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# Portfolio Liquidation under Factor Uncertainty

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# Portfolio liquidation under factor uncertainty\*

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## Abstract

We study an optimal liquidation problem under the ambiguity with respect to price impact parameters. Our main results show that the value function and the optimal trading strategy can be characterized by the solution to a semi-linear PDE with superlinear gradient, monotone generator and singular terminal value. We also establish an asymptotic analysis of the robust model for small amounts of uncertainty and analyze the effect of robustness on optimal trading strategies and liquidation costs. In particular, in our model ambiguity aversion is observationally equivalent to increased risk aversion. This suggests that ambiguity aversion increases liquidation rates.

**AMS Subject Classification:** 93E20, 91B70, 60H30.

**Keywords:** stochastic control, uncertainty, portfolio liquidation, singular terminal value, superlinear growth gradient

## 1 Introduction

Starting with the work of Almgren and Chriss [1] optimal portfolio liquidation strategies under various market regimes and price impact functions have been analyzed by many authors. Single player models have been analyzed by [3, 7, 24–26, 30, 39] among many others; multi-player models were analyzed in, e.g. [6, 21, 29]. From a mathematical perspective, the main characteristic of optimal liquidation models is the singular terminal condition of the value function that is induced by the liquidation constraint. The singularity becomes a major challenge when determining the value function and applying verification arguments.

In this paper we study a class of Markovian single-player portfolio liquidation problems where the investor is uncertain about the factor dynamics driving trading costs. The liquidation problem leads to a stochastic control problem of the form

$$\inf_{\xi} \sup_{Q \in \mathcal{Q}} \left( \mathbb{E}_Q \left[ \int_0^T \eta(Y_s) |\xi_s|^p + \lambda(Y_s) |X_s|^p ds \right] - \Upsilon(Q) \right) \quad (1.1)$$

subject to the state dynamics

$$\begin{aligned} dY_t &= b(Y_t)dt + \sigma(Y_t)dW_t, & Y_0 &= y \\ dX_t &= -\xi_t dt, & X_0 &= x \end{aligned} \quad (1.2)$$

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and the terminal state constraint

$$X_T = 0, \tag{1.3}$$

where  $\xi$  denotes the trading rate,  $X$  denotes the portfolio process,  $Y$  denotes a factor process that drives trading costs and  $\mathcal{Q}$  is a set of probability measures that are absolutely continuous with respect to a benchmark measure  $\mathbb{P}$ . The functions  $\eta$  and  $\lambda$  specify the instantaneous market impact from trading and the market risk of a portfolio holding, respectively. Instead of restricting the set of probability measures ex ante, we add a penalty term  $\Upsilon(Q)$  to the objective function. This approach was first introduced by Hansen and Sargent [27] and has since become a popular approach in both the economics and financial mathematics literature when analyzing optimal decision problems under model uncertainty.

The benchmark case where  $\mathcal{Q}$  contains a single element has been analyzed in [26, 28]. In this case, the value function can be described in terms of the unique nonnegative viscosity solution of polynomial growth of a semi-linear PDE with singular terminal value. The proof is based on an asymptotic expansion of the solution around the terminal time that shows that the value function converges to the instantaneous impact factor at the terminal time when properly rescaled.

If  $\mathcal{Q}$  contains more than one element, then the investor is uncertain about the dynamics of the factor process. For instance, the process  $\eta(Y_t)$  may be viewed as describing the inverse market depth, whose dynamics the investor may not be able to specify correctly. The market risk factor  $\lambda(Y_t)$ , on the other hand, can be linked to the volatility of the reference price process. If the price dynamics follows a stochastic volatility model, then factor uncertainty amounts to uncertainty about the volatility of the reference price.

Under factor uncertainty, additional regularity assumptions on the penalty function  $\Upsilon(Q)$  are required to guarantee that the optimization problem is tractable analytically. In order to guarantee analytical tractability we follow an approach that had first been introduced by Maenhout [35] when analyzing a class of portfolio allocation models for Merton-type investors under model uncertainty.<sup>1</sup> Specifically, we consider penalty functions with state-dependent ambiguity aversion parameters that satisfy a scaling property corresponding to homothetic preferences. The assumption of homothetic preferences does not only facilitate the mathematical analysis but it also has a clear economic implication. Our model with ambiguity aversion is observationally equivalent to a model without ambiguity aversion but increased risk aversion. An approach that is similar in spirit to the ones in [35] and in this paper has been followed by Björk et al. [9]. They studied an equilibrium model with mean-variance preferences and a (state-dependent) dynamic risk aversion parameter that is inversely proportional to wealth. For their choice of risk aversion the equilibrium monetary amount invested in the risky asset is proportional to current wealth.

Under our scaling property on the penalty function, we prove that the value function to our control problem can be characterized by the solution to a semi-linear PDE with superlinear gradient, monotone generator and singular terminal value. Our first main result is to show that this PDE admits a unique nonnegative viscosity solution of polynomial growth under standard assumptions on the factor process and the cost coefficients. Many authors including [4, 5, 13] studied the Lipschitz and Hölder regularity of viscosity solutions. In our setting, Hölder continuity and even  $C^{0,1}$ -regularity of the value function is not sufficient to guarantee admissibility of our martingale measure control. A particular asymptotic behavior of both the value function and its gradient at the terminal time is key to carry out the verification argument. Our second main result guarantees that, under an additional assumption on the penalty function and an additional boundedness condition on the market impact term, the viscosity solution to the HJB equation is of class  $C^{0,1}$  and that both the solution and its derivative have the desired asymptotic behavior at the terminal time. The proof is based on an asymptotic expansion of the solution near the terminal time as in [26, 28]. The difficulty is that now not only the value function but also its derivative needs to converge to the market impact term, respectively its derivative when properly rescaled. The precise asymptotic behavior of the solution allows us to obtain not only the optimal trading strategy but also the least favorable martingale measure in feedback form. It also allows us to establish our third main result, namely a first order approximation of both the value function and the optimal trading

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<sup>1</sup>The approach has been adapted by many authors, including [11, 17, 20, 36, 43], partly due to its analytical tractability but also due to the “embedded” equivalence between ambiguity and risk aversion.

strategy in terms of the solution to the benchmark model without uncertainty. The result shows that we can approximate the optimal strategy in a model with small uncertainty parameter in terms of the optimal strategy of the benchmark model and the first order approximation of the value function. As a byproduct we show that our model with factor uncertainty is observationally equivalent to a model without factor uncertainty but increased market risk. This suggests that factor uncertainty increases the rate of liquidation.

To the best of our knowledge, only few papers have studied the optimal liquidation problem under model uncertainty. Nyström et al. [37] and Cartea et al. [14, 15] considered problems of optimal liquidation with *limit orders* for a CARA, respectively a risk-neutral investor. In [37] it is assumed that the investor is uncertain about both the drift and the volatility of the underlying reference price process. They show that uncertainty may increase the bid-ask spread and hence reduce liquidity. In [14, 15] the investor is uncertain about the arrival rate of market orders, the fill probability of limit orders and the dynamics of the asset price. They show that ambiguity aversion with respect to each model factor has a similar effect on the optimal strategy, but the magnitude of the effect depends on time and inventory position in different ways depending on the source of uncertainty. In both papers strict liquidation is not required; instead open positions at the terminal time are penalized. This avoids the mathematical challenges resulting from the singular terminal value.

Lorenz and Schied [33] studied the drift dependence of optimal trade execution strategies under transient price impact with exponential resilience and strict liquidation constraint. They find an explicit solution to the problem of minimizing the expected liquidation costs when the unaffected price process is a square-integrable semimartingale. Later, Schied [42] analysed the impact on optimal trading strategies with respect to misspecification of the law of the unaffected price process in a model which only allows instantaneous price impact. Both papers studied the dependence of optimal liquidation strategies on model dynamics but did not consider the resulting robust control problem. Bismuth et al. [8] considered a portfolio liquidation model for a CARA investor that is uncertain about the drift of the reference price process but did not require a strict liquidation constraint. They do not consider a robust optimization problem either but dealt with the uncertainty by a general Bayesian prior for the drift, which allows them to solve the problem by dynamic programming techniques. All three papers focussed on misspecification of the reference price process and assumed that the market impact parameters are known. Our model is different; we analyze the effect of uncertainty about the model parameters, e.g. the market depth that we consider the most important impact factor.

In a recent paper, Popier and Zhou [40] analysed the optimal liquidation problem under drift and volatility uncertainty in a non-Markovian setting and characterized the value function by the solution of a second-order BSDE with monotone generator and singular terminal condition. In contrast to [40], we focus on the drift uncertainty about the factor model and add a penalty function in the spirit of convex risk measure theory. We also obtain much stronger regularity properties of the value function which allows us to study the effect of uncertainty on optimal trading strategies and costs in greater detail.

The remainder of this paper is organized as follows. In Section 2, we describe the modelling set-up, introduce the stochastic control problem and state our main results. The existence of viscosity solution to the HJBI equation is established in Section 3; the regularity of the viscosity solution is proved in Section 4. The verification argument is carried out in Section 5. Finally, Section 6 is devoted to an asymptotic analysis of the value function for small amounts of uncertainty.

*Notation and notational conventions.* We put

$$\langle y \rangle := (1 + |y|^2)^{1/2}.$$

Let  $I$  be a compact subset of  $\mathbb{R}$ . We denote by  $C_b(\mathbb{R}^d)$ ,  $C_b(I \times \mathbb{R}^d)$  the spaces of bounded continuous functions on  $\mathbb{R}^d$ , respectively,  $I \times \mathbb{R}^d$ . For a given  $n \geq 0$ , we define  $C_n(\mathbb{R}^d)$  (resp.  $C_n(I \times \mathbb{R}^d)$ ) to be the set of functions  $\phi \in C(\mathbb{R}^d)$  (resp.  $C(I \times \mathbb{R}^d)$ ) such that

$$\psi := \frac{\phi(y)}{1 + |y|^n} \in C_b(\mathbb{R}^d) \text{ (resp. } \psi := \frac{\phi(t, y)}{1 + |y|^n} \in C_b(I \times \mathbb{R}^d)).$$

A function  $\phi$  belongs to  $USC_n(I \times \mathbb{R}^d)$  (or  $LSC_n(I \times \mathbb{R}^d)$ ) if it has at most polynomial growth of order  $n$  in the second variable uniformly with respect to  $t \in I$  and is upper (lower) semi-continuous on  $I \times \mathbb{R}^d$ . We denote by  $C_b^1(\mathbb{R}^d)$  the set of all functions  $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$  which are bounded, continuous and continuously differentiable with bounded first derivative.  $C^{0,1}(I \times \mathbb{R}^d)$  denotes the set of all functions  $\phi : I \times \mathbb{R}^d \rightarrow \mathbb{R}$  which are continuous and continuously differentiable with respect to the second variable on  $I \times \mathbb{R}^d$ .

We denote by  $L_{\mathcal{F}}^{\infty}(0, T; \mathbb{R}^d)$  the set of progressively measurable  $\mathbb{R}^d$ -valued processes that are essentially bounded. The spaces  $L_{\mathcal{F}}^q(0, T; \mathbb{R}^d)$ ,  $H_{\mathcal{F}}^q(0, T; \mathbb{R}^d)$  denote the sets of all the progressively measurable  $\mathbb{R}^d$ -valued processes  $(Z_t)_{t \in [0, T]}$  satisfying that  $\mathbb{E}[\int_0^T |Z_t|^q dt] < \infty$ ,  $\mathbb{E}[(\int_0^T |Z_t|^2 dt)^{q/2}]^{1/q} < \infty$ , respectively; the subset of processes with continuous paths satisfying  $\mathbb{E}[\sup_{t \in [0, T]} |Z_t|^{q/2}]^{1/q} < \infty$  is denoted by  $S_{\mathcal{F}}^q(\Omega; C([0, T]; \mathbb{R}^d))$ . Whenever the notation  $T^-$  appears in the definition of a function space we mean the set of all functions whose restrictions satisfy the respective property when  $T^-$  is replaced by any  $s < T$ , e.g.,

$$C_n([0, T^-] \times \mathbb{R}^d) = \{u : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R} : u|_{[0, s] \times \mathbb{R}^d} \in C_n([0, s] \times \mathbb{R}^d) \text{ for all } s \in [0, T)\}.$$

Throughout, all equations and inequalities are to be understood in the a.s. sense. We adopt the convention that  $C$  is a constant that may vary from line to line and the operator  $D$  denotes the gradient with respect to the space variable.

## 2 Problem formulation and main results

Let  $T \in (0, \infty)$  and let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$  be a filtered probability space that satisfies the usual conditions and carries an  $\tilde{d}$ -dimensional standard Brownian motion  $W$  and an independent one-dimensional standard Brownian motion  $B$ .

In this paper we consider the problem of a large investor that needs to liquidate a given portfolio  $x \in \mathbb{R}$  within the time horizon  $[0, T]$ . Let  $t \in [0, T]$  be a given point in time and  $x \in \mathbb{R}$  be the portfolio position of the trader at time  $t$ . We denote by  $\xi_s \in \mathbb{R}$  the rate at which the agent trades at time  $s \in [t, T]$ . Given a trading strategy  $\xi$ , the portfolio position at time  $s \in [t, T]$  is given by

$$X_s = x - \int_t^s \xi_r dr, \quad s \in [t, T]$$

and the liquidation constraint is

$$X_T = 0. \tag{2.1}$$

In what follows we assume that all trading costs are driven by a factor process given by the  $d$ -dimensional Itô diffusion

$$\begin{cases} dY_s^{t,y} = b(Y_s^{t,y})ds + \sigma(Y_s^{t,y})dW_s, & s \in [t, T], \\ Y_t^{t,y} = y. \end{cases}$$

Our goal is to analyze the impact of uncertainty about the factor dynamics on optimal liquidation strategies and trading costs.

### 2.1 The benchmark model

In this section we briefly recall the liquidation model without factor uncertainty analyzed by Graewe et al. [26] against which our results shall be benchmarked. Following [26], we assume that the investor's transaction price  $P_s \in \mathbb{R}$  at time  $s \in [t, T]$  can additively decomposed into a fundamental asset price  $\tilde{P}_s$  and an instantaneous price impact term  $f(\xi_s)$  as

$$P_s = \tilde{P}_s - f(\xi_s)$$

where the fundamental asset price process  $\tilde{P}$  is given by a one-dimensional square-integrable Brownian martingale, which we assume to be of the form<sup>2</sup>

$$d\tilde{P}_s = \tilde{\sigma}(Y_s^{t,y})dB_s$$

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<sup>2</sup>See Example 2.3 below for a stochastic volatility model with uncertainty about the driver of the volatility process.

for some function  $\tilde{\sigma}$ . The investor aims at minimizing the difference between the book value of the portfolio and the expected proceeds from trading plus risk cost. We assume that the instantaneous impact factor is given by  $f(\xi_s) = \eta(Y_s^{t,y})|\xi_s|^{p-1} \text{sgn}(\xi_s)$  for some  $p > 1$  and some bounded function  $\eta$  that describes the inverse market depth and that the risk is measured by the integral of the  $p$ -th power of the value at risk of an open position over the trading period. The resulting cost functional is then given by

$$\begin{aligned} J(t, y, x, \xi) &= \text{book value} - \text{expected proceeds from trading} + \text{risk costs} \\ &= \mathbb{E}_{\mathbb{P}} \left[ \int_t^T \eta(Y_s^{t,y}) |\xi_s|^p ds + \int_t^T X_s d\tilde{P}_s + \int_t^T \lambda(Y_s^{t,y}) |X_s|^p ds \right] \\ &= \mathbb{E}_{\mathbb{P}} \left[ \int_t^T (\eta(Y_s^{t,y}) |\xi_s|^p + \lambda(Y_s^{t,y}) |X_s|^p) ds \right], \end{aligned} \quad (2.2)$$

where the last equality follows from the facts that  $X \in \mathcal{S}_{\mathcal{F}}^2(\Omega; C([t, T]; \mathbb{R}))$  and that  $\tilde{P}$  is a square-integrable martingale under  $\mathbb{P}$ .

For each initial state  $(t, y, x) \in [0, T) \times \mathbb{R}^d \times \mathbb{R}$  the value function of the investor's control problem is defined by

$$V_0(t, y, x) := \inf_{\xi \in \mathcal{A}(t, x)} J(t, y, x, \xi) \quad (2.3)$$

where the infimum is taken over the set  $\mathcal{A}(t, x)$  of all admissible controls, that is, over all the controls  $\xi$  that belong to  $L_{\mathcal{F}}^{2p}(t, T; \mathbb{R})$  and that satisfy the liquidation constraint (2.1). Under suitable assumptions on the model parameters it was shown in [26, 28] that the value function is given by  $V_0 = v_0|x|^p$  and that the optimal trading strategy is given by  $\xi_0^*(t, y, x) = \frac{v_0(t, y)^\beta}{\eta(y)^\beta} x$  where  $\beta = \frac{1}{p-1}$  and where  $v_0$  is the unique nonnegative viscosity solution of polynomial growth to the following PDE:

$$\begin{cases} -\partial_t v(t, y) - \mathcal{L}v(t, y) - F(y, v(t, y)) = 0, & (t, y) \in [0, T) \times \mathbb{R}^d, \\ \lim_{t \rightarrow T} v(t, y) = +\infty & \text{locally uniformly on } \mathbb{R}^d \end{cases} \quad (2.4)$$

where

$$\mathcal{L} := \frac{1}{2} \text{tr}(\sigma \sigma^* D^2) + \langle b, D \rangle, \quad F(y, v) := \lambda(y) - \frac{|v|^{\beta+1}}{\beta \eta(y)^\beta}.$$

## 2.2 The liquidation model under uncertainty

In order to analyze the impact of factor uncertainty on optimal liquidation strategies we introduce the class  $\mathcal{Q}$  of all probability measures  $Q \ll \mathbb{P}$  whose density with respect to the benchmark measure  $\mathbb{P}$  is given by

$$\frac{dQ}{d\mathbb{P}} = \mathcal{E} \left( \int_t^T \vartheta_s dW_s \right)_T, \quad Q\text{-a.s.}$$

for some progressively measurable process  $\vartheta$  satisfying that  $\int_t^T |\vartheta_s|^2 ds < \infty$ ,  $Q$ -a.s.. Here,  $\mathcal{E}(M)_t = \exp(M_t - \frac{\langle M \rangle_t}{2})$  denotes the Doléans-Dade exponential of a continuous semimartingale  $M$ .

Since our focus is on the impact of uncertainty about the factor dynamics on the optimal trading rules, we assume that the Brownian motions  $B$  and  $W$  are independent. In this case the unaffected price process is still a square-integrable martingale under every probability  $Q \in \mathcal{Q}$ . In view of (2.2), we thus obtain the same form for the cost function for every given probability  $Q$  in the set  $\mathcal{Q}$ :

$$J_Q(t, y, x, \xi) = \mathbb{E}_Q \left[ \int_t^T (\eta(Y_s^{t,y}) |\xi_s|^p + \lambda(Y_s^{t,y}) |X_s|^p) ds \right].$$

Following a standard approach in optimal decision making under model uncertainty introduced by Hansen and Sargent [27], we do not restrict the set of measures *a priori* but add a penalty term to the objective function. Specifically, every probability measure  $Q \in \mathcal{Q}$  receives a penalty

$$\Upsilon(Q) := \mathbb{E}_Q \left[ \int_t^T \frac{1}{\theta_s} |\vartheta_s|^m ds \right].$$

The nonnegative process  $\hat{\theta} = (\hat{\theta}_s)$  measures the degree of confidence in the reference model: the larger the process, the less deviations from the reference model are penalized. The case  $\hat{\theta}_s \equiv 0$  corresponds to the benchmark model without factor uncertainty. The case  $\hat{\theta}_s \equiv \hat{\theta}$  and  $m = 2$  corresponds to the entropic penalty function, see, e.g. [2, 10].

To the best of our knowledge, Maenhout [35] was the first to propose a state-dependent parameter  $\hat{\theta}$  when considering the robust portfolio optimization problem of a power-utility investor. He considered an uncertainty-tolerance parameter of the  $\hat{\theta}_s = \frac{\theta}{\mathcal{W}_s^{1-r}}$  where  $\theta$  is a positive constant,  $\mathcal{W}_s$  denotes the wealth of the investor at time  $s$  and  $r \in (0, 1)$  denotes the exponent in the power utility function. This choice of  $\hat{\theta}$  essentially corresponds to scaling the uncertainty-tolerance parameter by the value function. In his model, this leads to a solution that is invariant to the scale of wealth and is amenable to a rigorous mathematical analysis. Among other things, he found that for this choice of homothetic preferences the optimal solution under model uncertainty is observationally equivalent to the optimal solution without model uncertainty but increased risk aversion.

In our context, the approach of Maenhout [35] corresponds to the choice

$$\hat{\theta}_s := \frac{\theta}{a|X_s^\xi|^p}$$

and thus to the penalty functional<sup>3</sup>

$$\Upsilon(Q) := \mathbb{E}_Q \left[ \int_t^T \frac{1}{\theta} a |\vartheta_s|^m |X_s^\xi|^p ds \right],$$

where  $m \geq 2$ . The constant  $a := \frac{(m-1)^{m-1}}{m^m}$  is chosen for analytical convenience; this will become more clear in the following section. We thus model the costs associated with an admissible trading strategy  $\xi$  and probability measure  $Q \in \mathcal{Q}$  by

$$\tilde{J}(t, y, x; \xi, \vartheta) := \mathbb{E}_Q \left[ \int_t^T \left( \eta(Y_s^{t,y}) |\xi_s|^p + \lambda(Y_s^{t,y}) |X_s^\xi|^p - \frac{1}{\theta} a |\vartheta_s|^m |X_s^\xi|^p \right) ds \right].$$

We define the value function of the stochastic control problem for each initial state  $(t, y, x) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}$  as

$$V(t, y, x) := \inf_{\xi \in \mathcal{A}(t,x)} \sup_{Q \in \mathcal{Q}} \tilde{J}(t, y, x; \xi, \vartheta). \quad (2.5)$$

We assume throughout that  $p > 1, m \geq 2$ . Before presenting the main results, we first list our assumptions on the factor process in terms of some positive constants  $\underline{c}, \bar{C}$ .

