in A \otimes A \otimes A \otimes A\ (I)$ corresponds to $f_o^1 \otimes a_o^2 \otimes f_o^2 \otimes a_o^1 \in [A,I] \otimes A \otimes [A,I] \otimes A(I)$ under the isomorphisms

$[A,I] \otimes A \otimes [A,I] \otimes A \cong [A,A] \otimes [A,A] \cong A \otimes A \otimes A \otimes A$. Then $aczdbe = auzvbxy = <uzv>g_oxy = <<<z>f_o^1a_o^2a_1>f_1 1>f_o^2a_o^1 =$ $<z>f_o^1<a_o^2a_1> f_1<1>f_o^2a_o^1 = <<1>f_o^2a_o^2a_1>f_1<z>f_o^1a_o^1 = z = 1z1$ for all $z \in A(Z)$, hence $ac \otimes dbe = 1 \otimes 1$. Thus iii) for a monoid to be 2-Azumaya holds. i) and ii) hold by Proposition 1 , iv) by Proposition 11 .

Corollary 13: Let A and B be 2-Azumaya , then $A \otimes B$ is 2-Azumaya.

Proof: In view of the equivalence of a) and c) in Theorem 12 this follows from Proposition 3 and Proposition 10.

Theorem 14: Let $P \in C$ be a progenerator. Then $[P,P]$ is 2-Azumaya.

Proof: By Proposition 4 we get that $[P,P]$ is 1-Azumaya so that we only have to show that $[P,P]$ is separable. Let $f_o \otimes p_o$ be a dual basis for P and $p_1 \otimes f_1 \in P \otimes [P,I](I)$ with $<p_1>f_1 = 1$. Identify $[P,P]$ with $[P,I] \otimes P$ with the multiplication $(f \otimes p)(f' \otimes p') = f \otimes <p>f'p'$. Then define $a \otimes b := (f_o \otimes p_1) \otimes (f_1 \otimes p_o)$. For every $g \otimes q \in [P,I] \otimes P(X)$ we have

$$(g \otimes q)(f_o \otimes p_1) \otimes (f_1 \otimes p_o) = (g \otimes <q>f_op_1) \otimes (f_1 \otimes p_o) =$$

$$(g \otimes p_1) \otimes (f_1 \otimes <q>f_op_o) = (g \otimes p_1) \otimes (f_1 \otimes q) =$$

$(f_o\langle p_o\rangle g \boxtimes p_1) \boxtimes (f_1 \boxtimes q) = (f_o \boxtimes p_1) \boxtimes (f_1 \boxtimes \langle p_o\rangle gq) =$

$(f_o \boxtimes p_1) \boxtimes (f_1 \boxtimes p_o)(g \boxtimes q)$

so that b) i) of Proposition 6 holds. Furthermore

$(f_o \boxtimes p_1)(f_1 \boxtimes p_o) = f_o \boxtimes \langle p_1\rangle f_1 p_o = f_o \boxtimes p_o$,

which corresponds to $1 \in [P,P](I)$, shows b) ii) .

It may be interesting to have an explicit description of the element $c \boxtimes d \boxtimes e$ in the definition of 2-Azumaya for this case $[P,P]$. Let $f_o^i \boxtimes p_o^i$, $i = 1, 2, 3$ be copies of the dual basis of P . Then $c \boxtimes d \boxtimes e := (f_o^1 \boxtimes p_o^2) \boxtimes (f_o^3 \boxtimes p_o^1) \boxtimes (f_o^2 \boxtimes p_o^3)$ satisfies condition iii) for 2-Azumaya as is easily checked.

To define a Brauer group of 2-Azumaya monoids we need one more lemma.

<u>Lemma</u> 15: <u>Let</u> P <u>and</u> Q <u>be progenerators</u>. <u>Then</u> $P \boxtimes Q$ <u>is a progenerator and</u> $[P,P] \boxtimes [Q,Q] \cong [P \boxtimes Q, P \boxtimes Q]$ <u>as monoids</u>.

