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All-pay competition with captive consumers*

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Abstract

We study a game in which two firms compete in quality to serve a market consisting of consumers with different initial consideration sets. If both firms invest below a certain threshold, they only compete for those consumers already aware of their existence. Above this threshold, a firm is visible to all and the highest investment attracts all consumers. On the one hand, the existence of initially captive consumers introduces an anti-competitive element: holding fixed the behavior of its rival, a firm with a larger captive segment enjoys a higher payoff from not investing at all. On the other hand, the fact that a firm's initially captive consumers can still be attracted by very high quality introduces a pro-competitive element: a high investment becomes more profitable for the underdog when the captive segment of the dominant firm increases. The share of initially captive consumers therefore has a non-monotonic effect on the investment levels of both firms and on consumer surplus. We relate our findings to competition cases in digital markets.

Keywords: consideration set, regulation, all-pay auction, endogenous prize, digital markets

JEL-Code: D4, L1, L4

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1 Introduction

Consumers typically differ in the set of firms that they consider before making a purchasing decision and firms are not able to easily change a consumer's consideration set. When a product provides particularly high utility, however, most consumers become aware of its existence, be it through word-of-mouth recommendation, social networks or news reports. The firm offering this product is then considered by all consumers. In this paper, we study how the co-existence of initially captive market segments and of the possibility of reaching all consumers by providing exceptionally high utility affects competition.

We model this problem as a duopoly game in which firms simultaneously invest in their product to increase consumer utility. A share of the consumers considers both firms, while the other consumers are "captive" in the sense that they initially consider only a single firm. Captive consumers may, however, cease to be so if a product outside their initial consideration set provides a level of utility exceeding a certain threshold. A standard argument is that the existence of captive market segments is anti-competitive. This is partially true in our model: as there exists a contested consumer segment that is not captive to either firm, firms may indeed decide to compete only for those. Which thereby induces a positive reservation value from not investing at all, and decreases the incentives for a dominant firm to compete for the rest of the market. The presence of captive consumers however also introduces a pro-competitive element: The only way to become visible to the captive base of the competitor is to invest much more in one's product in order to provide *very* high utility.

We show that a higher share of the consumers initially considering one firm only may then actually induce firms to invest more and lead to higher consumer surplus. This is more likely to be the case if the share of captive consumers of the dominant firm is already high. In contrast, if a firm with a given captive segment manages to increase the threshold at which captive consumers become aware of the competing firm's offer, consumer welfare decreases unambiguously. These results relate to prominent competition cases in digital markets.

In 2013, Microsoft was fined by the European Commission for failing to make Windows users aware of competing web browsers despite being committed to do so since a 2009 settlement.[1] We look at the case through the lens of our model: Microsoft uses the dominant position of its operating system Windows to increase its share of captive consumers in the market for internet browsers by locking Windows users into the use of Microsoft's own browser Internet Explorer. Our theoretical analysis predicts that, in the presence of a competitor with a sufficiently high investment capacity—the ac-

[1] "Commission fines Microsoft for non-compliance with browser choice commitments," European Commission Press Release, 6 March 2013, IP/13/196.

tual competitor Google supposedly had this capacity—such an increase in the captive segment of the market leader may have actually led to higher incentives for the runner-up, here Google, to improve its own competing product. The intuition is that Google would know that being marginally better than Internet Explorer was not sufficient to attract Windows users. It also needed to be sufficiently good for Microsoft consumers to become aware of its existence. This prediction of our model is consistent with the fact that Google's browser Chrome actually overtook Microsoft Internet Explorer as the market leader before the 2013 ruling, despite Windows still enjoying more than 80% market share at the time, and before mobile phones became a major source of Internet browsing.[2]

In June 2017, Google—having further expanded its usership and the variety of its services—was fined for using its dominance as a search engine to demote comparison shopping services in search results that were competing with its own service.[3] At the source of the complaint was a website called Foundem arguing that Google showed its users the comparison website Froogle (now Google shopping) only, hiding the competitors. In our model, the behavior of Google corresponds to a firm that purposefully makes it more difficult for already captive consumers to learn about a possible alternative product. Moreover, in this case the competitor Foundem did not have sufficient investment capacity to respond to Google's practice by improving its own product to such an extent that it would have become visible to captive consumers. Both because Foundem was small and lacked resources and because the attention threshold created by Google was very high. These two elements suggest that the behavior of Google was unambiguously anti-competitive.

In 2018, Google was fined again for restricting Android device manufacturers in what they showed to consumers, in particular forcing them to pre-install Google search and Google's Chrome browser.[4] In our model, this is a much more ambiguous case than the previous ones: what Google does is to increase its captive share, a practice that may actually increase competition, in particular if there exists a competitor with sufficiently high investment capacity. The difference with the Windows example from above is that Google does not have such a high dominance with Android as Microsoft did in the market for Operating Systems. According to our model, it is thus not clear

[2]According to data by traffic analysis website StatCounter, Chrome became market leader in January 2012, when mobile browsing still represented only 8.49% of the traffic. In January 2012, Microsoft Windows had a 82.06% share of the market for Operating Systems (statcounter.com).

[3]"Commission fines Google €2.42 billion for abusing dominance as search engine by giving illegal advantage to own comparison shopping service" European Commission Press release, IP/17/1784, 27 June, 2017.

[4]"Commission fines Google €4.34 billion for illegal practices regarding Android mobile devices to strengthen dominance of Google's search engine" European Commission Press release, IP/18/4581, 18 July, 2018.

whether an increase in the loyal base of Android users would actually increase the investment incentives of competitors.

The question of the economic impact of consumers observing different sets of firms has received much attention in economics. Early models of price dispersion (Varian, 1980; Burdett and Judd, 1983) study consumers that are heterogeneous in the number of firms they observe. In Bulow and Klemperer (1998) (Appendix C), two firms compete for consumers, that can either be fully captive or uninformed. A general characterization of price competition in oligopoly settings with different exogenous consideration sets is provided by Armstrong and Vickers (2018).

The novelty of our approach is that we allow a firm to directly enter the consideration set of all consumers by providing a sufficiently high quality product. Among the more recent models using such dichotomy between informed and uninformed buyers, the one by De Cornière and Taylor (2019) is of particular interest to us, as the authors characterize the competitive impact of a dominant search engine guiding consumers towards a specific product. Our finding that captive consumers can increase competition resonates with their finding that biased intermediaries can improve consumer welfare. In their model, consumers reach one of two competing firms via intermediaries who are said to be biased if they direct consumers only to one firm thereby hiding the competitor. The reason behind their result that an intermediary bias on part of one firm may be beneficial to consumers is the possibility of congruence between firm and consumer payoffs (as opposed to conflicting payoffs). In case of congruence, the per-user profit of a firm increases in the utility it offers to consumers thereby giving the firm an incentive to improve consumer utility and thus welfare. Through the lens of our model, a biased intermediary leads to a share of consumers being captive to one firm with the competitor being unable to reach these consumers. In our paper, payoffs are neither congruent nor conflicting but we allow the competitor to overcome the bias of consumers with a sufficiently high investment and show that this induces a pro-competitive effect that can also increase consumer welfare. Our model thus provides an additional and complementary explanation for why a biased intermediary may benefit consumers.

By construction, our setting is similar to an all-pay auction (Baye *et al.*, 1996) but in contrast to the standard model, the levels of investment endogenously determine the number of and the size of the prizes for which the firms compete when choosing their investments.[5] This endogeneity of prizes is driven by the assumption that captive

[5] Siegel (2009) studies asymmetric players competing for a fixed number of prizes in a general setup allowing for various asymmetries between players. Another approach to multiple prize all-pay auctions is the Colonel Blotto game (Roberson, 2006), where consumers bid separately for different prizes. We do not consider which design of the prize structure would be optimal, e.g., in the sense of maximizing total expected investments. Moldovanu and Sela (2001) study the optimal allocation of a given number of prizes in a setting of imperfect information where the highest bidder gets the first prize, and the second bidder the second prize, and the goal is to maximize expected efforts of the bidders.

consumers remain captive only for low investments. If at least one of the two firms' investments exceeds a certain threshold, there is only one prize to be won, but there are two prizes, one for each firm, if both firms choose investments below this threshold. The unique equilibrium of our game is in mixed strategies. The intuition is similar to that of the standard all-pay auction: for every given level of investment of the winning firm, the competitor could win instead by choosing a marginally higher level. A crucial difference is that investments below the threshold can only win a share of the "prize", not-including the captive segment of the competitor. Depending on the size of the threshold, the equilibrium of the game is one of the following three.

In the limiting case where the utility threshold beyond which previously captive consumers consider a competitor is exactly zero, our model collapses to a symmetric all-pay auction: both firms always compete at high intensity, randomize over the same interval of investments in equilibrium, and make zero profit in expectation. On the other extreme, when the utility threshold is very high, the equilibrium is again symmetric but then no firm ever tries to reach the captive segment of the other: both firms randomize over the same interval of investments in equilibrium and make an expected profit equal to the value of their captive segment.

In the intermediate case, where the utility threshold is not too high but strictly positive, equilibrium bidding strategies are asymmetric. Both firms' equilibrium strategies contain investments below and above the threshold and, therefore, both firms expect to sometimes keep their captive consumers even when having invested less than the competitor or, at the extreme, when having not invested at all. Hence, each firm makes a strictly positive profit in expectation. The higher is the threshold, the higher is the probability mass that the firms put on investments below the threshold. In consequence, the probability of one firm obtaining a monopoly position decreases.

In this intermediate case, the support of the equilibrium mixed strategy exhibits a gap just below the minimum investment necessary to attract the competitor's captive consumers. Choosing an investment at or just above this threshold does not only increase the probability of winning, but also the prize of winning, which is then the whole population, including all captive consumers, instead of just a share of it.[6]

The starting point of our model is the well-known fact that consumers sometimes are biased in favor of certain options. Consumers may experience switching costs (Klemperer, 1987), or they may inspect competing firms in a certain order while bearing a search cost to observe an additional option (Arbatskaya, 2007, Armstrong *et al.*, 2009). Such biases lead to qualitatively similar results. For instance, with consumer switching costs, the firm with a larger base of captive consumers charges a higher (ripoff) price because it relies on the profits to be made on its captive segment. In our

[6] A similar feature of equilibrium strategies has been observed in models of price competition with consumer inventories (Hong *et al.*, 2002; Gangwar *et al.*, 2014; Ding and Zhang, 2018).

model, the firm with the larger segment of captive (attached) consumers invests less in its product in expectation and, therefore, offers a lower utility than the competitor on average. In both cases, the larger segment of captive consumers makes the firm lazy so that it provides less utility to consumers in expectation than the initially disadvantaged competitor with a smaller captive segment.

The observation that the firm with a larger share of captive consumers employs a less aggressive strategy resonates well also with earlier results from the literature on the effects of brand loyalty. For instance, Narasimhan (1984) finds that a firm with a larger loyal customer base is less aggressive in using discounts to attract further consumers. Similarly, Raju *et al.* (1990) find that a stronger brand uses price promotions less frequently than a weaker one. As in our model, the rationale is that a weaker competitor has more to gain from being aggressive.

