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NOTE

ON THE FACTORIZATION OF GRAPHS WITH EXACTLY ONE VERTEX OF INFINITE DEGREE*

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We give a necessary and sufficient condition for the existence of a 1-factor in graphs with exactly one vertex of infinite degree.

1. Introduction

The following well-known necessary and sufficient condition for the existence of a 1-factor in locally finite graphs is due to Tutte [5]:

Theorem A. *A locally finite graph $G = (V, E)$ has a 1-factor if and only if $C_1(V \setminus S) \leq |S|$ for all finite subset S of V . (See notations below.)*

In the present note, we extend this theorem to graphs with exactly one vertex of infinite degree. For bipartite graphs with exactly one vertex of infinite degree, our result reduces to a theorem due to Jung and Rado [4].

2. Notations and terminology

Graphs considered in this note are undirected without loops or multiple edges.

Let $G = (V, E)$ be a graph. A 1-factor, or *perfect matching*, of G is a set of pairwise disjoint edges of G containing all vertices. We say that G is *factorizable* if it contains at least one 1-factor.

A finite graph is *1-factor critical* if by deleting any vertex one obtains a factorizable graph. A 1-factor critical graph has clearly an odd number of vertices.

We denote by $C_1(G)$ the number of connected components with odd cardinalities of G , and by $C_{cr}(G)$ the number of connected components of G which are 1-factor critical.

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Given a subset S of V , we denote by $G[S]$ the subgraph of G induced by S . If no confusion results we abbreviate $C_1(G[S])$ and $C_{cr}(G[S])$ to $C_1(S)$ and $C_{cr}(S)$ respectively.

Given a vertex v , we denote by $A(v)$ the set of vertices adjacent to v in G . A graph is *locally finite* if $A(v)$ is finite for every vertex v .

3. Statement of the results

Theorem 3.1. *A graph $G = (V, E)$ with exactly one vertex v_0 of infinite degree is factorizable if and only if*

- (1.1) $C_1(V \setminus S) \leq |S|$ for all finite subsets S of V ,
- (1.2) $A(v_0) \not\subseteq \bigcup \{S \subseteq V \setminus \{v_0\} : S \text{ finite, } C_1(V \setminus [S \cup \{v_0\}]) = |S|\}$.

Corollary 3.2. *A graph $G = (V, E)$ with exactly one vertex v_0 of infinite degree is factorizable if and only if*

- (2.1) $C_{cr}(V \setminus S) \leq |S|$ for all finite subsets S of V ,
- (2.2) $A(v_0) \not\subseteq \bigcup \{S \subseteq V \setminus \{v_0\} : S \text{ finite, } C_{cr}(V \setminus [S \cup \{v_0\}]) = |S|\}$.

The following lemma proved in [1] is needed to prove Theorem 3.1 and Corollary 3.2:

Lemma 3.3. *Let $G = (V, E)$ be a locally finite graph and k a non-negative integer. If there exists a finite subset S of V such that $C_1(V \setminus S) \geq |S| + k$, then there exists a finite subset T of V such that $S \subseteq T$ and $C_{cr}(V \setminus T) \geq |T| + k$.*

From Lemma 3.3 a strengthening of Theorem A [1] follows:

Theorem B. *A locally finite graph $G = (V, E)$ is factorizable if and only if $C_{cr}(V \setminus S) \leq |S|$ for all finite subsets S of V .*

In the finite case, Theorem B is a well-known result, however we have been unable to find an explicit reference in the literature. The papers [2] and [3] can be given as implicit references.

4. Proof

Let $G = (V, E)$ be a graph with exactly one vertex v_0 of infinite degree.

(1) *If condition (1.1) holds for G and if G is not factorizable, then $G[V \setminus \{v_0\}]$ is factorizable.*

Since $G[V \setminus \{v_0\}]$ is locally finite, from Theorem A it is enough to prove that $C_1(V \setminus [S \cup \{v_0\}]) \leq |S|$ for all finite subset S of $V \setminus \{v_0\}$. Assume that there is a finite subset S of $V \setminus \{v_0\}$ such that

$$C_1(V \setminus [S \cup \{v_0\}]) \geq |S| + 1.$$

Since (1.1) holds for G we have $C_1(V \setminus [S \cup \{v_0\}]) = |S| + 1$. By Lemma 3.3 there is a finite subset T of V such that $S \cup \{v_0\} \subseteq T$ and $C_{cr}(V \setminus T) \geq |T|$. Since (1.1) holds for G we have $C_{cr}(V \setminus T) = |T|$, and every connected components of $G[V \setminus T]$ with odd cardinality is 1-factor critical.