Proof: Let $p_1 \boxtimes f_1$ resp. $q_1 \boxtimes g_1$ with $\langle p_1\rangle f_1 = 1$ resp. $\langle q_1\rangle g_1 = 1$ be given. Then form the element $(p_1 \boxtimes q_1) \boxtimes \varphi(f_1 \boxtimes g_1) \in (P \boxtimes Q) \boxtimes [P \boxtimes Q, I](I)$, where $\varphi: [P,I] \boxtimes [Q,I] \cong [P \boxtimes Q, I]$ is the isomorphism used in the proof of Proposition 5 . We get $(p_1 \boxtimes q_1)\varphi(f_1 \boxtimes g_1) = \langle p_1\rangle f_1 \langle q_1\rangle g_1 = 1$, hence $P \boxtimes Q$ is a progenerator in view of Proposition 5 .

Now we can define the Brauer group $\mathcal{B}_2(C)$, using 2-Azumaya monoids, in the same way as $\mathcal{B}_1(C)$. Since each 2-Azumaya monoid is 1-Azumaya and since each progenerator is faithfully projective we get a group homomorphism $\xi: \mathcal{B}_2(C) \longrightarrow \mathcal{B}_1(C)$. Since the notions of progenerator and faithfully projective coincide, if $I \in C$ is projective, the notions of 1-Azumaya and 2-Azumaya coincide by Theorem 12, b) and Proposition 1 . So does the equivalence relation used in the construction of the two Brauer groups and we get

Theorem 16: *The group homomorphism* $\xi\colon \mathcal{B}_2(\mathcal{C}) \longrightarrow \mathcal{B}_1(\mathcal{C})$ *is the identity in case* $I \in \mathcal{C}$ *is projective*.

Splitting Azumaya monoids by monoidal functors.

Now we want to discuss the behaviour of the Brauer groups under a monoidal functor. Let $\mathcal{C}$ and $\mathcal{D}$ be symmetric monoidal categories and $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a covariant functor. Denote the tensor products and the associativity, the symmetry and unity isomorphisms in $\mathcal{C}$ and $\mathcal{D}$ by the same signs $\otimes$, α, γ, λ, and ϱ . Assume that there are natural transformations

$\delta\colon FX \otimes FY \longrightarrow F(X \otimes Y)$

$\zeta\colon J \longrightarrow FI$

such that the following diagrams commute

$$\begin{array}{ccc} FX \otimes FI & \xleftarrow{1 \otimes \zeta} & FX \otimes J \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \varrho} \\ F(X \otimes I) & \xrightarrow{F(\varrho)} & FX \end{array}$$

$$\begin{array}{ccc} FI \otimes FX & \xleftarrow{\zeta \otimes 1} & J \otimes FX \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \lambda} \\ F(I \otimes X) & \xrightarrow{F(\lambda)} & FX \end{array}$$

$$\begin{array}{ccccc} FX \otimes (FY \otimes FZ) & \xrightarrow{1 \otimes \delta} & FX \otimes F(Y \otimes Z) & \xrightarrow{\delta} & F(X \otimes (Y \otimes Z)) \\ \downarrow{\scriptstyle \alpha} & & & & \downarrow{\scriptstyle F(\alpha)} \\ (FX \otimes FY) \otimes FZ & \xrightarrow{\delta \otimes 1} & F(X \otimes Y) \otimes FZ & \xrightarrow{\delta} & F((X \otimes Y) \otimes Z) \end{array} .$$

If $\mathcal{C}$ and $\mathcal{D}$ are symmetric we require in addition the commutativity of

$$\begin{array}{ccc} FX \boxtimes FY & \xrightarrow{\gamma} & FY \boxtimes FX \\ \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} \\ F(X \boxtimes Y) & \xrightarrow{F(\gamma)} & F(Y \boxtimes X) \end{array} .$$

Such a triple (F, δ, ζ) will be called a weakly monoidal functor.