We introduce the model and derive important properties of the equilibrium strategies in the simultaneous investment game in Section 2. We then derive the equilibrium in Section 3. We discuss the impact of the different parameters on equilibrium behavior in Section 4. We conclude in Section 5. For those results that do not follow directly from the text, formal proofs are collected in Appendix A.

2 The Model

We aim at modeling competition in markets where consumer utility and per-user revenue for firms are a combination of many factors. On the utility side, consumers may value high quality, low levels of advertising, low prices, or high levels of privacy. On the per-user revenue side, firms benefit from high prices as well as from high levels and targeting of advertisement. Moreover, providing quality is costly to the firms.

In building our model, we follow Armstrong and Vickers (2001) and De Cornière and Taylor (2019) in studying sellers competing in *utilities*. There are two firms, 1 and 2. Firm $i \in \{1, 2\}$ incurs a cost of $c(k_i)$ for each unit it invests into a combination of quality, price, and other factors corresponding to the utility its product provides to consumers, with c continuous, $c' \geq 0$ and $c'' \geq 0$.[7] In order to obtain closed form solutions, we use the linear function $c(k_i) = ck_i$ when solving for bidding strategies in Section 3. A firm investing k_i at cost $c(k_i)$ offers a utility $u = k_i \geq 0$ to all consumers choosing it. We therefore define k_i as both the firm's investment level and the utility it offers to consumers.

An important assumption is then to define whether quality investment and per-user revenue are conflicting or congruent (De Cornière and Taylor, 2019): do firms

[7] The model can be extended to allow for heterogeneous costs without providing much additional insight beyond what is known about cost asymmetries from the literature on all-pay contests (e.g. Siegel, 2009).

generate a higher or a lower per-user revenue when offering more utility? The former corresponds to interpreting a higher consumer utility as mostly the result of a lower price. The latter corresponds to interpreting a higher consumer utility as allowing for a higher price or a better targeting of user data to generate revenue. In this paper, we take the neutral view that per-user revenue is constant in per-user utility, and we normalize this revenue to 1. Per-user revenue is constant under the following two assumptions. First, we assume that a firm incurs a fixed cost to be able to offer a certain level of utility to its users and this cost only depends on the utility level but it is independent of the number of users. Second, we assume that the utility experienced by the users does not influence the per-user revenue.

In practice, this could correspond to a situation where firms compete to attract users to their respective online platforms by offering a high quality service, and each firm makes an exogenous per-user revenue. This is thus an assumption that fits particularly well those markets where the nominal price for buyers is often (close to) zero, and all revenue comes from a competitive advertising market. On the product market, our model represents an investment in R&D to provide a higher quality product, sold with a constant markup over the marginal cost of production. The cost of developing a certain product is thereby determined by its quality alone and does not depend on the number of users. For instance, the cost of setting up an online service depends largely on the cost of software development and acquisition of server capacity that is—in the short run—independent of the number of actual users.[8]

There is a mass one of consumers willing to buy exactly one unit from the firm within their consideration set that offers the highest utility. Before the game starts, some consumers have a singleton consideration set and are thus captive to one firm, while others have a consideration set that comprises both firms. By investing above a certain utility threshold, a firm can become prominent and enter the consideration set of all consumers.

Building on the specification of Armstrong and Vickers (2018) we assume that a fraction $\alpha_i \in (0,1)$ of consumers initially only has firm $i \in \{1,2\}$ in their consideration set. The remaining consumers $\alpha_{12} \in (0,1)$, which we also refer to as the contested segment, have both firms in their consideration set.

The novelty of our model is that consideration sets may change depending on firm behavior. A firm is considered by all consumers if it chooses to provide a level of utility above a certain threshold $\bar{k}$. If firm i chooses a utility $k_i < \bar{k}$, it is not considered by captive consumers of firm $j \neq i$. But if firm i chooses $k_i \geq \bar{k}$, firm i is considered not only by its own captive consumers and the contested segment but also by the captive

[8]The utility that users realize may depend on the number of other users positively due to beneficial network externalities or negatively in case of congestion. We abstract from these additional complications here to isolate the effect of endogenous consideration sets on investments.

Table 1: Firm payoffs for different investment choices.

	level of investment	
	$k_i, k_j < \bar{k}$	$k_i \geq \bar{k}$ or $k_j \geq \bar{k}$
$k_i > k_j$	$(l_i(k_i), o_j(k_j))$	$(w_i(k_i), 0)$
$k_j > k_i$	$(o_i(k_i), l_j(k_j))$	$(0, w_j(k_j))$

consumers of firm j.[9] We are interested in the set of Nash equilibria of this game. The structure of the game and frequencies of types are common knowledge.

Firm 1 and firm 2 simultaneously choose their investment levels. Firm i receives a payoff of

$$o_i(k_i) = \alpha_i \text{Prob}(k_j < \bar{k}) - c(k_i)$$

if it invests below the threshold ($k_i < \bar{k}$) and less than its competitor ($k_i < k_j$). We refer to the particular case where a firm does not invest $o_i(0)$ as firm i's *outside option*. The outside option of firm i depends on the probability with which firm j chooses an investment below $\bar{k}$, in which case captive consumers do not consider firm j's investment. Denote by

$$l_i(k_i) = 1 - \alpha_j - c(k_i)$$

the payoff of firm i if it invests below the threshold ($k_i < \bar{k}$) and more than its competitor ($k_i > k_j$). If firm i invests above the threshold $k_i \geq \bar{k}$, captive consumers consider both firms and the outside option of firm j drops to zero. In this case, firm i competes for all segments of the market and earns at best a payoff

$$w_i(k_i) = 1 - c(k_i).$$

We assume that each relevant consumer segment is shared equally by both firms in case of a tie. If at least one of the two firms invests $\bar{k}$ or more, the market becomes a winner-take-all market: the firm providing the highest utility then enters the consideration set of all consumers, who, in this case, have identical preferences. Thus, all consumers will buy from the firm that offers the highest utility. The market outcome will be a monopoly.

We use this notation to summarize the corresponding payoffs for all combinations of investments in table 1.

[9] In reality, it is unlikely that a utility threshold such as $\bar{k}$ determines with certainty which firm enters each consideration set. This simple modification of an otherwise standard setting allows us to represent the idea that investment can be either incremental or radical, and that only a radical investment can make a firm visible to a consumer who is not yet aware of this firm's product.

3 Equilibrium

3.1 Equilibrium properties

We now derive some general properties of the equilibrium investments. The game faced by the two firms resembles an all-pay auction where the bids are given by the investment levels and prizes are given by the market shares the two firms realize. If the investment of a firm exceeds the threshold $\bar{k}$, the market share of the winning firm increases discontinuously as compared to a winning bid just below the threshold because at this point the investment is just high enough to attract the competitor's captive segment in addition to the contested segment. We first state the results formally.

Lemma 1. *In equilibrium, the following holds:*

(i) The game does not have an equilibrium in pure strategies.

(ii) In equilibrium, at most one firm's investment strategy has a mass point at any given investment level.

(iii) Both firms' investment strategies are either continuous on the same connected support with upper bound $\bar{K} < \bar{k}$, or continuous on the same two disconnected supports, $(0, \delta)$ and $(\bar{k}, \bar{K})$, with $\delta < \bar{k}$ and $\bar{K} \leq k^{max}$.

(iv) If a firm bids a level of investment k with strictly positive probability, then $k \in \{0, \bar{k}\}$.

To understand the intutions behind these properties, let us first define the maximum possible level of investment at equilibrium as solving $c(k^{max}) = 1$: Obviously, it is never a best response for either firm to provide a utility greater than k^{max} because the cost would then exceed the highest possible revenue. As can be expected from the literature, the game does not have a pure-strategy equilibrium on the remaining range of possible investments (part (i) of Lemma 1). The intuition is that it is profitable to marginally outbid any deterministic investment of the competitor because this only marginally increases costs but ensures winning the entire market.

We then show that, even if the mixed strategies may contain mass points in equilibrium, these cannot be at the same investment levels for both firms (part (ii)). The intuition is that mass points in one firm's investment strategy imply that the expected profit of the other firm from certain investment levels changes discretely at the respective investment level and overbidding is profitable.

Further, we show that apart from any mass points, the densities of both firms equilibrium investment strategies coincide (part (iii)). This result relates to the well-known observation that the support of the equilibrium mixed strategy in an all-pay auction is connected and the density of equilibrium investments over this interval is constant and identical across contestants. In fact, if the attention threshold is so high that it never pays off to compete for the other firm's captive segment, the situation is akin to a symmetric all-pay auction where the two firms compete for the contested segment only.

In contrast to the standard result, the support of the equilibrium investment strategy may consist of two disconnected intervals if investments at or above $\bar{k}$ are within reach, meaning that these can be profitable as $\bar{k} < k^{\max}$. Then, firms bid above zero for two reasons: either to capture the contested segment only (below the threshold) or to capture the entire market (above the threshold). As winning above the threshold yields higher payoff than below, there is a gap in the investment strategies: no one wants to bid just below the threshold. This result is driven by the discontinuity in the prize of winning at the threshold $\bar{k}$. For investments below $\bar{k}$, a firm can be better off by choosing exactly $\bar{k}$ and capturing the entire market than by outbidding the competitor at the margin and winning only the contested segment. This tradeoff between marginally overbidding and bidding exactly $\bar{k}$ induces an endogenous gap between the highest "low intensity" investment δ and the lowest "high intensity" investment $\bar{k}$.

Finally, we show that the equilibrium investment strategies of both firms admit mass points only at zero and at the threshold beyond which the consideration of captive consumers is reached (part (iv) of Lemma 1). The intuition behind this property is the following. Each firm generally prefers bidding marginally above k' to bidding marginally below k' if the competitor invests k' with strictly positive probability (part (i) and (ii)). Therefore, any investment k' that is chosen with a strictly positive probability in equilibrium must be at the lower bound of an interval of investments in the support of the mixed strategy. By part (iii), this implies that only investments of 0 or $\bar{k}$ can be chosen with strictly positive probability.

A corollary of the above results it that, as long as the other firm invests below $\bar{k}$ with strictly positive probability, a firm has a positive reservation payoff from not providing any investment at all. The reservation payoff is equal to the revenue from the share of captive consumers multiplied by the probability that the competitor chooses an investment below $\bar{k}$: $o_1(0) = \text{Prob}(k_2 < \bar{k})\alpha_1$ for firm 1 and $o_2(0) = \text{Prob}(k_1 < \bar{k})\alpha_2$ for firm 2. This implies that firms do not choose investments up to the level at which they just break even. Instead, at the maximum investment, the expected profit conditional on this investment is equal to the reservation payoff in form of the expected profit from not investing at all, $o_1(0)$ or $o_2(0)$ as defined above.

3.2 The equilibrium

We now characterize the equilibrium of the game for different levels of the threshold $\bar{k}$. Let us assume without loss of generality that firm 1 enjoys a larger captive segment than firm 2, $\alpha_1 > \alpha_2$.[10] We are looking for closed-form equilibrium solutions and focus on the linear cost function $c(k_i) = ck_i$.

