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Note

Note on a Theorem of J. Folkman on Transversals of Infinite Families with Finitely Many Infinite Members

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In this note we show by a simple direct proof that Folkman's necessary and sufficient condition for an infinite family of sets with finitely many infinite members to have a transversal implies Woodall's condition. A short proof of Folkman's theorem results by combining with Woodall's proof.

Several authors, Brualdi and Scrimger [1], Folkman [2] and Woodall [5], have given necessary and sufficient conditions for an infinite family of sets with finitely many infinite members to have a transversal, generalising the well-known theorems of Hall [3] and Jung and Rado [4].

In each case the proof of necessity is straightforward. Woodall's proof of his theorem is one page long, and he also establishes simply the equivalence between his conditions and those of Brualdi and Scrimger. On the other hand the proof of Folkman's theorem in [2] is quite intricate. In this note we show by a simple direct proof that Folkman's condition implies Woodall's condition. A short proof of Folkman's theorem results by combining with Woodall's proof.

Throughout this note $A = (A_i/i \in I)$ is a family of subsets of a set E such that A_i is finite for all $i \in I_0$ with $I \backslash I_0$ finite. For $J \subseteq I$ we abbreviate $\bigcup_{i \in J} A_i$ to $A(J)$.

THEOREM 1 (J. Folkman [2]). *A has a transversal if and only if*

(1.1) *for every non-negative integer n and pair J, J′ of sets such that* $J' \subseteq J \subseteq I$, $A(J) \backslash A(J')$ *is finite and* $|A(J) \backslash A(J')| \leqslant |J \backslash J'| - n$ *there is a finite set* $J'' \subseteq J'$ *such that all finite sets L with* $J'' \subseteq L \subseteq J'$ *satisfy* $|A(L)| \geqslant |L| + n$.

Let X be a subset of E. If the subfamily $(A_i/i \in I_0)$ has a transversal

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disjoint from X (i.e., by the theorem of M. Hall, if $|A(J)\backslash X| \geqslant |J|$ for all finite $J \subseteq I_0$) we denote by $c(X)$ the union of the finite subsets J of I_0 such that $|A(J)\backslash X| = |J|$; otherwise we set $c(X) = \varnothing$.

THEOREM 2 (D. R. Woodall [5]). *A has a transversal if and only if*

(2.1) *the subfamily* $(A_i/i \in I_0)$ *has a transversal (i.e., by Hall's theorem* $|A(J)| \geqslant |J|$ *for all finite* $J \subseteq I_0$*);*

(2.2) *for all non-empty subsets* K *of* $I\backslash I_0$ *and all subsets* X *of* $A(K)$ *such that* $|X| < |K|$ *we have* $A(K)\backslash X \nsubseteq A(c(X))$.

Clearly (1.1) implies (2.1) (take $n = 0$ and $J = J' = L$). To show that (1.1) implies (2.2) we need the following easy lemma contained in both [2] and [5].

LEMMA. *Let* X *be a finite subset of* E *such that the family* $(A_i/i \in I_0)$ *has a transversal disjoint from* X.

Then any finite union (and intersection) of finite sets J *contained in* I_0 *satisfying* $|A(J)\backslash X| = |J|$ *also has this property. It follows that any finite subset of* $c(X)$ *is contained in a finite set* $L \subseteq c(X)$ *such that* $|A(L)\backslash X| = |L|$.

Proof of (1) $\Rightarrow$ (2.2). Suppose that for some $K \subseteq I\backslash I_0$ and $X \subseteq A(K)$ with $|X| < |K|$ we have $A(K)\backslash X \subseteq A(c(X))$.

Set $J' = c(X)$ and $J = c(X) \cup K$.

We have $A(J)\backslash A(J') = A(K)\backslash A(c(X)) \subseteq X\backslash A(c(X))$ and $J\backslash J' = K$ (since $c(X) \subseteq I_0$, $K \subseteq I\backslash I_0$). The hypothesis of (1.1) with $n = |K| - |X\backslash A(c(X))|$ is satisfied: hence there is a finite J'' contained in J' such that $|A(L)| \geqslant |L| + n$ for all finite sets L with $J'' \subseteq L \subseteq J'$.

We have $n \geqslant 1$, and hence $|A(J'')| \geqslant |J''| + n$ implies $J'' \neq \varnothing$: thus $c(X)$ containing J'' is also non-empty. By the lemma since J'' contained in $J' = c(X)$ is finite there is a finite set L with $J'' \subseteq L \subseteq J'$ such that $|A(L)\backslash X| = |L|$. Since $|A(L)| \geqslant |L| + n$ also, it follows that

$$\begin{aligned} |K| + |L| &= |X\backslash A(J')| + n + |L| \leqslant |X\backslash A(J')| + |A(L)| \\ &= |X\backslash A(J')| + |A(L) \cap X| + |A(L)\backslash X| \\ &\leqslant |X| + |L|, \end{aligned}$$

contradicting $|X| < |K|$.

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