On the other hand we prove that every connected component of $G[V \setminus T]$ with even or infinite cardinality is factorizable. Let C be such a component of $G[V \setminus T]$. Since v_0 belongs to T , $G[C]$ is locally finite. If $G[C]$ is not factorizable, from Theorem A there is a finite subset U of C such that $C_1(C \setminus U) \geq |U| + 1$. Therefore we have

$$C_1(V \setminus [T \cup U]) = C_1(V \setminus T) + C_1(G[C \setminus U]) \geq |S \cup T| + 1,$$

contradicting (1.1).

Since every connected component of $G[V \setminus T]$ with odd cardinality is 1-factor critical, the subgraph of G induced by T and the components of $G[V \setminus T]$ with odd cardinalities have a 1-factor. This 1-factor can be extended to a 1-factor of G , since the connected components of $G[V \setminus T]$ with even or infinite cardinalities are factorizable. The contradiction follows from the hypothesis that G is not factorizable, achieving the proof of (1).

(2) *If (1.1) holds for G and if G is not factorizable, then (1.2) does not hold for G .*

Let y be a vertex of $A(v_0)$. Put $G' = G[V \setminus \{v_0, y\}]$. Since G is not factorizable, G' is not factorizable. Since v_0 is not a vertex of G' , G' is locally finite and then from Theorem A there is a finite subset S of $V \setminus \{v_0, y\}$ such that $C_1(V \setminus [S \cup \{v_0, y\}]) \geq |S| + 1$. From (1) the subgraph $G[V \setminus \{v_0\}]$ is factorizable and then (1.1) holds for this subgraph. It follows that we have $C_1(V \setminus [S \cup \{v_0, y\}]) = |S| + 1$, i.e. $y \in \bigcup \{S \subseteq V \setminus \{v_0\} : S \text{ finite, } C_1(V \setminus [S \cup \{v_0\}]) = |S|\}$.

(3) *If (1.1) holds for G , then*

$$\begin{aligned} & \bigcup \{S \subseteq V \setminus \{v_0\} : S \text{ finite, } C_1(V \setminus [S \cup \{v_0\}]) = |S|\} \\ & \subseteq \bigcup \{S \subseteq V \setminus \{v_0\} : S \text{ finite, } C_{cr}(V \setminus [S \cup \{v_0\}]) = |S|\}. \end{aligned}$$

This results clearly from Lemma 3.3 with $k=0$.

(4) *If (1.1) holds for G and if (2.2) does not hold for G , then G is not factorizable.*

Assume that G has a 1-factor F . Then there is $y \in V$ such that $\{v_0, y\} \in F$. Since (2.2) does not hold for G , there is a finite subset S of $V \setminus \{v_0\}$ such that $y \in S$ and

$C_{cr}(V \setminus [S \cup \{v_0\}]) = |S|$. It follows that the subgraph $G[V \setminus \{v_0, y\}]$ does not satisfy (2.1), and then by Lemma 3.3 this subgraph does not satisfy (1.1). Therefore from Theorem A $G[V \setminus \{v_0, y\}]$ is not factorizable. On the other hand, since F is a 1-factor of G containing the edge $\{v_0, y\}$, $F \setminus \{v_0, y\}$ is a 1-factor of $G[V \setminus \{v_0, y\}]$, and the contradiction follows achieving the proof of (3).

(5) *If G is factorizable, then (1.1) holds for G .*

If G is factorizable and if S is a subset of V , every connected component of $G[V \setminus S]$ with odd cardinality is clearly joined to S by every 1-factor of G . Therefore (1.1) holds for G .

The proof of Theorem 3.1 and Corollary 3.2 is now complete. If G is factorizable, then by (4) and (5) conditions (1.1) and (2.2) hold. Therefore condition (1.2) holds by (3). From Lemma 3.3 with $k = 0$ it follows that condition (2.1) holds.

From (2), (3) and (4) it follows that conditions (1.1) and (1.2)—or conditions (2.1) and (2.2)—are sufficient for the existence of a 1-factor of G .

References

- [1] F. Bry and M. Las Vergnas, Matchings in locally finite graphs and Edmonds-Gallai decomposition, to appear.
- [2] J. Edmonds, Paths, trees and flowers, *Canad. J. Math.* 17 (1965) 449–467.
- [3] T. Gallai, Maximale Systeme unabhängiger Kanten, *Math. Kut. Int. Közl.* 9 (1964) 373–395.
- [4] R. Rado, Note on the transfinite case of Hall's theorem on representatives, *J. London Math. Soc.* 42 (1967) 321–324.
- [5] W.T. Tutte, The factorization of locally finite graphs, *Canad. J. Math.* 2 (1950) 44–49.