We consider three distinct cases. First, the threshold $\bar{k}$ that conditions the effectiveness of a firm's investment with respect to captive consumers may be very low so that both firms' investment capacity is high enough to compete for the entire population.[11] In this case, competition is intense with both firms choosing not only low but also high investments with positive probability. The market outcome is a monopoly with high probability (Proposition 1). Second, for intermediate levels of $\bar{k}$, firm 1, enjoying a larger captive segment, only engages in competition for the contested segment but never chooses high investments that would attract captive consumers of firm 2. Firm 2, however, gambles for a monopoly position by choosing investments of $\bar{k}$ with strictly positive probability to attract captive consumers of firm 1 (Proposition 2). Firm 2 has a higher investment capacity than firm 1 because its outside option is worse. Third, if $\bar{k}$ is very high, it is prohibitively costly for either firm to attract the competitor's captive consumers; neither firm has the capacity to invest beyond $\bar{k}$. Thus, both firms compete for the contested segment only (Proposition 3). We now turn to the detailed analysis of these cases one by one.

Consider first the case where the threshold is low, $\bar{k} < \frac{1}{c}\frac{\alpha_{12}+\alpha_1^2}{1-\alpha_2}$. Then, it is relatively easy to enter the consideration set of consumers in the competitor's captive segment. Moreover, both firms are in principle willing to choose investments high enough to do so. We show that in equilibrium both firms randomize over two disconnected intervals, one below and one above the threshold $\bar{k}$. In this equilibrium, firm 1 chooses to invest nothing with strictly positive probability because its larger share of captive consumers makes it compete less aggressively.

[10] For $\alpha_1 = \alpha_2 = \alpha$, the game has a unique symmetric equilibrium in which each firm is selected by the majority of consumers with a probability of one half. Competition is softer, expected investments lower, and profits higher if α is higher. See details in Appendix B.

[11] We loosely refer to a firm's investment capacity as the maximally profitable investment a firm might choose, which depends on the cost c but also on the size of the firm's captive segment and the equilibrium bidding behavior of the competitor.

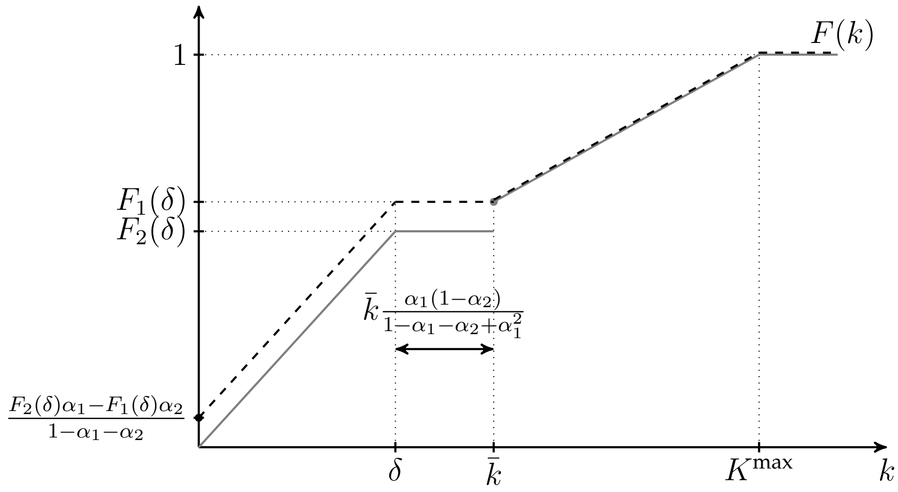


Figure 1: Proposition 1: Cumulative distribution functions if $\bar{k} < \frac{1}{c}\frac{1-\alpha_1-\alpha_2+\alpha_1^2}{1-\alpha_2}$. $\bar{K} = \frac{1}{c} - \bar{k}\frac{\alpha_1(1-\alpha_1)}{\alpha_{12}+\alpha_1^2}$. Dashed: firm 1, Gray solid: firm 2.

Proposition 1. *If $\bar{k} < \frac{1}{c}\frac{\alpha_{12}+\alpha_1^2}{\alpha_{12}+\alpha_1} = \bar{k}_l$ there exists a unique equilibrium. The cumulative distribution functions are given by*

$$F_1(k) = \begin{cases} \frac{c}{\alpha_{12}}k + \frac{c(\alpha_1-\alpha_2)\bar{k}}{(\alpha_{12}+\alpha_1^2)} & \text{if } k \in [0,\delta] \\ \frac{c(1-\alpha_2)\bar{k}}{\alpha_{12}+\alpha_1^2} & \text{if } \delta < k \leq \bar{k} \\ ck + \frac{c(1-\alpha_1)\alpha_1\bar{k}}{\alpha_{12}+\alpha_1^2} & \text{if } \bar{k} < k \leq \overline{K} \\ 1 & \text{if } k > \overline{K} \end{cases}$$

$$F_2(k) = \begin{cases} k\frac{c}{\alpha_{12}} & \text{if } k \in (0,\delta] \\ \frac{c(1-\alpha_1)\bar{k}}{\alpha_{12}+\alpha_1^2} & \text{if } \delta < k < \bar{k} \\ ck + \frac{c(1-\alpha_1)\alpha_1\bar{k}}{\alpha_{12}+\alpha_1^2} & \text{if } \bar{k} \leq k \leq \overline{K} \\ 1 & \text{if } k > \overline{K}, \end{cases}$$

with $\overline{K} = \frac{1}{c} - \bar{k}\frac{\alpha_1(1-\alpha_1)}{\alpha_{12}+\alpha_1^2} > \bar{k}$ and $\delta = \bar{k}\frac{(1-\alpha_1)\alpha_{12}}{\alpha_{12}+\alpha_1^2} < \bar{k}$.
In this equilibrium, firm 2 invests more in expectation and becomes a market leader with higher probability than firm 1. Both firms make an expected profit of $\Pi = F_2(\delta)\alpha_1 > 0$.

The cumulative distribution functions that characterize this equilibrium are such that, (i) at any interior point of both intervals, the two firms' investment strategies have the same density and (ii) both firms' investment strategies exhibit higher density on a given investment in the lower interval than in the upper interval. Moreover, firm 2 invests exactly $\bar{k}$ with strictly positive probability, while firm 1 invests exactly 0 with strictly positive probability. We represent the equilibrium strategies in Figure 1. Consumer surplus can be easily derived from these expressions and is formally stated in the proof of Propositions 1.

For low values of the threshold $\bar{k}$, firms have an incentive to sometimes invest aggressively to enter the consideration set of all consumers.[12] They can then attract the contested segment as well as the competitor's captive segment in case of winning. Competition is however never perfect and both firms always make a strictly positive profit in equilibrium. This results from the fact that they can rely on the competitor being complacent with positive probability. In such a case, investments fall short of the attention threshold so that captive consumers remain unaware of the competitor's offer. Thus, they buy from the unique firm in their consideration set, even if this firm did not invest and, therefore, provides lower utility than the competitor. The equilibrium with ranges of high and low investments crucially depends on the existence of a contested segment of consumers. If all consumers were captive to either of the two firms, firms would invest either at or above $\bar{k}$ or nothing at all as the endogenous threshold value δ from Proposition 1 collapses to zero.

Captive consumers have lower expected utility than the non-captive ones, and captive consumers of type 1 have a lower expected utility than captive consumers of type 2. The reason is that captive consumers only benefit from the investments of the other firm to the extent that those are above the threshold $\bar{k}$ whereas consumers from the contested segment always profit from higher investments.

A consequence of these equilibrium investment strategies is that firm 2, having the smallest segment of captive consumers, invests more aggressively and becomes a market leader more often in expectation. This is not simply a curiosity deriving from the mixed strategy equilibrium but results from the larger captive segment making firm 1 more complacent. We find qualitatively similar results in an environment where investments have probabilistic returns in terms of market size and the equilibrium is in pure strategies (see Appendix C for details).

For the parameter values corresponding to Proposition 1, both firms make the same expected profit, even if firm 1 appears to be the favored one due to its larger captive segment. To understand the logic, it is helpful to consider the problem of firm 1: to maximize its profit, it must be the case that no "obvious" overbidding strategy is available to firm 2. Hence, firm 1 wants to make firm 2 indifferent between all options in the support of the mixed strategy investment. In order to do so, firm 2 must believe that there is a sufficiently high probability P' that firm 1 invest some amount below $\bar{k}$. Similarly, firm 2 wants firm 1 to be indifferent between all options in the support of the mixed strategy investment. For firm 1 to be indifferent between investments above and below $\bar{k}$, it must believe that firm 2 invests some amount below $\bar{k}$ with a sufficiently high probability P''. Further, both firms must make the same

[12] If $\bar{k} = 0$, the model collapses to a classic all-pay auction as is easily seen from Proposition 1. If there is no attention threshold, captive consumers can never be kept and outside options collapse to zero, eliminating any asymmetry between the two firms.

profit in expectation because they are symmetric for investments above $\bar{k}$. However, as $\alpha_1 > \alpha_2$ and expected profits are determined by the outside options of both firms, $E[\Pi_1] = o_1(0) = \alpha_1 P'' = \alpha_2 P' = o_2(0) = E[\Pi_2]$, it must hold that $P'' < P'$. Thus, the mixed strategy of firm 1 must be less aggressive than that of firm 2 in order to make firm 2 indifferent between low and high investments. Firm 1 is thus content with less aggressive behavior because of firm 2's small captive segment.

At equilibrium, by definition, both firm 1 and firm 2 are indifferent between all investment levels in the support of the mixed strategy. Moreover, even if one of the two firms could commit ex-ante to a mixed strategy (using a randomization device), the one that would maximize each firm's expected surplus is the equilibrium one.

This result implies that the larger (dominant) firm is less likely to offer sufficiently high utility to enter all consumers' consideration sets. Instead, it counts on its large captive segment remaining unaware of the competitor and abstains from competition for the competitor's small captive segment.

Second, consider the case where the threshold, beyond which captive consumers change their consideration set, is sufficiently high for firm 1 not to find it worthwhile to attract the consideration of firm 2's captive share but firm 2 may still want to attract firm 1's captive segment. This asymmetry arises because firm 1 is more content with its larger captive segment, and firm 2 is more eager to escape its initially inferior market position.

Proposition 2. *If $\bar{k}_l < \bar{k} < \frac{\alpha_{12}+\alpha_1}{c} = \bar{k}_h$, there exists a unique equilibrium. The cumulative distribution functions are given by*

$$F_1(k) = \begin{cases} \frac{c}{\alpha_{12}}k + \frac{1-\alpha_2-c\bar{k}}{\alpha_{12}} & \text{if } k \in [0,\delta] \\ 1 & \text{if } k \geq \delta \end{cases}$$

$$F_2(k) = \begin{cases} \frac{c}{\alpha_{12}}k & \text{if } k \in (0,\delta] \\ \frac{c}{\alpha_{12}}\delta & \text{if } \delta \leq k \leq \bar{k} \\ 1 & \text{if } k \geq \bar{k}, \end{cases}$$

with $\delta = \bar{k} - \frac{\alpha_1}{c}$.
In this equilibrium, firm 2 invests more in expectation than firm 1 and becomes a market leader more often. The expected profit of firm 2 is $1-c\bar{k}$ and the expected profit of firm 1 is $\alpha_1 > 1-c\bar{k}$.