**Assumption 2.1.** (on the factor process)

(L.1) The drift function  $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$  is Lipschitz continuous and of linear growth, i.e. for each  $y \in \mathbb{R}^d$ ,

$$|b(x) - b(y)| \leq \bar{C}|x - y|, \quad |b(y)| \leq \bar{C}(1 + |y|).$$

(L.2) The volatility function  $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times \bar{d}}$  is Lipschitz continuous and of linear growth, i.e. for each  $y \in \mathbb{R}^d$ ,

$$|\sigma(x) - \sigma(y)| \leq \bar{C}|x - y|.$$

(L.3) The volatility function  $\sigma$  is uniformly bounded by  $\bar{C}$ .

(L.4) The drift and volatility functions  $b, \sigma$  belong to  $C^1$  and  $\sigma\sigma^*$  is uniformly positive definite.

<sup>3</sup>We may have  $\Upsilon(Q) = +\infty$  since  $Q$  is not equivalent but merely absolutely continuous with respect to  $\mathbb{P}$ .

Next, we list conditions on the cost coefficients. Conditions (F.1) and (F.2) are required to prove the existence of a viscosity solution to the HJB equation; the stronger condition (F.3) is required to establish differentiability of the viscosity solution and the verification result.

**Assumption 2.2.** (on the cost coefficients)

(F.1) The coefficients  $\eta, \lambda, 1/\eta : \mathbb{R}^d \rightarrow [0, \infty)$  are continuous. Moreover, there exists a constant  $k_0 \in (0, 1]$  such that for  $y \in \mathbb{R}^d$ ,

$$\lambda(y) \leq \bar{C} \langle y \rangle^{(1-k_0)m}$$

and

$$\underline{c} \langle y \rangle^{(1-pk_0)m} \leq \eta(y) \leq \bar{C} \langle y \rangle^{(1-k_0)m}.$$

Let  $\tilde{m} := (1 - k_0)m$ .

(F.2) The function  $\eta$  is twice continuously differentiable, and  $\|\frac{\mathcal{L}\eta}{\eta}\| \leq \bar{C}$ ,  $\left\| \frac{|D\eta|^{\alpha+1}}{\eta} \right\| \leq \bar{C}$  where  $\alpha := \frac{1}{m-1}$ .

(F.3) The function  $\lambda$  belongs to  $C_b^1(\mathbb{R}^d)$  and  $0 < \underline{c} \leq \eta \leq \bar{C}$ .

The assumptions on the diffusion coefficients are standard. Assumption (F.1) states that  $\lambda$  is of polynomial growth and that  $\eta$  can be bounded from below and above by polynomial growth functions, whose order may be negative. Under this assumption, we have that  $\langle y \rangle^{\tilde{m}(\beta+1)}/\eta^\beta$  is of polynomial growth of order  $m$ . Conditions similar to (F.2) and (F.3) have also been made in [28] and [26], respectively.

*Example 2.3.* The assumptions on the diffusion coefficients are satisfied for the two-dimensional diffusion process  $Y = (Y^1, Y^2)$  given by

$$dY_t^1 = -Y_t^1 dt + dW_t^1 \quad \text{and} \quad dY_t^2 = \mu dt + \sigma dW_t^2.$$

The Ornstein-Uhlenbeck process  $Y^1$  drives the market impact term while the arithmetic Brownian motion  $Y^2$  drives the market risk. Specifically, if we choose  $\eta = \tanh(-Y^1) + 2$ , then this process can be viewed as describing a stochastic liquidity process that fluctuates around a stationary level. Moreover, for the stochastic volatility model

$$d\tilde{P}_t = \tilde{\sigma}(Y_t^2) dB_t$$

for the reference price process the instantaneous volatility of the portfolio process is given by  $\tilde{\sigma}^2(Y_t^2)|X_t|^2$ . Hence, if  $\tilde{\sigma}$  is bounded and continuously differentiable with bounded derivative, then  $\lambda := \tilde{\sigma}^2$  satisfies the preceding assumptions.

## 2.3 The main results

If all the processes  $\vartheta$  take values in a compact set  $\Theta$  then all probability measures  $Q$  in  $\mathcal{Q}$  are equivalent to  $\mathbb{P}$ . In this case, the dynamic programming principle suggests that the value function satisfies the following Hamilton-Jacobi-Bellman-Issacs equation, cf. [19, Theorem 2.6]

$$-\partial_t V(t, y, x) - \mathcal{L}V(t, y, x) - \inf_{\xi \in \mathbb{R}} \sup_{\vartheta \in \Theta} \mathcal{H}(t, y, x, \xi, \vartheta, V) = 0, \quad (t, y, x) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}, \quad (2.6)$$

where  $\mathcal{H}$  is given by

$$\mathcal{H}(t, y, x, \xi, \vartheta, V) := \langle \sigma \vartheta, \partial_y V(t, y, x) \rangle - \xi \partial_x V(t, y, x) + c(y, x, \xi) - \frac{1}{\theta} a |\vartheta|^m |x|^p,$$

and

$$c(y, x, \xi) := \eta(y) |\xi|^p + \lambda(y) |x|^p.$$

In our case the set of probability measures is not restricted *a priori*. This suggests to characterize the value function (2.5) in terms of the solution to the modified HJBI equation

$$-\partial_t V(t, y, x) - \mathcal{L}V(t, y, x) - \inf_{\xi \in \mathbb{R}} \sup_{\vartheta \in \mathbb{R}^d} \mathcal{H}(t, y, x, \xi, \vartheta, V) = 0, \quad (t, y, x) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}. \quad (2.7)$$



Since the function  $\mathcal{H}$  separates additively into two terms that depend on  $\vartheta$  only and into two terms that depend  $\xi$  only,

$$\begin{aligned} \inf_{\xi \in \mathbb{R}} \sup_{\vartheta \in \mathbb{R}^{\bar{d}}} \mathcal{H}(t, y, x, \xi, \vartheta, V) &= \sup_{\vartheta \in \mathbb{R}^{\bar{d}}} \left\{ \langle \sigma \vartheta, \partial_y V(t, y, x) \rangle - \frac{1}{\theta} a |\vartheta|^m |x|^p \right\} \\ &\quad + \inf_{\xi \in \mathbb{R}} \left\{ -\xi \partial_x V(t, y, x) + c(y, x, \xi) \right\}. \end{aligned}$$

The structure of cost function suggests an ansatz of the form  $V(t, y, x) = v(t, y) |x|^p$ . In this case,

$$\begin{aligned} \vartheta^*(t, y) &:= \arg \max_{\vartheta \in \mathbb{R}^{\bar{d}}} \left\{ \langle \sigma \vartheta, Dv(t, y) \rangle - \frac{1}{\theta} a |\vartheta|^m \right\} \\ &= \theta^\alpha (1 + \alpha) |\sigma^*(y) Dv(t, y)|^{\alpha-1} \sigma^*(y) Dv(t, y), \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} \xi^*(t, y) &:= \arg \min_{\xi \in \mathbb{R}} \left\{ -p \xi v(t, y) |x|^{p-1} \operatorname{sgn}(x) + \eta(y) |\xi|^p \right\} \\ &= \frac{|v(t, y)|^\beta}{\eta(y)^\beta} x, \end{aligned} \tag{2.9}$$

where  $\alpha = \frac{1}{m-1}, \beta = \frac{1}{p-1}$ . Thus,

$$\inf_{\xi \in \mathbb{R}} \sup_{\vartheta \in \mathbb{R}^{\bar{d}}} \mathcal{H}(t, y, x, \xi, \vartheta, V) = \left( H(y, Dv(t, y)) + F(y, v(t, y)) \right) |x|^p$$

where

$$F(y, v) := \lambda(y) - \frac{|v|^{\beta+1}}{\beta \eta(y)^\beta}, \quad H(y, q) := \theta^\alpha |\sigma^*(y) q|^{\alpha+1}. \tag{2.10}$$

Similarly to the discussion in [26, Section 2.2], we expect the value function to be characterized by the following terminal value problem:

$$\begin{cases} -\partial_t v(t, y) - \mathcal{L}v(t, y) - H(y, Dv(t, y)) - F(y, v(t, y)) = 0, & (t, y) \in [0, T) \times \mathbb{R}^d, \\ \lim_{t \rightarrow T} v(t, y) = +\infty & \text{locally uniformly on } \mathbb{R}^d. \end{cases} \tag{2.11}$$

The problem reduces to the terminal value problem (2.4) in the absence of model uncertainty ( $H = 0$ ).

The following theorem guarantees the existence of a unique nonnegative viscosity solution to this singular problem under conditions (L.1)-(L.3), (F.1), (F.2) and  $\beta > \alpha$ , which corresponds to  $m > p$ . The additional assumption  $\beta > \alpha$  can also be found in [23] where the authors study the entire solutions of a similar kind of elliptic equation. The proof is given in Section 3.

**Theorem 2.4.** *Let  $m > p$ . Under Assumptions (L.1)-(L.3), (F.1) and (F.2), the singular terminal value problem (2.11) admits a unique nonnegative viscosity solution  $v$  in*

$$C_{\tilde{m}}([0, T^-] \times \mathbb{R}^d),$$

where  $\tilde{m}$  is introduced in condition (F.1).

Since the maximizer  $\vartheta^*$  in (2.8) depends on  $Dv$ , we expect the verification theorem to require the candidate value function  $v$  to be of class  $C^{0,1}$ . As it turns out the verification argument does not only require  $C^{0,1}$ -regularity of  $v$  but also requires the gradient to have a particular asymptotic behavior near the terminal time. In fact, we prove that uniformly in  $y$  as  $t \rightarrow T$  the function  $v$  satisfies

$$(T-t)^{1/\beta} v(t, y) = \eta(y) + O((T-t)^{1-\alpha/\beta}) \quad \text{and} \quad (T-t)^{1/\beta} Dv(t, y) = D\eta(y) + O((T-t)^{\frac{1}{2}-\alpha/\beta}).$$

Thus, under the additional assumption that  $\beta > 2\alpha$ , which corresponds to  $m > 2p - 1$ , we obtain that

$$\lim_{t \rightarrow T} (T - t)^{1/\beta} v(t, y) = \eta(y), \quad \text{and} \quad \lim_{t \rightarrow T} (T - t)^{1/\beta} Dv(t, y) = D\eta(y). \quad (2.12)$$

The proof of the following theorem is given in Section 4.

**Theorem 2.5.** *Let  $m > 2p - 1$ . Under Assumptions (L.1)-(L.4), (F.2)-(F.3), the unique nonnegative viscosity solution  $v$  to the singular terminal value problem (2.11) belongs to  $C^{0,1}([0, T^-] \times \mathbb{R}^d)$  and satisfies the asymptotics (2.12).*

The previously established regularity of the candidate value function is enough to carry out the verification argument, which is proven in Section 5.

**Theorem 2.6.** *Let  $m > 2p - 1$ . Under Assumptions (L.1)-(L.4), (F.2)-(F.3), let  $v \in C^{0,1}([0, T^-] \times \mathbb{R}^d)$  be the nonnegative viscosity solution to the singular terminal value problem (2.11). Then, the value function of the control problem (2.5) is given by  $V(t, y, x) = v(t, y)|x|^p$ , and the optimal control  $(\xi^*, \vartheta^*)$  is given in feedback form by*

$$\xi_s^* = \frac{v(s, Y_s^{t,y})^\beta}{\eta(Y_s^{t,y})^\beta} X_s^* \quad \text{and} \quad \vartheta_s^* = \theta^\alpha (1 + \alpha) |\sigma^*(Y_s^{t,y}) Dv(s, Y_s^{t,y})|^{\alpha-1} \sigma^*(Y_s^{t,y}) Dv(s, Y_s^{t,y}). \quad (2.13)$$

In particular, the resulting optimal portfolio process  $(X_s^*)_{s \in [t, T]}$  is given by

$$X_s^* = x \exp \left( - \int_t^s \frac{v(r, Y_r^{t,y})^\beta}{\eta(Y_r^{t,y})^\beta} dr \right). \quad (2.14)$$

*Remark 2.7.* The preceding results show that – as in [35] – the model with factor uncertainty is equivalent to the benchmark model (2.2) when the market risk factor  $\lambda$  is replaced

$$\lambda^H := \lambda + H(y, Dv(t, y)).$$

In particular, under model uncertainty the investor liquidates the asset at a faster rate.

We close this section with first order approximations of the value function and the optimal trading strategy for the model with uncertainty in terms of the solutions to the benchmark model without uncertainty. These results allow us to obtain the value function and optimal trading strategy based only on the benchmark model in the case of a small uncertainty-tolerance parameter. We first state our approximation result for the value function. The proof is given in Section 6.

**Theorem 2.8.** *Let  $m > 2p - 1$ . Let  $w = v(T - t)^{1/\beta}$  and  $w_0 = v_0(T - t)^{1/\beta}$  where  $v_0$  denotes the value function of the benchmark model. Under Assumptions (L.1)-(L.4), (F.2)-(F.3), we have that*

$$\lim_{\theta \rightarrow 0} \frac{w - w_0}{\theta^\alpha} = w_1, \quad (2.15)$$

on  $[0, T] \times \mathbb{R}^d$ , where  $w_1$  is a unique nonnegative solution to the PDE

$$\begin{cases} -\partial_t v(t, y) - \mathcal{L}v(t, y) - f_1(t, y, v(t, y)) = 0, & (t, y) \in [0, T] \times \mathbb{R}, \\ v(T, y) = 0, & y \in \mathbb{R}^d, \end{cases} \quad (2.16)$$

whose driver

$$f_1(t, y, v) = |\sigma^* Dv_0|^{1+\alpha} (T - t)^{1/\beta} - \frac{(\beta + 1)v_0^\beta}{\beta\eta^\beta} v + \frac{1}{\beta} \frac{v}{(T - t)}$$

depends on the solution to the benchmark model without factor uncertainty.

Theorem 2.8 allows us to derive a first order approximation of the optimal trading strategy under model uncertainty in terms of the solution to the benchmark model and the first order approximation to the value function.

**Corollary 2.9.** *Let  $m > 2p - 1$ . Let  $v_1 = \frac{w_1}{(T-t)^{1/\beta}}$  and let  $v_0$  and  $\xi^{0,*}$  be the value function and the optimal strategy in the benchmark model, respectively. Under Assumptions (L.1)-(L.4), (F.2)-(F.3),*

$$\lim_{\theta \rightarrow 0} \frac{\xi^* - \xi^{0,*}}{\theta^\alpha} = \tilde{\xi}, \text{ locally uniformly on } [t, T], \quad (2.17)$$

where  $\tilde{\xi} \in L^\infty(t, T; \mathbb{R})$  is defined by

$$\tilde{\xi}_s := \beta \xi_s^{0,*} \left( \frac{v_1(s, Y_s^{t,y})}{v_0(s, Y_s^{t,y})} - \int_t^s v_0(r, Y_r^{t,y})^{\beta-1} v_1(r, Y_r^{t,y}) dr \right), \quad s \in [t, T]. \quad (2.18)$$

The dependence of the relative error  $\| \frac{v - v_0 - \theta^\alpha v_1}{v} \|_\infty$  of the first order approximation for the value function is shown in Figure 1. Parameters are chosen as

$$b(y) = -y, \quad \sigma \equiv 1, \quad \eta(y) = \tanh(-y) + 2, \quad \sigma(y) = e^{-y^2}, \quad p = 2, \quad m = 5, \quad x = 1, y = 1, \quad T = 1.$$

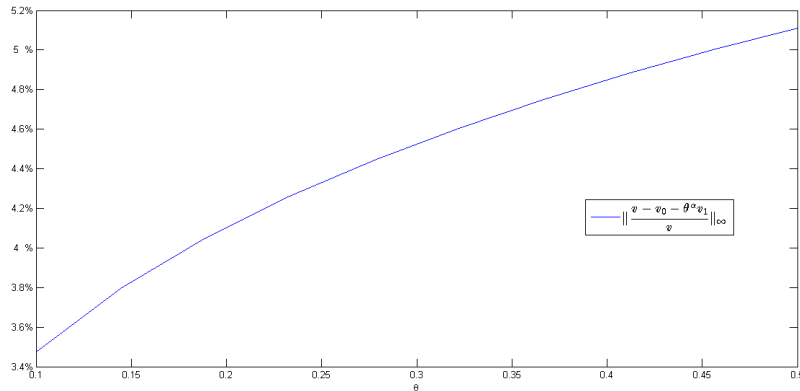


Figure 1: Relative error of the first order approximation: value function

Figure 2 displays the expected relative error  $\mathbb{E} \left[ \max_{t \in [0, \tau]} \left| \frac{\xi_t^* - \xi_t^{0,*} - \theta^\alpha \tilde{\xi}_t}{\xi_t^*} \right| \right]$  for the trading strategy where  $\tau = 0.9$ . The simulations suggest that both the value function and the optimal strategy are well approximated, even for relatively large uncertainty tolerance parameters with the relative errors staying within a 5% range for  $\theta \leq 0.5$ . The reason we consider the relative error for the trading strategy only away from the terminal time is the uncertain singularity arising in the absolute error that leads to the locally uniform convergence.

### 3 Viscosity solution

In this section, we prove Theorem 2.4. The proof uses modifications of arguments given in [28]. In a first step, we establish a comparison principle for semicontinuous viscosity solutions to (2.11). Due to the terminal state constraint we cannot follow the usual approach of showing that if a l.s.c. supersolution dominates an u.s.c. subsolution at the boundary, then it also dominates the subsolution on the entire domain. Instead, we prove that if some form of asymptotic dominance holds at the terminal time, then it holds near the terminal time.

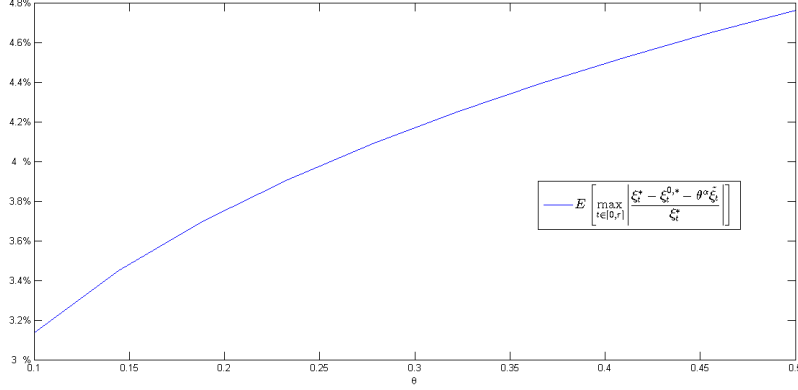


Figure 2: Relative error of the first order approximation: trading strategy

In a second step, we construct a smooth sub- and a supersolution to (2.11) satisfying the required assumptions. Using Perron's method, we can then establish the existence of an upper semi-continuous subsolution and of a lower semi-continuous supersolution, which are bounded by the respective smooth solutions. In particular, the semi-continuous solutions can be applied to the comparison principle. This establishes the existence of the desired continuous solution.

We start with the following comparison principle. The proof is given in Section A.2. We emphasize that the comparison principle will only be used to prove the existence of a viscosity solution. This justifies the rather strong assumptions (3.1) and (3.2) below.

**Proposition 3.1.** *Assume that Assumptions (L.1)-(L.3), (F.1) and (F.2) hold. Let  $\tilde{m}$  be as in condition (F.1). Fix  $\delta \in (0, T]$ . Let  $\bar{u} \in LSC_{\tilde{m}}([T - \delta, T^-] \times \mathbb{R}^d)$  and  $\underline{u} \in USC_{\tilde{m}}([T - \delta, T^-] \times \mathbb{R}^d)$  be a nonnegative viscosity super- and a viscosity subsolution to (2.11), respectively. If, uniformly on  $\mathbb{R}^d$ ,*

$$\limsup_{t \rightarrow T} \frac{\underline{u}(t, y)(T - t)^{1/\beta} - \eta(y)}{\langle y \rangle^{\tilde{m}}} \leq 0 \leq \liminf_{t \rightarrow T} \frac{\bar{u}(t, y)(T - t)^{1/\beta} - \eta(y)}{\langle y \rangle^{\tilde{m}}}, \quad (3.1)$$

and

$$\sqrt[\beta]{\frac{\frac{1}{2}\beta + 1}{\beta + 1}} \eta(y) \leq \underline{u}(t, y)(T - t)^{1/\beta}, \quad \bar{u}(t, y)(T - t)^{1/\beta} \leq C \langle y \rangle^{\tilde{m}}, \quad t \in [T - \delta, T), \quad (3.2)$$

for a constant  $C$ , then

$$\underline{u} \leq \bar{u} \quad \text{on} \quad [T - \delta, T) \times \mathbb{R}^d.$$

We are now going to construct smooth sub- and supersolutions to (2.11) that satisfy the conditions (3.1) and (3.2) of the above proposition. The supersolution will be defined in terms of the function

$$\hat{h}(t, y) := L(T - t) \langle y \rangle^{\tilde{m}} \quad (3.3)$$

where  $\tilde{m}$  is introduced in condition (F.1), and where the constant  $L$  will be determined later. Using the condition (F.1), we can find a constant  $C_0 > 0$  such that

$$\begin{aligned} & -\partial_t \hat{h}(t, y) - \mathcal{L}\hat{h}(t, y) - 2^\alpha \theta^\alpha \bar{C}^{\alpha+1} |D\hat{h}(t, y)|^{\alpha+1} - \lambda(y) \\ & \geq L \langle y \rangle^{\tilde{m}} - C_0 L (T - t) \langle y \rangle^{\tilde{m}} - C_0 L^{\alpha+1} (T - t)^{\alpha+1} \langle y \rangle^{\tilde{m}} - C_0 \langle y \rangle^{\tilde{m}}. \end{aligned} \quad (3.4)$$

Choosing  $L > 3C_0$  and then  $\tau = \frac{1}{L}$ , we get that

$$-\partial_t \hat{h}(t, y) - \mathcal{L}\hat{h}(t, y) - 2^\alpha \bar{C}^{\alpha+1} |D\hat{h}(t, y)|^{\alpha+1} - \lambda(y) \geq 0, \quad (t, y) \in [T - \tau, T) \times \mathbb{R}^d. \quad (3.5)$$

**Lemma 3.2.** *Suppose that Assumptions (L.1)-(L.3), (F.1) and (F.2) hold. Let  $\epsilon := 1 - \alpha/\beta$ . There exist constants  $K > 0, \delta \in (0, T]$  such that*

$$\check{v}(t, y) := \frac{\eta(y) - \eta(y) \|\frac{\mathcal{L}\eta}{\eta}\| (T-t)}{(T-t)^{1/\beta}} \quad (3.6)$$

and

$$\hat{v}(t, y) := \frac{\eta(y) + \eta(y)K(T-t)^\epsilon}{(T-t)^{1/\beta}} + \hat{h}(t, y) \quad (3.7)$$

are a nonnegative classical sub- and supersolution to (2.11) on  $[T-\delta, T] \times \mathbb{R}^d$ , respectively. Furthermore,  $\check{v}, \hat{v}$  satisfy the conditions (3.1) and (3.2).