Let $\pi: X \boxtimes [X,Y] \longrightarrow Y$ and $\tau: Y \longrightarrow [X, X \boxtimes Y]$ be front and back adjunction for the adjoint pair of functors $X \boxtimes -$ and $[X,-]$, if $[X,-]$ exists. Again we use the same notation in both categories $\mathcal{C}$ and $\mathcal{D}$. Let $\chi: \mathcal{C}(X \boxtimes Y, Z) \cong \mathcal{C}(Y, [X,Z])$ and $\omega: \mathcal{C}(Y, [X,Z]) \cong \mathcal{C}(X \boxtimes Y, Z)$ be the corresponding adjointness isomorphisms in $\mathcal{C}$ resp. also in $\mathcal{D}$. It is an easy exercise in diagram chasing for adjoint functors to show that there is a natural transformation $\Phi: F[X,Y] \longrightarrow [FX,FY]$ whenever $[X,Y]$ and $[FX,FY]$ exist, just take $\Phi = \chi(F(\pi)\delta)$. Furthermore the diagrams

$$\begin{array}{ccc} FX \boxtimes F[X,Y] & \xrightarrow{\delta} & F(X \boxtimes [X,Y]) \\ \downarrow{\scriptstyle 1 \boxtimes \Phi} & & \downarrow{\scriptstyle F(\pi)} \\ FX \boxtimes [FX,FY] & \xrightarrow{\pi} & FY \end{array} \qquad \begin{array}{ccc} FY & \xrightarrow{F(\tau)} & F[X, X \boxtimes Y] \\ \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\Phi} \\ [FX, FX \boxtimes FY] & \xrightarrow{[1,\delta]} & [FX, F(X \boxtimes Y)] \end{array} \text{ and}$$

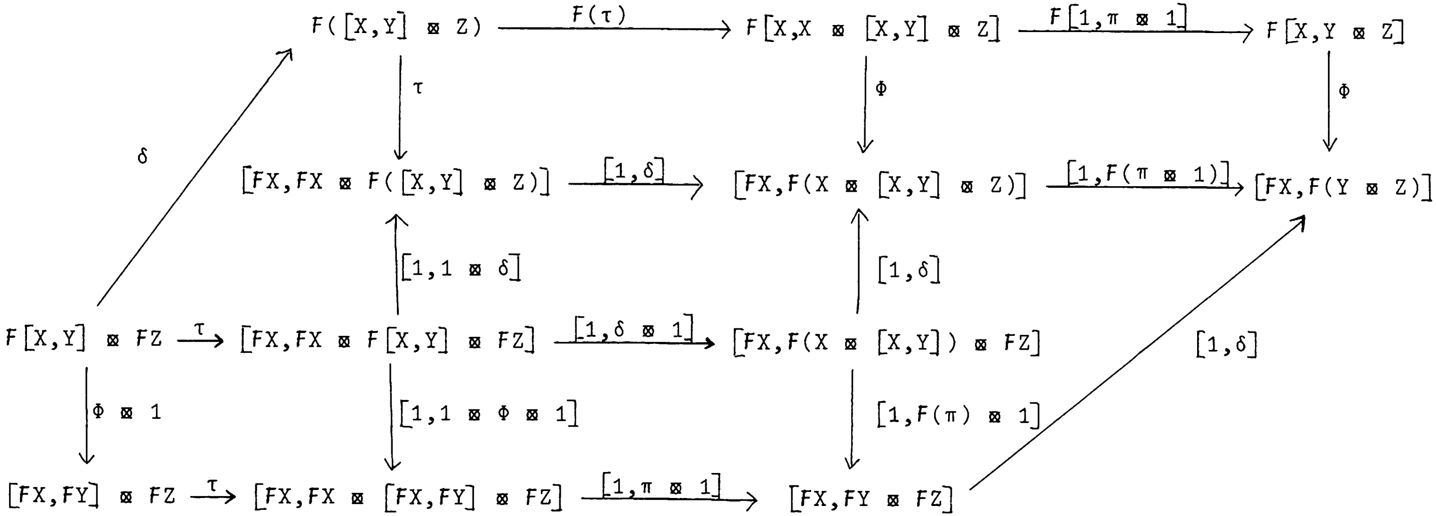
$$\begin{array}{ccc} J & \xrightarrow{j} & [FX,FX] \\ \downarrow{\scriptstyle\zeta} & & \downarrow{\scriptstyle\Phi} \\ FI & \xrightarrow{F(i)} & F[X,X] \end{array}$$

commute. Here $i: I \longrightarrow [P,P]$ is $\chi(\varrho)$ where $\varrho : X \boxtimes I \longrightarrow X$ and j is defined analogously in $\mathcal{D}$.