The distribution functions are such that, at any interior point of the interval, both firms invest with the same density. Moreover, firm 2 invests exactly $\bar{k}$ with strictly positive probability, while firm 1 invests exactly 0 with strictly positive probability. Consumer surplus can be easily derived from these expressions and is formally stated in the proof of Propositions 2.

The result from Proposition 2 is similar to that from Proposition 1 but here it is too costly for firm 1 to attract the captive segment of firm 2. Thus, firm 1 does not choose investments equal to or above $\bar{k}$ at all. As in the preceding arguments, for investments below $\bar{k}$, it is still the case that both density functions satisfy

$$(1) \qquad f_1(k) = f_2(k) = \frac{c}{\alpha_{12}}.$$

At or above $\bar{k}$, only firm 2 invests. As it does not face competition at or above $\bar{k}$, firm 2 chooses an investment exactly equal to $\bar{k}$ with strictly positive probability and does never choose any strictly higher investment to save on investment costs. This level is sufficient not only to outbid firm 1 but to also attract firm 1's captive segment with certainty. Firm 1 in contrast decides not to invest at all with a strictly positive probability and otherwise randomizes over relatively low investment levels.

Proposition 2 shows that, in this equilibrium, expected profits of both firms differ. Firm 1 benefits from its larger base, invests less and makes a higher expected profit than firm 2. The asymmetry is twofold: firm 2 invests more aggressively and wins the market more often, but firm 1 actually makes the highest profit in expectation because it has a larger base of captive consumers and lower investments.

We now turn to the third case, where $\bar{k}$ is so high that neither firm finds it profitable to compete for the consideration of its competitor's captive segment. Thus, both firms refrain from full competition. The winner of this softened competition enjoys a dominant market position but never achieves a monopoly. Even though the two firms have differently sized captive segments, they behave identically and end up dominating the market with equal probability.

Proposition 3. *If $\bar{k} > \bar{k}_h$, there exists a unique equilibrium. The cumulative distribution functions are given by:*

$$F_i(k) = \begin{cases} \frac{c}{\alpha_{12}}k & \text{for all } k \in [0, \frac{\alpha_{12}}{c}] \\ 1 & \text{for } k \geq \frac{\alpha_{12}}{c} \end{cases} \quad \text{for } i = 1, 2$$

In this equilibrium, both firms invest the same amount in expectation and become a market leader with equal probability. The expected profit of firm 2 is α_2 and the expected profit of firm 1 is $\alpha_1 > \alpha_2$.

The distribution functions are such that, at any point, both firms invest with the same density. No firm invests at any point with strictly positive probability. No firm ever chooses an investment that would attract the consideration of the competitor's captive segment. Therefore, expected payoffs equal the firms' outside options and market leadership is reached with equal probability. As the outside options are given by the captive segments, firm 1 makes a strictly larger profit in expectation than firm 2.

Consumer surplus can be easily derived from these expressions and is formally stated in the proof of Proposition 3.

While it is obvious that competition for the entire population is not profitable for $\bar{k} > k^{max} = \frac{1}{c}$, it is not clear this is also true here because $\bar{k}_h < k^{max}$. *A priori*, high investments can be profitable for $\bar{k}_h < \bar{k} < k^{max}$ if the success probability of reaching the entire population is high enough. However, in equilibrium, the expected profit from attracting previously captive consumers would not outweigh the investment cost, so that neither firm chooses investments equal to or above $\bar{k}$. As a consequence, $\lim_{\varepsilon \to 0} F(\bar{k} - \varepsilon) = 1$ and captive consumers do not consider the competitor's offer.

Finally, the equilibrium of the investment game is unique. Indeed, Propositions 1, 2, and 3 each characterize an equilibrium that is unique for the range of $\bar{k}$ to which the respective proposition applies, and the ranges of $\bar{k}$ given in Propositions 1 to 3 constitute a partition of the admissible range for $\bar{k}$. The equilibrium we have characterized is also "stable" in the sense that best responses to any small perturbation to the equilibrium probabilities would bring the game back to equilibrium.[13]

4 Implications

In this section, we discuss some comparative statics and their policy implications. We start by showing that the size of the dominant firm's captive segment α_1 has a non-monotonic effect on both firms' levels of investment, consumer surplus and firm profit. Then, we study the effect of changing the difficulty of reaching the competitor's captive consumers' consideration sets captured by our threshold $\bar{k}$.

Firm strategies that affect the size of the dominant firm's captive segment α_1 or the attention threshold $\bar{k}$ have different policy implications. The former relates to the ability of a large firm to further increase the number of consumers it captures. This corresponds for instance to the strategy of a large firms to acquire a competitor with the objective of redirecting the acquired consumers to its own services, a practice widely used by companies such as Microsoft, Facebook, or Google for instance. Our analysis shows that such acquisitions are not necessarily anti-competitive, if there exists at least one competitor with sufficiently high investment capacity, and if the share of captive consumers of the dominant firm is already large. The picture is different when it come to attempts to increase $\bar{k}$ in the sense that a firm increases its ability, once it has captured consumers, to keep them unaware of the competition. An example would be the case of

[13]Consider a level of investment $k' < \bar{k}$ that is chosen by both firms with density $f(k') = \frac{c}{\alpha_{12}}$ in equilibrium. Suppose firm 1 instead chose to put slightly more density at k', say $f(k') = \phi > \frac{c}{\alpha_{12}}$. Then, firm 2 would want to put more weight on the investment level marginally above k', as marginally outbidding an investment of k' would yield an expected benefit of $\phi\alpha_{12} > c$. This change in firm 2's investment strategy, however, would induce a strict decrease in the expected profit of firm 1. Thus, firm 1 is better off by sticking to the proposed equilibrium strategy.

a search engine hiding the result of services from its competitors far away in the search results to lure captive consumers into using its own service. Our analysis shows that a higher $\bar{k}$ always hurts consumers.

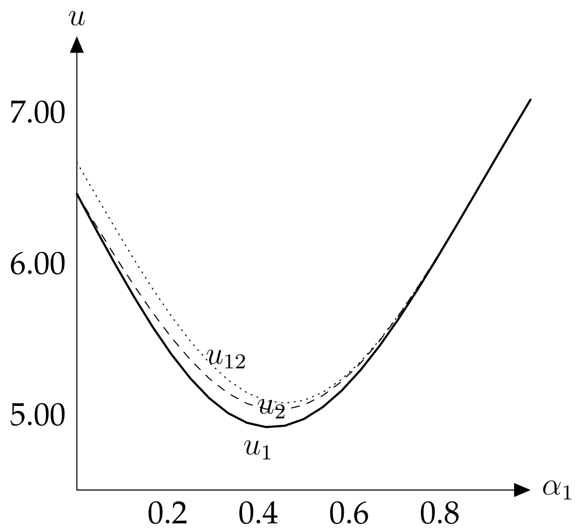
4.1 The share of captive consumers of the dominant firm has a non-monotonic effect on both firms' investments

In this section, we focus on the setting where the dominant firm (firm 1), perhaps using its market power and ability to redirect consumers among its different services, increases its share of captive consumers thereby shrinking the contested segment. To isolate the effect, we assume that the smaller firm (firm 2) has no captive consumers ($\alpha_2 = 0$) and study the impact of an increase in α_1. We further assume that the threshold $\bar{k}$ is sufficiently small for the equilibrium to be characterized by Proposition 1.

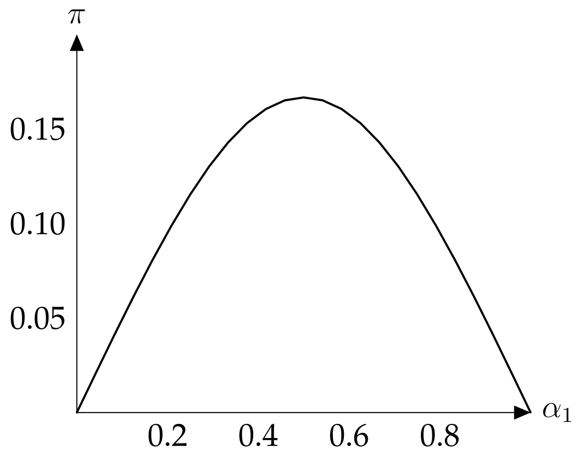
Proposition 4. *Suppose that $\alpha_2 = 0$ and $\bar{k} < \frac{1-\alpha_1(1-\alpha_1)}{c}$. Then, the share of captive consumers of firm 1 has a non-monotonic effect on both firms' investments. An increase in α_1 has a negative effect for α_1 small and a positive effect for α_1 large. Consumer surplus is thus decreasing in α_1 for the lowest values of α_1 and increasing for the highest ones.*

When the share of firm 1's captive consumers, who initially consider only the product of the dominant firm 1, increases, this has two effects. First, we observe an anti-competitive effect because the dominant firm 1 invests less as α_1 increases. The reason is that firm 1's outside option, that is its payoff when not investing at all, increases directly through the increase in α_1. This makes firm 1 more complacent. As investments are strategic complements, firm 2 also invests less. Second, there is a pro-competitive effect because an increasing share of captive consumers implies that an increasing share of the market can be reached only by investing above the threshold. Thus, these high investments become more profitable for the underdog, firm 2 as α_1 increases. Due to the complementarity of investments, not only firm 2 but also firm 1 invests more.

In Figure 2, we plot an illustrative example of the case where both firms have enough investment capacity to compete for the entire market – we choose parameter values ensuring that the equilibrium always corresponds to the one in Proposition 1. The first effect dominates when few consumers are captive to firm 1, whereas the second effect dominates when already most consumers are captive to firm 1. When the dominant firm becomes too dominant, it leaves no other option to the other firm but to compete very aggressively for the entire market leading to high expected consumer welfare and low expected profits. As investments are strategic complements this results in both firms investing more. When the dominant firm 1 is not too dominant, however, a higher share of captive consumers softens competition by segmenting the market. This decrease in competition lowers expected consumer welfare but increases expected profits.



(a) Expected utility in equilibrium.



(b) Expected profit.

Figure 2: Comparative statics with respect to the share of captive consumers of the dominant firm, within the equilibrium of Proposition 1. Illustration with $\alpha_2 = 0$, $c = 0.1$, $\bar{k} = 5$.

4.2 Comparing the different equilibria

We now relax the previously made assumptions that the threshold $\bar{k}$ is low enough for the equilibrium in proposition 1 to hold and that $\alpha_2 = 0$, and discuss how changes in $\bar{k}$ change the investment behavior of the two firms across the three possible equilibria. We illustrate the effect of $\bar{k}$ on expected investments and on consumer welfare (defined as the sum of individual consumer utilities) in Figure 3. The vertical dotted lines represent the values of $\bar{k}$ that delimit the zones corresponding to Propositions 1 to 3.

When the investment threshold $\bar{k}$ is small but strictly positive (part (i) of Figure 3, Proposition 1) both firms compete for the captive segment of their competitor with positive probability but not with certainty. Whenever both firms choose investments below $\bar{k}$, competition is softened because it is restricted to the contested segment. Both firms can then serve their captive segments even at an investment of zero. As a consequence, both firms include zero investment in their investment strategy and make strictly positive profits in expectation. Competition is dampened if $\bar{k}$ increases and firm 1 puts increasingly more mass on not investing at all. As a result, both $E(k_1)$ and $E(k_2)$ decrease with $\bar{k}$ but the decrease is much stronger for firm 1 so that the gap between the two increases with $\bar{k}$. Thus, while part (i) applies, both firms invest less as it gets harder to enter the consideration set of the competitor's captive consumers, leading to decreases in expected consumer utility.