*Proof.* In view of (F.2), the quantity  $\|\frac{\mathcal{L}\eta}{\eta}\|$  is well-defined and finite; hence  $\delta_0 := 1/\|\frac{\mathcal{L}\eta}{\eta}\| > 0$ . It has been shown in [28] that  $\check{v}$  is a subsolution to (2.11) on  $[T-\delta_0, T] \times \mathbb{R}^d$  when  $H = 0$ . Since  $H$  is nonnegative, we know that  $\check{v}$  is still a subsolution on  $[T-\delta_0, T] \times \mathbb{R}^d$ . We now verify that  $\hat{v}$  is a nonnegative classical supersolution to (2.11) on  $[T-\delta_1, T] \times \mathbb{R}^d$  for small  $\delta_1$ . To this end, we first obtain by a direct computation that

$$\begin{aligned} -\partial_t \hat{v}(t, y) - \mathcal{L} \hat{v}(t, y) &= -\frac{\eta(y) + K(1-\beta\epsilon)\eta(y)(T-t)^\epsilon + \beta\mathcal{L}\eta(y)(T-t)(1+K(T-t)^\epsilon)}{\beta(T-t)^{(\beta+1)/\beta}} \\ &\quad - \partial_t \hat{h}(t, y) - \mathcal{L} \hat{h}(t, y). \end{aligned}$$

Assuming that  $K\delta_1^\epsilon \leq 1$  and  $\delta_1 \leq 1$ , we see that  $K(T-t)^\epsilon \leq 1$  and  $(T-t)^{1-\epsilon} \leq 1$  for  $t \in [T-\delta_1, T]$ . Thus,

$$\begin{aligned} -\partial_t \hat{v}(t, y) - \mathcal{L} \hat{v}(t, y) &\geq -\frac{\eta(y) + K(1-\beta\epsilon)\eta(y)(T-t)^\epsilon + 2\beta\bar{C}\eta(y)(T-t)^\epsilon}{\beta(T-t)^{(\beta+1)/\beta}} \\ &\quad - \partial_t \hat{h}(t, y) - \mathcal{L} \hat{h}(t, y). \end{aligned} \quad (3.8)$$

Recalling the definition of  $H$  and  $F$  in (2.10),

$$\begin{aligned} -H(y, D\hat{v}(t, y)) &\geq -2^\alpha \theta^\alpha \bar{C}^{\alpha+1} \frac{|D\eta|^{\alpha+1} [1 + K(T-t)^\epsilon]^{\alpha+1}}{(T-t)^{(1+\alpha)/\beta}} - 2^\alpha \theta^\alpha \bar{C}^{\alpha+1} |D\hat{h}(t, y)|^{\alpha+1} \\ &\geq -2^\alpha \theta^\alpha \bar{C}^{\alpha+1} \left\| \frac{|D\eta|^{\alpha+1}}{\eta} \right\| \left\| \eta(y) \frac{[1 + K(T-t)^\epsilon]^{\alpha+1}}{(T-t)^{(1+\alpha)/\beta}} - 2^\alpha \theta^\alpha \bar{C}^{\alpha+1} |D\hat{h}(t, y)|^{\alpha+1} \right\| \\ &\geq -2^{2\alpha+1} \theta^\alpha \bar{C}^{\alpha+2} \frac{\eta(y)}{(T-t)^{(1+\alpha)/\beta}} - 2^\alpha \theta^\alpha \bar{C}^{\alpha+1} |D\hat{h}(t, y)|^{\alpha+1}. \end{aligned} \quad (3.9)$$

Applying Bernoulli's inequality in the form  $(u+v+w)^{\beta+1} \geq u^{\beta+1} + (\beta+1)u^\beta v$  for  $u, v, w \geq 0$  to the term  $|\hat{v}(t, y)|^{\beta+1}$  in  $F$ , we obtain

$$-F(y, \hat{v}(t, y)) \geq -\lambda(y) + \frac{\eta(y)^{\beta+1} + (\beta+1)\eta(y)^\beta \eta(y)K(T-t)^\epsilon}{\beta\eta(y)^\beta (T-t)^{(\beta+1)/\beta}}. \quad (3.10)$$

Hence, adding (3.8), (3.9) and (3.10) and using (3.5) yields,

$$\begin{aligned} &-\partial_t \hat{v}(t, y) - \mathcal{L} \hat{v}(t, y) - H(y, D\hat{v}(t, y)) - F(y, \hat{v}(t, y)) \\ &\geq \eta(y) \frac{(1+\epsilon)K - 2\bar{C} - 2^{2\alpha+1}\theta^\alpha \bar{C}^{\alpha+2}}{(T-t)^{(1+\alpha)/\beta}} \\ &\quad - \partial_t \hat{h}(t, y) - \mathcal{L} \hat{h}(t, y) - 2^\alpha \theta^\alpha \bar{C}^{\alpha+1} |D\hat{h}(t, y)|^{\alpha+1} - \lambda(y) \\ &\geq \eta(y) \frac{(1+\epsilon)K - 2\bar{C} - 2^{2\alpha+1}\theta^\alpha \bar{C}^{\alpha+2}}{(T-t)^{(1+\alpha)/\beta}}. \end{aligned} \quad (3.11)$$

Choosing  $K \geq \frac{2\bar{C} + 2^{2\alpha+1}\theta^\alpha \bar{C}^{\alpha+2}}{1+\epsilon}$  and then  $\delta_1 = \min\{1, \frac{1}{L}, \sqrt[\beta]{\frac{1}{K}}\}$ , we conclude that

$$-\partial_t \hat{v}(t, y) - \mathcal{L} \hat{v}(t, y) - H(y, D\hat{v}(t, y)) - F(y, \hat{v}(t, y)) \geq 0, \quad (t, y) \in [T-\delta_1, T] \times \mathbb{R}^d.$$

Next, we prove that  $\check{v}, \hat{v}$  satisfy the asymptotic behavior (3.1) and (3.2). Recalling the definition of  $\check{v}, \hat{v}$  and using the condition (F.1), we have

$$\begin{aligned} (T-t)^{1/\beta} \check{v}(t, y) &= \eta(y) + \langle y \rangle^{\tilde{m}} O(T-t), & \text{uniformly in } y \text{ as } t \rightarrow T. \\ (T-t)^{1/\beta} \hat{v}(t, y) &= \eta(y) + \langle y \rangle^{\tilde{m}} O((T-t)^\epsilon), & \text{uniformly in } y \text{ as } t \rightarrow T. \end{aligned} \quad (3.12)$$

From this, we see that

$$\lim_{t \rightarrow T} \frac{\check{v}(t, y)(T-t)^{1/\beta} - \eta(y)}{\langle y \rangle^{\tilde{m}}} = \lim_{t \rightarrow T} \frac{\hat{v}(t, y)(T-t)^{1/\beta} - \eta(y)}{\langle y \rangle^{\tilde{m}}} = 0, \quad \text{uniformly on } \mathbb{R}^d, \quad (3.13)$$

which verifies the condition (3.1). The upper bound in (3.2) can be obtained using the condition (F.1) again. Moreover, for the lower bound in (3.2), choosing  $\delta := \min\{\delta_0(1 - \sqrt[\beta]{\frac{1}{2}\beta+1}), \delta_1\}$ , we have that for all  $(t, y) \in [T-\delta, T) \times \mathbb{R}^d$ ,

$$\hat{v}(t, y)(T-t)^{1/\beta} \geq \eta(y) \geq \check{v}(t, y)(T-t)^{1/\beta} = \eta(y) - \eta(y) \left\| \frac{\mathcal{L}\eta}{\eta} \right\| (T-t) \geq \sqrt[\beta]{\frac{1}{2}\beta+1} \eta(y).$$

□

*Remark 3.3.* Due to the presence of the gradient term  $H$ , an additional term (3.9) needs to be dominated and thus we make the choice that  $\epsilon = 1 - \alpha/\beta$ . If  $H = 0$ , we can choose  $\epsilon = 1$  as in [28].

We are now ready to prove the existence result.

*Proof of Theorem 2.4.* In order to apply Perron's method, we set

$$\mathcal{S} = \{u \mid u \text{ is a subsolution of (2.11) on } [T-\delta, T) \times \mathbb{R}^d \text{ and } u \leq \hat{v}\}.$$

Since  $\check{v} \in \mathcal{S}$ , the set  $\mathcal{S}$  is non-empty. Thus, the function

$$v(t, y) = \sup\{u(t, y) : u \in \mathcal{S}\}$$

is well-defined and satisfies that  $\check{v} \leq v$ . Classical arguments<sup>4</sup> show that the upper semi-continuous envelope  $v^*$  of  $v$  is a viscosity subsolution to (2.11). From [44, Lemma A.2], the lower semi-continuous envelope  $v_*$  of  $v$  is a viscosity supersolution to (2.11). Since  $\check{v} \leq v_* \leq v^* \leq \hat{v}$ , we have for all  $(t, y) \in [T-\delta, T) \times \mathbb{R}^d$  that

$$\sqrt[\beta]{\frac{1}{2}\beta+1} \eta(y) \leq v_*(t, y)(T-t)^{1/\beta}, v^*(t, y)(T-t)^{1/\beta} \leq C \langle y \rangle^{\tilde{m}},$$

and

$$\begin{aligned} \frac{\check{v}(t, y)(T-t)^{1/\beta} - \eta(y)}{\langle y \rangle^{\tilde{m}}} &\leq \frac{v_*(t, y)(T-t)^{1/\beta} - \eta(y)}{\langle y \rangle^{\tilde{m}}} \leq \frac{v^*(t, y)(T-t)^{1/\beta} - \eta(y)}{\langle y \rangle^{\tilde{m}}} \\ &\leq \frac{\hat{v}(t, y)(T-t)^{1/\beta} - \eta(y)}{\langle y \rangle^{\tilde{m}}}. \end{aligned}$$

Hence, it follows from (3.13) that

$$\lim_{t \rightarrow T} \frac{v_*(t, y)(T-t)^{1/\beta} - \eta(y)}{\langle y \rangle^{\tilde{m}}} = \lim_{t \rightarrow T} \frac{v^*(t, y)(T-t)^{1/\beta} - \eta(y)}{\langle y \rangle^{\tilde{m}}} = 0, \quad \text{uniformly on } \mathbb{R}^d. \quad (3.14)$$

From our comparison principle [Proposition 3.1] we can thus conclude that  $v^* \leq v_*$  on  $[T-\delta, T) \times \mathbb{R}^d$ , which shows that  $v$  is the desired viscosity solution to (2.11) that belongs to  $C_{\tilde{m}}([T-\delta, T^-] \times \mathbb{R}^d)$ .

<sup>4</sup>The standard Perron method of finding viscosity solutions for elliptic PDEs can be found in [16]. We refer to [44, Appendix A] for the proof of this method for parabolic equations.

Next, we find a sub- and supersolution to (2.11) on  $[0, T - \delta] \times \mathbb{R}^d$  with terminal value  $v(T - \delta, \cdot)$  at  $t = T - \delta$ . Obviously, 0 is a subsolution of (2.11). We now conjecture that there exists a constant  $\bar{L} > 0$  such that  $\bar{w} := \bar{L}\langle y \rangle^{\bar{m}}$  is a viscosity supersolution to (2.11). In fact, since  $v \leq \hat{v}$  at  $t = T - \delta$ , we see that

$$v(T - \delta, y) \leq \frac{\bar{C}}{\delta^{1/\beta}} \eta(y) + \langle y \rangle^{\bar{m}} \leq \left( \frac{\bar{C}^2}{\delta^{1/\beta}} + 1 \right) \langle y \rangle^{\bar{m}}, \quad y \in \mathbb{R}^d.$$

Let  $h(y) := \langle y \rangle^{\bar{m}}$ . In view of the condition (F.1), we have that

$$\begin{aligned} & -\partial_t \bar{w}(t, y) - \mathcal{L}\bar{w}(t, y) - H(y, D\bar{w}) - F(y, \bar{w}(t, y)) \\ & \geq -\bar{L}\mathcal{L}h(y) - \theta^\alpha \bar{C}^{\alpha+1} \bar{L}^{\alpha+1} |Dh|^{\alpha+1} - \lambda(y) + \bar{L}^{\beta+1} \frac{h(y)^{\beta+1}}{\beta \eta(y)^\beta} \\ & \geq h(y) \left( \frac{1}{\beta \bar{C}^\beta} \bar{L}^{\beta+1} - C_0 \bar{L} - C_0 \bar{L}^{\alpha+1} - C_0 \right), \end{aligned}$$

where the constants  $C_0$  and  $\bar{C}$  are chosen as in (3.4) and (F.1), respectively. Choosing  $\bar{L}$  large enough, we have that

$$-\partial_t w(t, y) - \mathcal{L}w(t, y) - H(y, Dw) - F(y, w(t, y)) > 0.$$

Furthermore,  $\bar{w}^{\beta+1}/\eta^\beta$  is of polynomial growth of order  $m$ . Combining the general comparison principle [Proposition A.1] with Perron's method, we obtain a viscosity solution  $v \in C_{\bar{m}}([0, T - \delta] \times \mathbb{R}^d)$ . From the comparison principle for continuous viscosity solutions [Lemma A.3], we get a unique global viscosity solution  $v \in C_{\bar{m}}([0, T^-] \times \mathbb{R}^d)$ .  $\square$

## 4 Regularity of the viscosity solution

This section is devoted to the proof of Theorem 2.5. We assume throughout that Assumptions (L.1)-(L.4) and (F.2)-(F.3) are satisfied and that  $\beta > 2\alpha$ . In this case  $\bar{m} = 0$  and the viscosity solution  $v$  obtained in the previous section belongs to  $C_b([0, T^-] \times \mathbb{R}^d)$ .

Unlike in [28], continuity is not enough to carry out our verification argument. We need to prove that  $v$  is of class  $C^{0,1}$  and satisfies (2.12). The desired regularity of the viscosity solution *away from the terminal time* can be established using classical PDE results or the standard link between viscosity solutions and forward-backward SDEs once the desired regularity *near the terminal time* has been established. The key challenge is thus to prove that there exists some  $\delta > 0$  such that for  $(t, y) \in [T - \delta, T] \times \mathbb{R}^d$  the gradient  $Dv(t, y)$  exists and satisfies

$$|Dv(t, y)| \leq \frac{C}{(T - t)^{1/\beta}}.$$

### 4.1 Regularity near the terminal time

In view of the definition of  $\epsilon$  in Lemma 3.2, we know that  $\epsilon = 1 - \frac{\alpha}{\beta} \in (\frac{1}{2}, 1)$ . Recalling the asymptotic behavior (3.12) of the super- and subsolution, the viscosity solution  $v$  constructed in the previous section is of the form

$$v(T - t, y) = \frac{\eta(y) + \tilde{u}(t, y)}{t^{1/\beta}}, \quad (4.1)$$

for some function  $\tilde{u}$  that satisfies

$$\tilde{u}(t, y) = O(t^\epsilon) \text{ uniformly in } y \text{ as } t \rightarrow 0.$$

We choose the following equivalent ansatz:

$$v(T - t, y) = \frac{\eta(y)}{t^{1/\beta}} + \frac{u(t, y)}{t^{1+1/\beta}}, \quad u(t, y) = O(t^{1+\epsilon}) \text{ uniformly in } y \text{ as } t \rightarrow 0. \quad (4.2)$$

It is worth pointing out that if  $H = 0$ , we can choose  $\epsilon = 1$  in (4.1) and (4.2). Plugging the asymptotic ansatz into (2.11) results in a semilinear parabolic equation for  $u$  with finite initial condition. The proof of the following lemma is similar to [26, Lemma 4.1] and hence omitted.

**Lemma 4.1.** *If, for some  $\delta > 0^5$ , a function  $u$  satisfies*

$$|u(t, y)| \leq t\eta(y), \quad t \in [0, \delta], y \in \mathbb{R}^d, \quad (4.3)$$

*and solves the equation*

$$\begin{cases} \partial_t u(t, y) = \mathcal{L}u(t, y) + F_0(t, y, u(t, y), Du(t, y)), & t \in (0, \delta], y \in \mathbb{R}^d, \\ u(0, y) = 0, & y \in \mathbb{R}^d, \end{cases} \quad (4.4)$$

*where*

$$\begin{aligned} F_0(t, y, u, Du) = & t\mathcal{L}\eta(y) + t^p\lambda(y) - \frac{\eta(y)}{\beta} \sum_{k=2}^{\infty} \binom{\beta+1}{k} \left(\frac{u}{t\eta(y)}\right)^k \\ & + \theta^\alpha t^\epsilon \left| \sigma^*(y) \left(\frac{Du}{t} + D\eta\right) \right|^{\alpha+1}, \end{aligned}$$

*then a local solution  $v \in C^{0,1}([T-\delta, T^-] \times \mathbb{R}^d)$  to problem (2.11) is given by*

$$v(t, y) = \frac{\eta(y)}{(T-t)^{1/\beta}} + \frac{u(T-t, y)}{(T-t)^{1+1/\beta}}.$$

The case  $H = 0$  has been solved under additional regularity assumptions in [26] using an analytic semigroup approach. Due to the presence of  $H$  in our case, we need to choose  $\epsilon < 1$ , which renders the analysis more complex. In particular, the locally Lipschitz continuity in [26, Lemma 4.5] no longer holds in our case. Instead, we solve equation (4.4) using the weak continuous semigroup approach introduced in [18, Section 4] in order to obtain a solution in a space of functions with the desired asymptotic behavior near the initial time.

In a first step we introduce the transition semigroup. Under Assumptions (L.1) and (L.2), the operator

$$P_{t,s}[\varphi](y) = \mathbb{E}[\varphi(Y_s^{t,y})], \quad \varphi \in C_b(\mathbb{R}^d), 0 \leq t \leq s$$

is well-defined and satisfies the Markov property  $P_{t,r} = P_{t,s}P_{s,r}$  for  $0 \leq t \leq s \leq r$ . Since  $b$  and  $\sigma$  are independent of the time variable,

$$P_{t,s}[\varphi](y) = P_{0,s-t}[\varphi](y).$$

For convenience, we denote

$$P_t[\varphi](y) = \mathbb{E}[\varphi(Y_t^{0,y})], \quad \varphi \in C_b(\mathbb{R}^d). \quad (4.5)$$

For every  $\varphi \in C_b(\mathbb{R}^d)$ ,

$$|P_t[\varphi](y)| \leq \|\varphi\|, \quad (t, y) \in [0, T] \times \mathbb{R}^d. \quad (4.6)$$

Furthermore, from [18, Theorem 4.65], we have the following proposition.

**Proposition 4.2.** *Suppose that Assumptions (L.1)-(L.4) hold and let  $\varphi \in C_b(\mathbb{R}^d)$ . Then for every  $0 \leq t \leq T$ , the function  $y \rightarrow P_t[\varphi](y)$  is continuously differentiable on  $\mathbb{R}^d$ . Moreover, there exists a constant  $M > 0$  such that for every  $\varphi \in C_b(\mathbb{R}^d)$  and for  $0 \leq t \leq T$ ,*

$$|DP_t[\varphi](y)| \leq \frac{M}{t^{1/2}} \|\varphi\|, \quad y \in \mathbb{R}^d. \quad (4.7)$$

Next, we introduce the notion of a mild solution of our modified PDE.

**Definition 4.3.** We say that a function  $u : [0, \delta] \times \mathbb{R}^d \rightarrow \mathbb{R}$  is a mild solution of the PDE (4.4) if the following conditions are satisfied:

---

<sup>5</sup>For convenience, we use here the same symbol as in Section 3. We can always choose the smaller one to define  $\delta$ .



(i)  $u \in C_b^{0,1}([0, \delta] \times \mathbb{R}^d)$ .

(ii) for every  $t \in [0, T]$  and  $y \in \mathbb{R}^d$ ,

$$u(t, y) = \int_0^t P_{t-s}[F_0(s, \cdot, u(s, \cdot), Du(s, \cdot))](y) ds. \quad (4.8)$$

We prove the existence of a mild solution to (4.4) by a contraction argument. To this end, we need to choose an appropriate weighted norm on  $C_b^{0,1}([0, \delta] \times \mathbb{R}^d)$  to cope with the singularity in  $F_0$ . Recalling the ansatz (4.2) and the property (4.7), we consider the space

$$\Sigma := \left\{ u \in C_b^{0,1}([0, \delta] \times \mathbb{R}^d) : \|u(t, \cdot)\| + \|t^{1/2} Du(t, \cdot)\| = O(t^{1+\epsilon}) \text{ as } t \rightarrow 0 \right\} \quad (4.9)$$

endowed with the weighted norm

$$\|u\|_\Sigma = \sup_{(t,y) \in (0,\delta] \times \mathbb{R}^d} \left( \frac{|u(t,y)|}{t^{1+\epsilon}} + \frac{|Du(t,y)|}{t^{1/2+\epsilon}} \right).$$

It is easy to verify that the vector space  $\Sigma$  endowed with the norm  $\|\cdot\|_\Sigma$  is a Banach space.