Omitting special arrows for the associativity α we get the commutative diagram on the next page.

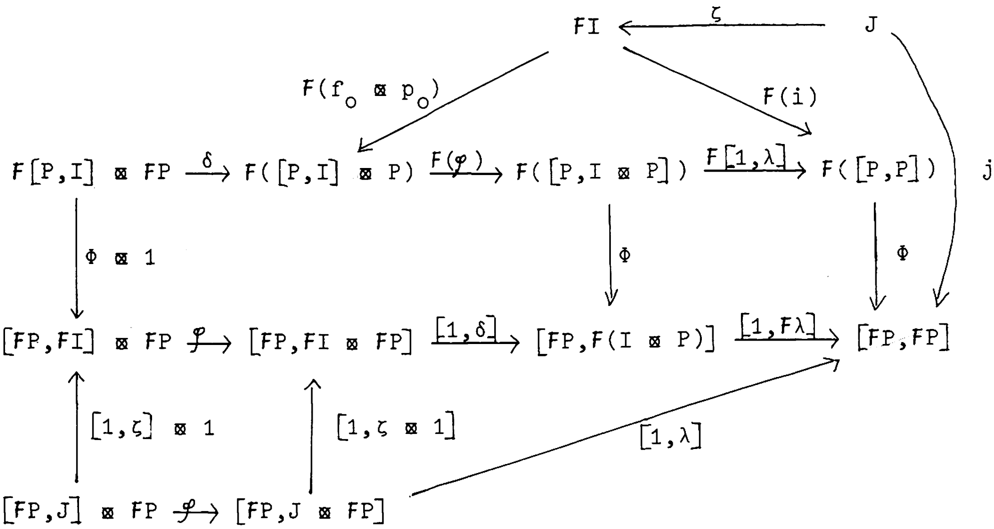
If we abbreviate $[1, \pi \boxtimes 1]\tau$ by $\varphi : [X,Y] \boxtimes Z \longrightarrow [X, Y \boxtimes Z]$ then the following diagram (the outer frame of the given diagram) commutes

$$\begin{array}{ccccc} F[X,Y] \boxtimes FZ & \xrightarrow{\delta} & F([X,Y] \boxtimes Z) & \xrightarrow{F(\varphi)} & F([X, Y \boxtimes Z]) \\ \downarrow{\scriptstyle \Phi \boxtimes 1} & & & & \downarrow{\scriptstyle\Phi} \\ [FX,FY] \boxtimes FZ & \xrightarrow{\varphi} & [FX, FY \boxtimes FZ] & \xrightarrow{[1,\delta]} & [FX, F(Y \boxtimes Z)] \end{array} .$$


F([X,Y] ⊠ Z)
F(τ)
F[X,X ⊠ [X,Y] ⊠ Z]
F[1,π ⊠ 1]
F[X,Y ⊠ Z]
τ
Φ
Φ
δ
[FX,FX ⊠ F([X,Y] ⊠ Z)]
[1,δ]
[FX,F(X ⊠ [X,Y] ⊠ Z)]
[1,F(π ⊠ 1)]
[FX,F(Y ⊠ Z)]
[1,1 ⊠ δ]
[1,δ]
F[X,Y] ⊠ FZ
τ
[FX,FX ⊠ F[X,Y] ⊠ FZ]
[1,δ ⊠ 1]
[FX,F(X ⊠ [X,Y]) ⊠ FZ]
[1,δ]
Φ ⊠ 1
[1,1 ⊠ Φ ⊠ 1]
[1,F(π) ⊠ 1]
[FX,FY] ⊠ FZ
τ
[FX,FX ⊠ [FX,FY] ⊠ FZ]
[1,π ⊠ 1]
[FX,FY ⊠ FZ]

Theorem 17: Let $F: C \longrightarrow \mathcal{D}$ be a weakly monoidal functor. Assume that $\zeta: J$ —— FI is an isomorphism and that $\zeta: F[P,I] \boxtimes FP \longrightarrow F([P,I] \boxtimes P)$ is an isomorphism for all finite objects $P \in C$. If P is finite in C and if $[FP,-]$ exists in $\mathcal{D}$ then FP is finite in $\mathcal{D}$.