In part (ii) of Figure 3 (Proposition 2), the impact of $\bar{k}$ on expected investments is ambiguous, but the impact on consumer welfare is clearly negative. For these intermediate values, firm 1, having the larger captive segment, invests increasingly aggressively as $\bar{k}$ increases. This is because in this range, and as opposed to Proposition 1, the probability mass allocated to the boundary points, $P(k_1 = 0) = P(k_2 = \bar{k})$, decreases with the threshold $\bar{k}$ because it becomes less attractive for the firm to compete for the other's captive segment. This implies that the investment strategies of firms become more and more symmetric as $\bar{k}$ increases because competition becomes more and more restricted to the contested segment, and, there, both firms act symmetrically. In contrast, the expected profits of firms 1 and 2 become more and more asymmetric as the profit effect of the asymmetric captive segments kicks in.

When $\bar{k}$ reaches the level at which firms decide to only compete for the contested segment (part (iii) of Figure 3, Proposition 3), the expected investments in both types of equilibrium are the same and the expected level of investment remains constant for further increases in $\bar{k}$. Firms compete only for the contested segment of consumers who anyway consider both firms. Therefore, investment behavior is independent of $\bar{k}$, both firms behave symmetrically but make asymmetric expected profits as given by the size of their captive segments.

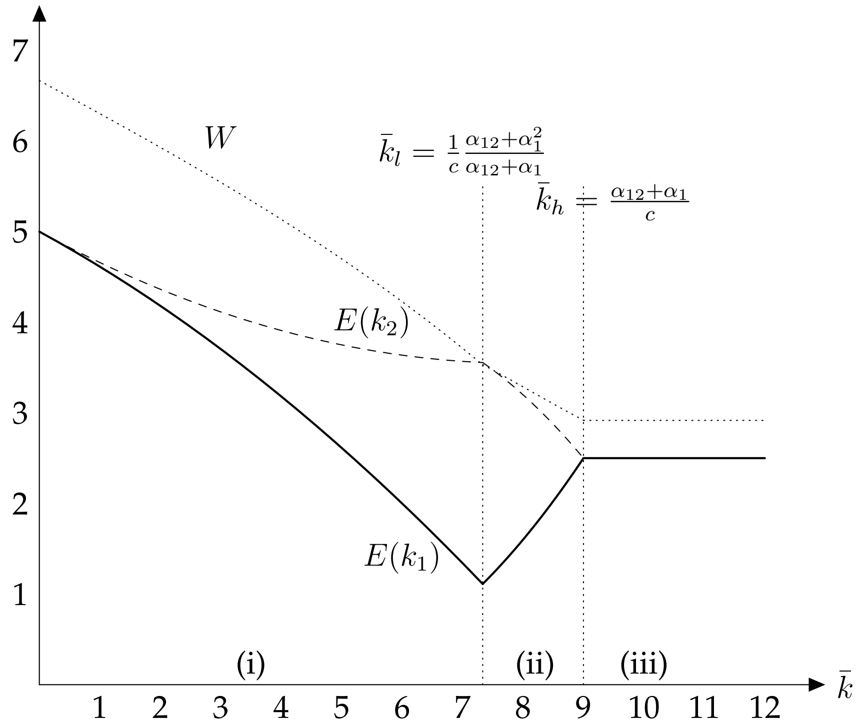


Figure 3: Expected equilibrium investment in the equilibria corresponding to Propositions 1 in part (i), 2 in (ii), and 3 in (iii). Illustration with $\alpha_1 = 0.4$, $\alpha_2 = 0.1$ and $c = 0.1$. W is the sum of individual utilities, $W = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_{12} u_{12}$.

5 Conclusion

When two players bid to win a discrete prize and both winning and losing bids are forfeited, it is well known that, in equilibrium, players randomize continuously over a connected interval of bids and make zero profit in expectation. In this paper, we analyze a game between two firms that resembles this classic all-pay auction but differs in the following way. We assume that each of the two competing firms may serve a segment of captive consumers who can only be served by the competitor if it invests above a certain threshold. If the investment falls short of this threshold, captive consumers do not consider the product of the competing firm. We find that the existence of these captive segments induces asymmetries in the probabilities of one or the other firm dominating the market. The reason is that the outside option of serving only its own captive segment in a shared market is less attractive for a firm with a small captive segment than it is for the competitor with a larger one.

The effect of the share of captive consumers on total expected investments is non-monotone: if the underdog has sufficient resources to target the whole market, a higher share of captive consumers of the dominant firm may lead to higher investments. This may well have been the case for Internet Explorer competing with Google Chrome. When Microsoft chose to breach its 2009 promise to make Windows users aware of the existence of competing browsers to Internet Explorer, Google was already big enough to compete aggressively, and eventually overtook most of the market before regulators forced Microsoft to place Google Chrome such that it would enter the consideration set of Windows users. Hence, a key question for a regulator is to identify whether a competitor with the potential to overtake the whole market exists. Facing a weak underdog, a regulator may do well to prevent the dominant firm from keeping consumers unaware of alternatives to its products. Facing a strong underdog, the case is much more balanced.

An important limitation of our model is that we focus on the case where per-user revenue and consumer utility are neither congruent nor conflicting. Using the results in De Cornière and Taylor (2019), we can see how considering the other cases would influence our equilibria. Under congruence, the benefit from winning a share of the market increases with investment. Hence, competition would more often be at levels of utility above our threshold, with monopoly a more natural outcome. Under conflicting payoffs and revenues, firms would have a higher incentive not to invest above the threshold, as this would mean lower per-user revenue. Hence, we would expect lower competition and a monopoly outcome becomes less likely.

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Appendix

A Proofs

Proof of Lemma 1

Part (i)

Proof. The proof is by contradiction. Suppose there is pure strategy equilibrium in which firms 1 and 2 choose investments $k_1, k_2 < k^{max}$ with certainty. Suppose first that $k_1 = k_2$. Obviously, each firm could profitably deviate to marginally overbidding the other as this would only marginally increase cost but discretely increases the chance of winning. Thus, this cannot be an equilibrium. Suppose instead $k_1 < k_2$. We distinguish two cases. If $k_2 \geq \bar{k}$, firm 1 can profitably deviate to investing just marginally above k_2 which would imply winning the entire market with certainty and by $k_2 < k^{max}$ would yield a positive profit. Now consider the case $k_2 < \bar{k}$. This can only be an equilibrium if firm 1 chooses $k_1 = 0$ because it looses in any case. But then, firm 2 would want to just marginally overbid 0 which could in turn be profitably outbid by firm 1. Thus, these investments do not constitute an equilibrium either. The analogous arguments hold if we exchange subscripts 1 and 2. Therefore, the equilibrium must be in mixed strategies. □

Part (ii)

Proof. The proof is by contradiction. Suppose firm 1 invests according to an equilibrium investment strategy F_1 and as part of it chooses $k_1 = k'$ with strictly positive probability, $P_1(k') > 0$. Suppose further that also firm 2 chooses k' with positive probability, i.e., $P_2(k') > 0$ is part of firm 2's equilibrium strategy F_2. Denote by $E[\Pi_2]$ the expected profit of firm 2 given this strategy. Note that firm 2 can discretely increase its expected profit by switching to a mixed strategy F_2' that differs from F_2 only in that firm 2 reallocates probability mass from k' to an investment of $k_2 = k' + \epsilon$ for any $\epsilon > 0$ small enough. Thus, both firms will not allocate positive probability to the same investment. □

Part (iii)

Proof. (i) We first show that, if there is a gap in the support of the mixed strategy of a firm, the gap must be an interval containing $\bar{k}$. Suppose there is a gap $[k', k'']$ in the support of firm $i's$ strategy between with $k', k'' \in (0, \bar{k})$, $k' < k''$, and $F_i(k') = F_i(k'')$. Note that firm j then strictly prefers investing k' over investing k'' because the expected profit is the same but the expected cost is lower for the lower investment. Thus, firm j prefers to marginally overbid firm i at k' over any higher investment and in particular over investing k''. This in turn implies that firm i also strictly prefers to invest marginally above firm j's investment over investing k'' because k'' is more expensive but does not increase the chance of winning. Thus, the condition that a firms has the same expected profit over the support of her mixed strategy would be violated. The same reasoning applies to any pair $k', k'' > \bar{k}$. Thus, if there is a gap, it must be the case that $k' < \bar{k}$ and $k'' \geq \bar{k}$.

(ii) It follows that any gap must have as an upper bound $\bar{k}$. Else, following the same logic as above, a firm would strictly prefer bidding $\bar{k}$ over some $k'' > \bar{k}$. Hence, if there is a gap, it must be in some interval $[\delta, \bar{k}]$.

(iii) The fact that firms randomize over the same intervals is a standard property: if an investment k is part of only one firm's mixed strategy support, this firm would be better off investing less (at the top of the other firm's support). The fact that the bottom of the lower investment support is zero is also standard: else investing zero with strictly positive probability would be a profitable deviation.

□

Part (iv)

Proof. For a mass point to be an equilibrium strategy, it must satisfy two properties. First, by part (ii) of this Lemma, a firm does not invest k with strictly positive probability in equilibrium if k is in the support of the other firm's strategy. Else, the other firm would be better off marginally outbidding k than bidding just below it. Second, the same holds if a value marginally below k is in the support of the other firm, for a similar reason. There needs to be a gap in the support of the mixed strategy of player i below an investment k for firm i to invest k with strictly positive probability in any equilibrium. This leaves only two possibilities: $k = 0$ (as no one invests below 0) and $k = \bar{k}$ (if there is a gap in the support of the investment strategy of the other firm below $\bar{k}$). □

Proof of Proposition 1

Proof. We first construct the equilibrium, then verify that indeed neither firm has an incentive to deviate from the proposed investment strategy, and finally show that no other equilibrium exists.