**Lemma 4.4.** *Suppose that  $\beta > 2\alpha$  and that Assumptions (L.1)-(L.4) and (F.2)-(F.3) hold. Let  $R > 0$  and  $\delta \in (0, \epsilon^{-\frac{1}{\beta}} \sqrt{\underline{c}}/R \wedge 1]$ . Define the closed ball  $\bar{B}_\Sigma(R) := \{u \in \Sigma : \|u\|_\Sigma \leq R\}$ . For every  $u \in \bar{B}_\Sigma(R)$ , the function*

$$f_0(t, y) := F_0(t, y, u(t, y), Du(t, y))$$

*is continuous on  $[0, \delta] \times \mathbb{R}^d$ .*

*Proof.* For  $u \in \bar{B}_\Sigma(R)$ , we may decompose  $f_0(t, y)$  in the following way:

$$f_0(t, y) = t\mathcal{L}\eta(y) + t^p\lambda(y) - (p-1)\eta(y)g_0(t, y) + \theta^\alpha t^\epsilon g_1(t, y). \quad (4.10)$$

where

$$g_0(t, y) = \sum_{k=2}^{\infty} \binom{\beta+1}{k} \left( \frac{u(t, y)}{t\eta(y)} \right)^k \quad \text{and} \quad g_1(t, y) = \left| \sigma^*(y) \left( \frac{Du(t, y)}{t} + D\eta(y) \right) \right|^{\alpha+1}.$$

The assumption  $\delta \leq \epsilon^{-\frac{1}{\beta}} \sqrt{\underline{c}}/R$  guarantees that the series converges since then

$$\left| \frac{u(t, y)}{t\eta(y)} \right| \leq \frac{t^{1+\epsilon}R}{t\underline{c}} \leq \frac{\delta^\epsilon R}{\underline{c}} \leq 1, \quad t \in [0, \delta], y \in \mathbb{R}^d.$$

Moreover,

$$\left| \frac{Du(t, y)}{t} \right| \leq \frac{t^{\frac{1}{2}+\epsilon}R}{t} \leq \delta^{\epsilon-\frac{1}{2}}R \leq \underline{c}, \quad t \in [0, \delta], y \in \mathbb{R}^d. \quad (4.11)$$

In view of (4.10) it is sufficient to prove that  $g_0$  and  $g_1$  are continuous in  $t$ , uniformly with respect to  $y$  on every compact subset of  $\mathbb{R}^d$ . In fact, by the mean value theorem, we have for  $0 \leq t \leq s \leq \delta, y \in \mathbb{R}^d$  that

$$\begin{aligned} |g_1(t, y) - g_1(s, y)| &\leq \left| \left| \sigma^*(y) \left( \frac{Du(t, y)}{t} + D\eta(y) \right) \right|^{\alpha+1} - \left| \sigma^*(y) \left( \frac{Du(s, y)}{s} + D\eta(y) \right) \right|^{\alpha+1} \right| \\ &\leq (\alpha+1)\bar{C}^{\alpha+1}(\underline{c} + \bar{C})^\alpha \left| \frac{Du(t, y)}{t} - \frac{Du(s, y)}{s} \right|. \end{aligned}$$

In order to establish the continuity of  $g_0$ , notice that for every  $k \geq 2$  and  $0 \leq t \leq s \leq \delta, y \in \mathbb{R}^d$  it holds that

$$\begin{aligned}
& \left| \left( \frac{u(t, y)}{t\eta(y)} \right)^k - \left( \frac{u(s, y)}{s\eta(y)} \right)^k \right| \\
& \leq \frac{1}{\underline{c}^k} \left| \frac{u(t, y)}{t} - \frac{u(s, y)}{s} \right| \sum_{l=0}^{k-1} \left| \frac{u(t, y)}{t} \right|^l \left| \frac{u(s, y)}{s} \right|^{k-1-l} \\
& \leq \frac{R^{k-1}}{\underline{c}^k} \left| \frac{u(t, y)}{t} - \frac{u(s, y)}{s} \right| \sum_{l=0}^{k-1} t^{\epsilon l} s^{\epsilon(k-1-l)} \\
& \leq \frac{kR^{k-1}}{\underline{c}^k} \left| \frac{u(t, y)}{t} - \frac{u(s, y)}{s} \right| s^{(k-1)\epsilon} \\
& \leq \frac{k}{\underline{c}} \left( \frac{Rs^\epsilon}{\underline{c}} \right)^{k-1} \left| \frac{u(t, y)}{t} - \frac{u(s, y)}{s} \right|.
\end{aligned} \tag{4.12}$$

Using the identity  $k \binom{\beta+1}{k} = (\beta+1) \binom{\beta}{k-1}$ , we get that

$$|g_0(t, y) - g_0(s, y)| \leq (\beta+1) \max\{2^\beta - 1, \beta\} \frac{Rs^\epsilon}{\underline{c}^2} \left| \frac{u(t, y)}{t} - \frac{u(s, y)}{s} \right|.$$

The claim now follows from the fact that the maps  $(t, y) \mapsto \frac{u(t, y)}{t}, \frac{Du(t, y)}{t}$  are continuous on  $[0, \delta] \times \mathbb{R}^d$ .  $\square$

The following lemma can be established using similar arguments as above.

**Lemma 4.5.** *Suppose that  $\beta > 2\alpha$  and that Assumptions (L.1)-(L.4) and (F.2)-(F.3) hold. For every  $R > 0$  there exists a constant  $L > 0$  independent of  $\delta \in (0, \epsilon^{-\frac{1}{\beta}} \underline{c}/R]$  such that*

$$\begin{aligned}
& |F_0(t, y, u(t, y), Du(t, y)) - F_0(t, y, v(t, y), Dv(t, y))| \\
& \leq Lt^\epsilon \left( \frac{|u(t, y) - v(t, y)|}{t} + \frac{|Du(t, y) - Dv(t, y)|}{t} \right), \quad u, v \in \overline{B}_\Sigma(R), \quad t \in [0, \delta], y \in \mathbb{R}^d.
\end{aligned}$$

We are now ready to carry out the fixed point argument.

**Theorem 4.6.** *Let  $\beta > 2\alpha$ . Under Assumptions (L.1)-(L.4) and (F.2)-(F.3), there exists a constant  $\delta > 0$  such that Equation (4.4) admits a mild solution  $u$  in the space  $\Sigma$  defined in (4.9).*

*Proof.* Let us define the operator

$$\Gamma[u](t, y) := \int_0^t P_{t-s} [F_0(s, \cdot, u(s, \cdot), Du(s, \cdot))](y) ds \tag{4.13}$$

STEP 1: THE MAP  $\Gamma$  IS WELL DEFINED ON THE CLOSED BALL  $\overline{B}_\Sigma(R)$ . Let  $u \in \overline{B}_\Sigma(R)$ . By Lemma 4.4 and [18, Proposition 4.67]<sup>6</sup>, we see that  $\Gamma[u] \in C_b([0, \delta] \times \mathbb{R}^d)$  and  $D\Gamma[u] \in C_b([0, \delta] \times \mathbb{R}^d)$ . In order to see the continuity of  $D\Gamma[u]$  at  $t = 0$ , we differentiate (4.13) to obtain that

$$D\Gamma[u](t, y) = \int_0^t DP_{t-s} [F_0(s, \cdot, u(s, \cdot), Du(s, \cdot))](y) ds, \quad (t, y) \in [0, \delta] \times \mathbb{R}^d. \tag{4.14}$$

By Proposition 4.2,

$$|D\Gamma[u](t, y)| \leq \int_0^t M \frac{\|f_0\|}{(t-s)^{1/2}} ds = \sqrt{t} M \|f_0\|.$$

From this, we conclude that the map  $(t, y) \mapsto D\Gamma[u](t, y)$  belongs to  $C_b([0, \delta] \times \mathbb{R}^d)$ .

<sup>6</sup>The strong continuity in this proposition is equivalent to the standard continuity in finite-dimensional space.

STEP 2: CONTRACTION PROPERTY OF  $\Gamma$  ON  $\overline{B}_\Sigma(R)$  FOR A SUITABLE CHOICE OF  $R, \delta$ . Let

$$B(a, b) := \int_0^1 r^{a-1}(1-r)^{b-1} dr$$

be the Beta function with  $a, b > 0$ . We choose

$$R = 2(1 + MB_0) (\|\mathcal{L}\eta\| + \|\lambda\| + \theta^\alpha \|\sigma^* D\eta\|^{\alpha+1}),$$

and

$$\delta = \min\left\{\epsilon^{-\frac{1}{4}}\sqrt{\underline{c}/R}, \epsilon^{-\frac{1}{4}}\sqrt{1/(2L(1 + MB_1))}, 1\right\},$$

where  $L > 0$  is the Lipschitz constant given by Lemma 4.5 and

$$B_0 := B\left(1 + \epsilon, \frac{1}{2}\right), \quad B_1 := B\left(2\epsilon + \frac{1}{2}, \frac{1}{2}\right).$$

Let  $u, v \in \overline{B}_\Sigma(R)$ . By Lemma 4.5, we have for  $(t, y) \in [0, \delta] \times \mathbb{R}^d$  that

$$\begin{aligned} & |\Gamma[u](t, y) - \Gamma[v](t, y)| \\ &= \left| \int_0^t P_{t-s} [F_0(s, \cdot, u(s, \cdot), Du(s, \cdot)) - F_0(s, \cdot, v(s, \cdot), Dv(s, \cdot))](y) ds \right| \\ &\leq \int_0^t \|F_0(s, y, u(s, \cdot), Du(s, \cdot)) - F_0(s, \cdot, v(s, \cdot), Dv(s, \cdot))\| ds \\ &\leq \int_0^t L s^\epsilon \left( \frac{\|u(s, \cdot) - v(s, \cdot)\|}{s} + \frac{\|Du(s, \cdot) - Dv(s, \cdot)\|}{s} \right) ds \\ &= \int_0^t L \left( s^{2\epsilon} \frac{\|u(s, \cdot) - v(s, \cdot)\|}{s^{1+\epsilon}} + s^{2\epsilon-1/2} \frac{\|Du(s, \cdot) - Dv(s, \cdot)\|}{s^{1/2+\epsilon}} \right) ds \\ &\leq L t^{2\epsilon+1/2} \|u - v\|_\Sigma. \end{aligned}$$

Similarly,

$$\begin{aligned} & |D\Gamma[u](t, y) - D\Gamma[v](t, y)| \\ &= \left| \int_0^t DP_{t-s} [F_0(s, \cdot, u(s, \cdot), Du(s, \cdot)) - F_0(s, \cdot, v(s, \cdot), Dv(s, \cdot))](y) ds \right| \\ &\leq M \int_0^t \frac{1}{(t-s)^{1/2}} \|F_0(s, y, u(s, \cdot), Du(s, \cdot)) - F_0(s, \cdot, v(s, \cdot), Dv(s, \cdot))\| ds \\ &\leq \int_0^t ML \frac{1}{(t-s)^{1/2}} \left( s^{2\epsilon-1/2} \|u - v\|_\Sigma \right) ds \\ &\leq ML B_1 t^{2\epsilon} \|u - v\|_\Sigma. \end{aligned}$$

Hence

$$\|\Gamma[u] - \Gamma[v]\|_\Sigma \leq \frac{1}{2} \|u - v\|_\Sigma.$$

STEP 3:  $\Gamma$  MAPS  $\overline{B}_\Sigma(R)$  INTO ITSELF. Note that  $s^k \leq 1$  for all  $k > 0$  and  $s \in [0, \delta]$  since  $\delta \leq 1$ . Hence, it holds for every  $t \in [0, \delta]$  that

$$\begin{aligned} |\Gamma[0](t, y)| &= \left| \int_0^t P_{t-s} [F_0(s, \cdot, 0, 0)](y) ds \right| \\ &\leq \int_0^t \|s\mathcal{L}\eta + s^p\lambda + \theta^\alpha s^\epsilon |\sigma^* D\eta|^{\alpha+1}\| ds \\ &\leq t^{1+\epsilon} (\|\mathcal{L}\eta\| + \|\lambda\| + \|\sigma^* D\eta\|^{\alpha+1}), \end{aligned}$$

and

$$\begin{aligned}
|D\Gamma[0](t, y)| &= \left| \int_0^t DP_{t-s}[F_0(s, \cdot, 0, 0)](y) ds \right| \\
&\leq \int_0^t \frac{1}{(t-s)^{1/2}} M \|s\mathcal{L}\eta + s^p\lambda + \theta^\alpha s^\epsilon |\sigma^* D\eta|^{\alpha+1}\| ds \\
&\leq t^{1+\epsilon-1/2} MB_0 (\|\mathcal{L}\eta\| + \|\lambda\| + \theta^\alpha \|\sigma^* D\eta\|^{\alpha+1}).
\end{aligned}$$

Thus,

$$\|\Gamma[u]\|_\Sigma \leq \|\Gamma[u] - \Gamma[0]\|_\Sigma + \|\Gamma[0]\|_\Sigma \leq R.$$

Hence,  $\Gamma$  is a contraction from  $\overline{B}_\Sigma(R)$  to itself and has a unique fixed point  $u$  in  $\overline{B}_\Sigma(R)$ .  $\square$

## 4.2 Regularity away from the terminal time

In this section we finish the proof of Theorem 2.5. We also provide a standard link between our viscosity solution and a class of singular FBSDEs that will be useful for proving the verification argument.

*Proof of Theorem 2.5.* Combining Lemma 4.1 with Theorem 4.6, we know that there exists a mild solution  $w \in C_b^{0,1}([T-\delta, T^-] \times \mathbb{R}^d)$  defined by

$$w(t, y) := \frac{\eta(y)}{(T-t)^{1/\beta}} + \frac{u(T-t, y)}{(T-t)^{1+1/\beta}}, \quad \text{on } [T-\delta, T) \times \mathbb{R}^d.$$

to the equation (2.11). By [12, Theorem 15], this implies the existence of a solution  $w$  in the space  $C_b^{0,1}([0, T-\delta] \times \mathbb{R}^d)$  to the PDE

$$\begin{cases} -\partial_t v(t, y) - \mathcal{L}v(t, y) - \lambda(y) + \frac{|v(t, y)|^{\beta+1}}{\beta\eta(y)^\beta} - \theta^\alpha |\sigma^*(y) Dv(t, y)|^{\alpha+1} = 0, & (t, y) \in [0, T-\delta) \times \mathbb{R}^d, \\ v(T-\delta, y) = w(T-\delta, y), & y \in \mathbb{R}^d. \end{cases} \quad (4.15)$$

Altogether, this yields a solution  $w \in C_b^{0,1}([0, T^-] \times \mathbb{R}^d)$  to the PDE (2.11). Since  $u$  belongs to the space  $\Sigma$  defined in (4.9) it follows from the boundedness of  $D\eta$  derived from (F.2) and (F.3) and the fact that  $\epsilon - \frac{1}{2} = \frac{1}{2} - \alpha/\beta > 0$  that  $w$  satisfies (2.12). Since  $v$  is a viscosity solution to the PDE (2.11) we deduce from Lemma A.3 that  $v \equiv w$  on  $[0, T) \times \mathbb{R}^d$ . Hence  $v$  satisfies the desired regularity properties.  $\square$

*Remark 4.7.* Our global regularity result uses [12, Theorem 15] whose proof is based on probabilistic arguments. Alternatively, one can use PDE arguments to obtain the existence of a global smooth solution. Classical a priori estimates in [31] show that the gradient of  $v$  is bounded if it exists. Under the additional assumption that the diffusion operator  $\mathcal{L}$  generates an analytic semi-group in  $C(\mathbb{R}^d)$  (which excludes Ornstein-Uhlenbeck processes) one can then use results established in [34, Chapter 7] to show that the solution to our HJB equation is a classical solution away from the terminal time; see [26, Proof of Theorem 2.9] for details.

Since the gradient of the terminal condition of the PDE (4.15) is bounded, classical PDE results show that the gradient of the solution is uniformly bounded on the entire domain  $[0, T-\delta] \times \mathbb{R}^d$ . The same result follows from the classical link between viscosity solutions to PDEs and FBSDEs. The next result is standard, see [12, Theorem 15] and [41, Theorem 3.6] for details. Both the FBSDE representation of our viscosity solution and the global gradient bound will be very useful when proving the verification argument.

**Corollary 4.8.** *Suppose that  $\beta > 2\alpha$  and that Assumptions (L.1)-(L.4) and (F.2)-(F.3) hold. There exist processes  $(U^{t,y}, Z^{t,y}) \in S_{\mathcal{F}}^\infty(t, T^-; \mathbb{R}) \times H_{\mathcal{F}}^q(t, T^-; \mathbb{R}^{1 \times \bar{d}})$  for all  $q \geq 2$  satisfying*

$$U_s^{t,y} = v(s, Y_s^{t,y}), \quad Z_s^{t,y} = Dv(s, Y_s^{t,y})\sigma(s, Y_s^{t,y}) \quad (4.16)$$

and for any  $t \leq r \leq s < T$ ,

$$U_r^{t,y} = U_s^{t,y} + \int_r^s F(Y_\rho^{t,y}, U_\rho^{t,y}) + \theta^\alpha |Z_\rho^{t,y}|^{1+\alpha} d\rho - \int_r^s Z_\rho^{t,y} dW_\rho. \quad (4.17)$$

Furthermore, there exists a constant  $C > 0$  such that

$$|Z_r^{t,y}| \leq \begin{cases} \frac{C}{(T-r)^{1/\beta}}, & r \in [T-\delta, T]; \\ C, & r \in [t, T-\delta]. \end{cases} \quad (4.18)$$

## 5 Verification

This section is devoted to the verification argument. We first prove admissibility of the strategy  $\xi^*$  by using the estimates of the nonnegative viscosity solution  $v$  derived from the proof of Theorem 2.4. Subsequently, we show that  $(\xi^*, \vartheta^*)$  is a saddle point of the cost function and is indeed optimal. The proof uses a change of measure argument. Since the viscosity solution belongs to  $C_b^{0,1}([0, T^-] \times \mathbb{R}^d)$  and satisfies the asymptotics (2.12), the optimal density  $\vartheta^*$  has sufficient integrability for the corresponding stochastic exponential to be a true martingale.

**Lemma 5.1.** *The feedback control  $\xi^*$  given by (2.13) is admissible, and the portfolio process  $(X_s^*)_{s \in [t, T]}$  is monotone.*

*Proof.* From the construction of the viscosity solution in the proof of Theorem 2.4, we have that  $\check{v} \leq v \leq \hat{v}$  on  $[T-\delta, T]$  where  $\check{v}$  and  $\hat{v}$  were introduced in (3.6) and (3.7), respectively. Moreover, under condition (F.3) the function  $\hat{h}$  introduced in (3.3) reduces to  $\hat{h}(r, y) = L(T-r)$  because  $\tilde{m} = 0$ . Hence, for  $r \in [T-\delta, T]$

$$\frac{1 - \|\frac{\mathcal{L}\eta}{\eta}\|(T-r)}{(T-r)^{1/\beta}} \eta(Y_r^{t,y}) \leq v(r, Y_r^{t,y}) \leq \frac{1 + K(T-r)^\epsilon}{(T-r)^{1/\beta}} \eta(Y_r^{t,y}) + \hat{h}(r, Y_r^{t,y}).$$

For  $s \in [T-\delta, T]$ ,

$$\begin{aligned} |X_s^*| &\leq |x| \exp\left(-\int_t^s \frac{v(r, Y_r^{t,y})^\beta}{\eta(Y_r^{t,y})^\beta} dr\right) \\ &\leq |x| \exp\left(-\int_{T-\delta}^s \frac{\left(1 - \|\frac{\mathcal{L}\eta}{\eta}\|(T-r)\right)^\beta}{(T-r)} dr\right) \\ &\leq |x| \exp\left(\int_{T-\delta}^s \frac{1 - \left(1 - \|\frac{\mathcal{L}\eta}{\eta}\|(T-r)\right)^\beta}{(T-r)} dr\right) \cdot \exp\left(-\int_{T-\delta}^s \frac{1}{T-r} dr\right) \\ &\leq C|x| \frac{T-s}{\delta} \xrightarrow{s \rightarrow T} 0. \end{aligned}$$

The last inequality holds because  $\lim_{r \rightarrow T} \frac{1 - \left(1 - \|\frac{\mathcal{L}\eta}{\eta}\|(T-r)\right)^\beta}{(T-r)} = \beta \|\frac{\mathcal{L}\eta}{\eta}\|$ . As a result,  $X_T^* = 0$ .

For controls  $\xi^*$  given by (2.13), the process  $(X_s^*)_{s \in [t, T]}$  is obviously monotone. It remains to establish the integrability of  $\xi^*$ . In fact, since  $1/\eta, v$  are bounded on  $\mathbb{R}^d$  and  $[0, T-\delta] \times \mathbb{R}^d$ , respectively, we see

that

$$\begin{aligned}
\sup_{t \leq s < T} |\xi_s^*| &\leq \sup_{t \leq s \leq T-\delta} |\xi_s^*| + \sup_{T-\delta \leq s < T} |\xi_s^*| \\
&= \sup_{t \leq s \leq T-\delta} \frac{v(s, Y_s^{t,y})^\beta}{\eta(Y_s^{t,y})^\beta} |X_s^*| + \sup_{T-\delta \leq s < T} \frac{v(s, Y_s^{t,y})^\beta}{\eta(Y_s^{t,y})^\beta} |X_s^*| \\
&\leq |x| \sup_{t \leq s \leq T-\delta} \frac{v(s, Y_s^{t,y})}{\eta(Y_s^{t,y})} + \sup_{T-\delta \leq s < T} \frac{\left(1 + KT^\epsilon + \frac{LT^{1+1/\beta}}{\eta(Y_s^{t,y})}\right)^\beta}{T-s} \cdot C|x| \frac{T-s}{\delta} \\
&< +\infty.
\end{aligned}$$

It follows that  $\xi^* \in L^\infty_{\mathcal{F}}(t, T; \mathbb{R})$  and hence that  $\xi^*$  is admissible. □

The following lemma shows that for any  $\xi \in \mathcal{A}(t, x)$  the expected residual costs vanish as  $s \rightarrow T$  under a particular class of equivalent measure.