Proof: Since finiteness is equivalent to the fact that $i: I \longrightarrow [P,P]$ can be factored through $[P,I] \boxtimes P$ the following commutative diagram shows that $j: J \longrightarrow [FP,FP]$ can be factored through $[FP,J] \boxtimes FP$



thus FP is finite.

Corollary 18: Under the assumption of Theorem 17 is the morphism $\Phi: F[P,X] \longrightarrow [FP,FX]$ an isomorphism for all $X \in C$ and all finite $P \in C$.

Proof: Let $f_o \boxtimes p_o: I \longrightarrow [P,I] \boxtimes P$ be the dual basis for P and $\overline{f}_o \boxtimes \overline{p}_o: J \longrightarrow [FP,J] \boxtimes FP$ be the dual basis for FP. Define $\Psi: [FP,FX] \longrightarrow F[P,X]$ to be

$$[FP,FX] \cong FI \boxtimes [FP,FX] \xrightarrow{F(f_o \boxtimes p_o) \boxtimes 1} F([P,I] \boxtimes P) \boxtimes [FP,FX] \xrightarrow{\delta^{-1} \boxtimes 1}$$
$$F[P,I] \boxtimes FP \boxtimes [FP,FX] \xrightarrow{1 \boxtimes \pi} F[P,I] \boxtimes FX \longrightarrow F[P,X] \; .$$

Omitting some of the obvious isomorpisms we get a commutative dia - gram

$$\begin{array}{ccccc}
[FP,FX] & & & & \\
 & \searrow^{\delta^{-1} F(f_o \boxtimes p_o) \boxtimes 1} & & & \\
 & & F[P,I] \boxtimes FP \boxtimes [FP,FX] \xrightarrow{1 \boxtimes \pi} & F[P,I] \boxtimes FX \longrightarrow & F[P,X] \\
 & {\scriptstyle \bar{f}_o \boxtimes \bar{p}_o \boxtimes 1} \searrow & \downarrow{\scriptstyle \Phi \boxtimes 1 \boxtimes 1} & \downarrow{\scriptstyle \Phi \boxtimes 1} & \downarrow{\scriptstyle \Phi} \\
 & & [FP,FI] \boxtimes FP \boxtimes [FP,FX] \xrightarrow{1 \boxtimes \pi} & [FP,FI] \boxtimes FX \cong & [FP,FX]
\end{array}$$

where the left triangle commutes by the construction of $\bar{f}_o \boxtimes \bar{p}_o$ in Theorem 17 and right square commutes in the same way as the middle of the diagram in the proof of Theorem 17 does. If we look at the lower part of our diagram we see that the morphism $[FP,FX] \longrightarrow [FP,FX]$ is the identity since $\bar{f}_o\langle\bar{p}_o\rangle g = g$ for all $g \in [FP,FX](Y)$. The upper part is $\Phi\Psi$, hence $\Phi\Psi$ = id. Conversely the commutative diagram

$$\begin{array}{ccccc}
F[P,X] \longrightarrow & F[P,I] \boxtimes FP \boxtimes F[P,X] & \xrightarrow{1 \boxtimes \delta} & F[P,I] \boxtimes F(P \boxtimes [P,X]) & \\
\downarrow{\scriptstyle \Phi} & \downarrow{\scriptstyle 1 \boxtimes 1 \boxtimes \Phi} & & \downarrow{\scriptstyle 1 \boxtimes F(\pi)} & \\
[FP,FX] \longrightarrow & F[P,I] \boxtimes FP \boxtimes [FP,FX] & \xrightarrow{1 \boxtimes \pi} & F[P,I] \boxtimes FX & \longrightarrow F[P,X]
\end{array}$$

shows $\Psi\Phi$ = id .