Characterization: For every investment of firm 2 below $\bar{k}$ which is contained in the support of the equilibrium strategy, the following condition has to hold (with ϵ arbitrarily small):

$$F_2(k)\alpha_{12} + \lim_{\varepsilon\to 0} F_2(\bar{k}-\varepsilon)\alpha_1 - ck = \lim_{\varepsilon\to 0} F_2(\bar{k}-\varepsilon)\alpha_1 \quad \Rightarrow \quad F_2(k) = \frac{c}{\alpha_{12}}k \tag{2}$$

and for every investment equal to or above $\bar{k}$

$$F_2(k) - ck = \lim_{\varepsilon\to 0} F_2(\bar{k}-\varepsilon)\alpha_1 \quad \Rightarrow \quad F_2(k) = ck + \lim_{\varepsilon\to 0} F_2(\bar{k}-\varepsilon)\alpha_1 \tag{3}$$

If firm 1 chooses zero with positive probability, firm 2's mixed strategy must not contain an atom at zero. However, firm 2 must also be indifferent between all investment levels in the support of its equilibrium mixed strategy. Denote firm 2's expected profit by $E[\Pi_2]$. Then, for all $k < \bar{k}$

$$\begin{aligned} & F_1(k)\alpha_{12} + \lim_{\varepsilon\to 0} F_1(\bar{k}-\varepsilon)\alpha_2 - ck = E[\Pi_2] \\ \Rightarrow \quad & F_1(k) = \frac{c}{\alpha_{12}}k + \frac{E[\Pi_2] - \lim_{\varepsilon\to 0} F_1(\bar{k}-\varepsilon)\alpha_2}{\alpha_{12}} \end{aligned} \tag{4}$$

For every investment at $\bar{k}$ or above having a lower investment than the competitor implies also losing their share of captive consumers.

$$F_1(k) - ck = E[\Pi_2] \quad \Rightarrow \quad F_1(k) = ck + E[\Pi_2] \tag{5}$$

From lines (2) to (5) it follows that firm 1's and firm 2's distribution functions have the same slopes. This is true in both the low and the high investment range. Since the slope is higher for investments below $\bar{k}$ than for investments above $\bar{k}$, there exists $\delta \in (0, \bar{k})$ such that for both firms

$$F_1(k) = F_1(\delta) \text{ and } F_2(k) = F_2(\delta) \text{ for all } k \in [\delta, \bar{k}) \tag{6}$$

and therefore $\lim_{\varepsilon\to 0} F_1(\bar{k}-\varepsilon) = F_1(\delta)$ and $\lim_{\varepsilon\to 0} F_2(\bar{k}-\varepsilon) = F_2(\delta)$.

Neither firm has an incentive to strictly exceed the maximum investment of the other. This would increase the cost but not the probability of winning. Thus, there exists a unique $\overline{K}$ such that $F_1(\overline{K}) = F_2(\overline{K}) = 1$ and for all $\varepsilon > 0$, $F_1(\overline{K}-\varepsilon) < 1$ and $F_2(\overline{K}-\varepsilon) < 1$. Since the distribution functions of firms 1 and 2 also have identical slopes for $k \geq \bar{k}$, the distribution functions of both firms are identical for $k \geq \bar{k}$:

$$F_1(k) = F_2(k) \text{ for all } k \geq \bar{k} \tag{7}$$

Combining Equations (3), (5), and (7) yields $E[\Pi_2] = F_2(\delta)\alpha_1$. Starting with Line (4) and plugging in yields for $k < \bar{k}$

$$F_1(k) = \frac{c}{\alpha_{12}}k + \frac{F_2(\delta)\alpha_1}{\alpha_{12}} - \frac{F_1(\delta)\alpha_2}{\alpha_{12}}. \tag{8}$$

We solve (8) for $F_2(\delta)$ at $k = \delta$ and obtain

$$F_2(\delta) = F_1(\delta)\frac{\alpha_{12} + \alpha_2}{\alpha_1} - \frac{c}{\alpha_1}\delta.$$

We plug in from line (2) and solve for $F_1(\delta)$ to obtain

$$F_1(\delta) = c\delta\left(\frac{\alpha_1}{\alpha_{12}(\alpha_{12} + \alpha_2)} + \frac{1}{\alpha_{12} + \alpha_2}\right). \tag{9}$$

The flat part in the distribution functions (equation (6)) implies together with the different shares of captive consumers that firm 2 chooses an investment equal to $\bar{k}$ with a positive probability while firm 1's strategy has an atom at zero. Since the two firms cannot have an atom at the same investment level (Lemma 1, part (ii)), and since neither firm chooses δ with positive probability in equilibrium (Lemma 1, part (iv)), the distribution function of firm 1 must take the same value at δ and $\bar{k}$. In addition, at $\bar{k}$ the distribution functions of both firms take identical values. Thus, the following holds

$$F_1(\delta) = F_1(\bar{k}) = F_2(\bar{k}) \tag{10}$$

We can rewrite (3) using (2) as

$$F_2(\bar{k}) = c\bar{k} + \frac{c}{\alpha_{12}}\delta\alpha_1. \tag{11}$$

Taking line (10) and plugging in from line (9) on the left-hand side and from line (11) on the right-hand side, we arrive at

$$\begin{aligned} & c\delta\left(\frac{\alpha_1}{\alpha_{12}(\alpha_{12} + \alpha_2)} + \frac{1}{\alpha_{12} + \alpha_2}\right) = c\bar{k} + \frac{c}{\alpha_{12}}\delta\alpha_1 \\ \Leftrightarrow\quad & \delta = \bar{k}\frac{\alpha_{12}(\alpha_{12} + \alpha_2)}{\alpha_{12} + \alpha_1 - \alpha_1(\alpha_{12} + \alpha_2)} = \bar{k}\frac{(1 - \alpha_1)\alpha_{12}}{\alpha_{12} + \alpha_1^2}. \end{aligned} \tag{12}$$

It is easily verified that

$$(1 - \alpha_1)\alpha_{12} < \alpha_{12} + \alpha_1^2 \Rightarrow \delta < \bar{k}.$$

Finally, we derive the maximum investment levels. Suppose $K > \bar{k}$. Since the distribution functions stay constant at one for all investment levels above the maximum level chosen, we obtain the following condition

$$c\overline{K} + F_2(\delta)\alpha_1 = 1 \Leftrightarrow c\overline{K} = 1 - \frac{\alpha_1 c}{\alpha_{12}}\delta = 1 - \alpha_1 c\bar{k}\frac{(1-\alpha_1)\alpha_{12}}{\alpha_{12}(\alpha_{12}+\alpha_1^2)} \tag{13}$$

where δ has been derived in Equation (12). Rewriting (13) yields the maximum investment level

$$\overline{K} = \frac{1}{c} - \alpha_1\bar{k}\frac{1-\alpha_1}{\alpha_{12}+\alpha_1^2}.$$

As by assumption $\alpha_1 + \alpha_2 + \alpha_{12} = 1$, we replace in the above results to state Proposition 1.

For the derivation of the maximum investment, we have assumed $\overline{K} > \bar{k}$. This is indeed true if

$$\frac{1}{c} - \alpha_1\bar{k}\frac{1-\alpha_1}{\alpha_{12}+\alpha_1^2} > \bar{k} \Leftrightarrow \bar{k} < \frac{1}{c}\frac{\alpha_{12}+\alpha_1^2}{1-\alpha_2}. \tag{14}$$

Equilibrium verification: The above computations establish that both firms are indifferent between all levels of investment in their support such that it does not pay to reshuffle probability mass within interior investments. Hence, it suffices to show that there is no strictly profitable deviation for either firm to investments outside the support or at the boundaries. Note that by construction no firm has an incentive to deviate to an investment in the gap or above $\bar{K}$, as this would yield strictly lower expected profit. Note further that firm 1 would be strictly worse off to invest $\bar{k}$ with strictly positive probability than what she already gets by investing marginally above $\bar{k}$ (as firm 2 invests exactly $\bar{k}$ with strictly positive probability). The same holds for firm 2 investing exactly 0, as it would get strictly lower profit then by investing just above 0.

Uniqueness: By the above construction, the slopes of the distributions over the two intervals and the value of δ are the only ones satisfying the condition of equal profit over the intervals. We also know from Lemma 1, part (ii) and (iv) that the only other possibility in terms of a mass point satisfying the condition that both firms need to invest with total probability of 1 would be to have firm 1 investing $\bar{k}$ with strictly positive probability and firm 2 investing 0 with strictly positive probability. However, in any equilibrium over two intervals, with $\bar{K}$ the upper bound of the upper interval, it must hold by Lemma 1, part (iii) that $F_1(\bar{K}) = F_2(\bar{K}) = 1$, the profit of both firms must be identical. This does not hold if $\alpha_1 > \alpha_2$, $F_2(0) > 0$ and $F_1(0) = 0$. Hence, Proposition 1 characterizes the unique equilibrium in which firms randomize over two disconnected intervals. Furthermore, the indifference conditions prohibit any equilibrium with a connected equilibrium support for low $\bar{k}$. Thus, the equilibrium is unique.

Investments, market leadership, and consumer surplus: Using the distribution functions from above, we observe that $F_1(\delta) > F_2(\delta)$ so that firm 2 has a higher investment than firm 1 more often than the reverse. We compute expected investments as

$$\begin{aligned}
E[k_1] &= \int_0^\delta \frac{c}{\alpha_{12}} x dx + \int_{\bar{k}}^{\overline{K}} cx dx \\
&= \tfrac{c(1-\alpha_1)^2\alpha_{12}\bar{k}^2}{2\left(\alpha_{12}+\alpha_1^2\right)^2} + \tfrac{1}{2}c\left(\tfrac{(\alpha_{12}-\alpha_1((1+\alpha_1)c\bar{k}-\alpha_1))^2}{c^2\left(\alpha_{12}+\alpha_1^2\right)^2} - \bar{k}^2\right) \\
E[k_2] &= \int_0^\delta \frac{c}{\alpha_{12}} x dx + \int_{\bar{k}}^{\overline{K}} cx dx + \text{Prob}(k_2 = \bar{k})\bar{k} \\
&= \tfrac{c(1-\alpha_1)^2\alpha_{12}\bar{k}^2}{2\left(\alpha_{12}+\alpha_1^2\right)^2} + \tfrac{1}{2}c\left(\tfrac{(\alpha_{12}-\alpha_1((1+\alpha_1)c\bar{k}-\alpha_1))^2}{c^2\left(\alpha_{12}+\alpha_1^2\right)^2} - \bar{k}^2\right) + \tfrac{c(\alpha_1-\alpha_2)\bar{k}^2}{\alpha_{12}+\alpha_1^2}
\end{aligned}$$

It is easily verified that $E[k_1] < E[k_2]$. By the properties of the mixed strategy equilibrium, the expected profit of each firm $i = 1, 2$ equals its expected profit conditional on investing zero. This corresponds to its outside option $o_i(0)$ which is the value of its captive segment multiplied with the probability of the competitor investing below $\bar{k}$.

Expected consumer surplus is given by

$$\begin{aligned}
u_1 &= F_2(\delta) \int_0^\delta \frac{c}{a_{12}} k dk + \int_{\bar{k}}^K cF_2(k) k dk + \int_{\bar{k}}^K cF_1(k) k dk \\
u_2 &= F_1(\delta) \int_0^\delta \frac{c}{a_{12}} k dk + \int_{\bar{k}}^K cF_2(k) k dk + \int_{\bar{k}}^K cF_1(k) k dk \\
u_{12} &= \int_0^\delta \frac{c}{a_{12}} F_2(k) k dk + \int_0^\delta \frac{c}{a_{12}} F_1(k) k dk + \int_{\bar{k}}^K cF_2(k) k dk + \int_{\bar{k}}^K cF_1(k) k dk.
\end{aligned}$$

A.1 Proof of Proposition 2

Proof. We first construct the equilibrium and then verify that indeed neither firm has an incentive to deviate from the proposed investment strategy, and finally show that no other equilibrium exists.

Characterization: Suppose that both firms randomize over $(0, \delta)$ for some $\delta \in (0, \bar{k})$. Suppose further that firm 1 chooses zero with positive probability and firm 2 chooses $\bar{k}$ with positive probability. Finally, suppose that firm 1 chooses investments below or equal to δ with certainty (we verify this later), i.e., $F_1(\delta)$ whereas firm 2 also chooses $\bar{k}$ such that $F_2(\delta) < 1$. We now derive the value for $\delta \in (0, \bar{k})$.