**Lemma 5.2.** *For every  $\xi \in \mathcal{A}(t, x)$  and every  $Q \in \mathcal{Q}$  satisfying*

$$\mathbb{E} \left[ e^{q \int_t^T |\vartheta_r|^2 dr} \right] < \infty, \quad \text{for every } q > 0,$$

it holds that

$$\mathbb{E}_Q [v(s, Y_s^{t,y}) |X_s^\xi|^p] \rightarrow 0, \quad s \rightarrow T. \quad (5.1)$$

*Proof.* Set  $\pi_s = \mathcal{E}(\int_t^s \vartheta_r dW_r)$ . For  $k > 1, s \in [t, T]$ , by the Hölder inequality, we have that

$$\begin{aligned}
\mathbb{E} [(\pi_s)^k] &= \mathbb{E} \left[ e^{k \int_t^s \vartheta_r dW_r - k^2 \int_t^s |\vartheta_r|^2 dr} \cdot e^{(k^2 - k/2) \int_t^s |\vartheta_r|^2 dr} \right] \\
&\leq \left( \mathbb{E} \left[ \mathcal{E} \left( 2k \int_t^s \vartheta_r dW_r \right) \right] \right)^{1/2} \cdot \left( \mathbb{E} \left[ e^{(2k^2 - k) \int_t^s |\vartheta_r|^2 dr} \right] \right)^{1/2} < \infty.
\end{aligned}$$

Since  $X_s^\xi = X_T^\xi + \int_s^T \xi_r dr = \int_s^T \xi_r dr$ , using Hölder inequality again, we obtain

$$|X_s^\xi|^p \leq (T-s)^{1/\beta} \int_s^T |\xi_r|^p dr.$$

Close to the terminal time the upper estimate  $v(s, Y_s^{t,y}) \leq \frac{C}{(T-s)^{1/\beta}}$  holds; away from the terminal time,  $v$  is bounded. Hence this estimate holds everywhere and so

$$\begin{aligned}
\mathbb{E}_Q [v(s, Y_s^{t,y}) |X_s^\xi|^p] &= \mathbb{E} [\pi_s v(s, Y_s^{t,y}) |X_s^\xi|^p] \\
&\leq C \mathbb{E} \left[ \pi_s \int_s^T |\xi_r|^p dr \right] \\
&\leq C \left( (T-s) \mathbb{E} [(\pi_s)^2] \mathbb{E} \left[ \int_s^T |\xi_r|^{2p} dr \right] \right)^{1/2}.
\end{aligned}$$

Letting  $s \rightarrow T$ , the desired result (5.1) follows since  $\xi \in L^2_{\mathcal{F}}(t, T; \mathbb{R})$ . □

Our verification argument will be based on the following probabilistic representation of the viscosity solution to (2.11).

We are now ready to carry out the verification argument. We will show that  $v(\cdot, \cdot) \cdot |\cdot|^p$  is indeed equal to the value function of our control problem and that the candidate strategy is optimal on the whole time interval.

*Proof of Theorem 2.6.* For fixed  $t \leq s < T$ , by Corollary 4.8 we have that

$$U_t^{t,y} = U_s^{t,y} + \int_t^s (F(Y_r^{t,y}, U_r^{t,y}) + |Z_r^{t,y}|^{1+\alpha}) dr - \int_t^s Z_r^{t,y} dW_r.$$

This allows us to apply to  $U_r^{t,y}|X_r^\xi|^p$  the integration by parts formula on  $[t, s]$  and to get that

$$\begin{aligned} U_t^{t,y}|x|^p &= U_s^{t,y}|X_s^\xi|^p + \int_t^s \{(F(Y_r^{t,y}, U_r^{t,y}) + \theta^\alpha |Z_r^{t,y}|^{1+\alpha})|X_r^\xi|^p \\ &\quad + p\xi_r U_r^{t,y} \operatorname{sgn}(X_r^\xi)|X_r^\xi|^{p-1}\} dr - \int_t^s Z_r^{t,y}|X_r^\xi|^p dW_r. \end{aligned}$$

Denote  $W_r^\vartheta = W_r - \int_t^r \vartheta_\rho d\rho$ . Thus,

$$\begin{aligned} U_t^{t,y}|x|^p &= U_s^{t,y}|X_s^\xi|^p + \int_t^s \{(F(Y_r^{t,y}, U_r^{t,y}) + \theta^\alpha |Z_r^{t,y}|^{1+\alpha} - \vartheta_r Z_r^{t,y})|X_r^\xi|^p \\ &\quad + p\xi_r U_r^{t,y} \operatorname{sgn}(X_r^\xi)|X_r^\xi|^{p-1}\} dr - \int_t^s Z_r^{t,y}|X_r^\xi|^p dW_r^\vartheta. \end{aligned} \quad (5.2)$$

In what follows, we show that  $(\xi^*, \vartheta^*)$  is a saddle point of the functional  $\tilde{J}$ , i.e.

$$\tilde{J}(t, y, x; \xi^*, \vartheta) \leq \tilde{J}(t, y, x; \xi^*, \vartheta^*) \leq \tilde{J}(t, y, x; \xi, \vartheta^*).$$

STEP 1:  $\tilde{J}(t, y, x; \xi^*, \vartheta^*) \leq \tilde{J}(t, y, x; \xi, \vartheta^*)$  FOR EVERY  $\xi$ .

Set  $\pi_s^* = \mathcal{E}(\int_t^s \vartheta_r^* dW_r)$ . From the definition of  $\vartheta^*$  in (2.13), we see that  $|\vartheta_r^*| \leq (1 + \alpha)\theta^\alpha |Z_r^{t,y}|^\alpha$ . Using the estimate in (4.18),

$$\begin{aligned} \int_t^T |\vartheta_s^*|^2 ds &\leq \int_{T-\delta}^T |\vartheta_s^*|^2 ds + \int_t^{T-\delta} |\vartheta_s^*|^2 ds \\ &\leq (1 + \alpha)^2 \theta^{2\alpha} \left( \int_{T-\delta}^T \frac{C^{2\alpha}}{(T-s)^{2\alpha/\beta}} ds + \int_t^{T-\delta} C^{2\alpha} ds \right) \\ &\leq (1 + \alpha)^2 \theta^{2\alpha} C^{2\alpha} (\delta^{1-2\alpha/\beta} + T) < +\infty. \end{aligned} \quad (5.3)$$

Hence  $\mathbb{E}[(\pi_s^*)^k] < +\infty$  for every  $k > 1$  and the Novikov condition implies that  $\pi^*$  is indeed a positive martingale. Setting  $dQ^* = \pi_T^* d\mathbb{P}$ , by the Girsanov theorem  $W^{\vartheta^*}$  is a Brownian motion under  $Q^*$ . This allows us to show that the stochastic integral in (5.2) is a  $Q^*$ -martingale. Since  $Z^{t,y}$  is bounded away from the terminal time and

$$\mathbb{E}[\sup_{t \leq r \leq s} |X_r^\xi|^{2p}] \leq C \mathbb{E} \left[ \int_t^T |\xi_r|^{2p} dr \right],$$

we have that

$$\begin{aligned} \mathbb{E}_{Q^*} \left[ \int_t^s |Z_r^{t,y}|^2 |X_r^\xi|^{2p} dr \right]^{1/2} &= \mathbb{E} \left[ (\pi_s^*)^2 \int_t^s |Z_r^{t,y}|^2 |X_r^\xi|^{2p} dr \right]^{1/2} \\ &\leq \left( \mathbb{E} \left[ (\pi_s^*)^2 \sup_{t \leq r \leq s} |X_r^\xi|^{2p} \right]^{3/4} \right)^{2/3} \left( \mathbb{E} \left[ \int_t^s |Z_r^{t,y}|^2 dr \right]^{3/2} \right)^{1/3} \\ &\leq \left( \mathbb{E} \left[ \frac{(\pi_s^*)^6}{4} + \frac{3 \sup_{t \leq r \leq s} |X_r^\xi|^{2p}}{4} \right] \right)^{2/3} \left( \mathbb{E} \left[ T^{3/2} \sup_{t \leq r \leq s} |Z_r^{t,y}|^3 \right] \right)^{1/3} \\ &< +\infty. \end{aligned}$$

Set

$$c(y, x, \xi) := \eta(y)|\xi|^p + \lambda(y)|x|^p, \quad C(y, x, \xi, \vartheta) := c(y, x, \xi) - \frac{1}{\theta} |\vartheta|^m |x|^p.$$

By (2.9), we have that

$$\begin{aligned}
U_t^{t,y}|x|^p &= \mathbb{E}_{Q^*} [U_s^{t,y}|X_s^\xi|^p] + \mathbb{E}_{Q^*} \left[ \int_t^s C(Y_r^{t,y}, X_r^\xi, \xi_r, \vartheta_r^*) dr \right] \\
&\quad + \mathbb{E}_{Q^*} \left[ \int_t^s \left\{ F(Y_r^{t,y}, U_r^{t,y})|X_r^\xi|^p + p\xi_r U_r^{t,y} \operatorname{sgn}(X_r^\xi)|X_r^\xi|^{p-1} - c(Y_r^{t,y}, X_r^\xi, \xi_r) \right\} dr \right] \\
&\leq \mathbb{E}_{Q^*} [U_s^{t,y}|X_s^\xi|^p] + \mathbb{E}_{Q^*} \left[ \int_t^s C(Y_r^{t,y}, X_r^\xi, \xi_r, \vartheta_r^*) dr \right].
\end{aligned} \tag{5.4}$$

Since  $U_t^{t,y}$  is nonnegative, we can obtain that

$$\mathbb{E}_{Q^*} \left[ \int_t^s \frac{1}{\theta} |\vartheta_r^*|^m |X_r^\xi|^p dr \right] \leq \mathbb{E}_{Q^*} [U_s^{t,y}|X_s^\xi|^p] + \mathbb{E}_{Q^*} \left[ \int_t^s c(Y_r^{t,y}, X_r^\xi, \xi_r) dr \right].$$

The right hand side is finite as  $s$  goes to  $T$  by Lemma 5.2 together with the admissibility of  $\xi$  and the boundedness of  $\eta, \lambda$ . In view of Lemma 5.2, letting  $s \rightarrow T$  in (5.4) we get

$$v(t, y)|x|^p \leq \tilde{J}(t, y, x; \xi, \vartheta^*).$$

Finally note that the equality holds in (5.4) if  $\xi = \xi^*$ . This yields

$$\begin{aligned}
v(t, y)|x|^p &= \mathbb{E}_{Q^*} [v(s, Y_s^{t,y})|X_s^{\xi^*}|^p] + \mathbb{E}_{Q^*} \left[ \int_t^s C(Y_r^{t,y}, X_r^{\xi^*}, \xi_r^*, \vartheta_r^*) dr \right] \\
&\rightarrow \tilde{J}(t, y, x; \xi^*, \vartheta^*) \quad \text{as } s \rightarrow T.
\end{aligned}$$

Thus,

$$v(t, y)|x|^p = \tilde{J}(t, y, x; \xi^*, \vartheta^*) \leq \tilde{J}(t, y, x; \xi, \vartheta^*).$$

STEP 2.  $\tilde{J}(t, y, x; \xi^*, \vartheta) \leq \tilde{J}(t, y, x; \xi^*, \vartheta^*)$  FOR EVERY  $\vartheta$ .

Let us introduce the sequence of stopping times

$$\tau_n := \inf \left\{ r \in [t, T] : \int_t^r |\vartheta_\rho|^2 d\rho > n \right\}.$$

Put  $\vartheta_r^n = \vartheta_r \mathbb{1}_{r \leq \tau_n}$  and define  $W_r^n = W_r + \int_t^r \vartheta_r^n dr$ . From the definition of  $\tau_n$ , it follows that

$$\int_t^T |\vartheta_r^n|^2 dr = \int_t^{\tau_n} |\vartheta_r|^2 dr \leq n. \tag{5.5}$$

Therefore, defining  $\pi_s^n = \mathcal{E}(\int_t^s \vartheta_r^n dW_r)$ , the Novikov condition implies that  $\mathbb{E}[\pi_T^n] = 1$ . Setting  $dQ^n = \pi_T^n d\mathbb{P}$ , by the Girsanov theorem  $W^{\vartheta^n}$  is a Brownian motion under  $Q^n$ . Moreover,  $\mathbb{E}[(\pi_s^n)^k] < +\infty$  for every  $k > 1$ .

As discussed before, we can show that the stochastic integrals  $\int_t^s Z_r^{t,y} |X_r^{\xi^*}|^p dW_r^{\vartheta^n}$  are  $Q^n$ -martingales for any  $n \in \mathbb{R}$ . Together with (2.8), we have that

$$\begin{aligned}
U_t^{t,y}|x|^p &= \mathbb{E}_{Q^n} [U_s^{t,y}|X_s^{\xi^*}|^p] + \mathbb{E}_{Q^n} \left[ \int_t^s C(Y_r^{t,y}, X_r^{\xi^*}, \xi_r^*, \vartheta_r^n) dr \right] \\
&\quad + \mathbb{E}_{Q^n} \left[ \int_t^s \left\{ (\theta^\alpha |Z_r^{t,y}|^{1+\alpha} - \vartheta_r^n Z_r^{t,y} + \frac{1}{\theta} |\vartheta_r^n|^m) |X_r^{\xi^*}|^p \right\} dr \right] \\
&\geq \mathbb{E}_{Q^n} [U_s^{t,y}|X_s^{\xi^*}|^p] + \mathbb{E}_{Q^n} \left[ \int_t^s C(Y_r^{t,y}, X_r^{\xi^*}, \xi_r^*, \vartheta_r^n) dr \right].
\end{aligned} \tag{5.6}$$

Letting  $s \rightarrow T$  we get

$$U_t^{t,y}|x|^p \geq \mathbb{E}_{Q^n} \left[ \int_t^T C(Y_r^{t,y}, X_r^{\xi^*}, \xi_r^*, \vartheta_r^n) dr \right] \tag{5.7}$$



by Lemma 5.2. We are now going to show that

$$\begin{aligned} U_t^{t,y}|x|^p &\geq \mathbb{E}_Q \left[ \int_t^T C(Y_r^{t,y}, X_r^{\xi^*}, \xi_r^*, \vartheta_r) dr \right] \\ &= \mathbb{E}_Q \left[ \int_t^T c(Y_r^{t,y}, X_r^{\xi^*}, \xi_r^*) dr \right] - \mathbb{E}_Q \left[ \int_t^T \frac{1}{\theta} |\vartheta_r|^m |X_r^{\xi^*}|^p dr \right]. \end{aligned} \quad (5.8)$$

If  $\mathbb{E}_Q \left[ \int_t^T \frac{1}{\theta} |\vartheta_r|^m |X_r^{\xi^*}|^p dr \right]$  is infinite, this inequality holds naturally since  $c(Y_r^{t,y}, X_r^{\xi^*}, \xi_r^*)$  is bounded on  $[t, T]$ . Hence, we assume w.l.o.g. that  $\mathbb{E}_Q \left[ \int_t^T \frac{1}{\theta} |\vartheta_r|^m |X_r^{\xi^*}|^p dr \right]$  is finite. For  $r \in [t, T]$ , we have that

$$\begin{aligned} \mathbb{E} \left[ \pi_r^n |\vartheta_r^n|^m |X_r^{\xi^*}|^p \right] &\geq \mathbb{E} \left[ \mathbb{E} \left[ \pi_r^n |\vartheta_r^{n-1}|^m |X_r^{\xi^*}|^p | \mathcal{F}_{r \wedge \tau_{n-1}} \right] \right] \\ &\geq \mathbb{E} \left[ \mathbb{E} \left[ \pi_r^n | \mathcal{F}_{r \wedge \tau_{n-1}} \right] |\vartheta_r^{n-1}|^m |X_r^{\xi^*}|^p \right] \\ &= \mathbb{E} \left[ \pi_r^{n-1} |\vartheta_r^{n-1}|^m |X_r^{\xi^*}|^p \right]. \end{aligned}$$

The monotone convergence theorem thus yields

$$\mathbb{E}_{Q^n} \left[ \int_t^T \frac{1}{\theta} |\vartheta_r^n|^m |X_r^{\xi^*}|^p dr \right] \xrightarrow{n \rightarrow \infty} \mathbb{E}_Q \left[ \int_t^T \frac{1}{\theta} |\vartheta_r|^m |X_r^{\xi^*}|^p dr \right].$$

Using the boundedness of  $c(Y_r^{t,y}, X_r^{\xi^*}, \xi_r^*)$  on  $[t, T]$  again, we apply the dominated convergence theorem to get that

$$\mathbb{E}_{Q^n} \left[ \int_t^T c(Y_r^{t,y}, X_r^{\xi^*}, \xi_r^*) dr \right] \xrightarrow{n \rightarrow \infty} \mathbb{E}_Q \left[ \int_t^T c(Y_r^{t,y}, X_r^{\xi^*}, \xi_r^*) dr \right].$$

Letting  $n$  goes to infinity in (5.7), we obtain the inequality (5.8). Recall that  $\tilde{J}(t, y, x; \xi^*, \vartheta^*) = v(t, y)|x|^p$ , we conclude that

$$\tilde{J}(t, y, x; \xi^*, \vartheta) \leq \tilde{J}(t, y, x; \xi^*, \vartheta^*).$$

□

*Remark 5.3.* It was shown that  $(\xi^*, \vartheta^*)$  is a saddle point of the functional  $\tilde{J}$ , thus  $(\xi^*, \vartheta^*)$  is indeed a solution of the robust control problem (2.5). However,  $\tilde{J}$  is not convex in  $\xi$  for fixed  $\vartheta$ . So the saddle point  $(\xi^*, \vartheta^*)$  may not be unique.

## 6 Asymptotic analysis

In Section 2, we provided both theoretical results and numerical examples on the first order approximations of the value function and the optimal trading strategy for the model with uncertainty. In this section, we give the proofs of Theorem 2.8 and Corollary 2.9. The main idea is to construct a super- and subsolution to (2.11) by an asymptotic expansion around the benchmark solution and then to apply the comparison principle [Lemma A.3].

The following lemma extends the results in [26, Theorem 2.9]. The proof is given in the Appendix A.3.

**Lemma 6.1.** *Let  $\beta > 2\alpha$ . Under Assumptions (L.1)-(L.4), (F.2)-(F.3), the terminal value problem (2.4) admits a unique nonnegative solution  $v_0$  in  $C^{0,1}([0, T^-] \times \mathbb{R}^d)$ . The solution satisfies the following estimates:*

$$\frac{\underline{c}}{(T-t)^{1/\beta}} \leq v_0 \leq \frac{C_0}{(T-t)^{1/\beta}}, \quad |Dv_0| \leq \frac{C_0}{(T-t)^{1/\beta}}, \quad (t, y) \in [0, T] \times \mathbb{R}^d,$$

for some constant  $C_0 > 0$ .

The next lemma establishes the existence of a unique nonnegative solution to the terminal value problem (2.16) and provides *a priori* estimates on the solution and its derivative.

**Lemma 6.2.** *Let  $\beta > 2\alpha$ . Under Assumptions (L.1)-(L.4), (F.2)-(F.3), the terminal value problem (2.16) admits a unique nonnegative viscosity solution  $w_1$  in  $C^{0,1}([0, T] \times \mathbb{R}^d)$ . Moreover, the following estimates hold:*

$$0 \leq w_1 \leq C_1(T-t)^{1-\alpha/\beta}, \quad |Dw_1| \leq C_1(T-t)^{1/2-\alpha/\beta}, \quad (t, y) \in [0, T] \times \mathbb{R}^d,$$

for some constant  $C_1 > 0$ .

*Proof.* Set  $A := |\sigma^* Dv_0|^{1+\alpha}$  and  $B := \frac{(\beta+1)v_0^\beta}{\beta\eta^\beta}$ . Let  $\delta_0 := 1/\|\frac{\mathcal{L}\eta}{\eta}\|$ . Using similar arguments to [26, Corollary 3.2] and [28, Proposition 3.5], we know that for  $(t, y) \in [T - \delta_0, T] \times \mathbb{R}^d$ ,

$$\frac{v_0(t, y)^\beta}{\eta(y)^\beta} \geq \frac{1 - \|\frac{\mathcal{L}\eta}{\eta}\|(T-t)}{T-t}.$$

Hence, for  $\delta := \frac{\beta}{2(\beta+1)}\delta_0$ ,

$$B(t, y) = \frac{(\beta+1)v_0(t, y)^\beta}{\beta\eta(y)^\beta} \geq \frac{1 + \beta/2}{\beta(T-t)}, \quad (t, y) \in [T - \delta, T] \times \mathbb{R}^d, \quad (6.1)$$

and so

$$-B(t, y) + \frac{1}{\beta(T-t)} \leq \frac{1}{\beta\delta} \mathbb{1}_{t \in [0, T-\delta]} - \frac{1}{2(T-t)} \mathbb{1}_{t \in [T-\delta, T]} \leq \frac{1}{\beta\delta} \quad (t, y) \in [0, T] \times \mathbb{R}^d. \quad (6.2)$$

Using the estimates on  $Dv_0$  in Lemma 6.1 along with the fact that  $\beta > 2\alpha$ , we have that

$$\mathbb{E} \left[ \int_0^T \left( A(s, Y_s^{t,y})(T-s)^{1/\beta} \right)^2 ds \right] \leq \int_0^T \frac{C}{(T-s)^{2\alpha/\beta}} ds < +\infty. \quad (6.3)$$

By (6.2) and (6.3), it follows from the Feynman-Kac formula [38, Theorem 3.2] that

$$w_1(t, y) := \mathbb{E} \left[ \int_t^T \exp \left( \int_t^s \left( -B(r, Y_r^{t,y}) + \frac{1}{\beta(T-r)} \right) dr \right) A(s, Y_s^{t,y})(T-s)^{1/\beta} ds \right]$$

is the unique viscosity solution to the terminal value problem (2.16) on  $[0, T] \times \mathbb{R}^d$ . Moreover, we have for  $(t, y) \in [0, T] \times \mathbb{R}^d$  that

$$\begin{aligned} w_1(t, y) &\leq \mathbb{E} \left[ \int_t^T \exp \left( \int_t^s \frac{1}{\beta\delta} dr \right) A(s, Y_s^{t,y})(T-s)^{1/\beta} ds \right] \\ &\leq \int_t^T e^{T/(\beta\delta)} \frac{C}{(T-s)^{\alpha/\beta}} ds \\ &\leq C_1(T-t)^{1-\alpha/\beta} \end{aligned} \quad (6.4)$$

for some constant  $C_1$ .