<u>Corollary 19: Under the assumptions of Theorem 17 if</u> P <u>is a pro - generator then</u> FP <u>is a progenerator. If</u> F <u>preserves difference cokernels and</u> P <u>is faithfully projective then</u> FP <u>is faithfully projective</u>.

Proof: Let P be finite. P is a progenerator iff there is a mor - phism $f: I \longrightarrow P \boxtimes [P,I]$ such that

$$\begin{array}{ccc} & & I \\ & \overset{f}{\swarrow} & \downarrow 1 \\ P \otimes [P,I] & \xrightarrow{\pi} & I \end{array}$$

commutes. Now the diagram

$$\begin{array}{ccc} & & J \\ & \overset{F(f)\zeta}{\swarrow} & \wr\!\!\| \, \zeta \\ F(P \otimes [P,I]) & \xrightarrow{F(\pi)} & FI \\ \wr\!\!\| \; (1 \otimes [1,\zeta^{-1}])(1 \otimes \Phi)\delta^{-1} & & \wr\!\!\| \, \zeta^{-1} \\ FP \otimes [FP,J] & \xrightarrow{\pi} & J \end{array}$$

commutes hence FP is a progenerator. In the case of a faithfully projective P we have to replace $P \otimes [P,I]$ by $P \otimes_{[P,P]} [P,I]$ and $FP \otimes [FP,J]$ by $FP \otimes_{[FP,FP]} [FP,J] \cong F(P \otimes_{[P,P]} [P,I])$. The last isomorphism is a conse - quence of the fact that F preserves difference cokernels.

<u>Theorem 20: Let $F: \mathcal{C} \longrightarrow \mathcal{D}$ be a weakly monoidal functor such that $\zeta: J \longrightarrow FI$ is an isomorphism, $\delta: FX \otimes FP \longrightarrow F(X \otimes P)$ is an iso - morphism for all $X \in \mathcal{C}$ and for all finite $P \in \mathcal{C}$, $[FP,-]$ exists for all finite $P \in \mathcal{C}$ and F preserves difference cokernels. Then F induces homomorphisms of Brauer groups $\mathcal{B}_i(F): \mathcal{B}_i(\mathcal{C}) \longrightarrow \mathcal{B}_i(\mathcal{D})$ for $i = 1, 2$ such that</u>

$$\begin{array}{ccc} \mathcal{B}_2(\mathcal{C}) & \xrightarrow{\mathcal{B}_2(F)} & \mathcal{B}_2(\mathcal{D}) \\ \downarrow \xi & & \downarrow \xi \\ \mathcal{B}_1(\mathcal{C}) & \xrightarrow{\mathcal{B}_1(F)} & \mathcal{B}_1(\mathcal{D}) \end{array}$$

<u>commutes.</u>

Proof: Let A be a monoid in $\mathcal{C}$. Then FA is a monoid in $\mathcal{D}$ with the multiplication $FA \otimes FA \xrightarrow{\delta} F(A \otimes A) \xrightarrow{F(\mu)} FA$ and unit $J \xrightarrow{\zeta} FI \xrightarrow{F(\eta)} FA$. If A is i-Azumaya, i = 1, 2, then FA is faithfully projective resp. a progenerator by Corollary 19, Proposi - tion 1 and Theorem 12. So we only have to show that $\psi: FA \otimes FA \longrightarrow [FA,FA]$ is an isomorphism. ψ is induced by

$T\colon A \boxtimes A \boxtimes A(X) \ni a \boxtimes b \boxtimes c \longmapsto bac \in A(X)$, so that $\psi = \chi(T)$ where $\chi\colon \mathcal{C}(X \boxtimes Y, Z) \cong \mathcal{C}(Y, [X,Z])$.