As firm 1 invests only below $\bar{k}$, firm 2 could ensure profit $1 - c\bar{k}$ by investing $\bar{k}$ with certainty. Thus, the distribution function of firm 1 must fulfill for all $k \leq \delta$

$$F_1(k)\alpha_{12} + \alpha_2 - ck = 1 - c\bar{k} \Rightarrow F_1(k) = \frac{c}{\alpha_{12}}k + \frac{1 - \alpha_2 - c\bar{k}}{\alpha_{12}} \tag{15}$$

By assumption $\bar{k} < \frac{1-\alpha_2}{c}$ and thus $\frac{1-\alpha_2-c\bar{k}}{\alpha_{12}} > 0$. Note that choosing $\bar{k}$ also yields an expected profit equal to $1 - c\bar{k}$ for firm 2.

Firm 1 obtains an expected profit equal to its outside option $o_1(0)$ which is given by its captive segment multiplied by the probability that firm 2 invests less than $\bar{k}$, $F_2(\delta)\alpha_1$. For the distribution function of firm 2 and investments $k \leq \delta$ the following must hold:

$$F_2(k)\alpha_{12} + F_2(\delta)\alpha_1 - ck = F_2(\delta)\alpha_1 \Leftrightarrow F_2(k) = \frac{c}{\alpha_{12}}k$$

The investment level δ is such that the distribution function of firm 1 just reaches 1 at this level

$$\frac{c}{\alpha_{12}}\delta + \frac{1 - \alpha_2 - c\bar{k}}{\alpha_{12}} = 1 \Leftrightarrow \delta = \bar{k} - \frac{\alpha_1}{c} \tag{16}$$

If $\bar{k} < \frac{1-\alpha_2}{c}$, then $\delta < \frac{\alpha_{12}}{c}$.

Finally, we derive the probability with which firm 2 chooses $\bar{k}$.

$$\text{Prob}(k_2 = \bar{k}) = 1 - \frac{c}{\alpha_{12}}\delta = 1 - 1 + \frac{1 - \alpha_2}{\alpha_{12}} - \frac{c}{\alpha_{12}}\bar{k} = \frac{1 - \alpha_2}{\alpha_{12}} - \frac{c}{\alpha_{12}}\bar{k}$$

From line (15) also

$$\text{Prob}(k_1 = 0) = \frac{1 - \alpha_2}{\alpha_{12}} - \frac{c}{\alpha_{12}}\bar{k} = \text{Prob}(k_2 = \bar{k})$$

By $\bar{k} < \frac{1-\alpha_2}{c}$, it holds that $\text{Prob}(k_2 = \bar{k}) > 0$. Moreover,

$$\alpha_{12} > 0 \Rightarrow \alpha_{12} + \alpha_2 > \alpha_2 \Rightarrow (1 - \alpha_1)^2 > \alpha_2(1 - \alpha_1) \Rightarrow \alpha_{12} + \alpha_1^2 > \alpha_1 - \alpha_1\alpha_2$$
$$\Rightarrow \frac{\alpha_{12} + \alpha_1^2}{1 - \alpha_2} > \alpha_1$$

and therefore

$$\bar{k} > \frac{1}{c}\frac{\alpha_{12} + \alpha_1^2}{1 - \alpha_2} \Rightarrow \bar{k} > \frac{\alpha_1}{c}$$

so that $\delta < \bar{k}$.

By $\bar{k} > \frac{1}{c}\frac{\alpha_{12}+\alpha_1^2}{1-\alpha_2}$ firm 1 does indeed not want to deviate to choosing $\bar{k}$:

$$\bar{k} > \frac{1}{c}\frac{\alpha_{12}+\alpha_1^2}{1-\alpha_2} \Rightarrow c\bar{k}(\alpha_{12} + \alpha_1) > \alpha_{12} + \alpha_1^2 \Leftrightarrow -\frac{\alpha_1^2}{\alpha_{12}} + \frac{c}{\alpha_{12}}\bar{k}\alpha_1 > 1 - c\bar{k} \Leftrightarrow F_2(\delta)\alpha_1 > 1 - c\bar{k}$$

Equilibrium verification: From the above derivations, both firms are indifferent between all levels of investment in the support of their equilibrium investment strategies by construction. Hence, it suffices to show that there is no strictly profitable deviation for either firm. Note that no firm has an incentive to deviate to an investment strictly above $\bar{k}$, as this would yield strictly lower expected profit. Firm 1 would be strictly worse off to invest $\bar{k}$ as it would get the same expected profit as firm 2, lower than what it gets at equilibrium. Further, firm 2 cannot gain from investing exactly 0, as it would then get strictly lower profit than by investing just above 0.

Uniqueness: By the above construction, the cumulative distribution functions and the value of δ are the only ones satisfying the indifference condition for randomization of investments over a single connected interval. We also know from Lemma 1 part (ii) and (iv) that the only other possibility in terms of mass points satisfying the condition that both firms need to invest with total probability of 1 would be to have firm 1 invest $\bar{k}$ with strictly positive probability and firm 2 invest 0 with strictly positive probability. However, for such a mixed strategy to be an equilibrium it must also be true that firm 1 is indifferent between investing just above 0 and exactly $\bar{k}$,

$$\alpha_1 + F_2(0) = 1 - c\bar{k},$$

and that firm 2 weakly prefers to invest 0 over investing $\bar{k}$,

$$F_1(\delta)\alpha_2 \geq 1 - c\bar{k}.$$

As $\alpha_1 > \alpha_2$, $F_1(\delta) < 1$ and $F_2(0) > 0$, this leads to a contradiction. Hence, the above equilibrium is the unique one where both firms randomize over the same connected interval, when the total probability mass allocated below δ (remember there is only one possible slope for the distribution at equilibrium) is strictly below 1. Note further that there cannot be an equilibrium where firms randomize over two disconnected intervals as the one described in Proposition 1 is the only one that exists but the condition on $\bar{k}$ is not fulfilled here. Thus, the equilibrium we characterized here is unique for the set range of $\bar{k}$.

Investments, market leadership, and consumer surplus: Using the distribution functions from above, we observe that $F_1(\delta) > F_2(\delta)$, and we compute expected investments as

$$\begin{aligned}
E[k_1] &= \int_0^{\delta} \frac{c}{\alpha_{12}} x dx = \frac{c(\frac{\alpha_1}{c} - \bar{k})^2}{2\alpha_{12}} \\
E[k_2] &= \int_0^{\delta} \frac{c}{\alpha_{12}} x dx + \text{Prob}(k_2 = \bar{k})\bar{k} = \frac{c(\frac{\alpha_1}{c} - \bar{k})^2}{2\alpha_{12}} + \bar{k}\frac{\alpha_1 + \alpha_{12} - c\bar{k}}{\alpha_{12}}
\end{aligned}$$

where obviously $E[k_1] < E[k_2]$.

By the properties of the mixed strategy equilibrium, the expected profit of each firm $i = 1, 2$ equals its expected profit conditional on investing zero which is its outside option $o_i(0)$ which is given by its captive segment multiplied with the probability of the competitor investing below $\bar{k}$.

Expected consumer surplus is given by

$$\begin{aligned}
u_1 &= F_2(\delta) \int_0^\delta \frac{c}{a_{12}} k dk + \bar{k} \frac{1 - \alpha_2 - c\bar{k}}{\alpha_{12}} \\
u_2 &= \int_0^\delta \frac{c}{a_{12}} k dk + \bar{k} \frac{1 - \alpha_2 - c\bar{k}}{\alpha_{12}} \\
u_{12} &= \int_0^\delta \frac{c}{a_{12}} F_2(k) k dk + \int_0^\delta \frac{c}{a_{12}} F_1(k) k dk + \bar{k} \frac{1 - \alpha_2 - c\bar{k}}{\alpha_{12}}.
\end{aligned}$$

□

A.2 Proof of Proposition 3

Proof. Let $\bar{k} > \frac{1-\alpha_2}{c}$. We first construct the equilibrium, verify that neither firm has an incentive to deviate, and finally show that no other equilibrium exists.

Characterization: Suppose that neither firm chooses an investment high enough to steal captive consumers from its competitor. The outside option of firm $i = 1, 2$ is to keep its captive segment and receive a profit of $o_i(0) = \alpha_i$. The prize of winning is then the value of additionally attracting the contested segment α_{12}. This observation implies that both firms are symmetric at the margin. Moreover, the captive segments can be disregarded since they are not at stake. Both firms compete until their expected profits from competition are zero, in which case their expected profit is determined only by their captive segment. Thus, in equilibrium, the following must hold for all $k_i < \bar{k}$ for $i = 1, 2$ and $j \neq i$:

(17) $$F_j(k_i)\alpha_{12} - ck_i = 0 \Leftrightarrow F_j(k_i) = \frac{c}{\alpha_{12}} k_i.$$

As the distribution function of investments cannot exceed 1, both firms randomize continuously over $[0, \frac{\alpha_{12}}{c}]$ and do not invest any higher amounts. The cumulative distribution function is as follows for firm $i = 1, 2$:

$$F_i(k) = \begin{cases} \frac{c}{\alpha_{12}} k & \text{for all } k \in [0, \frac{\alpha_{12}}{c}] \\ 1 & \text{for } k \geq \frac{\alpha_{12}}{c} \end{cases}$$

Each firm must be indifferent at equilibrium between all investments in $[0, \frac{\alpha_{12}}{c}]$, and none of the two firms chooses zero with strictly positive probability because this would not be consistent with the indifference condition in (17). The expected payoff of firm 1 and 2 is equal to its outside option, $E[\Pi_1] = o_1(0) = \alpha_1$ and $E[\Pi_2] = o_2(0) = \alpha_2$, respectively.

Equilibrium verification: By the above, both firms are indifferent between all levels of investment in their support by construction. Hence, it suffices to show that there is no strictly profitable deviation for either firm. Suppose firm i considered deviating to an investment at $\bar{k}$, sufficient to capture the entire population. Then, firm i would make an expected profit of $F(\frac{\alpha_{12}}{c}) - c\bar{k} = 1 - c\bar{k} < 1 - (1 - \alpha_2) = \alpha_2 < \alpha_1$ such that this deviation is not profitable for firm $i = 1, 2$. As a consequence, no investment level at or above $\bar{k}$ forms part of the equilibrium mixed strategy. We also know that any investment between $\frac{\alpha_{12}}{c}$ and $\bar{k}$ would yield strictly lower profit, hence there is no profitable deviation.

Uniqueness: We have shown above that the slopes of the distributions and the value of δ as specified in the equilibrium characterization are the unique ones satisfying the condition of equal profit for randomization over a single connected interval. Hence, the above equilibrium is the unique one on a single interval when the total probability mass allocated below δ is equal to 1. Note further that there cannot be an equilibrium where firms randomize over two disconnected intervals as the one described in Proposition 1 is the only one that exists but the condition on $\bar{k}$ is not fulfilled here. According to Lemma 1, no further equilibrium types are admissible. Thus, the equilibrium we characterized here is unique for the set range of $\bar{k}$.