Next, we study the derivative of  $w_1$ . For any  $\varepsilon \in (0, T)$ , restricting the PDE (2.16) to  $[0, T - \varepsilon]$ ,

$$\begin{cases} -\partial_t v(t, y) - \mathcal{L}v(t, y) - f_1(t, y, v(t, y)) = 0, & (t, y) \in [0, T - \varepsilon] \times \mathbb{R}^d, \\ v(T - \varepsilon, y) = w_1(T - \varepsilon, y) & y \in \mathbb{R}^d, \end{cases}$$

Since  $A, B$  are bounded on  $[0, T - \varepsilon]$ , it follows from the Bismut-Elworthy formula [22, Theorem 4.2] that  $w_1(t, \cdot)$  is differentiable for  $t \in [0, T - \varepsilon]$  and

$$|Dw_1(t, y)| \leq \frac{C}{(T - \varepsilon - t)^{1/2}} \|w_1(T - \varepsilon, \cdot)\| + \int_t^{T - \varepsilon} \frac{C}{(s - t)^{1/2}} \left( (T - s)^{1/\beta} \|A(s, \cdot)\| + \left( \|B(s, \cdot)\| + \frac{1}{\beta(T - s)} \right) \|w_1(s, \cdot)\| \right) ds$$

for  $(t, y) \in [0, T - \varepsilon] \times \mathbb{R}^d$ . Using the estimates on  $v_0, w_1$  we get that

$$\begin{aligned} |Dw_1(t, y)| &\leq \frac{C}{(T - \varepsilon - t)^{1/2}} \varepsilon^{1 - \alpha/\beta} + C \int_t^T \frac{1}{(s - t)^{1/2}} (T - s)^{-\alpha/\beta} ds \\ &\leq \frac{C}{(T - \varepsilon - t)^{1/2}} (T - t)^{1 - \alpha/\beta} + C(T - t)^{1/2 - \alpha/\beta}, \quad (t, y) \in [0, T - \varepsilon] \times \mathbb{R}^d, \end{aligned}$$

where  $C$  is independent of  $\varepsilon$ . By letting  $\varepsilon$  go to zero, we see that (by an adjustment of  $C_1$  if necessary)

$$|Dw_1(t, y)| \leq C_1 (T - t)^{1/2 - \alpha/\beta}, \quad (t, y) \in [0, T] \times \mathbb{R}^d. \quad (6.5)$$

□

By the transformation  $v_1 = \frac{1}{(T - t)^{1/\beta}} w_1$ , we know that  $v_1$  is a solution to the equation

$$-\partial_t v(t, y) - \mathcal{L}v(t, y) - |\sigma Dv_0|^{1+\alpha} + \frac{(\beta + 1)v_0^\beta}{\beta\eta^\beta} v = 0, \quad (t, y) \in [0, T] \times \mathbb{R}. \quad (6.6)$$

Moreover, since  $\beta > 2\alpha$ , there exists a constant  $C_2 > 0$  such that for  $(t, y) \in [0, T] \times \mathbb{R}^d$ ,

$$\begin{aligned} 0 \leq v_1 &\leq C_1 (T - t)^{1 - (1 + \alpha)/\beta} \leq C_2 (T - t)^{-1/\beta}, \\ |Dv_1| &\leq C_1 (T - t)^{1/2 - (1 + \alpha)/\beta} \leq C_2 (T - t)^{-1/\beta}. \end{aligned} \quad (6.7)$$

Armed with these estimates, we are now ready to prove the asymptotic result.

*Proof of Theorem 2.8.* Let  $\delta$  be as in (6.1) and set  $b := \frac{\bar{C}^\beta}{(\beta + 1)\underline{c}^\beta \delta^{1/\beta}}$ . Our goal is to find two constants  $L_1 > 0, L_2 < 0$  such that

$$u_i = v_0 + \theta^\alpha v_1 + \theta^{2\alpha} L_i \left( b + \frac{1}{(T - t)^{1/\beta}} \right), \quad i = 1, 2$$

is a supersolution ( $i = 1$ ), respectively a subsolution ( $i = 2$ ) to (2.11). For  $i = 1, 2$ ,

$$\begin{aligned} &-\theta^\alpha |\sigma^* Du_i|^{1+\alpha} + \frac{u_i^{\beta+1}}{\beta\eta^\beta} - \lambda \\ &= -\theta^\alpha |\sigma^* (Dv_0 + \theta^\alpha Dv_1)|^{1+\alpha} + \frac{\left( v_0 + \theta^\alpha v_1 + \theta^{2\alpha} L_i \left( b + \frac{1}{(T - t)^{1/\beta}} \right) \right)^{\beta+1}}{\beta\eta^\beta} - \lambda \\ &= -\theta^\alpha |\sigma^* Dv_0|^{1+\alpha} + \frac{v_0^{\beta+1} + (\beta + 1)\theta^\alpha v_0^\beta v_1}{\beta\eta^\beta} - \lambda + \theta^{2\alpha} \frac{L_i}{\beta(T - t)^{1/\beta+1}} + \mathcal{I}_i, \end{aligned}$$

where  $\mathcal{I}_i := \mathcal{I}_i^0 + \mathcal{I}_i^1 + \mathcal{I}_i^2$  and  $\mathcal{I}_i^0, \mathcal{I}_i^1, \mathcal{I}_i^2$  are given by

$$\begin{aligned} \mathcal{I}_i^0 &:= -\theta^{2\alpha} L_i \frac{1}{\beta(T - t)^{1/\beta+1}}; \\ \mathcal{I}_i^1 &:= \frac{\left( v_0 + \theta^\alpha v_1 + \theta^{2\alpha} L_i \left( b + \frac{1}{(T - t)^{1/\beta}} \right) \right)^{\beta+1}}{\beta\eta^\beta} - \frac{v_0^{\beta+1} + (\beta + 1)\theta^\alpha v_0^\beta v_1}{\beta\eta^\beta}; \\ \mathcal{I}_i^2 &:= \theta^\alpha |\sigma^* Dv_0|^{1+\alpha} - \theta^\alpha |\sigma^* (Dv_0 + \theta^\alpha Dv_1)|^{1+\alpha}. \end{aligned}$$

It is sufficient to prove that  $\mathcal{I}_1 > 0$  (supersolution) and that  $\mathcal{I}_2 < 0$  (subsolution) on  $[0, T] \times \mathbb{R}^d$ .

The second order Taylor approximation around  $v_0$  in the first summand of  $\mathcal{I}_i^1$  yields a function  $\zeta$  satisfying  $\min\{v_0, u_i\} \leq \zeta \leq \max\{v_0, u_i\}$  such that

$$\mathcal{I}_i^1 = \theta^{2\alpha} L_i \frac{1}{\beta \eta^\beta} (\beta + 1) v_0^\beta \left( b + \frac{1}{(T-t)^{1/\beta}} \right) + \frac{1}{2\eta^\beta} (\beta + 1) \zeta^{\beta-1} \left( \theta^\alpha v_1 + \theta^{2\alpha} L_i \left( b + \frac{1}{(T-t)^{1/\beta}} \right) \right)^2.$$

The mean value theorem along with the triangle inequality also yields a constant  $\tilde{C}_0 > 0$  such that

$$\begin{aligned} |\mathcal{I}_i^2| &\leq \theta^{2\alpha} \bar{C}^\alpha (|Dv_0|^\alpha + |Dv_0 + \theta^\alpha Dv_1|^\alpha) |Dv_1| \\ &\leq \theta^{2\alpha} \tilde{C}_0 (T-t)^{(1+\alpha)/\beta} \\ &\leq \theta^{2\alpha} \frac{\tilde{C}_0 T^{1-\alpha/\beta}}{(T-t)^{1/\beta+1}}. \end{aligned}$$

STEP 1: CONSTRUCTION OF SUPERSOLUTION. Using the lower bound of  $v_0$  in Lemma 6.1, we have that for  $t \in [0, T - \delta]$ ,

$$\frac{\eta(y)^\beta}{(\beta + 1) v_0(t, y)^\beta (T-t)^{1/\beta+1}} \leq \frac{\bar{C}^\beta}{(\beta + 1) \underline{c}^\beta (T-t)^{1/\beta}} \leq \frac{\bar{C}^\beta}{(\beta + 1) \underline{c}^\beta \delta^{1/\beta}} = b.$$

Set  $c := \min\{\frac{1}{2}, \frac{(\beta+1)\underline{c}^\beta}{\beta \bar{C}^\beta}\}$ . The preceding inequality along with the inequality (6.1) yields that for  $t \in [0, T]$ ,

$$-\frac{1}{\beta(T-t)^{1/\beta+1}} + \frac{1}{\beta\eta^\beta} (\beta + 1) v_0^\beta \left( b + \frac{1}{(T-t)^{1/\beta}} \right) \geq c \frac{1}{(T-t)^{1/\beta+1}}. \quad (6.8)$$

Since the second term in the definition of  $\mathcal{I}_1^1$  is nonnegative, we have that

$$\mathcal{I}_1 \geq c \theta^{2\alpha} \frac{L_1}{(T-t)^{1/\beta+1}} - \theta^{2\alpha} \frac{\tilde{C}_0 T^{1-\alpha/\beta}}{(T-t)^{1/\beta+1}}.$$

Choosing  $L_1 > \frac{\tilde{C}_0 T^{1-\alpha/\beta}}{c}$ , we obtain that  $\mathcal{I}_1 > 0$ .

STEP 2: CONSTRUCTION OF SUBSOLUTION. Using the lower bound of  $v_0$  in Lemma 6.1 again and choosing  $L_2 < 0, \theta > 0$  such that  $\theta^{2\alpha} |L_2| (T^{1/\beta} b + 1) \leq \frac{\underline{c}}{2}$ , we obtain that  $u_2 \geq \frac{\underline{c}}{2(T-t)^{1/\beta}} \geq 0$ . Different from Step 1, an additional estimate on the second term in the definition of  $\mathcal{I}_2^1$  is needed to obtain that  $\mathcal{I}_2 < 0$ . Since  $\min\{v_0, u_2\} \leq \zeta \leq \max\{v_0, u_2\}$ , we see that  $\zeta(T-t)^{1/\beta}$  can be bounded both from below and above. Therefore, there exists a constant  $\tilde{C}_1 > 0$  such that

$$\frac{1}{2\eta^\beta} (\beta + 1) \zeta^{\beta-1} \left( \theta^\alpha v_1 + \theta^{2\alpha} L_i \left( b + \frac{1}{(T-t)^{1/\beta}} \right) \right)^2 \leq \theta^{2\alpha} \frac{\tilde{C}_1}{(T-t)^{1/\beta+1}}.$$

By the inequality (6.8) and the nonpositivity of  $L_2$ , we have that

$$-\frac{L_2}{\beta(T-t)^{1/\beta+1}} + \frac{1}{\beta\eta^\beta} (\beta + 1) v_0^\beta L_2 \left( b + \frac{1}{(T-t)^{1/\beta}} \right) \leq c \frac{L_2}{(T-t)^{1/\beta+1}}.$$

Thus,

$$\mathcal{I}_2 \leq c \theta^{2\alpha} \frac{L_2}{(T-t)^{1/\beta+1}} + \theta^{2\alpha} \frac{\tilde{C}_0 T^{1-\alpha/\beta}}{(T-t)^{1/\beta+1}} + \theta^{2\alpha} \frac{\tilde{C}_1}{(T-t)^{1/\beta+1}} < 0$$

if we first choose

$$L_2 < -\frac{\tilde{C}_1 + \tilde{C}_0 T^{1-\alpha/\beta}}{c}$$

and then

$$\theta < \min\{1, \sqrt[2\alpha]{\underline{c}/(2|L_2|(T^{1/\beta} b + 1))}\}.$$

Hence  $u_2$  is a nonnegative viscosity subsolution to (2.11). By Lemma A.3, we then have that  $u_2 \leq v \leq u_1$ . Thus, the desired equality (2.15) follows from

$$\theta^\alpha w_1 + \theta^{2\alpha} L_2 (b(T-t)^{1/\beta} + 1) \leq w - w_0 \leq \theta^\alpha w_1 + \theta^{2\alpha} L_1 (b(T-t)^{1/\beta} + 1).$$

□

Based on Theorem 2.8 we can now derive the first order approximation of the optimal trading strategy.

*Proof of Corollary 2.9.* From the preceding result, we have that on  $[0, T) \times \mathbb{R}^d$ ,

$$v - v_0 = \theta^\alpha v_1 + \theta^{2\alpha} \tilde{v}^\theta$$

where for some small  $\theta_0 \in (0, 1)$  there exists a constant  $K_0 > 0$  satisfying that  $|\tilde{v}^\theta(t, y)| \leq \frac{K_0}{(T-t)^{1/\beta}}$  for  $(t, y) \in [0, T) \times \mathbb{R}^d$ ,  $\theta < \theta_0$ . Assume that  $\theta < \theta_0$  in the sequel.

The second order Taylor approximation of power function around  $v_0$  yields a function  $\zeta$  satisfying  $v_0 \leq \zeta \leq v$  such that

$$\begin{aligned} v^\beta - v_0^\beta &= \beta v_0^{\beta-1} (\theta^\alpha v_1 + \theta^{2\alpha} \tilde{v}^\theta) + \frac{1}{2} \beta (\beta - 1) \zeta^{\beta-2} (\theta^\alpha v_1 + \theta^{2\alpha} \tilde{v}^\theta)^2 \\ &= \theta^\alpha \beta v_0^{\beta-1} v_1 + \theta^{2\alpha} \left( \beta v_0^{\beta-1} \tilde{v}^\theta + \frac{1}{2} \beta (\beta - 1) \zeta^{\beta-2} (v_1 + \theta^\alpha \tilde{v}^\theta)^2 \right). \end{aligned}$$

Recalling the estimates in Lemma 6.1 and (6.7), we have that on  $[0, T) \times \mathbb{R}^d$ ,

$$\begin{aligned} v &\leq \frac{C_0 + C_2 + K_0}{(T-t)^{1/\beta}}, \\ v_0^{\beta-1} v_1 &\leq \frac{\max\{C_0^{\beta-1}, \underline{c}^{\beta-1}\} C_2}{T-t}, \\ v_0^{\beta-1} |\tilde{v}^\theta| &\leq \frac{\max\{C_0^{\beta-1}, \underline{c}^{\beta-1}\} K_0}{T-t}, \\ \zeta^{\beta-2} (v_1 + \theta^\alpha \tilde{v}^\theta)^2 &\leq \frac{\max\{(C_0 + C_2 + K_0)^{\beta-2}, \underline{c}^{\beta-2}\} (C_2 + K_0)^2}{T-t}. \end{aligned} \tag{6.9}$$

Therefore, we obtain that for  $r \in [t, T)$ ,

$$v(r, Y_r^{t,y})^\beta - v_0(r, Y_r^{t,y})^\beta = \theta^\alpha \beta v_0(r, Y_r^{t,y})^{\beta-1} v_1(r, Y_r^{t,y}) + \theta^{2\alpha} O\left(\frac{1}{T-r}\right). \tag{6.10}$$

Let

$$\Phi(s) := \int_t^s \frac{v(r, Y_r^{t,y})^\beta - v_0(r, Y_r^{t,y})^\beta}{\eta(Y_r^{t,y})^\beta} dr \geq 0.$$

Using the second order Taylor approximation of exponential function around 0 yields a function  $\tilde{\zeta}$  satisfying  $0 \leq \tilde{\zeta} \leq \Phi$  such that

$$\begin{aligned} \exp(-\Phi(s)) - 1 &= -\Phi(s) + \frac{1}{2} \exp(-\tilde{\zeta}(s)) (-\Phi(s))^2 \\ &= -\int_t^s \left( \theta^\alpha \beta v_0(r, Y_r^{t,y})^{\beta-1} v_1(r, Y_r^{t,y}) + \theta^{2\alpha} O\left(\frac{1}{T-r}\right) \right) dr \\ &\quad + \frac{1}{2} \exp(-\tilde{\zeta}(s)) \left( \int_t^s \left( \beta \theta^\alpha v_0(r, Y_r^{t,y})^{\beta-1} v_1(r, Y_r^{t,y}) + \theta^{2\alpha} O\left(\frac{1}{T-r}\right) \right) dr \right)^2. \end{aligned}$$

In view of the estimate (6.9), we have that

$$\begin{aligned} &\exp(-\Phi(s)) - 1 \\ &= \theta^\alpha \left( -\int_t^s \beta v_0(r, Y_r^{t,y})^{\beta-1} v_1(r, Y_r^{t,y}) dr \right) + \theta^{2\alpha} O\left( \ln \frac{T-t}{T-s} + \left( \ln \frac{T-t}{T-s} \right)^2 \right). \end{aligned} \tag{6.11}$$

We know that the optimal strategies  $\xi^*, \xi^{0,*}$  belong to  $L^\infty_{\mathcal{F}}(t, T; \mathbb{R})$  and are given by

$$\begin{aligned}\xi_s^* &= \frac{v(s, Y_s^{t,y})^\beta}{\eta(Y_s^{t,y})^\beta} X_s^* = x \frac{v(s, Y_s^{t,y})^\beta}{\eta(Y_s^{t,y})^\beta} \exp\left(-\int_t^s \frac{v(s, Y_r^{t,y})^\beta}{\eta(Y_r^{t,y})^\beta} dr\right), \\ \xi_s^{0,*} &= x \frac{v_0(s, Y_s^{t,y})^\beta}{\eta(Y_s^{t,y})^\beta} \exp\left(-\int_t^s \frac{v_0(s, Y_r^{t,y})^\beta}{\eta(Y_r^{t,y})^\beta} dr\right).\end{aligned}$$

Similarly to the proof of Lemma 5.1, we obtain that

$$\exp\left(-\int_t^s \frac{v_0(s, Y_r^{t,y})^\beta}{\eta(Y_r^{t,y})^\beta} dr\right) = O(T-s).$$

Together with (6.10) and (6.11), it follows that

$$\begin{aligned}& \xi_s^* - \xi_s^{0,*} \\ &= x \frac{v(s, Y_s^{t,y})^\beta}{\eta(Y_s^{t,y})^\beta} \exp(-\Phi(s)) \exp\left(-\int_t^s \frac{v_0(s, Y_r^{t,y})^\beta}{\eta(Y_r^{t,y})^\beta} dr\right) - x \frac{v_0(s, Y_s^{t,y})^\beta}{\eta(Y_s^{t,y})^\beta} \exp\left(-\int_t^s \frac{v_0(s, Y_r^{t,y})^\beta}{\eta(Y_r^{t,y})^\beta} dr\right) \\ &= \theta^\alpha x \frac{\beta v_0(s, Y_s^{t,y})^{\beta-1}}{\eta(Y_r^{t,y})^\beta} \exp\left(-\int_t^s \frac{v_0(s, Y_r^{t,y})^\beta}{\eta(Y_r^{t,y})^\beta} dr\right) \\ &\quad \left( v_1(s, Y_s^{t,y}) - v_0(s, Y_s^{t,y}) \int_t^s v_0(r, Y_r^{t,y})^{\beta-1} v_1(r, Y_r^{t,y}) dr \right) + \theta^{2\alpha} O\left(1 + \ln \frac{T-t}{T-s} + \left(\ln \frac{T-t}{T-s}\right)^2\right) \\ &= \theta^\alpha \beta \xi_s^{0,*} \left( \frac{v_1(s, Y_s^{t,y})}{v_0(s, Y_s^{t,y})} - \int_t^s v_0(r, Y_r^{t,y})^{\beta-1} v_1(r, Y_r^{t,y}) dr \right) + \theta^{2\alpha} O\left(1 + \ln \frac{T-t}{T-s} + \left(\ln \frac{T-t}{T-s}\right)^2\right) \\ &= \theta^\alpha \tilde{\xi}_s + \theta^{2\alpha} O\left(1 + \ln \frac{T-t}{T-s} + \left(\ln \frac{T-t}{T-s}\right)^2\right).\end{aligned}$$

where  $\tilde{\xi}$  is defined in (2.18). Hence, we conclude that

$$\lim_{\theta \rightarrow 0} \frac{\xi^* - \xi^{0,*}}{\theta} = \tilde{\xi}, \quad \text{locally uniformly on } [t, T].$$

The fact that  $\tilde{\xi} \in L^\infty_{\mathcal{F}}(t, T; \mathbb{R})$  follows from the estimates in Lemma 6.1 and (6.7) that imply that

$$\begin{aligned}\sup_{s \in [t, T]} |\tilde{\xi}_s| &\leq \sup_{s \in [t, T]} \beta \xi_s^{0,*} \left( \frac{v_1(s, Y_s^{t,y})}{v_0(s, Y_s^{t,y})} + \int_t^T v_0(r, Y_r^{t,y})^{\beta-1} v_1(r, Y_r^{t,y}) dr \right) \\ &\leq \beta \|\xi^{0,*}\|_\infty \left( \frac{C_2}{\underline{c}} + \max\{C_0^{\beta-1}, \underline{c}^{\beta-1}\} C_1 \int_t^T (T-r)^{-\alpha/\beta} dr \right) \\ &= \beta \|\xi^{0,*}\|_\infty \left( \frac{C_2}{\underline{c}} + \max\{C_0^{\beta-1}, \underline{c}^{\beta-1}\} C_1 (T-t)^{1-\alpha/\beta} \right) < \infty.\end{aligned}$$

□

## A Appendix

### A.1 Comparison principle

In this section, we state and prove comparison principles for solutions to PDEs with superlinear gradient term. Both finite and singular terminal values will be considered. We refer to [32] as an important reference for PDEs with superlinear gradient term. Let us now consider the general PDE

$$\begin{cases} -\partial_t v(t, y) - \mathcal{L}v(t, y) - H(y, Dv(t, y)) - F(y, v(t, y)) = 0, & (t, y) \in [0, T) \times \mathbb{R}^d, \\ v(T, y) = \phi(y), & y \in \mathbb{R}^d. \end{cases} \quad (\text{A.1})$$

A comparison principle for such PDEs is obtained in [32] under a Lipschitz continuity assumption of  $F$  on  $v$ . This condition is not satisfied in our case; we only have monotonicity. Additional assumptions on the solution are thus required to establish a comparison principle. However, we can make a weaker assumption on the coefficients than (F.1) and (F.2).

(F.4) The coefficients  $\eta, \lambda, 1/\eta : \mathbb{R}^d \rightarrow [0, \infty)$  are continuous and  $\lambda$  is of polynomial growth of order  $m$ .

We first introduce two subsets of functions having superlinear growth. For a given  $r > 0$ , a function  $h : I \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  belongs to  $\mathcal{SSG}_r^\pm$  if and only if

$$\liminf_{|y| \rightarrow \infty} \frac{\pm h(t, y)}{|y|^r} \geq 0.$$

Notice that  $h \in \mathcal{SSG}_r^+$  (resp.,  $\mathcal{SSG}_r^-$ ) if, for any  $\varepsilon > 0$ , there exists  $C_\varepsilon = C_\varepsilon(h) > 0$  such that

$$h(t, y) \geq -\varepsilon|y|^r - C_\varepsilon \text{ (resp., } h(t, y) \leq \varepsilon|y|^r + C_\varepsilon), (t, y) \in I \times \mathbb{R}^d.$$

We define  $\mathcal{SSG}_r = \mathcal{SSG}_r^+ \cap \mathcal{SSG}_r^-$ . Notice that  $h \in \mathcal{SSG}_r$  if and only if

$$\lim_{|y| \rightarrow \infty} \frac{|h(t, y)|}{|y|^r} = 0$$

for every  $t \in I$ .