Now the diagram

$$\begin{array}{ccc} FA \boxtimes FA & \xrightarrow{\chi(T)} & [FA,FA] \\ \downarrow \delta & & \uparrow \Phi \\ F(A \boxtimes A) & \xrightarrow{F(\chi(T))} & F[A,A] \end{array}$$

commutes, since

$$\begin{array}{ccc} FA \boxtimes (FA \boxtimes FA) & \xrightarrow{T} & FA \\ \downarrow 1 \boxtimes \delta & & \uparrow F(T) \\ FA \boxtimes F(A \boxtimes A) & \xrightarrow{\delta} & F(A \boxtimes (A \boxtimes A)) \end{array}$$

commutes so that by applying χ we get

$\Phi \circ F(\chi(T)) \circ \delta = \chi(F(T) \circ \delta) \circ \delta = [1, F(T)] \circ \chi(\delta) \circ \delta = \chi(T)$. The first identity results from the commutativity of

$$\begin{array}{ccccc} \mathcal{C}(X \boxtimes Y, Z) & \xrightarrow{F} & \mathcal{D}(F(X \boxtimes Y), FZ) & \xrightarrow{\mathcal{D}(\delta,1)} & \mathcal{D}(FX \boxtimes FY, FZ) \\ \downarrow \chi & & & & \downarrow \chi \\ \mathcal{C}(Y, [X,Z]) & \xrightarrow{F} & \mathcal{D}(FY, F[X,Z]) & \xrightarrow{\mathcal{D}(1,\Phi)} & \mathcal{D}(FY, [FX,FZ]) \end{array} .$$

Now $\Phi\colon F[A,A] \longrightarrow [FA,FA]$ is an isomorphism by Corollary 18 and $F(\chi(T)) = F(\psi)$ is an isomorphism since A is Azumaya. Since δ is an isomorphism, too, we get that $\psi = \chi(T)\colon FA \boxtimes FA \longrightarrow [FA,FA]$ is an isomorphism.

If $P \in \mathcal{C}$ is faithfully projective or a progenerator in $\mathcal{C}$ then as above FP is faithfully projective or a progenerator in $\mathcal{D}$ and $F[P,P] \cong [FP,FP]$ as monoids using the first commutative diagram we proved for Φ .

Thus if A and B are i-Azumaya, then FA and FB are i-Azumaya. If A and B are equivalent w.r.t. $\mathcal{B}_i(\mathcal{C})$, then so are FA and FB . Finally we have $F(A \boxtimes B) \cong FA \boxtimes FB$ and $FI = J$ so that F induces homomorphisms $\mathcal{B}_i(F)\colon \mathcal{B}_i(\mathcal{C}) \longrightarrow \mathcal{B}_i(\mathcal{D})$ such that the diagram in the theorem commutes.

If $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is a functor satisfying the conditions of Theorem 20 then we define the kernel of $\mathcal{B}_i(F)$ as $\mathcal{B}_i(\mathcal{C},F)$ so that we get exact sequences

$$0 \longrightarrow \mathcal{B}_i(\mathcal{C},\mathcal{F}) \longrightarrow \mathcal{B}_i(\mathcal{C}) \longrightarrow \mathcal{B}_i(\mathcal{D})$$

for $i = 1, 2$. $\mathcal{B}_i(\mathcal{C},\mathcal{F})$ containes those elements $[A]$ of $\mathcal{B}_i(\mathcal{C})$ with $[\mathcal{F}A] = [[P,P]]$ for some $P \in \mathcal{D}$ which is faithfully projective resp. a progenerator. These i-Azumaya monoids A are called $\mathcal{F}$-split. From Theorem 20 follows immediatly a homomorphism

$$\xi : \mathcal{B}_2(\mathcal{C},\mathcal{F}) \longrightarrow \mathcal{B}_1(\mathcal{C},\mathcal{F}) .$$

If $\mathcal{C}$ is a symmetric monoidal closed category with difference kernels and difference kokernels and $K \in \mathcal{C}$ is commutative monoid, then ${}_K\mathcal{C}$ is again a symmetric monoidal closed category with $\otimes_K$ as tensor product and K as basic object. Then the functor $\mathcal{C} \ni X \longmapsto K \otimes X \in {}_K\mathcal{C}$ has all properties required in Theorem 20 hence there are homomorphisms $\mathcal{B}_i(\mathcal{C}) \longrightarrow \mathcal{B}_i({}_K\mathcal{C})$ with kernels $\mathcal{B}_i(K/\mathcal{C})$.

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