Investments, market leadership, and consumer surplus: Using the distribution functions from above, the expected investment in equilibrium equals

$$E[k_i] = \int_0^{\frac{\alpha_{12}}{c}} \frac{c}{\alpha_{12}} x dx = \frac{1}{2}\frac{\alpha_{12}}{c} \text{ for } i = 1, 2$$

per firm. In total, the two firms invest $\frac{\alpha_{12}}{c}$. Since equilibrium mixed strategies and investments are identical, both firms have the same probability of winning of $\frac{1}{2}$. The expected profit of each firm equals its expected profit conditional on investing zero which is the value of its captive segment.

Expected consumer surplus is given by

$$u_1 = \int_0^{\delta} \frac{c}{a_{12}} k dk$$
$$u_2 = \int_0^{\delta} \frac{c}{a_{12}} k dk$$
$$u_{12} = \int_0^{\delta} \frac{c}{a_{12}} F_2(k) k dk + \int_0^{\delta} \frac{c}{a_{12}} F_1(k) k dk.$$

□

A.3 Proof of Proposition 4

Investment levels:

Using the investment levels found in the proof of Proposition 1, replacing α_2 by 0 and α_{12} by $1 - \alpha_1$ and taking the derivative with respect to α_1 we find

$$\frac{dE(k_1)}{d\alpha_1} = \frac{\bar{k}(\alpha_1(4 + \alpha_1 c\bar{k}) - c\bar{k} - 2)}{2((\alpha_1 - 1)\alpha_1 + 1)^2}$$

and

$$\frac{dE(k_2)}{d\alpha_1} = \frac{\bar{k}(\alpha_1(4 - \alpha_1 c\bar{k}) - c\bar{k} - 2)}{2((\alpha_1 - 1)\alpha_1 + 1)^2}.$$

Taking $\alpha_1 \to 0$ we find

$$\lim_{\alpha_1 \to 0} \frac{dE(k_1)}{d\alpha_1} = -\frac{1}{2}\bar{k}(c\bar{k} + 2) < 0$$

and

$$\lim_{\alpha_1 \to 0} \frac{dE(k_2)}{d\alpha_1} = -\frac{1}{2}\bar{k}(2 - c\bar{k}) < 0,$$

as we have assumed $c\bar{k} < 1$. Similarly, we find

$$\lim_{\alpha_1 \to 1} \frac{dE(k_1)}{d\alpha_1} = \lim_{\alpha_1 \to 1} \frac{dE(k_2)}{d\alpha_1} = \bar{k} > 1.$$

Consumer surplus:

Using the expected surplus as defined in Proposition 1, replacing α_2 by 0 and α_{12} by $1-\alpha_1$ and taking the derivative with respect to α_1 we find:

$$\frac{du_1}{d\alpha_1} = \frac{(2\alpha_1-1)\bar{k}}{((\alpha_1-1)\alpha_1+1)^2} - \frac{(\alpha_1-1)(\alpha_1+1)(\alpha_1(\alpha_1(2\alpha_1-5)+6)-1)c^2\bar{k}^3}{2((\alpha_1-1)\alpha_1+1)^4} \tag{18}$$

$$\frac{du_2}{d\alpha_1} = \frac{\alpha_1(\alpha_1((4-3\alpha_1)\alpha_1-3)+2)c^2\bar{k}^3}{2((\alpha_1-1)\alpha_1+1)^4} + \frac{(2\alpha_1-1)\bar{k}}{((\alpha_1-1)\alpha_1+1)^2} \tag{19}$$

$$\frac{du_{12}}{d\alpha_1} = \frac{(2\alpha_1-1)\bar{k}}{((\alpha_1-1)\alpha_1+1)^2} - \frac{(\alpha_1-1)\left(\alpha_1\left(2\alpha_1^3+8\alpha_1+3\right)-1\right)c^2\bar{k}^3}{6((\alpha_1-1)\alpha_1+1)^4} \tag{20}$$

Taking $\alpha_1 \to 0$ we find

$$\lim_{\alpha_1\to 0}\frac{du_1}{d\alpha_1} = -\frac{1}{2}\bar{k}\left(c^2\bar{k}^2+2\right) < 0 \tag{21}$$

$$\lim_{\alpha_1\to 0}\frac{du_2}{d\alpha_1} = -\bar{k} < 0 \tag{22}$$

$$\lim_{\alpha_1\to 0}\frac{du_{12}}{d\alpha_1} = -\frac{1}{6}\bar{k}\left(c^2\bar{k}^2+6\right) < 0 \tag{23}$$

Similarly, we find

$$\lim_{\alpha_1\to 1}\frac{du_1}{d\alpha_1} = \frac{du_2}{d\alpha_1} = \frac{du_{12}}{d\alpha_1} = \bar{k} > 1.$$

B The symmetric case

The equilibrium characterization for the symmetric case is not, in principle, different from the one in the asymmetric case. In this section, we briefly discuss the results for $\alpha_1 = \alpha_2 = \alpha$ as a special case. Note first, that the equilibrium properties derived in Lemma 1 also apply to the symmetric case. From the three different types of equilibria that can arise in the asymmetric case, only two can also occur with symmetric shares of captive consumers: The intermediate case of Proposition 2 is ruled out with $\alpha_1 = \alpha_2 = \alpha$ and, accordingly, $\alpha_{12} = 1-2\alpha$ because then $\bar{k}_l = \frac{1-\alpha}{c} = \bar{k}_h$.

The equilibria as defined by Propositions 1 and 3, however, also apply to the symmetric case without further ado. The main text only concentrates on the asymmetric case as the more interesting one. Assuming that the shares of captive consumers are identical for both firms, $\alpha_1 = \alpha_2 = \alpha$, the contested segment is of size $1-2\alpha$. Using these expressions in the equilibrium characterizations from Propositions 1 and 3, we obtain the following result for the symmetric case:

Proposition B1. *Suppose $\alpha_1 = \alpha_2 = \alpha$. Then, the equilibrium is unique and symmetric. Each firm captures the contested segment with probability $\frac{1}{2}$.*

(i) If $\bar{k} < (1-\alpha)\frac{1}{c}$, both firms randomize over $[0, \delta]$ and $[\bar{k}, \frac{1}{c} - \bar{k}\frac{\alpha}{1-\alpha}]$ with $\delta = \bar{k}\frac{1-2\alpha}{1-\alpha} < \bar{k}$ using identical probability density functions over investments. The density of investments is $f(k) = \frac{c}{1-2\alpha}$ for $0 \leq k \leq \delta$ and $f(k) = c$ for $\bar{k} \leq k \leq \frac{1}{c} - \bar{k}\frac{\alpha}{1-\alpha}$. Each firm makes an expected profit of $\bar{k}c\frac{\alpha}{1-\alpha} < \alpha$. Expected investment are $\frac{1}{2}(\frac{1}{c} - \bar{k}\frac{\alpha}{1-\alpha} - c\bar{k}^2\frac{(1-2\alpha)\alpha}{(1-\alpha^2)})$ for each of the two firms.

(ii) If $\bar{k} > (1-\alpha)\frac{1}{c}$ both firms randomize identically and uniformly over the interval $[0, \frac{1-2\alpha}{c}]$. The density of investments is $f(k) = \frac{c}{1-2\alpha}$ for $0 \leq k \leq \frac{1-2\alpha}{c}$. Each firm makes an expected profit of α. Expected investments are $\frac{1}{2}\frac{1-2\alpha}{c}$ for each firm.

Proof. The proof follows from replacing α_1 and α_2 by α and α_{12} by $1 - 2\alpha$ in the proofs of Proposition 1 for part (i) and of Proposition 3 for part (ii). □

It is easily seen from this characterization that firms investments are decreasing in the share of the captive consumers whereas expected profits are increasing it it. This is intuitive as a larger share of captive consumers implies that the contested segment that drives competition is decreasing so that each firm becomes more complacent. As consumer surplus crucially depends on investments, consumer surplus is decreasing in the size of the captive shares α. The probability with which captive consumers switch is unaffected by α within regime (i), where one fraction of captive consumers always switches, and in regime (ii), where captive consumers always consider only one firm. An increase in α reduces the cutoff value for the attention threshold $\bar{k}$ beyond which firms do not compete for the contested segment anymore.

C Probabilistic setting

In this section, we show that the fact that investment is deterministic with a discrete threshold $\bar{k}$ is not crucial to our results. Consider two firms, 1 and 2 choosing a level of investment e_i, with $i \in \{1, 2\}$, at cost $c(e_i)$ with $c' > 0$, $c'' > 0$ and $c(0) = 0$. Firms compete for consumers from a population of mass one. This population consists of three types of consumers, t_1, t_2, and t_u. Types t_1 and t_2 occur with frequency α_1 and α_2, respectively, in the population and the remaining part are of type t_u, α_{12}. The structure of the game and frequencies of types are common knowledge.

Different from the main part of the text, we assume captive consumers of firm i (types t_1 and t_2) bear a switching cost $\bar{k}$ if they join the other firm. Hence, the utility of a consumer visiting a firm i is equal to e_i, minus the switching cost when it applies. Consumers of type t_u do not experience switching costs.

Suppose all types of customers intend to join the firm that maximizes their utility but may make mistakes and join the "wrong" firm. We employ the commonly used ratio-form contest success function which imposes that the probability of choosing one firm over the other equals its share in total investments.[14]

Consumers who are not captive to either firm (the contested segment) choose firm i with a probability

$$p^i_{t_u}(e_i, e_j) = \frac{e_i}{e_i + e_j}. \tag{24}$$

The captive consumers of type t_i choose the firm i with a probability

$$p^i_{t_i}(e_i, e_j) = \frac{e_i + \bar{k}}{e_i + e_j + \bar{k}}. \tag{25}$$

Therefore, the captive consumers of type t_j choose firm i with a probability

$$p^i_{t_j}(e_i, e_j) = 1 - p^i_i(e_i, e_j) = \frac{e_i}{e_i + e_j + \bar{k}}. \tag{26}$$

Firm 1 chooses the level of investment that maximizes her expected profit

$$E(\Pi_1) = \alpha_{12} p^1_{t_u}(e_1, e_2) + \alpha_1 p^1_{t_1}(e_1, e_2) + \alpha_2 p^1_{t_2}(e_1, e_2) - c(e_1). \tag{27}$$

Solving the first-order condition of the profit maximization with respect to e_a yields

$$c'(e_1) = \frac{\alpha_{12} e_2}{(e_1 + e_2)^2} + \frac{e_2(\alpha_1 + \alpha_2) + \alpha_2 \bar{k}}{(e_1 + e_2 + \bar{k})^2}. \tag{28}$$

Solving the same way for firm b yields

$$c'(e_2) = \frac{\alpha_{12} e_1}{(e_1 + e_2)^2} + \frac{e_1(\alpha_1 + \alpha_2) + \alpha_1 \bar{k}}{(e_1 + e_2 + \bar{k})^2}. \tag{29}$$

We immediately observe that:

(i) The equilibrium level of investment decreases in the cost-efficiency (the c function).

(ii) The firm that invests the most in equilibrium is the firm with the smallest captive segment.

(iii) Assuming no firm has a majority of captive consumers, the largest firm is, in expectation, the one with the smallest captive segment.

[14] Jia (2008) shows how such a contest success function can be derived from a model where the realized benefits from given investments are subject to stochastic shocks which are drawn independently from an inverse exponential distribution.

□