**Proposition A.1.** *Assume that (L.1)-(L.3) and (F.4) hold and that  $\phi \in C_m(\mathbb{R}^d)$ . Let  $v \in LSC([0, T] \times \mathbb{R}^d) \cap \mathcal{SSG}_m^+$  and  $u \in USC([0, T] \times \mathbb{R}^d) \cap \mathcal{SSG}_m^-$  be a nonnegative viscosity super- and a nonnegative viscosity subsolution to (A.1). Suppose that there exists  $\hat{C} > 0$  such that for all  $(t, y) \in [0, T] \times \mathbb{R}^d$ ,*

$$u^{\beta+1}(t, y), v^{\beta+1}(t, y) \leq \hat{C}\eta^\beta(y)\langle y \rangle^m. \quad (\text{A.2})$$

Then,

$$u \leq v \quad \text{on } [0, T] \times \mathbb{R}^d.$$

*Proof.* STEP 1: LINEARIZATION. For  $\rho \in (0, 1)$ , it is easy to verify that  $\tilde{v} := \rho v$  is a viscosity supersolution of the following PDE:

$$\begin{cases} -\partial_t \tilde{v}(t, y) - \mathcal{L}\tilde{v}(t, y) - \rho H(y, \frac{D\tilde{v}(t, y)}{\rho}) - \rho F(y, \frac{\tilde{v}(t, y)}{\rho}) = 0, & (t, y) \in [0, T] \times \mathbb{R}^d, \\ \tilde{v}(T, y) = \rho\phi(y), & y \in \mathbb{R}^d. \end{cases}$$

In what follows, we show that  $w := u - \tilde{v}$  is a viscosity subsolution of the following extremal PDE:

$$-\partial_t w(t, y) - \mathcal{L}w(t, y) - \left(\frac{1-\rho}{2}\right)^{-\alpha} \bar{C}^{\alpha+1} |Dw|^{\alpha+1} - (1-\rho) \left[ \lambda(\bar{y}) + \frac{1+\beta}{\beta} \hat{C} \langle y \rangle^m \right] = 0, \quad (\text{A.3})$$

for  $(t, y) \in [0, T] \times \mathbb{R}^d \cap \{w > 0\}$ .

Let  $\varphi \in C^2([0, T] \times \mathbb{R}^d)$  be a test function and  $(\bar{t}, \bar{y}) \in [0, T] \times \mathbb{R}^d \cap \{w > 0\}$  be a local maximum of  $w - \varphi$ . We may assume that this maximum is strict in the set  $[\bar{t} - r, \bar{t} + r] \times \bar{B}_r(\bar{y}) \subset [0, T] \times \mathbb{R}^d$  for small  $r \in (0, 1)$ ; we choose  $[0, r] \times \bar{B}_r(\bar{y})$  if  $\bar{t} = 0$ . Let

$$\Phi(t, x, y) := \frac{|x - y|^2}{2\varepsilon} + \varphi(t, y)$$

and

$$M_\varepsilon := \max_{t \in [\bar{t} - r, \bar{t} + r], x, y \in \bar{B}_r(\bar{y})} (u(t, x) - \tilde{v}(t, y) - \Phi(t, x, y)).$$

This maximum is attained at a point  $(t_\varepsilon, x_\varepsilon, y_\varepsilon)$  and is strict. We know that

$$\frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon} \rightarrow 0 \text{ and } M_\varepsilon \rightarrow u(\bar{t}, \bar{y}) - \tilde{v}(\bar{t}, \bar{y}) - \varphi(\bar{t}, \bar{y}) \text{ as } \varepsilon \rightarrow 0.$$

We now apply [16, Theorem 8.3]. In terms of their notation we have that  $k = 2, u_1 = u, u_2 = -\tilde{v}, \varphi(t, x, y) = \Phi(t, x, y)$ . Moreover, we recall the property that  $\bar{\mathcal{P}}^{2,-}(\tilde{v}) = -\bar{\mathcal{P}}^{2,+}(-\tilde{v})$ . Then, setting  $p_\varepsilon = \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}$ , we have that

$$\begin{aligned} \partial_x \Phi(t_\varepsilon, x_\varepsilon, y_\varepsilon) &= p_\varepsilon, \\ -\partial_y \Phi(t_\varepsilon, x_\varepsilon, y_\varepsilon) &= p_\varepsilon - D\varphi(t_\varepsilon, y_\varepsilon) \end{aligned}$$

and that

$$A = D^2 \Phi(t_\varepsilon, x_\varepsilon, y_\varepsilon) = \begin{pmatrix} \frac{I}{\varepsilon} & -\frac{I}{\varepsilon} \\ -\frac{I}{\varepsilon} & \frac{I}{\varepsilon} + D^2 \varphi(t_\varepsilon, y_\varepsilon) \end{pmatrix}.$$

From this we conclude that for every  $\iota > 0$ , there exist  $a_1, a_2 \in \mathbb{R}, X, Y \in \mathcal{S}^d$  such that

$$(a_1, p_\varepsilon, X) \in \bar{\mathcal{P}}^{2,+} u(t_\varepsilon, x_\varepsilon), \quad (a_2, p_\varepsilon - D\varphi(t_\varepsilon, y_\varepsilon), Y) \in \bar{\mathcal{P}}^{2,-} \tilde{v}(t_\varepsilon, y_\varepsilon),$$

such that  $a_1 - a_2 = \partial_t \Phi(t_\varepsilon, x_\varepsilon, y_\varepsilon) = \varphi_t(t_\varepsilon, x_\varepsilon)$  and such that

$$-\left(\frac{1}{\iota} + \|A\|\right)I \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq A + \iota A^2. \quad (\text{A.4})$$

From the definition of viscosity solution, we obtain that

$$-a_1 - b(x_\varepsilon)p_\varepsilon - \frac{1}{2} \text{tr} [\sigma \sigma^*(x_\varepsilon)X] - F(x_\varepsilon, u(x_\varepsilon)) \leq H(x_\varepsilon, p_\varepsilon)$$

and that

$$\begin{aligned} & -a_2 - b(y_\varepsilon)(p_\varepsilon - D\varphi(t_\varepsilon, y_\varepsilon)) - \frac{1}{2} \text{tr} [\sigma \sigma^*(y_\varepsilon)Y] - \rho F(y_\varepsilon, \frac{\tilde{v}(y_\varepsilon)}{\rho}) \\ & \geq \rho H(y_\varepsilon, \frac{p_\varepsilon - D\varphi(t_\varepsilon, y_\varepsilon)}{\rho}). \end{aligned}$$

Subtracting the two inequalities, we have

$$\begin{aligned} & -\partial_t \varphi_t(t_\varepsilon, y_\varepsilon) + b(y_\varepsilon)(p_\varepsilon - D\varphi(t_\varepsilon, y_\varepsilon)) - b(x_\varepsilon)p_\varepsilon \\ & + \frac{1}{2} \text{tr} [\sigma \sigma^*(y_\varepsilon)Y] - \frac{1}{2} \text{tr} [\sigma \sigma^*(x_\varepsilon)X] \\ & + \rho F(y_\varepsilon, \frac{\tilde{v}(y_\varepsilon)}{\rho}) - F(x_\varepsilon, u(x_\varepsilon)) \leq H(x_\varepsilon, p_\varepsilon) - \rho H(y_\varepsilon, \frac{p_\varepsilon - D\varphi(t_\varepsilon, y_\varepsilon)}{\rho}). \end{aligned}$$

We are now going to estimate the terms involving the drift, the volatility, and the functions  $F$  and  $H$  separately.

- Since  $b$  is Lipschitz continuous,

$$\begin{aligned} b(y_\varepsilon)(p_\varepsilon - D\varphi(t_\varepsilon, y_\varepsilon)) - b(x_\varepsilon)p_\varepsilon &= -b(y_\varepsilon)D\varphi(t_\varepsilon, y_\varepsilon) + (b(y_\varepsilon) - b(x_\varepsilon))p_\varepsilon \\ &\geq -b(y_\varepsilon)D\varphi(t_\varepsilon, y_\varepsilon) - \bar{C}\varepsilon^{-1}|x_\varepsilon - y_\varepsilon|^2. \end{aligned}$$

- In order to estimate the volatility term we denote by  $(e_i)_{1 \leq i \leq \bar{d}}$  the canonical basis of  $\mathbb{R}^{\bar{d}}$ . By using



(A.4) and the Lipschitz continuity of  $\sigma$ , we obtain

$$\begin{aligned}
& \operatorname{tr} [\sigma \sigma^*(x_\varepsilon) X] - \operatorname{tr} [\sigma \sigma^*(y_\varepsilon) Y] \\
&= \sum_{i=1}^{\bar{d}} \langle X \sigma(x_\varepsilon) e_i, \sigma(x_\varepsilon) e_i \rangle - \sum_{i=1}^{\bar{d}} \langle Y \sigma(y_\varepsilon) e_i, \sigma(y_\varepsilon) e_i \rangle \\
&\leq \sum_{i=1}^{\bar{d}} \langle D^2 \varphi(t_\varepsilon, y_\varepsilon) \sigma(y_\varepsilon) e_i, \sigma(y_\varepsilon) e_i \rangle + \frac{1}{\varepsilon} |\sigma(x_\varepsilon) - \sigma(y_\varepsilon)|^2 + \omega\left(\frac{\iota}{\varepsilon^2}\right) \\
&\leq \operatorname{tr} [\sigma \sigma^*(y_\varepsilon) D^2 \varphi(t_\varepsilon, y_\varepsilon)] + \bar{C}^2 \varepsilon^{-1} |x_\varepsilon - y_\varepsilon|^2 + \omega\left(\frac{\iota}{\varepsilon^2}\right)
\end{aligned}$$

where  $\omega$  is a modulus of continuity which is independent of  $\iota$  and  $\varepsilon$ .

- We now estimate  $\tilde{F} := \rho F(y_\varepsilon, \frac{\tilde{v}}{\rho}) - F(x_\varepsilon, u)$ . To this end, we first observe that

$$u(t_\varepsilon, x_\varepsilon) - \tilde{v}(t_\varepsilon, y_\varepsilon) - \varphi(t_\varepsilon, y_\varepsilon) \geq M_\varepsilon \geq u(\bar{t}, \bar{y}) - \tilde{v}(\bar{t}, \bar{y}) - \varphi(\bar{t}, \bar{y}).$$

Since  $(\bar{t}, \bar{y}) \in \{w > 0\}$  and  $\varphi$  is continuous, we can fix  $r$  small enough to obtain that

$$u(t_\varepsilon, x_\varepsilon) - \tilde{v}(t_\varepsilon, y_\varepsilon) \geq 0.$$

Recalling the definition of  $F$  in (2.10), the fact that  $F(y, \cdot)$  is decreasing on  $\mathbb{R}_+$  and the fact that  $\rho(1 - \rho^\beta) < (1 + \beta)(1 - \rho)$  for  $0 < \rho < 1$ , this yields

$$\begin{aligned}
\tilde{F} &= \rho F(y_\varepsilon, \frac{\tilde{v}}{\rho}) - F(y_\varepsilon, u) + F(y_\varepsilon, u) - F(x_\varepsilon, u) \\
&\geq (\rho - 1) \lambda(y_\varepsilon) + \frac{|u|^{\beta+1}}{\beta \eta(y_\varepsilon)^\beta} - \rho^{-\beta} \frac{|\tilde{v}|^{\beta+1}}{\beta \eta(y_\varepsilon)^\beta} \\
&\quad - \omega_R(|x_\varepsilon - y_\varepsilon|) \\
&= (\rho - 1) \lambda(y_\varepsilon) + \frac{|u|^{\beta+1}}{\beta \eta(y_\varepsilon)^\beta} - \frac{|\tilde{v}|^{\beta+1}}{\beta \eta(y_\varepsilon)^\beta} \\
&\quad - \rho(1 - \rho^\beta) \frac{|v|^{\beta+1}}{\beta \eta(y_\varepsilon)^\beta} - \omega_R(|x_\varepsilon - y_\varepsilon|) \\
&\geq -(1 - \rho) \lambda(y_\varepsilon) - (1 + \beta)(1 - \rho) \frac{|v|^{\beta+1}}{\beta \eta(y_\varepsilon)^\beta} - \omega_R(|x_\varepsilon - y_\varepsilon|) \\
&\geq -(1 - \rho) \left[ \lambda(y_\varepsilon) + \frac{1 + \beta}{\beta} \hat{C} \langle y_\varepsilon \rangle^m \right] - \omega_R(|x_\varepsilon - y_\varepsilon|)
\end{aligned} \tag{A.5}$$

where  $\omega_R$  denotes the modulus of continuity with  $R := |\bar{y}| + r$ .

- We finally estimate  $\tilde{H} := H(x_\varepsilon, p_\varepsilon) - \rho H(y_\varepsilon, \frac{p_\varepsilon - D\varphi(t_\varepsilon, y_\varepsilon)}{\rho})$ . By convexity, we have, for  $z_1, z_2 \in \mathbb{R}^d$ , that

$$|z_1|^{\alpha+1} - \rho \left| \frac{z_2}{\rho} \right|^{\alpha+1} \leq (1 - \rho) \left| \frac{z_1 - z_2}{1 - \rho} \right|^{\alpha+1}.$$

Hence,

$$\begin{aligned}
& H(x_\varepsilon, p_\varepsilon) - \rho H(y_\varepsilon, \frac{p_\varepsilon - D\varphi(t_\varepsilon, y_\varepsilon)}{\rho}) \\
&\leq (1 - \rho) \theta^\alpha \left| \frac{\sigma(x_\varepsilon) p_\varepsilon - \sigma(y_\varepsilon) (p_\varepsilon - D\varphi(t_\varepsilon, y_\varepsilon))}{1 - \rho} \right|^{\alpha+1} \\
&\leq \left(\frac{1 - \rho}{2}\right)^{-\alpha} \bar{C}^{\alpha+1} \left( |D\varphi(t_\varepsilon, y_\varepsilon)|^{\alpha+1} + (|x_\varepsilon - y_\varepsilon| \cdot |p_\varepsilon|)^{\alpha+1} \right)
\end{aligned}$$

where (L.2), (L.3) are used in the last inequality. If necessary, we can choose  $\bar{C}$  large enough to satisfy that  $\theta^\alpha |\sigma|^{\alpha+1} \leq \bar{C}^{\alpha+1}$ .

Denoting a generic modulus of continuity independent of  $\iota$  and  $\varepsilon$  by  $\omega$ , we thus get

$$\begin{aligned} & -\partial_t \varphi(t_\varepsilon, y_\varepsilon) - \mathcal{L}\varphi(t_\varepsilon, y_\varepsilon) - \left(\frac{1-\rho}{2}\right)^{-\alpha} \bar{C}^{\alpha+1} |D\varphi(t_\varepsilon, y_\varepsilon)|^{\alpha+1} \\ & - (1-\rho) \left[ \lambda(y_\varepsilon) + \frac{1+\beta}{\beta} \hat{C} \langle y_\varepsilon \rangle^m \right] \leq \omega(\varepsilon) + \omega\left(\frac{\iota}{\varepsilon^2}\right). \end{aligned}$$

Letting first  $\iota$  go to 0 and then sending  $\varepsilon$  to 0, we finally conclude the desired viscosity subsolution property of  $w$ .

**STEP 2: SMOOTH STRICT SUPERSOLUTION.** We are now going to construct smooth strict supersolutions to (A.3) on  $[T-\tau, T]$  for some small  $\tau > 0$ . To this end, let

$$\psi(t, y) := (1-\rho)C \langle y \rangle^m e^{L(T-t)}$$

where  $L, C > 0$  will be chosen later. Since  $\lambda, \phi \in C_m(\mathbb{R}^d)$  and  $u \in \mathcal{SSG}_m^-([0, T] \times \mathbb{R}^d)$ , we choose a large enough constant  $\bar{C}$  such that for  $\zeta = \lambda, \phi$

$$\zeta(y) \leq \bar{C} \langle y \rangle^m, \quad y \in \mathbb{R}^d,$$

and such that

$$u(t, y) \leq \bar{C} \langle y \rangle^m, \quad (t, y) \in [0, T] \times \mathbb{R}^d. \quad (\text{A.6})$$

Note that

$$D \langle y \rangle^m = m \langle y \rangle^{m-2} y, \quad D^2 \langle y \rangle^m = m \langle y \rangle^{m-4} (\langle y \rangle^2 I + (m-2)y \otimes y).$$

Since  $b, \sigma$  grow at most linearly,

$$\begin{aligned} \mathcal{L}\psi(t, y) & \leq (1-\rho)C e^{L(T-t)} [\bar{C}(1+|y|)|D \langle y \rangle^m| + \bar{C}^2(1+|y|)^2 |D^2 \langle y \rangle^m|] \\ & \leq (1-\rho)C e^{L(T-t)} [2m\bar{C} \langle y \rangle^m + 2m(m-1)\bar{C}^2 \langle y \rangle^m] \\ & \leq [2m\bar{C} + 2m(m-1)\bar{C}^2] \psi(t, y). \end{aligned}$$

Recalling that  $(m-1)(\alpha+1) = m$ , we have

$$\begin{aligned} & \left(\frac{1-\rho}{2}\right)^{-\alpha} \bar{C}^{\alpha+1} |D\psi(t, y)|^{\alpha+1} \\ & = \left(\frac{1-\rho}{2}\right)^{-\alpha} \bar{C}^{\alpha+1} \cdot (1-\rho)^{\alpha+1} C^{\alpha+1} e^{(\alpha+1)L(T-t)} |D \langle y \rangle^m|^{\alpha+1} \\ & \leq [2^\alpha m^{\alpha+1} \bar{C}^{\alpha+1} C^\alpha e^{\alpha L(T-t)}] \psi(t, y) \end{aligned}$$

By condition (F.4),

$$(1-\rho) \left[ \lambda(y) + \frac{1+\beta}{\beta} \hat{C} \langle y \rangle^m \right] \leq (1-\rho) \frac{1+2\beta}{\beta} \bar{C} \langle y \rangle^m \leq \frac{1+2\beta}{\beta} \frac{\bar{C}}{C} \psi(t, y)$$

Choosing  $C > \max\{2m\bar{C} + 2m(m-1)\bar{C}^2, 2^\alpha m^{\alpha+1} \bar{C}^{\alpha+1}, \frac{1+2\beta}{\beta} \bar{C}\}$ , we have

$$\begin{aligned} & -\partial_t \psi(t, y) - \mathcal{L}\psi(t, y) - \left(\frac{1-\rho}{2}\right)^{-\alpha} \bar{C}^{\alpha+1} |D\psi(t, y)|^{\alpha+1} - (1-\rho) \left[ \lambda(y) + \frac{1+\beta}{\beta} \hat{C} \langle y \rangle^m \right] \\ & > \psi(t, y) \left[ L - C - 1 - C^{\alpha+1} e^{\alpha L(T-t)} \right]. \end{aligned}$$

Then taking  $L > C + 1 + C^{\alpha+1} e$ , we get

$$-\partial_t \psi(t, y) - \mathcal{L}\psi(t, y) - \left(\frac{1-\rho}{2}\right)^{-\alpha} \bar{C}^{\alpha+1} |D\psi(t, y)|^{\alpha+1} - (1-\rho) \left[ \lambda(y) + \frac{1+\beta}{\beta} \hat{C} \langle y \rangle^m \right] > 0$$

for all  $y \in \mathbb{R}^d$  and  $t \in [T-\tau, T]$ , where  $\tau = \frac{1}{\alpha L}$ .

STEP 3: CONCLUSIONS. Since  $w \in USC([T-\tau, T] \times \mathbb{R}^d) \cap \mathcal{SSG}_m^-$ , the function  $w - \psi$  attains its maximum at some point  $(t, y) \in [T-\tau, T] \times \mathbb{R}^d$ . We claim that  $t = T$ . Indeed, suppose to the contrary that  $\bar{t} < T$ . Then, since  $w$  is a viscosity subsolution of (A.3), by taking  $\psi$  as a test function,

$$-\partial_t \psi(t, y) - \mathcal{L}\psi(t, y) - \left(\frac{1-\rho}{2}\right)^{-\alpha} \bar{C}^{\alpha+1} |D\psi(t, y)|^{\alpha+1} - (1-\rho) \left[ \lambda(y) + \frac{1+\beta}{\beta} \hat{C} \langle y \rangle^m \right] \leq 0.$$

This contradicts the fact that  $\psi$  is a strict supersolution. Thus, for all  $(t, y) \in [T-\tau, T] \times \mathbb{R}^d$ ,

$$w(t, y) - \psi(t, y) \leq w(T, y) - \psi(T, y) \leq (1-\rho)\phi(y) - (1-\rho)C\langle y \rangle^m \leq 0$$

where the last inequality follows from  $C > \bar{C}$ . In particular,  $w(t, y) \leq \psi(t, y)$ . Letting  $\rho \rightarrow 1$ , we get  $u \leq v$  on  $[T-\tau, T] \times \mathbb{R}^d$ .

The preceding argument can be iterated on time intervals of the same length  $\tau$ . Indeed, let us choose  $C, L, \tau$  as in Step 2 and put

$$\psi(t, y) := (1-\rho)C\langle y \rangle^m e^{L(T-\tau-t)}$$

on  $[T-2\tau, T-\tau]$ . It follows by (A.6) and the previously established inequality  $u \leq v$  on  $[T-\tau, T] \times \mathbb{R}^d$  that for all  $y \in \mathbb{R}^d$ ,

$$w(T-\tau, y) = u(T-\tau, y) - \tilde{v}(T-\tau, y) \leq (1-\rho)u(T-\tau, y) \leq (1-\rho)\bar{C}\langle y \rangle^m.$$

Following the same arguments as above, we obtain that for all  $(t, y) \in [T-2\tau, T-\tau] \times \mathbb{R}^d$ ,

$$w(t, y) - \psi(t, y) \leq w(T-\tau, y) - \psi(T-\tau, y) \leq (1-\rho)\bar{C}\langle y \rangle^m - (1-\rho)C\langle y \rangle^m \leq 0.$$

These arguments can be iterated to complete the proof.  $\square$

*Remark A.2.* It is worth noting that the constant  $\hat{C}$  in (A.3) is exactly derived from the upper bound of  $v$  in (A.2) when estimating  $\tilde{F}$  in (A.5). We show below that using the constant derived from the upper bound of  $u$  instead is also feasible. To this end, we estimate  $\tilde{F}$  in the following way:

$$\begin{aligned} \tilde{F} &= \rho F(x_\varepsilon, \frac{\tilde{v}}{\rho}) - F(x_\varepsilon, u) + \rho F(y_\varepsilon, \frac{\tilde{v}}{\rho}) - \rho F(x_\varepsilon, \frac{\tilde{v}}{\rho}) \\ &\geq (\rho-1)\lambda(x_\varepsilon) + \frac{|u|^{\beta+1}}{\beta\eta(x_\varepsilon)^\beta} - \rho^{-\beta} \frac{|\tilde{v}|^{\beta+1}}{\beta\eta(x_\varepsilon)^\beta} - \omega_R(|x_\varepsilon - y_\varepsilon|) \\ &\geq -(1-\rho)\lambda(x_\varepsilon) - (1-\rho) \frac{1+\beta}{\beta} \hat{C} \langle x_\varepsilon \rangle^m - \omega_R(|x_\varepsilon - y_\varepsilon|), \end{aligned} \quad (\text{A.7})$$

In the last inequality we used the facts that  $u^{\beta+1}(t, y) \leq \hat{C}\eta^\beta(y)\langle y \rangle^m$  on  $[0, T] \times \mathbb{R}^d$  and  $\rho^{-\beta} - 1 \leq (\beta+1)(1-\rho)$  for  $\rho \in (\sqrt[\beta+1]{\frac{\beta}{\beta+1}}, 1)$ .

The next lemma establishes a comparison principle for continuous solutions to (2.11) when imposed with a singular terminal time. The proof uses the shifting argument given in [26].

**Lemma A.3.** *Assume that (L.1)-(L.3), (F.1) and (F.2) hold. Let  $\tilde{m}$  be as in condition (F.1). Let  $\underline{v}, \bar{v} \in C_{\tilde{m}}([0, T^-] \times \mathbb{R}^d)$  be a nonnegative viscosity sub- and a nonnegative viscosity supersolution to (2.11), respectively, such that*

$$\lim_{t \rightarrow T} \bar{v}(t, y) = +\infty \quad \text{locally uniformly on } \mathbb{R}^d.$$

Then,

$$\underline{v} \leq \bar{v} \quad \text{in } [0, T] \times \mathbb{R}^d.$$

In particular, there exists at most one nonnegative viscosity solution in  $C_{\tilde{m}}([0, T^-] \times \mathbb{R}^d)$  to (2.11).

*Proof.* Due to the time-homogeneity of the PDE in (2.11), viscosity (super-/sub-)solutions stay viscosity (super-/sub-)solutions when shifted in time. For any  $\delta > 0$ , we define the difference function  $w : [0, T - \delta] \times \mathbb{R}^d \rightarrow \mathbb{R}$  by

$$w(t, y) := \underline{v}(t, y) - \rho \bar{v}(t + \delta, y).$$

Under assumptions (F.1) and (F.2), we have that  $\underline{v}, \bar{v}$  belong to  $\mathcal{SSG}_m$  and satisfy the condition (A.2) in Proposition A.1 on  $[0, T] \times \mathbb{R}^d$ . Hence, we can use the similar argument as in the proof of Proposition A.1 to obtain that  $w$  is a viscosity subsolution of the following PDE:

$$-\partial_t u(t, y) - \mathcal{L}u(t, y) - \left(\frac{1-\rho}{2}\right)^{-\alpha} \bar{C}^{\alpha+1} |Du|^{\alpha+1} - (1-\rho) \left[ \lambda(\bar{y}) + \frac{1+\beta}{\beta} \hat{C}\langle y \rangle^m \right] = 0, \quad (\text{A.8})$$

for  $(t, y) \in [0, T - \delta] \times \mathbb{R}^d \cap \{w > 0\}$  and  $\lim_{t \rightarrow T - \delta} w(t, y) \leq (1-\rho)\underline{v}(T - \delta, y)$  for  $y \in \mathbb{R}^d$ . In fact, Remark A.2 shows that we can get around the difficulty of the singularity of  $\bar{v}(\cdot + \delta, \cdot)$  at time  $t = T - \delta$  in this step. Following Steps 2 and 3 in the proof of Proposition A.1, we have that  $\underline{v}(t, y) \leq \bar{v}(t + \delta, y)$  on  $[0, T - \delta] \times \mathbb{R}^d$ . Finally, by letting  $\delta \rightarrow 0$  we conclude that  $\underline{v} \leq \bar{v}$  on  $[0, T] \times \mathbb{R}^d$  by continuity of  $\bar{v}$ .  $\square$

## A.2 Proof of Proposition 3.1

Under assumptions (F.1), (F.2) and (3.2), the functions  $(t, y) \mapsto (T-t)^{1/\beta} \underline{u}(t, y), (T-t)^{1/\beta} \bar{u}(t, y)$  satisfy the condition (A.2) in Proposition A.1. Let us fix  $\rho \in \left( \sqrt[\beta]{\frac{\frac{1}{4}\beta+1}{\frac{1}{2}\beta+1}}, 1 \right)$  and consider the difference

$$w := \underline{u} - \rho \bar{u} \in USC_{\bar{m}}([T - \delta, T^-] \times \mathbb{R}^d) \subset \mathcal{SSG}_m([T - \delta, T^-] \times \mathbb{R}^d).$$

The proof of the following lemma is similar to that of Proposition A.1.

**Lemma A.4.** *The function  $w$  is a viscosity subsolution to*

$$\begin{aligned} & -\partial_t w(t, y) - \mathcal{L}w(t, y) - \left(\frac{1-\rho}{2}\right)^{-\alpha} \bar{C}^{\alpha+1} |Dw|^{\alpha+1} - l(t, y)w(t, y) \\ & - (1-\rho) \left[ \lambda(y) + \frac{1+\beta}{\beta} \frac{\hat{C}\langle y \rangle^m}{(T-t)^{1/\beta+1}} \right] = 0, \quad (t, y) \in [T - \delta, T] \times \mathbb{R}^d \end{aligned} \quad (\text{A.9})$$

where

$$l(t, y) := \frac{F(y, \underline{u}(t, y)) - F(y, \rho \bar{u}(t, y))}{\underline{u}(t, y) - \rho \bar{u}(t, y)} \mathbb{1}_{\underline{u}(t, y) \neq \rho \bar{u}(t, y)}.$$

The next lemma constructs a local smooth strict supersolution to (A.9).

**Lemma A.5.** *There exists  $L, C, \tau > 0$  such that*

$$\chi(t, y) := (1-\rho) \frac{e^{L(T-t)} C \langle y \rangle^m}{(T-t)^{1/\beta}}$$

satisfies

$$\begin{aligned} \mathcal{J}[\chi] & := -\partial_t \chi(t, y) - \mathcal{L}\chi(t, y) - \left(\frac{1-\rho}{2}\right)^{-\alpha} \bar{C}^{\alpha+1} |D\chi(t, y)|^{\alpha+1} + \frac{1+\frac{1}{4}\beta}{\beta(T-t)} \chi(t, y) \\ & - (1-\rho) \left[ \lambda(y) + \frac{1+\beta}{\beta} \frac{\hat{C}\langle y \rangle^m}{(T-t)^{1/\beta+1}} \right] > 0, \quad (t, y) \in [T - \tau, T] \times \mathbb{R}^d. \end{aligned} \quad (\text{A.10})$$

*Proof.* Set  $\psi(t, y) := (1 - \rho)e^{L(T-t)}C\langle y \rangle^m$ . Analogous to the proof of Proposition A.1, we have

$$\begin{aligned}\mathcal{L}\chi(t, y) &\leq [2m\bar{C} + 2m(m-1)\bar{C}^2] \frac{\psi(t, y)}{(T-t)^{1/\beta}}, \\ \left(\frac{1-\rho}{2}\right)^{-\alpha} \bar{C}^{\alpha+1} |D\chi(t, y)|^{\alpha+1} &\leq [2^\alpha m^{\alpha+1} \bar{C}^{\alpha+1} C^\alpha e^{\alpha L(T-t)}] \frac{\psi(t, y)}{(T-t)^{(1+\alpha)/\beta}}, \\ (1-\rho) \left[ \lambda(y) + \frac{1+\beta}{\beta} \frac{\hat{C}\langle y \rangle^m}{(T-t)^{1/\beta+1}} \right] &\leq \frac{\bar{C}}{C} \psi(t, y) + \frac{1+\beta}{\beta} \frac{\bar{C}}{C} \frac{\psi(t, y)}{(T-t)^{1/\beta+1}}.\end{aligned}$$

Choosing  $C > \max\{2m\bar{C} + 2m(m-1)\bar{C}^2, 2^\alpha m^{\alpha+1} \bar{C}^{\alpha+1}, 8^{\frac{1+\beta}{\beta}} \bar{C}\}$ , we obtain that

$$\begin{aligned}\mathcal{J}[\chi] &> \frac{L\psi}{(T-t)^{1/\beta}} - \frac{\psi}{\beta(T-t)^{1/\beta+1}} - \frac{C\psi}{(T-t)^{1/\beta}} - C^{\alpha+1} e^{\alpha L(T-t)} \frac{\psi}{(T-t)^{(1+\alpha)/\beta}} \\ &\quad + \frac{1+\frac{1}{4}\beta}{\beta(T-t)^{1/\beta+1}} \psi - \psi - \frac{\psi}{8(T-t)^{1/\beta+1}} \\ &> \psi \left[ \frac{L-C-T^{1/\beta}}{(T-t)^{1/\beta}} + \frac{1-8C^{\alpha+1} e^{\alpha L(T-t)} (T-t)^{1-\alpha/\beta}}{8(T-t)^{1/\beta+1}} \right]\end{aligned}$$

Taking  $L > C + T^{1/\beta}$  and then choosing  $\tau = \min\{\frac{1}{\alpha L}, (8C^{\alpha+1} e^1)^{(\alpha-\beta)/\alpha}\}$ , we get  $\mathcal{J}[\chi] > 0$  for all  $(t, y) \in [T-\tau, T) \times \mathbb{R}^d$ .  $\square$

The following lemma is key to the proof of the comparison principle.

**Lemma A.6.** *Let  $\tau$  be as in Lemma A.5. The function*

$$\Phi(t, y) := w(t, y) - \chi(t, y)$$

*is either nonpositive or attains its supremum at some point  $(\bar{t}, \bar{y})$  in  $[T-\tau, T) \times \mathbb{R}^d$ .*

*Proof.* Suppose that the supremum of  $\Phi$  on  $[T-\tau, T) \times \mathbb{R}^d$  is positive and denote by  $(t_k, y_k)$  a sequence in  $[T-\tau, T) \times \mathbb{R}^d$  approaching the supremum point. For the choice of  $C$  in Lemma A.5,  $\eta(y) < C\langle y \rangle^m$  for all  $y \in \mathbb{R}^d$ . Thus, the representation

$$\Phi(t, y) = \frac{\left[ \frac{\underline{u}(t, y)(T-t)^{1/\beta}}{\langle y \rangle^{\tilde{m}}} - \frac{\rho \bar{u}(t, y)(T-t)^{1/\beta}}{\langle y \rangle^{\tilde{m}}} \right] \langle y \rangle^{\tilde{m}} - (1-\rho)e^{L(T-t)}C\langle y \rangle^m}{(T-t)^{1/\beta}},$$

along with Condition (3.1) and the fact that  $\tilde{m} < m$  yields

$$\limsup_{t \rightarrow T} \Phi(t, y) = -\infty, \text{ uniformly on } \mathbb{R}^d.$$

Hence  $\lim_k t_k < T$ . Furthermore,  $\lim_k |y_k| < \infty$  because  $w \in \mathcal{SSG}_m^-$ . As a result, the supremum is attained at some point  $(\bar{t}, \bar{y})$  because  $\Phi$  is upper semicontinuous. This proves the assertion.  $\square$

We are now ready to prove the comparison principle.

*Proof of Proposition 3.1. STEP 1: COMPARISON ON  $[T-\tau, T)$ .* Let  $\tau$  be as in Lemma A.5. We claim that the function  $\Phi$  introduced in Lemma A.6 is nonpositive. It then follows that  $\underline{u} \leq \bar{u}$  in  $[T-\tau, T) \times \mathbb{R}^d$  by letting  $\rho \rightarrow 1$ . In view of Lemma A.6, we just need to consider the case where  $\Phi$  attains its supremum at some point  $(\bar{t}, \bar{y}) \in [T-\tau, T) \times \mathbb{R}^d$ . Since  $\chi$  is smooth and  $w$  is a viscosity subsolution to (A.9),

$$\begin{aligned}-\partial_t \chi(\bar{t}, \bar{y}) - \mathcal{L}\chi(\bar{t}, \bar{y}) - \left(\frac{1-\rho}{2}\right)^{-\alpha} \bar{C}^{\alpha+1} |D\chi|^{\alpha+1} - l(\bar{t}, \bar{y})w(\bar{t}, \bar{y}) \\ - (1-\rho) \left[ \lambda(\bar{y}) + \frac{1+\beta}{\beta} \frac{\hat{C}\langle \bar{y} \rangle^m}{(T-\bar{t})^{1/\beta+1}} \right] \leq 0.\end{aligned}\tag{A.11}$$

By the mean value theorem and in view of condition (3.2),

$$\begin{aligned}
l(t, y) &= \frac{F(y, \underline{u}(t, y)) - F(y, \rho \bar{u}(t, y))}{\underline{u}(t, y) - \rho \bar{u}(t, y)} \mathbb{I}_{\underline{u}(t, y) \neq \bar{u}(t, y)} \\
&\leq \partial_u F(y, \rho \sqrt{\frac{\frac{1}{2}\beta + 1}{\beta + 1}} \frac{\eta(y)}{(T-t)^{1/\beta}}) \\
&\leq -\frac{1 + \frac{1}{4}\beta}{\beta(T-t)}.
\end{aligned} \tag{A.12}$$

Thus, comparing (A.10) with (A.11) yields

$$l(\bar{t}, \bar{y})w(\bar{t}, \bar{y}) > -\frac{1 + \frac{1}{4}\beta}{\beta(T-t)}\chi(\bar{t}, \bar{y}) \geq l(\bar{t}, \bar{y})\chi(\bar{t}, \bar{y}). \tag{A.13}$$

Since  $l \leq 0$ , we can conclude that  $\Phi(\bar{t}, \bar{y}) \leq 0$ , and so  $\Phi \leq 0$ .

STEP 2: COMPARISON ON  $[T-\delta, T]$ . If  $\tau > \delta$ , then the proof is finished. Else, we can proceed as follows. From the condition (3.2),

$$\underline{u}(t, y), \bar{u}(t, y) \leq \frac{\hat{C}}{\tau^{1/\beta}}\eta(y), \quad t \in [T-\delta, T-\tau].$$

Since we have already shown that  $\underline{u}(T-\tau, \cdot) \leq \bar{u}(T-\tau, \cdot)$ , an application of our general comparison principle [Proposition A.1] shows that  $\underline{u} \leq \bar{u}$  on  $[T-\delta, T] \times \mathbb{R}^d$ .  $\square$

### A.3 Proof of Lemma 6.1

The existence of a classical solution  $v_0$  to (2.4) along with the stated estimates on  $v_0$  has been proved in [26]; the gradient was not given in [26]. In what follows we analyze the  $C^{0,1}$  regularity of  $v_0$  under weaker assumptions. As discussed in [26], we can plug the asymptotic ansatz

$$v(T-t, y) = \frac{\eta(y)}{t^{1/\beta}} + \frac{u(t, y)}{t^{1+1/\beta}}, \quad u(t, y) = O(t^2) \text{ uniformly in } y \text{ as } t \rightarrow 0. \tag{A.14}$$

into (2.4) and consider instead the PDE

$$\begin{cases} \partial_t u(t, y) = \mathcal{L}u(t, y) + f(t, y, u(t, y)), & t > 0, y \in \mathbb{R}^d, \\ u(0, y) = 0, & y \in \mathbb{R}^d. \end{cases} \tag{A.15}$$

where

$$f(t, y, u) := t\mathcal{L}\eta(y) + t^p\lambda(y) - \frac{\eta(y)}{\beta} \sum_{k=2}^{\infty} \binom{\beta+1}{k} \left(\frac{u}{t\eta(y)}\right)^k.$$

We now show that this PDE admits a mild solution in  $C^{0,1}([0, \delta] \times \mathbb{R}^d)$ . To this end we consider the space

$$E := \{u \in C_b^{0,1}([0, \delta] \times \mathbb{R}^d) : \|u(t, \cdot)\| + \|t^{1/2}Du(t, \cdot)\| = O(t^2) \text{ as } t \rightarrow 0\}$$

endowed with the weighted norm

$$\|u\|_E = \sup_{0 < t \leq \delta, y \in \mathbb{R}^d} \|t^{-2}u(t, y)\|$$

and define the operator

$$\Gamma[u](t, y) = \int_0^t P_{t-s}[f(s, \cdot, u(s, \cdot))](y)ds.$$

Let  $R > 0$  and  $\delta \in (0, \underline{c}/R]$ . Using arguments given in [26, Section 4], we see that for every  $u$  in the closed ball  $\bar{B}_E(R) := \{u \in E : \|u\|_E \leq \underline{c}/\delta\}$ , the function  $f(\cdot, u(\cdot))$  belongs to  $C_b([0, \delta] \times \mathbb{R}^d)$ . In particular, the map  $\Gamma$  is well defined on  $\bar{B}_E(R)$ . Moreover, there exists a constant  $L > 0$  independent of  $\delta$  such that

$$|f(t, y, u(t, y)) - f(t, y, v(t, y))| \leq L|u(t, y) - v(t, y)|, u, v \in \bar{B}_E(R), (t, y) \in [0, \delta] \times \mathbb{R}^d.$$

Now we are ready to carry out the fixed point argument.

Let  $B(a, b) := \int_0^1 r^{a-1}(1-r)^{b-1}dr$  be the Beta function with  $a, b > 0$ . We choose

$$R = 2(1 + MB_0)(\|\mathcal{L}\eta\| + \|\lambda\|),$$

and

$$\delta = \min\{\underline{c}/R, (2L(1 + MB_1)), 1\},$$

where  $L > 0$  is the Lipschitz constant given by Lemma 4.5 and  $B_0 := B(2, \frac{1}{2}), B_1 := B(3, \frac{1}{2})$ .

Let  $u, v \in \bar{B}_\Sigma(R)$ . For  $(t, y) \in [0, \delta] \times \mathbb{R}^d$ ,

$$\begin{aligned} |\Gamma[u](t, y) - \Gamma[v](t, y)| &= \left| \int_0^t P_{t-s} [f(s, \cdot, u(s, \cdot)) - f(s, \cdot, v(s, \cdot))](y) ds \right| \\ &\leq \int_0^t \|f(s, \cdot, u(s, \cdot)) - f(s, \cdot, v(s, \cdot))\| ds \\ &\leq \int_0^t L \|u(s, \cdot) - v(s, \cdot)\| ds \\ &\leq \delta L t^2 \|u - v\|_E. \end{aligned}$$

Similarly,

$$\begin{aligned} |D\Gamma[u](t, y) - D\Gamma[v](t, y)| &= \left| \int_0^t DP_{t-s} [f(s, \cdot, u(s, \cdot)) - f(s, \cdot, v(s, \cdot))](y) ds \right| \\ &\leq M \int_0^t \frac{1}{(t-s)^{1/2}} \|f(s, \cdot, u(s, \cdot)) - f(s, \cdot, v(s, \cdot))\| ds \\ &\leq \int_0^t ML \frac{1}{(t-s)^{1/2}} (s^2 \|u - v\|_E) ds \\ &\leq \delta t^{3/2} ML B_1 \|u - v\|_E. \end{aligned}$$

Hence

$$\|\Gamma[u] - \Gamma[v]\|_\Sigma \leq \frac{1}{2} \|u - v\|_E.$$

To show that  $\Gamma$  maps  $\bar{B}_E(R)$  into itself, note that  $\delta \leq 1$  implies  $s^k \leq 1$  for all  $k > 0$  and  $s \in [0, \delta]$ . Hence, for every  $t \in [0, \delta]$

$$\begin{aligned} |\Gamma[0](t, y)| &= \left| \int_0^t P_{t-s} [f(s, \cdot, 0)](y) ds \right| \\ &\leq \int_0^t \|s\mathcal{L}\eta + s^p\lambda\| ds \\ &\leq t^2 (\|\mathcal{L}\eta\| + \|\lambda\|) \end{aligned}$$

and

$$\begin{aligned} |D\Gamma[0](t, y)| &= \left| \int_0^t DP_{t-s} [F_0(s, \cdot, 0)](y) ds \right| \\ &\leq \int_0^t \frac{1}{(t-s)^{1/2}} M \|s\mathcal{L}\eta + s^p\lambda\| ds \\ &\leq t^{3/2} MB_0 (\|\mathcal{L}\eta\| + \|\lambda\|) \end{aligned}$$

Thus,

$$\|\Gamma[u]\|_E \leq \|\Gamma[u] - \Gamma[0]\|_E + \|\Gamma[0]\|_E \leq R.$$

The operator  $\Gamma$  is therefore a contraction from  $\overline{B}_E(R)$  to itself. Hence, it has a unique fixed point  $u$  in  $\overline{B}_E(R)$ . We conclude that Equation (A.15) admits a mild solution in  $C_b^{0,1}([0, \delta] \times \mathbb{R}^d)$ .

In view of the ansatz (A.14),  $v_0$  is a solution to (2.4) in  $C_b^{0,1}([T - \delta, T^-] \times \mathbb{R}^d)$  and there exists a constant  $C > 0$  such that for  $(t, y) \in [T - \delta, T) \times \mathbb{R}^d$ ,

$$|Dv_0| \leq \frac{C}{(T - t)^{1/\beta}}.$$

The  $C^{0,1}$ -regularity of  $v_0$  along with the boundedness of  $Dv_0$  on  $[0, T - \delta] \times \mathbb{R}^d$  can be obtained by [12, Theorem 15]. To conclude, for a constant  $C_0 > 0$ ,

$$|Dv_0| \leq \frac{C_0}{(T - t)^{1/\beta}}, \quad (t, y) \in [0, T) \times \mathbb{R}^d. \quad (\text{A.16})$$

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